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On Morse – Smale 3-Diffeomorphisms with a Given Tuple of Sink Points Periods

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Abstract—In investigating dynamical systems with chaotic attractors, many aspects of global behavior of a flow or a diffeomorphism with such an attractor are studied by replacing a nontrivial attractor by a trivial one [1, 2, 11, 14]. Such a method allows one to reduce the original system to a regular system, for instance, of a Morse – Smale system, matched with it. In most cases, the possibility of such a substitution is justified by the existence of Morse – Smale diffeomorphisms with partially determined periodic data, the complete understanding of their dynamics and the topology of manifolds, on which they are defined. With this aim in mind, we consider Morse – Smale diffeomorphisms f with determined periods of the sink points, given on a closed smooth 3-manifold. We have shown that, if the total number of these sinks is k, then their nonwandering set consists of an even number of points which is at least $2k$. We have found necessary and sufficient conditions for the realizability of a set of sink periods in the minimal nonwandering set. We claim that such diffeomorphisms exist only on the 3-sphere and establish for them a sufficient condition for the existence of heteroclinic points. In addition, we prove that the Morse – Smale 3-diffeomorphism with an arbitrary set of sink periods can be implemented in the nonwandering set consisting of $2k + 2$ points. We claim that any such a diffeomorphism is supported by a lens space or the skew product $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$.

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1. INTRODUCTION AND FORMULATION OF RESULTS

In the present paper, we consider the class of Morse – Smale diffeomorphisms (MS-diffeomorphisms) $f : M^3 \to M^3$ defined on a smooth closed connected 3-manifold M^3. We denote this class by $MS(M^3)$. The results obtained here describe the properties of these diffeomorphisms under the condition that the periods of sink orbits are determined.

For any finite set A, we will denote by $|A|$ its cardinality. Let $r \in \mathbb{N}$,

$$\kappa = (k_1, \dots, k_r)$$

be an r-tuple of natural numbers such that $k_1 \leqslant \dots \leqslant k_r$ and

$$k = k_1 + \dots + k_r.$$

For any tuple κ, let MS_κ denote the set of MS-diffeomorphisms f whose tuple of sink periods is exactly κ. Let Ω_f denote the nonwandering set of f. The set

$$\Omega_q,\ q \in \{0, 1, 2, 3\}$$

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is defined as a subset of Ω_f, consisting of points with the *Morse index* q (it means that the unstable manifold of these points has the topological dimension q). Let

$$C_q = |\Omega_q|.$$

In this paper, we get the following results about diffeomorphisms from MS_κ.

Lemma 1. *For any $f \in MS_\kappa$ there exists an integer $n_f \geqslant 0$ such that*

$$|\Omega_f| = 2(k + n_f). \tag{1.1}$$

For any $n \in \mathbb{N} \cup \{0\}$, $MS_{\kappa,n}$ is defined as the subset (possibly empty) of MS_κ, consisting of diffeomorphisms f such that $|\Omega_f| = 2(k + n)$.

Lemma 2. *A diffeomorphism $f \in MS_\kappa$ belongs to $MS_{\kappa,n}$ iff $C_2 = n$.*

Since systems with a minimum number of points in a nonwandering set are of great scientific interest [1, 2], we will focus on considering the class $MS_{\kappa,0}$. The following statement gives us some necessary conditions for an MS-diffeomorphism to belong to $MS_{\kappa,0}$.

Lemma 3. *Let $f : M^3 \to M^3$ belong to $MS_{\kappa,0}$. Then*

1) *$C_3 = 1$, $C_1 = k - 1$;*

2) *M^3 is a 3-sphere $\mathbb{S}^3$.*

The next result shows that the set $MS_{\kappa,0}$ may be empty for some tuples κ.

Theorem 1. *The set $MS_{\kappa,0}$ is non-empty iff the tuple $\kappa = (k_1, \dots, k_r)$ has the following property:*

$$\exists\, i \in \{1, \dots, r\},\, m_i \in \mathbb{N} \cup \{0\} :\, k_i = 2^{m_i}. \tag{1.2}$$

Moreover, for any κ satisfying (1.2), the set $MS_{\kappa,0}$ contains both orientation-reversing and orientation-preserving diffeomorphisms.

A surprising consequence of this theorem is the following result.

Theorem 2. *Let κ satisfy (1.2). Then any diffeomorphism $f \in MS_{\kappa,0}$ has the following properties:*

a) if $k_1 \geqslant 2$, then Ω_f contains a fixed saddle point σ such that $f|_{W^u_\sigma}$ is orientation-reversing;

b) if $k_1 \geqslant 3$, then the wandering set of f contains heteroclinic points.

Below we show that, in contrast to the class $MS_{\kappa,0}$, every κ can be realized by some diffeomorphism $f \in MS_{\kappa,1}$. For this reason, we investigate the periodic data of diffeomorphisms from this class.

Lemma 4. *Let $f : M^3 \to M^3$ be in $MS_{\kappa,1}$. Then the periodic data of f has one of the following forms:*

1) *$C_0 = k$, $C_1 = k$, $C_2 = 1$, $C_3 = 1$;*

2) *$C_0 = k$, $C_1 = k - 1$, $C_2 = 1$, $C_3 = 2$.*

Theorem 3. *For any κ, the set $MS_{\kappa,1}$ is never empty and contains both types of diffeomorphisms: given on orientable and non-orientable 3-manifolds.*

According to Lemma 2, the nonwandering set of $f \in MS_{\kappa,1}$ has a unique saddle point σ with a one-dimensional stable manifold. In the general case, the set $\mathrm{cl}(W^s_\sigma)$ is not a submanifold of the ambient manifold M^3 (see, for example, [7]). In the case where it is, the following result takes place.

Theorem 4. *Let a diffeomorphism $f : M^3 \to M^3$ be in $MS_{\kappa,1}$ and $\mathrm{cl}(W^s_\sigma)$, $\sigma \in \Omega_2$ be a submanifold of M^3. Then M^3 is homeomorphic either to a lens space or to the skew product $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$.*[1)]

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1) The skew product $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$ is defined as the space obtained from $\mathbb{S}^2 \times [0, 1]$ by gluing its boundary spheres via an orientation-reversing homeomorphism.

2. NECESSARY DEFINITIONS AND FACTS

2.1. Orbit Spaces

This part is devoted to the topology of the orbit space for some homeomorphism f defined on the topological manifold X. In presenting the material, we mainly follow the monograph [13].

Let $f: X \to X$ be a homeomorphism of a topological space X. Denote by

$$\mathcal{Z}_f$$

the cyclic (finite or infinite) group of integers k such that $\{f^k,\, k \in \mathcal{Z}_f\}$ is the set of all different powers of the homeomorphism f. We use the notation

$$X/f$$

for the space of f-orbits on X and

$$p_{X/f}: X \to X/f$$

for the natural projection equipped the space X/f with the quotient topology. For each connected component $\hat{X}$ of X/f, let us denote by $m_{\hat{X}}$ the number of connected components in the preimage $p_{X/f}^{-1}(\hat{X})$.

Recall that *a fundamental domain of the f-action on X* is a closed set $D_f \subset X$ such that there is a subset $\tilde{D}_f \subset D_f$ with the following properties:

1) $\mathrm{cl}(\tilde{D}_f) = D_f$;

2) $f^k(\tilde{D}_f) \cap \tilde{D}_f = \varnothing$ for all $k \in (\mathcal{Z}_f \setminus \{0\})$;

3) $\bigcup\limits_{k\in\mathcal{Z}_f} f^k(\tilde{D}_f) = X$.

Denote by $D_f/_{\sim_f}$ the quotient space under the smallest equivalence relation $\sim_f$ on D_f for which $x \sim_f f(x)$.

It is said that

- a homeomorphism f *acts freely* on X if $f^k(x) \neq x$ for any point $x \in X$ and any element $k \in (\mathcal{Z}_f \setminus \{0\})$;
- a homeomorphism f *acts discontinuously* on X if for any compact $K \subset X$ the set of elements $k \in \mathcal{Z}_f$ for which $f^k(K) \cap K \neq \varnothing$ is finite.

In the case of free and discontinuous action, the orbit space X/f and the projection $p_{X/f}$ have the following properties.

Proposition 1 ([13]). *Let a homeomorphism* (*diffeomorphism*) *f act freely and discontinuously on a topological* (*smooth*) *n-manifold X. Then*

1) *$p_{X/f}: X \to X/f$ is a cover, inducing the structure of a topological* (*smooth*) *n-manifold on the orbit space X/f;*

2) *for a fundamental domain D_f of the f-action on X, the spaces $D_f/_{\sim_f}$ and X/f are homeomorphic;*

3) *for each connected component $\hat{X}$ of X/f, the map*

$$\eta_{\hat{X}}: \pi_1(\hat{X}) \to m_{\hat{X}}\mathcal{Z}_f,$$

assigning to each class $[\hat{c}] \in \pi_1(\hat{X})$ the integer $k \in \mathcal{Z}_f$ such that the lift c of a loop $\hat{c}$ connects the point x to the point $f^k(x)$, is an epimorphism.

Let us denote by $\eta_{X/f}$ the map made up of epimorphisms $\eta_{\hat{X}}$ (from the Cartesian product of fundamental groups to the Cartesian product of the corresponding subgroups of $\mathbb{Z}$).

Proposition 2 ([13]). *Let homeomorphisms f and f' act freely and discontinuously on manifolds X and X', respectively, and let X/f, X'/f' be connected. Then*

1) *if $h : X \to X'$ is a homeomorphism such that $hf = f'h$, then the map $\hat{h} : X/f \to X'/f'$ defined by*

$$\hat{h}p_{X/f} = p_{X'/f'}h \tag{2.1}$$

is a homeomorphism and $\eta_{X/f} = \eta_{X'/f'}\hat{h}_*$;

2) *if $\hat{h} : X/f \to X'/f'$ is a homeomorphism such that $\eta_{X/f} = \eta_{X'/f'}\hat{h}_*$, then equality (2.1) defines a homeomorphism $h : X \to X'$ for which $hf = f'h$, and this homeomorphism is unique up to the choice of a pair of points $x \in X$, $h(x) \in X'$ in each connected component of X, X'.*

2.2. Tori and Klein Bottles Embeddings in $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$

First of all, we say that the character "≅" means "isomorphic", "homeomorphic", or "diffeomorphic", depending on the context.

Let X be a topological space; let $A \subset X$ be its subset. We denote by

$$j_A : A \to X$$

the inclusion map.

Consider two topological spaces, X and Y. Let $A \subset X$ and $B \subset Y$ be their subsets, let $g : A \to B$ be some homeomorphism, and let $\sim$ be the minimal equivalence relation on $X \sqcup Y$ for which $a \sim g(a)$ for all $a \in A$. The quotient space by this equivalence relation is said to be obtained by *gluing* the space Y to the space X along the map g, written

$$X \cup_g Y.$$

A topological (*smooth*) *embedding* of a topological (smooth) manifold X into a topological (smooth) manifold Y is defined as a map $e : X \to Y$ such that $e : X \to e(X)$ is a homeomorphism (diffeomorphism).

For example, *a knot* in Y is the image $e(\mathbb{S}^1)$ of an embedding $e : \mathbb{S}^1 \to Y$ (or this embedding itself).

A topological embedding $e : X \to Y$ of an m-manifold X into an n-manifold Y ($m \leqslant n$) is said to be *locally flat at point $e(x)$, $x \in X$,* if the point $e(x)$ lies in the domain of a chart (U, ψ) of the manifold Y such that either $\psi\big(U \cap e(X)\big) = \mathbb{R}^m$ (here $\mathbb{R}^m \subset \mathbb{R}^n$ is the set of those points, whose last $n - m$ coordinates are 0) or $\psi\big(U \cap e(X)\big) = \mathbb{R}^m_+$ (here $\mathbb{R}^m_+ \subset \mathbb{R}^m$ is the set of points with nonnegative last coordinate). The embedding e is said to be *tame* and the manifold X is said to be *tamely embedded* into Y if e is locally flat at all points of $e(X)$, and $Z = e(X)$ is called *an m-submanifold* of Y. Otherwise, the embedding e is called *wild* and the manifold X is called *wildly embedded*. The point $e(x)$, at which e is not locally flat, is called *a point of wildness*.

A tubular neighborhood of an m-submanifold Z of an n-manifold Y is a submanifold $N(Z) \subset Y$ containing Z and equipped with a continuous map $p : N(Z) \to Z$ such that for any point $z \in Z$ there exist a neighborhood $U \ni z$ and a diffeomorphism $\phi_U : U \times \mathbb{D}^{n-m} \to p^{-1}(U)$ for which $p\phi_U(u, t) = u$ (see Fig. 1). The existence of a tubular neighborhood of any smooth submanifold $Z \subset \operatorname{int} Y$ follows from [9], for example.

An $(n - 1)$-submanifold Z of an n-manifold Y is said to be *two-sided* if its tubular neighborhood is homeomorphic to $Z \times [-1, 1]$ and *one-sided*, otherwise.

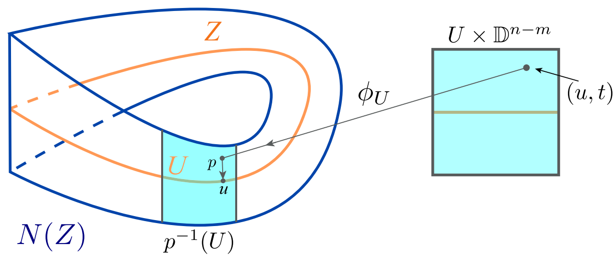

Fig. 1. A tubular neighborhood $N(Z)$ of a submanifold Z.

Proposition 3 ([10, Lemma 3], [9]). *Let X be an n-manifold with nonempty boundary ∂X and let $g : \partial X \to \partial X \times \{0\}$ be a homeomorphism. Then*

$$X \cup_g (\partial X \times [0,1]) \cong X.$$

For $q \in \mathbb{N}$, we consider

$$\mathbb{D}^q = \{(x_1, \ldots, x_q) \in \mathbb{R}^q : \ x_1^2 + \ldots, +x_q^2 \leqslant 1\} \ - \ \text{a}\, q\text{-ball (disc)},$$

$$\mathbb{S}^{q-1} = \partial \mathbb{D}^q \ - \ \text{a}\,(q-1)\text{-sphere}$$

and for $\nu \in \{-,+\}$, we define the involution $\iota_{q,\nu} : \mathbb{D}^q \to \mathbb{D}^q$ as follows:

$$\iota_{q,\nu}(x_1, x_2, \ldots, x_q) = (\nu x_1, x_2, \ldots, x_q). \tag{2.2}$$

Identifying the boundary disks $\mathbb{D}^q \times \{0\}$, $\mathbb{D}^q \times \{1\}$ of $\mathbb{D}^q \times [0,1]$ by the relation $(d,1) \sim (\iota_{q,\nu}(d), 0)$, we obtain the manifold

$$(\mathbb{D}^q \times \mathbb{S}^1)_\nu$$

with the boundary

$$(\mathbb{S}^{q-1} \times \mathbb{S}^1)_\nu,$$

which is

- the direct product $\mathbb{D}^q \times \mathbb{S}^1$ with the boundary $\mathbb{S}^{q-1} \times \mathbb{S}^1$, if $\nu = +$;
- the skew product $\mathbb{D}^q \tilde{\times} \mathbb{S}^1$ with the boundary $\mathbb{S}^{q-1} \tilde{\times} \mathbb{S}^1$, if $\nu = -$.

For some natural number $n > q$ and a sign $\mu \in \{-,+\}$, let us consider the n-manifold $\mathbb{S}^{q-1} \times \mathbb{D}^{n-q} \times [0,1]$. After identifying $(s,d,1) \sim (\iota_{q,\nu}(s), \iota_{n-q,\mu}(d), 0)$, we get the n-manifold

$$(\mathbb{S}^{q-1} \times \mathbb{D}^{n-q} \times \mathbb{S}^1)_{\nu,\mu}.$$

Here is the notation list for the manifolds most commonly used throughout this article:

- 2-torus $\mathbb{T}_+ = \mathbb{S}^1 \times \mathbb{S}^1$;
- Klein bottle $\mathbb{T}_- = \mathbb{S}^1 \tilde{\times} \mathbb{S}^1$;
- $N_\mu(\mathbb{T}_\nu) = (\mathbb{S}^1 \times \mathbb{D}^1 \times \mathbb{S}^1)_{\nu,\mu}$, here
  - $N_+(\mathbb{T}_\nu) = \mathbb{T}_\nu \times [-1,1]$ is a tubular neighborhood of a two-sided $\mathbb{T}_\nu$,
  - $N_-(\mathbb{T}_+)$ is a tubular neighborhood of a one-sided torus,
  - $N_-(\mathbb{T}_-)$ is a tubular neighborhood of a one-sided Klein bottle;
- solid torus $\mathbb{W}_+ = \mathbb{D}^2 \times \mathbb{S}^1$;

- solid Klein bottle $\mathbb{W}_- = \mathbb{D}^2 \tilde{\times} \mathbb{S}^1$;
- $\mathbb{P}_+ = \mathbb{S}^2 \times \mathbb{S}^1$;
- $\mathbb{P}_- = \mathbb{S}^2 \tilde{\times} \mathbb{S}^1$.

Let us recall that *a meridian* of a solid torus (Klein bottle) $\mathbb{W}_+$ ($\mathbb{W}_-$) is such a knot that is noncontractible on its boundary $\mathbb{T}_+$ ($\mathbb{T}_-$), but is contractible in the solid torus (Klein bottle) itself.

Proposition 4 ([3, 21]). *Let $g : \partial\mathbb{W}_+ \to \partial\mathbb{W}_+$ be a homeomorphism. Then*

- *g can be extended to a homeomorphism $g : \mathbb{W}_+ \to \mathbb{W}_+ \iff g$ maps a meridian into a meridian;*
- *$\mathbb{W}_+ \cup_g \mathbb{W}_+ \cong \mathbb{P}_+ \iff g$ maps a meridian into a meridian.*

The 3-manifold $L_{p,q} = \mathbb{W}_+ \cup_g \mathbb{W}_+$, where $g : \partial\mathbb{W}_+ \to \partial\mathbb{W}_+$ is a homeomorphism, mapping a meridian of the solid torus into some knot with the homotopy type $\langle p, q\rangle$, is called *a lens space*. In particular,

$$\mathbb{S}^3 \cong L_{1,0},\ \mathbb{S}^2 \times \mathbb{S}^1 \cong L_{0,1}.$$

Proposition 5 ([16, 17]). *Let $g : \partial\mathbb{W}_- \to \partial\mathbb{W}_-$ be a homeomorphism. Then*

- *g always maps a meridian into a meridian;*
- *g can always be extended to a homeomorphism $g : \mathbb{W}_- \to \mathbb{W}_-$;*
- *$\mathbb{W}_- \cup_g \mathbb{W}_- \cong \mathbb{P}_-$.*

Let us elaborate on the properties of embeddings $e : \mathbb{T}_\nu \to \mathbb{P}_\pm$.

Note that $\pi_1(\mathbb{P}_\pm) \cong \mathbb{Z}$, so we will say that a connected subset $A \subset \mathbb{P}_\pm$ is *homotopically nontrivial* if $j_{A*}\big(\pi_1(A)\big) \neq 0$. In this case, there is a natural $m \in \mathbb{N}$ such that

$$j_{A*}\big(\pi_1(A)\big) = m\mathbb{Z}. \tag{2.3}$$

We call some set *m-turning* if it has property (2.3).

Proposition 6 ([4, Theorem 4]). *Every m-turning torus $T_+ \subset \mathbb{P}_+$ bounds an m-turning solid torus.*

To describe a 1-turning Klein bottle in $\mathbb{P}_+$, we consider a rotation $\mathcal{R}_{AB}(t)$ of the 2-sphere $\mathbb{S}^2$ by the angle πt around the axis AB, here $A = (0, 1, 0)$ and $B = (0, -1, 0)$. We choose the biggest circle S_0 on $\mathbb{S}^2$, passing through A and B. Then the sought Klein bottle (see Fig. 2) can be defined, for example, by the formula

$$\mathcal{K} = \bigcup_{t\in[0,1]} \mathcal{R}_{AB}(S_0) \times \{e^{i2\pi t}\}.$$

By construction, a tubular neighborhood $N(\mathcal{K})$ of $\mathcal{K}$ is diffeomorphic to $N_-(\mathbb{T}_-)$ and the manifold $\mathcal{Y} = \mathbb{P}_+ \setminus \operatorname{int} N(\mathcal{K})$ is a 2-turning solid torus (see Fig. 2).

Proposition 7 ([6, Proposition 1.4]). *For every Klein bottle $T_- \subset \mathbb{P}_+$ there exists a homeomorphism $h : \mathbb{P}_+ \to \mathbb{P}_+$ such that $h(T_-) = \mathcal{K}$. Moreover, any two Klein bottles in $\mathbb{P}_+$ have a nonempty intersection.*

An immediate consequence of Propositions 6 and 7 is the following fact.

Proposition 8. *Let $T_\nu \subset \mathbb{P}_+$ be a homotopically nontrivial torus or Klein bottle. Then T_ν is one of the following types:*

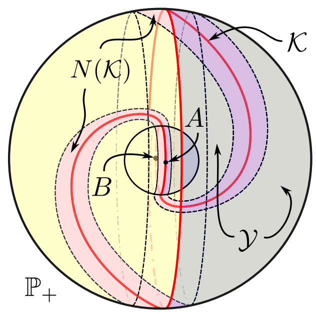

Fig. 2. Klein bottle $\mathcal{K}$ in $\mathbb{P}_+$.

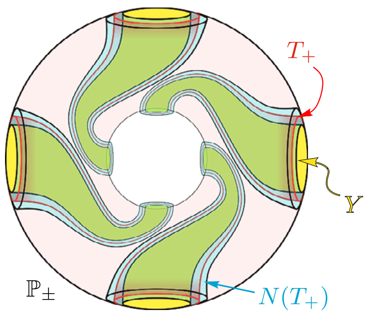

Fig. 3. 4-turning two-sided torus $T_+ \subset \mathbb{P}_\pm$.

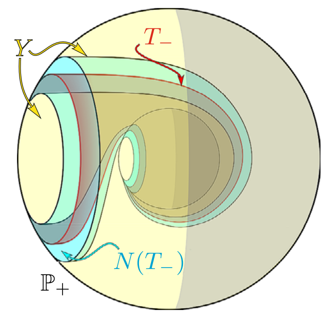

Fig. 4. Klein bottle $T_- \subset \mathbb{P}_+$.

1) *T_+ is a two-sided m-turning ($m \in \mathbb{N}$) torus and the manifold $\mathbb{P}_+ \setminus \operatorname{int} N(T_+)$ consists of two connected components, at least one of which, Y, is an m-turning solid torus* (*see Fig.* 3)*;*
2) *T_- is a one-sided 1-turning Klein bottle, $N(T_-) \cong N_-(\mathbb{T}_-)$ and $Y = \mathbb{P}_+ \setminus \operatorname{int} N(T_-)$ is a 2-turning solid torus* (*see Fig.* 4)*.*

We do not know of any generalization of this result to $\mathbb{P}_-$, so we provide our proof here.

Lemma 5. *Let $T_\nu \subset \mathbb{P}_-$ be a homotopically nontrivial torus or Klein bottle. Then T_ν is one of the following types:*

1) *T_+ is a two-sided $2k$-turning ($k \in \mathbb{N}$) torus and the manifold $\mathbb{P}_- \setminus \operatorname{int} N(T_+)$ consists of two connected components, exactly one of which Y is a $2k$-turning solid torus* (*see Fig.* 3)*;*
2) *T_+ is a one-sided 1-turning torus, $N(T_+) \cong N_-(\mathbb{T}_+)$ and $Y = \mathbb{P}_- \setminus \operatorname{int} N(T_+)$ is a 2-turning solid torus* (*see Fig.* 6)*;*
3) *T_- is a two-sided $(2k-1)$-turning ($k \in \mathbb{N}$) Klein bottle and the manifold $\mathbb{P}_- \setminus \operatorname{int} N(T_-)$ consists of two connected components, at least one of which, Y, is a $(2k-1)$-turning solid Klein bottle* (*see Fig.* 5)*.*

Proof. Let us represent the manifold $\mathbb{P}_-$ as the orbit space

$$\mathbb{P}_- = (\mathbb{S}^2 \times \mathbb{R})/\mathcal{T}$$

of the action of a diffeomorphism $\mathcal{T} : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}$ defined (see formula (2.2) for $\iota_{3,-}$) as follows:

$$\mathcal{T}(s,t) = \left(\iota_{3,-}(s), t + \frac{1}{2}\right).$$

Then

$$\mathbb{P}_+ = (\mathbb{S}^2 \times \mathbb{R})/\mathcal{T}^2.$$

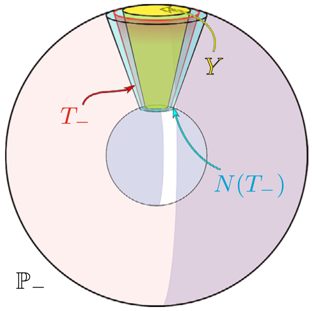

Fig. 5. Two-sided 1-turning Klein bottle $T_- \subset \mathbb{P}_-$.

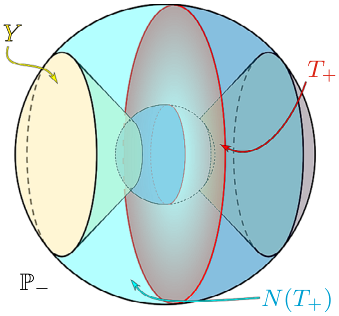

Fig. 6. One-sided torus $T_+ \subset \mathbb{P}_-$.

In this case, the mapping

$$\tau = p_{\mathbb{P}_+} \mathcal{T} p_{\mathbb{P}_+}^{-1} : \mathbb{P}_+ \to \mathbb{P}_+ \tag{2.4}$$

is an involution such that $\mathbb{P}_- = \mathbb{P}_+/\tau$, and its corresponding natural projection

$$r = p_{\mathbb{P}_-} p_{\mathbb{P}_+}^{-1} : \mathbb{P}_+ \to \mathbb{P}_- \tag{2.5}$$

is a two-fold cover. In addition, we note that

$$T_\nu \subset \mathbb{P}_\pm \text{ is an } m\text{-turning surface } \iff p_{\mathbb{P}_\pm}^{-1}(T_\nu) \text{ consists of } m \text{ connected components.} \tag{2.6}$$

Let $T_+ \subset \mathbb{P}_-$ be an m-turning torus. Then it follows from formulas (2.4)–(2.6) that $r^{-1}(T_+)$ consists of either i) one m-turning torus $\tilde{T}$ if m is odd or ii) two $\frac{m}{2}$-turning tori $\tilde{T}, \tau(\tilde{T})$ if m is even. Due to Proposition 8, there is a two-sheeted cover of a tubular neighborhood $N(T_+) \subset \mathbb{P}_-$ by one or two copies of $N_+(\mathbb{T}_+)$, respectively, for each of the cases.

In case i), the torus $\tilde{T}$ bounds a solid torus in $\mathbb{P}_+$ by virtue of Proposition 6. If $\tilde{T}$ bounds two solid tori and the involution τ maps one of them into the other, then, due to Proposition 4, $m = 1$. Therefore, $N(T_+) \cong N_-(\mathbb{T}_+)$ and $Y = \mathbb{P}_- \setminus \operatorname{int} N(T_+)$ is a 2-turning solid torus. Thus, we have obtained case 2) of this lemma.

Otherwise, keeping in mind that $\tilde{T}$ bounds a solid torus, we conclude that every component $\bar{T}$ of $p_{\mathbb{P}_-}^{-1}(T_+)$ bounds a solid cylinder, not intersecting with the other components. It follows from the definition of an m-turning surface that there is a compact cylinder $\bar{T}_m \subset \bar{T}$, a lift of $\tilde{T}$, such that $p_{\mathbb{P}_-}(\bar{T}_m) = T_+$ and one of its boundary circles maps into the other under $\mathcal{T}^m$, the mth power of $\mathcal{T}$. Since m is odd, we immediately get that T_+ is a Klein bottle, which is a contradiction. Thus, there are no $(2k-1)$-turning tori in $\mathbb{P}_-$.

In case ii), each of the tori $\tilde{T}, \tau(\tilde{T})$ bounds one solid torus and the involution τ maps one solid torus to another. Then $N(T_+) \cong N_+(\mathbb{T}_+)$ and the manifold $\mathbb{P}_- \setminus \operatorname{int} N(T_+)$ consists of two connected components, exactly one of which Y is an m-turning solid torus. Thus, case 1) is realized.

Let $T_- \subset \mathbb{P}_-$ be an m-turning Klein bottle. Since a two-fold cover of a Klein bottle can be done only using either torus or two distinct Klein bottles as the covering space, then, due to the uniqueness of Klein bottle in $\mathbb{P}_+$ (see Proposition 7) and formulas (2.4)–(2.6), the manifold $r^{-1}(T_-)$ is one m-turning torus $\tilde{T}$. It follows directly from this that m is odd, that is, $m = 2k-1$, $k \in \mathbb{N}$. By virtue of Proposition 6, $\tilde{T}$ bounds a solid torus in $\mathbb{P}_+$ which is invariant under involution τ. Therefore, $N(T_-) \cong N_+(\mathbb{T}_-)$ and the manifold $\mathbb{P}_- \setminus \operatorname{int} N(T_-)$ consists of two connected components, at least one of which, Y, is a $(2k-1)$-turning solid Klein bottle. □

2.3. Morse – Smale Diffeomorphisms

In this section, we will remind the reader of the general information about Morse – Smale diffeomorphisms (see, for example, [12] for more details). Here and below, M^n is a closed connected n-manifold with a metric d; $f : M^n \to M^n$ is a diffeomorphism.

A point $x \in M^n$ is said to be *wandering* for f if there is an open neighborhood U_x of x such that $f^k(U_x) \cap U_x = \varnothing$ $\forall k \in \mathbb{N}$. Otherwise, the point x is said to be *nonwandering*. The set of all nonwandering points of f is called the *nonwandering set*, usually denoted by Ω_f. The simplest example of a nonwandering point is a *periodic point*, i. e., a point $p \in M^n$ such that there exists some natural number m for which $f^m(p) = p$. The smallest such number is said to be the *period* of the point p; written as m_p.

A periodic point $p \in \Omega_f$ with a period m_p is said to be *hyperbolic* if the absolute value of each eigenvalue of the Jacobi matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)\Big|_p$ is not equal to 1. The number q_p of the eigenvalues of the Jacobi matrix, whose absolute values are greater than 1, is called the *Morse index* of a point p. Points with the Morse index n (0) are called *sources* (*sinks*), the other points are called *saddles*. The sets

$$W^s_p = \{x \in M^n : \lim_{k\to+\infty} d(f^{km_p}(x), p) = 0\},$$

$$W^u_p = \{x \in M^n : \lim_{k\to-\infty} d(f^{km_p}(x), p) = 0\}$$

are called the *stable* and the *unstable manifolds* of a point p, respectively. The sets W^s_p, W^u_p are also called *invariant manifolds* of a point p.

For an f-invariant closed subset B of Ω_f we define the sets W^u_B, W^s_B as follows:

$$W^u_B = \bigcup_{p\in B} W^u_p, \quad W^s_B = \bigcup_{p\in B} W^s_p.$$

If p and r are hyperbolic periodic points of f with $W^u_p \cap W^s_r \neq \varnothing$, then the intersection $W^u_p \cap W^s_r$ is called a *heteroclinic intersection*. 0-dimensional path-connected components of heteroclinic intersection are called *heteroclinic points*, 1-dimensional components are called *heteroclinic curves*, and components of higher dimension are called *heteroclinic manifolds*.

Some diffeomorphism $f : M^n \to M^n$ is called a *Morse – Smale diffeomorphism* ($f \in MS(M^n)$) if 1) its nonwandering set Ω_f is finite (thus consists of periodic points) and hyperbolic and 2) for any two periodic points p and r the unstable manifold W^u_p intersects the stable manifold W^s_r transversally. Here and below, $f \in MS(M^n)$.

A compact f-invariant set $A \subset M^n$ is called an *attractor* of a diffeomorphism f if it has a compact neighborhood U_A such that

$$f(U_A) \subset \operatorname{int} U_A, \ A = \bigcap_{k\geqslant 0} f^k(U_A).$$

In this definition, the neighborhood U_A is said to be *trapping*. A *repeller* R of f is defined to be an attractor for f^{-1}.

For any hyperbolic periodic point $p \in \Omega_f$ of $f \in MS(M^n)$ we use the following notation:

- m_p is the period of p;
- q_p is the Morse index of p;
- (ν_p, μ_p) is the orientation type of p, i. e., $\nu_p = +$ $(-)$ if the map $f^{m_p}|_{W^u_p}$ preserves (reverses) orientation and $\mu_p = +$ $(-)$ if the map $f^{m_p}|_{W^s_p}$ preserves (reverses) orientation;
- ℓ^u_p (ℓ^s_p) is a path-connected component of the set $W^u_p \setminus p$ ($W^s_p \setminus p$) which is called an unstable (stable) *separatrix* of p;

- $\mathcal{O}_p$ is the orbit of p for which we set $m_{\mathcal{O}_p} = m_p$, $q_{\mathcal{O}_p} = q_p$, $\nu_{\mathcal{O}_p} = \nu_p$, $\mu_{\mathcal{O}_p} = \mu_p$.

Proposition 9 ([13, Theorem 2.1, Proposition 2.3]). *Let $f \in MS(M^n)$. Then*

1) $M^n = \bigcup\limits_{p\in\Omega_f} W^u_p$;

2) *for any $p \in \Omega_f$, W^u_p is a smooth submanifold of M^n and is diffeomorphic to $\mathbb{R}^{q_p}$;*

3) $cl(\ell^u_p) \setminus (\ell^u_p \cup p) = \bigcup\limits_{r\in\Omega_f : \ell^u_p \cap W^s_r \neq \varnothing} W^u_r$ *for any unstable separatrix ℓ^u_p of $p \in \Omega_f$;*

4) *if the separatrix ℓ^u_p does not take part in heteroclinic intersections, then there is a unique sink ω such that*
$$cl(\ell^u_p) = p \cup \ell^u_p \cup \omega,$$
with $cl(\ell^u_p)$ being homeomorphic to either a closed interval if $q_p = 1$ or the sphere $\mathbb{S}^{q_p}$ if $q_p > 1$.

The same proposition holds for stable manifolds.

Let us denote by
$$\Omega_q,\ q \in \{0, \ldots, n\}$$
the subset of the nonwandering set Ω_f of f, consisting of points with the Morse index q. Let $C_q = |\Omega_q|$. The symbol $\beta_q = \beta_q(M^n)$ denotes the *qth Betti number*, i. e.,
$$\beta_q(M^n) = \operatorname{rank} H_q(M^n, \mathbb{Z}).$$
By $\chi(M^n)$ we denote the Euler characteristic of M^n, i. e.,
$$\sum_{q=0}^{n} (-1)^q \beta_q = \chi(M^n).$$

Proposition 10 (Lefschetz – Hopf Theorem, [23, 24]). *For any $f \in MS(M^n)$, the following relations hold:*
$$C_0 \geqslant \beta_0, \quad C_1 - C_0 \geqslant \beta_1 - \beta_0, \quad C_2 - C_1 + C_0 \geqslant \beta_2 - \beta_1 + \beta_0, \quad \ldots,$$
$$\sum_{q=0}^{n} (-1)^q C_q = \chi(M^n).$$

Since M^n is connected, $\beta_0 = 1$. All other Betti numbers are nonnegative, and if n is odd, then $\chi(M^n) = 0$. Given this, the immediate consequence of the Proposition 10 is the following fact.

Proposition 11. *For any diffeomorphism $f \in MS(M^n)$, there are the following estimates from below:*
$$C_0 \geqslant 1,\ C_n \geqslant 1; \tag{2.7}$$
$$|\Omega_f| \geqslant 2C_0; \tag{2.8}$$
$$C_0 + C_2 + \cdots + C_{n-1} = C_1 + C_3 + \cdots + C_n, \text{ for an odd } n. \tag{2.9}$$

Proposition 12 ([18, Theorem 1]). *Let $f \in MS(M^n)$, $n \geqslant 3$, and let all its saddle points be of the Morse index 1. Then $M^n \cong \mathbb{S}^n$.*

2.4. Linearizing Neighborhood

In this section, following [7, 8, 12], we show how to reduce the study of local dynamics of Morse–Smale diffeomorphisms $f \in MS(M^n)$ to the study of linear systems in $\mathbb{R}^n$.

Let us fix $q \in \{0, 1, \dots, n\}$, $\nu, \mu \in \{-, +\}$ and denote by $a_{q,\nu,\mu} : \mathbb{R}^n \to \mathbb{R}^n$ the diffeomorphism defined as

$$a_{q,\nu,\mu}(x_1, x_2, \dots, x_n) = \left(\nu 2x_1, 2x_2, \dots, 2x_q, \mu\frac{x_{q+1}}{2}, \frac{x_{q+2}}{2}, \dots, \frac{x_n}{2}\right).$$

We call $a_{q,\nu,\mu}$ the *canonical diffeomorphism*. The stable $\mathcal{W}^u_q$ and the unstable $\mathcal{W}^s_q$ manifolds of its unique fixed point O are the coordinate hyperplanes $O_{x_1 \dots x_q}$ and $O_{x_{q+1} \dots x_n}$, respectively. The restrictions $a^u_{q,\nu} = a_{q,\nu,\mu}|_{\mathcal{W}^u_q}$ and $a^s_{q,\mu} = a_{q,\nu,\mu}|_{\mathcal{W}^s_q}$ are called *the canonical expansion and contraction*, respectively, i. e.,

$$a^u_{q,\nu}(x_1, \dots, x_q) = (\nu 2x_1, \dots, 2x_q), \ \ \forall (x_1, \dots, x_n) \in \mathcal{W}^u_q,$$

$$a^s_{q,\mu}(x_{q+1}, \dots, x_n) = \left(\mu\frac{x_{q+1}}{2}, \dots, \frac{x_n}{2}\right), \ \ \forall (x_1, \dots, x_n) \in \mathcal{W}^s_q.$$

One can check directly that the diffeomorphism $a^u_{q,\nu}$ acts freely and discontinuously on the smooth q-manifold $\mathcal{W}^u_q \setminus O$. In order to identify the topology of the manifold

$$\hat{\mathcal{W}}^u_{q,\nu} = (\mathcal{W}^u_q \setminus O)/a^u_{q,\nu},$$

one can note that a fundamental domain of the $a^u_{q,\nu}$-action on $\mathcal{W}^u_q \setminus O$ can be chosen to be homeomorphic to

$$\mathbb{S}^{q-1} \times [0, 1].$$

Then, using the notation from Section 2.2, we obtain from Propositions 1, 2 the following result (see Figs. 7, 8).

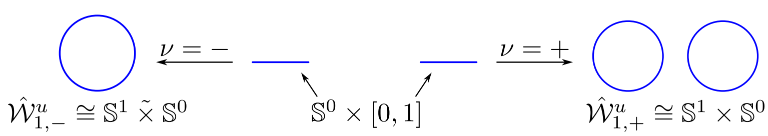

Fig. 7. Manifolds $\hat{\mathcal{W}}^u_{1,-}$, $\hat{\mathcal{W}}^u_{1,+}$.

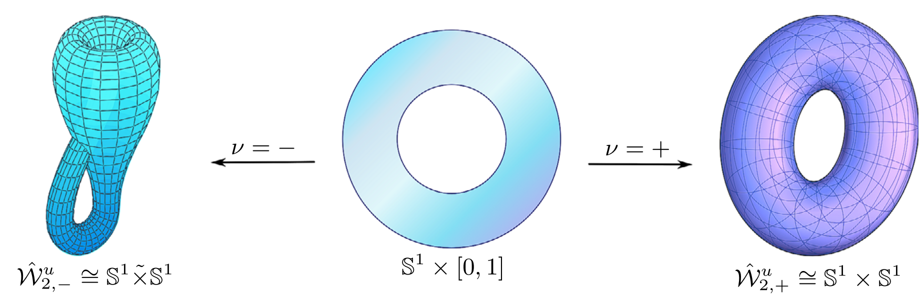

Fig. 8. Manifolds $\hat{\mathcal{W}}^u_{2,-}$, $\hat{\mathcal{W}}^u_{2,+}$.

Proposition 13 ([13, Proposition 2.5]). *The orbit space $\hat{\mathcal{W}}^u_{q,\nu}$ is a smooth closed q-manifold. Moreover,*

- $\hat{\mathcal{W}}^u_{q,\nu} \cong (\mathbb{S}^{q-1} \times \mathbb{S}^1)_\nu$*;*
- $\eta_{\hat{\mathcal{W}}^u_{q,\nu}}\big(\pi_1(\hat{\mathcal{W}}^u_{q,\nu})\big) = 2\mathbb{Z}$ *if* $q = 1$, $\nu = -$ *and* $\eta_{\hat{\mathcal{W}}^u_{q,\nu}}\big(\pi_1(\hat{\mathcal{W}}^u_{q,\nu})\big) = \mathbb{Z}$ *for all other* q, ν.

The same is true for the orbit space $\hat{\mathcal{W}}^s_{q,\mu} = (\mathcal{W}^s_q \setminus O)/a^s_{q,\mu}$, since

$$\hat{\mathcal{W}}^s_{q,\mu} \cong \hat{\mathcal{W}}^u_{n-q,\mu}.$$

For $q \in \{1, \ldots, n-1\}$, we assume

$$\mathcal{N}_q = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1^2 + \cdots + x_q^2)(x_{q+1}^2 + \cdots + x_n^2) \leqslant 1\}.$$

It can be checked directly that $\mathcal{N}_q$ is an $a_{q,\nu,\mu}$-invariant set which is a closed neighborhood for both $\mathcal{W}^s_q$ and $\mathcal{W}^u_q$. Let

$$\mathcal{N}^u_q = \mathcal{N}_q \setminus \mathcal{W}^s_q,\, \mathcal{N}^s_q = \mathcal{N}_q \setminus \mathcal{W}^u_q.$$

It is readily verified that $a_{q,\nu,\mu}$ acts freely and discontinuously on the smooth n-manifold $\mathcal{N}^u_q$ with boundary, producing the orbit space

$$\hat{\mathcal{N}}^u_{q,\nu,\mu} = \mathcal{N}^u_q / a_{q,\nu,\mu}$$

with the fundamental domain (see an example of this in Fig. 9) homeomorphic to

$$\mathbb{S}^{q-1} \times \mathbb{D}^{n-q} \times [0,1].$$

Again, using the notation from Section (2.2), due to Propositions 1, 2, we obtain the following result.

Proposition 14. *The orbit space $\hat{\mathcal{N}}^u_{q,\nu,\mu}$ is a smooth n-manifold with boundary. Moreover,*

- $\hat{\mathcal{N}}^u_{q,\nu,\mu} \cong (\mathbb{S}^{q-1} \times \mathbb{D}^{n-q} \times \mathbb{S}^1)_{\nu,\mu}$;
- $\eta_{\hat{\mathcal{N}}^u_{q,\nu,\mu}}\big(\pi_1(\hat{\mathcal{N}}^u_{q,\nu,\mu})\big) = 2\mathbb{Z}$ *if* $q = 1$, $\nu = -$ *and* $\eta_{\hat{\mathcal{N}}^u_{q,\nu,\mu}}\big(\pi_1(\hat{\mathcal{N}}^u_{q,\nu,\mu})\big) = \mathbb{Z}$ *for all other* q, ν.

The same is true for the orbit space $\hat{\mathcal{N}}^s_{q,\nu,\mu} = \mathcal{N}^s_q / a_{q,\nu,\mu}$, since

$$\hat{\mathcal{N}}^s_{q,\nu,\mu} \cong \hat{\mathcal{N}}^u_{n-q,\mu,\nu}.$$

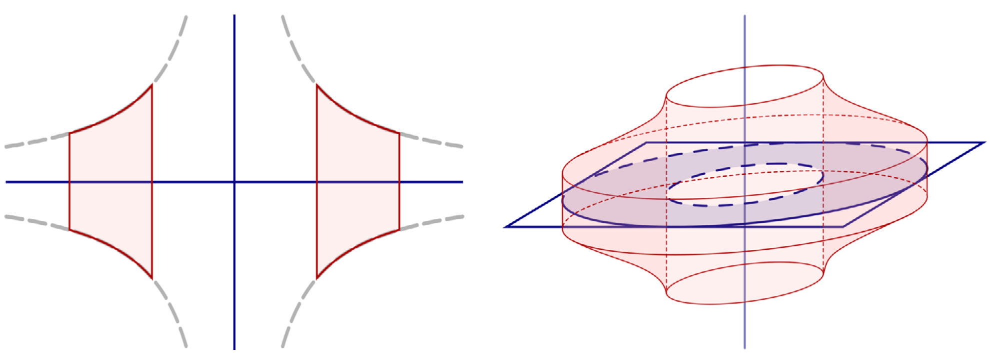

Fig. 9. The fundamental domain of the $a_{1,\nu,\mu}$-action on $\mathcal{N}^u_1$ for $n = 2$ (on the left) and that of the $a_{2,\nu,\mu}$-action on $\mathcal{N}^u_2$ for $n = 3$ (on the right).

Let us introduce the pair of $a_{q,\nu,\mu}$-invariant transversal foliations in $\mathcal{N}_q$ as follows:

$$\mathcal{F}^u_q = \bigcup_{(c_{q+1},\ldots,c_n)\in O_{x_{q+1}\ldots x_n}} \{(x_1, \ldots, x_n) \in \mathcal{N}_q : (x_{q+1}, \ldots, x_n) = (c_{q+1}, \ldots, c_n)\},$$

$$\mathcal{F}^s_q = \bigcup_{(c_1,\ldots,c_q)\in O_{x_1\ldots x_q}} \{(x_1, \ldots, x_n) \in \mathcal{N}_q : (x_1, \ldots, x_q) = (c_1, \ldots, c_q)\}.$$

Let

$$\hat{\mathcal{F}}^{uu}_{q,\nu,\mu} = p_{\hat{\mathcal{N}}^u_{q,\nu,\mu}}(\mathcal{F}^u_q), \;\; \hat{\mathcal{F}}^{su}_{q,\nu,\mu} = p_{\hat{\mathcal{N}}^u_{q,\nu,\mu}}(\mathcal{F}^s_q);$$

$$\hat{\mathcal{F}}^{us}_{q,\nu,\mu} = p_{\hat{\mathcal{N}}^s_{q,\nu,\mu}}(\mathcal{F}^u_q), \;\; \hat{\mathcal{F}}^{ss}_{q,\nu,\mu} = p_{\hat{\mathcal{N}}^s_{q,\nu,\mu}}(\mathcal{F}^s_q).$$

Since every leaf of the foliation $\mathcal{F}^u_q \setminus \mathcal{W}^u_q$ ($\mathcal{F}^s_q \setminus \mathcal{W}^s_q$) is homeomorphic to $\mathbb{D}^q$ ($\mathbb{D}^{n-q}$) and does not contain points from the same orbit of $a_{q,\nu,\mu}$, each leaf of the foliation $\hat{\mathcal{F}}^{us}_{q,\nu,\mu}$ ($\hat{\mathcal{F}}^{su}_{q,\nu,\mu}$) is also homeomorphic to $\mathbb{D}^q$ ($\mathbb{D}^{n-q}$) and intersects transversally the manifold $\hat{\mathcal{W}}^s_{q,\mu}$ ($\hat{\mathcal{W}}^u_{q,\nu}$) at a single point (see Figs. 10, 11).

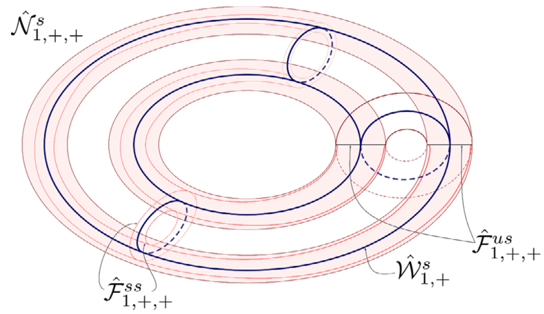

Fig. 10. Manifold $\hat{\mathcal{N}}^s_{1,+,+} \cong \mathbb{T}^2 \times [-1,1]$, $n=3$, with the two-dimensional foliation $\hat{\mathcal{F}}^{ss}_{1,+,+}$ and the one-dimensional one $\hat{\mathcal{F}}^{us}_{1,+,+}$.

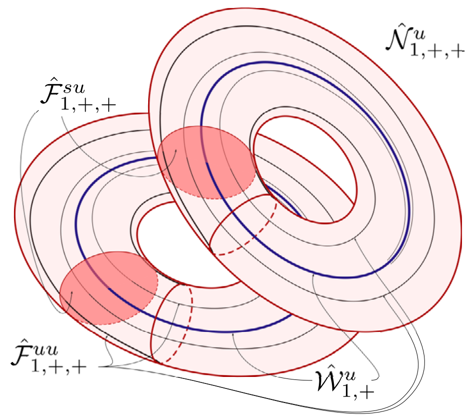

Fig. 11. Manifold $\hat{\mathcal{N}}^s_{s,+,+} \cong \mathbb{D}^2 \times \mathbb{S}^1 \times \mathbb{S}^0$, $n=3$, with the 2-dimensional foliation $\hat{\mathcal{F}}^{su}_{1,+,+}$ and the 1-dimensional one, $\hat{\mathcal{F}}^{uu}_{1,+,+}$.

Since $\mathcal{N}^u_q \setminus W^u_O = \mathcal{N}^s_q \setminus W^s_O$, the formula

$$\hat{g}_{q,\nu,\mu} = p_{\hat{\mathcal{N}}^s_{q,\nu,\mu}} \left(p_{\hat{\mathcal{N}}^u_{q,\nu,\mu}}|_{N^u_q \setminus W^u_O} \right)^{-1}$$

defines the *transition diffeomorphism*

$$\hat{g}_{q,\nu,\mu} : \hat{\mathcal{N}}^u_{q,\nu,\mu} \setminus \hat{\mathcal{W}}^u_{q,\nu} \to \hat{\mathcal{N}}^s_{q,\nu,\mu} \setminus \hat{\mathcal{W}}^s_{q,\mu}. \tag{2.10}$$

Moreover,

$$\hat{g}_{q,\nu,\mu}(\hat{\mathcal{F}}^{uu}_{q,\nu,\mu} \setminus \hat{\mathcal{W}}^u_{q,\nu}) = \hat{\mathcal{F}}^{us}_{q,\nu,\mu} \setminus \hat{\mathcal{W}}^s_{q,\nu}, \;\; \hat{g}_{q,\nu,\mu}(\hat{\mathcal{F}}^{su}_{q,\nu,\mu} \setminus \hat{\mathcal{W}}^u_{q,\nu}) = \hat{\mathcal{F}}^{ss}_{q,\nu,\mu} \setminus \hat{\mathcal{W}}^s_{q,\nu}.$$

It immediately implies that each leaf of the foliation $\hat{\mathcal{F}}^{uu}_{q,\nu,\mu}$ ($\hat{\mathcal{F}}^{ss}_{q,\nu,\mu}$) is homeomorphic to the punctured disc $\mathbb{D}^q \setminus O$ ($\mathbb{D}^{n-q} \setminus O$).

Let us establish the connection between the linear models constructed above and the invariant sets of a Morse – Smale diffeomorphism.

Proposition 15 ([13, Theorem 2.3]). *Let $f \in MS(M^n)$ and let p be its periodic point. Then*

- *f acts freely and discontinuously on $W^u_{\mathcal{O}_p} \setminus \mathcal{O}_p$;*
- *$f^{m_p}|_{W^u_p \setminus p}$ and $a^u_{q_p,\nu_p}$ are smoothly conjugate;*
- *the orbit space $\hat{W}^u_p = (W^u_{\mathcal{O}_p} \setminus \mathcal{O}_p)/f = (W^u_p \setminus p)/f^{m_p}$ is diffeomorphic to $\hat{\mathcal{W}}^u_{q_p,\nu_p}$;*
- *$\eta_{\hat{W}^u_p} = m_p \eta_{\hat{\mathcal{W}}^u_{q_p,\nu_p}}$.*

Let p be a saddle periodic point of $f \in MS(M^n)$. We say that the neighborhood N_p of p is *linearizing* if there is a homeomorphism $\upsilon_p : N_p \to \mathcal{N}_{q_p}$, conjugating $f^{m_p}|_{N_p}$ and $a_{q_p,\nu_p,\mu_p}|_{\mathcal{N}_{q_p}}$ such that $F^u_p = \upsilon_p^{-1}(\mathcal{F}^u_{q_p})$, $F^s_p = \upsilon_p^{-1}(\mathcal{F}^s_{q_p})$ are two transverse smooth foliations.

Proposition 16 ([5]). *Any saddle periodic point p of $f \in MS(M^n)$ has a linearizing neighborhood.*

Let

$$N^u_p = N_{\mathcal{O}_p} \setminus W^s_{\mathcal{O}_p},\ N^s_p = N_{\mathcal{O}_p} \setminus W^u_{\mathcal{O}_p}.$$

Proposition 17 ([13, Theorem 2.4]). *Let $f \in MS(M^n)$ and let p be its saddle periodic point. Then*

- *f acts freely and discontinuously on N^u_p;*
- *the orbit space $\hat{N}^u_p = N^u_p/f$ is diffeomorphic to $\hat{\mathcal{N}}^u_{q_p,\nu_p,\mu_p}$;*
- *$\eta_{\hat{N}^u_p} = m_p \eta_{\hat{\mathcal{N}}^u_{q_p,\nu_p,\mu_p}}$.*

The same is true for $\hat{N}^s_p = N^s_p/f$. Let

$$\hat{F}^{uu}_p = p_{\hat{N}^u_p}(F^u_p),\ \ \hat{F}^{su}_p = p_{\hat{N}^u_p}(F^s_p);$$

$$\hat{F}^{us}_p = p_{\hat{N}^s_p}(F^u_p),\ \ \hat{F}^{ss}_p = p_{\hat{N}^s_p}(F^s_p).$$

For a periodic orbit $\mathcal{O}_p$, the neighborhood $N_{\mathcal{O}_p} = \bigsqcup\limits_{i=0}^{m_p-1} f^i(N_p)$, equipped with the map $\upsilon_{\mathcal{O}_p}$: $N_{\mathcal{O}_p} \to \bigsqcup\limits_{i=0}^{m_p-1} \mathcal{N}_{q_p}$ constructed from homeomorphisms

$$\upsilon_p f^{-i} : f^i(N_p) \to \mathcal{N}_{q_p},\ i \in \{0, m_p - 1\},$$

is called a *linearizing neighborhood* of the orbit $\mathcal{O}_p$. In this definition, the map $\upsilon_{\mathcal{O}_p}$ is said to be *linearizing* as well.

2.5. Classification of Morse – Smale 3-Diffeomorphisms

In the current section, we present some classification results for three-dimensional MS-diffeomorphisms following [7]. Most of the facts are formulated and proved there under the assumption that M^3 is orientable, but all of them, with some refinement of the formulations, are also valid in the non-orientable case.

2.5.1. Necessary and sufficient conditions of topological conjugacy

Let $f \in MS(M^3)$.

Suppose

$$A_f = W^u_{\Omega_0\cup\Omega_1}, \quad R_f = W^s_{\Omega_2\cup\Omega_3}, \quad V_f = M^3 \setminus (A_f \cup R_f).$$

Proposition 18 ([15, Theorem 1]). *Let $f \in MS(M^3)$. Then*

- *the sets A_f and R_f are a connected attractor and a connected repeller of f respectively, and their topological dimension is not greater than 1;*
- *V_f is a smooth connected 3-manifold, on which f acts freely and discontinuously, and*
$$V_f = W^s_{A_f\cap\Omega_f} \setminus A_f = W^u_{R_f\cap\Omega_f} \setminus R_f;$$
- *$\hat{V}_f = V_f/f$ is a closed connected smooth 3-manifold and $\eta_{\hat{V}_f}\big(\pi_1(\hat{V}_f)\big) = \mathbb{Z}$.*

Let

$$p_f = p_{\hat{V}_f},\ \eta_f = \eta_{\hat{V}_f},\ \hat{L}^s_f = p_f(W^s_{\Omega_1} \setminus A_f),\ \hat{L}^u_f = p_f(W^u_{\Omega_2} \setminus R_f).$$

The collection

$$S_f = (\hat{V}_f, \eta_f, \hat{L}^s_f, \hat{L}^u_f)$$

is called the *scheme of the diffeomorphism* $f \in MS(M^3)$.

Proposition 19 ([7, Theorem 1]). *Diffeomorphisms $f, f' \in MS(M^3)$ are topologically conjugate iff their schemes are equivalent, i.e., there exists a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ such that*

- *$\eta_f = \eta_{f'}\hat{\varphi}_*$;*
- *$\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$, $\hat{\varphi}(\hat{L}^u_f) = \hat{L}^u_{f'}$.*

2.5.2. Scheme surgery

To solve the realization problem for Morse–Smale 3-diffeomorphisms, it was proved in [8] that any scheme of such a diffeomorphism possesses some characteristic property. This property will allow us to define an abstract scheme that is implemented by some diffeomorphism.

To illustrate this property let us consider an orientation-preserving Morse–Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ depicted in Fig. 12. Its nonwandering set Ω_f consists of a fixed source α, one saddle periodic orbit $\mathcal{O}_\sigma = \{\sigma, f(\sigma), f^2(\sigma)\}$ of the Morse index 1, one fixed sink ω_1, and one periodic sink orbit $\mathcal{O}_{\omega_2} = \{\omega_2, f(\omega_2), f^2(\omega_2)\}$. By definition, we construct V_f by deleting the sets $A_f = W^u_{\mathcal{O}_\sigma} \cup \{\omega_1\} \cup \mathcal{O}_{\omega_2}$ and $R_f = \alpha$ from $\mathbb{S}^3$. The fundamental domain D_f of the f-action on V_f is homeomorphic to $\mathbb{S}^2 \times [0, 1]$, therefore, $\hat{V}_f \cong \mathbb{P}_+$.

As V_f is connected, the f-action induces the epimorphism $\eta_f : \pi_1(\hat{V}_f) \to \mathbb{Z}$. As the saddle σ has period 3, the covering map $p_f : V_f \to \hat{V}_f$ projects $W^s_{\mathcal{O}_\sigma} \setminus \mathcal{O}_\sigma$ to a 3-turning torus T in $\hat{V}_f$ (see Fig. 13). So, $\hat{L}^s_f = T$, while $\hat{L}^u_f = \varnothing$. As torus $\hat{L}^s_f$ is homotopically nontrivial, its fundamental group admits generators a, b such that $\eta_f(a) = 3$, $\eta_f(b) = 0$.

Also, consider the sink basins $W^s_{\omega_1}$, $W^s_{\mathcal{O}_{\omega_2}}$ with the wandering subsets $V_1 = W^s_{\omega_1} \setminus \omega_1$, $V_2 = W^s_{\mathcal{O}_{\omega_2}} \setminus \mathcal{O}_{\omega_2}$ and orbit spaces $\hat{V}_1 = V_1/f$, $\hat{V}_2 = V_2/f$. By Propositions 13 and 15, $\hat{V}_1 \cong \hat{V}_2 \cong \mathbb{P}_+$ and f-action induces the epimorphisms $\eta_1 : \pi_1(\hat{V}_1) \to \mathbb{Z}$, $\eta_2 : \pi_1(\hat{V}_2) \to 3\mathbb{Z}$, as ω_1 is fixed, ω_2 has period 3.

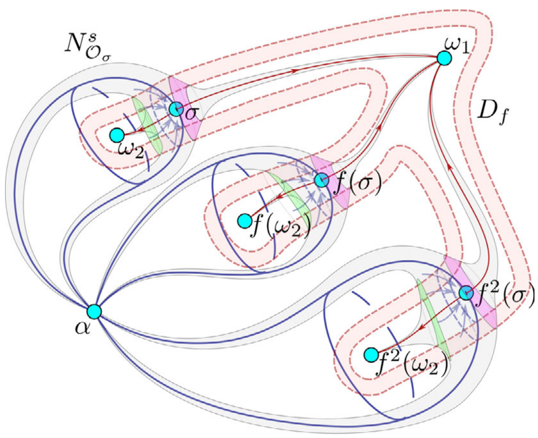

Fig. 12. The phase portrait of f.

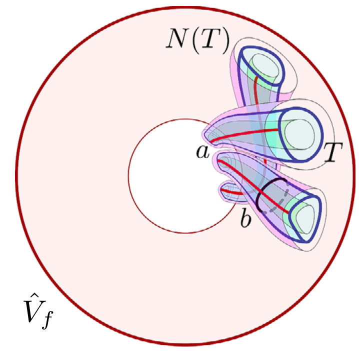

Fig. 13. The characteristic space $\hat{V}_f$.

On the other hand, we can get $(\hat{V}_1, \eta_1)$, $(\hat{V}_2, \eta_2)$ from $(\hat{V}_f, \eta_f, T)$ directly. To this end, it is enough to do on $\hat{V}_f$ a *surgery along the torus* T in the following way. Take a tubular neighborhood $N(T) \cong \mathbb{T}^2 \times [-1, 1]$, delete its interior from $\hat{V}_f$ and attach two copies of solid tori W_1, W_2 (gluing meridians with the generator b) to the boundary of the resulting manifold (see Fig. 14). Thus, we get a manifold $(\hat{V}_f)_T$ which, by Proposition 6, consists of two copies of $\mathbb{P}_+$. Moreover, every component of $(\hat{V}_f)_T$ inherits *surgical epimorphisms* η_T to $\mathbb{Z}$ and $3\mathbb{Z}$, respectively, which continue η_f. So we have $\big((\hat{V}_f)_T, \eta_T\big) = (\hat{V}_1, \eta_1) \sqcup (\hat{V}_2, \eta_2)$.

To explain the surgery above dynamically, let us notice that

$$V_1 \sqcup V_2 = (V_f \setminus \text{int } N^s_{\mathcal{O}_{\sigma_1}}) \cup N^u_{\mathcal{O}_{\sigma_1}},$$

where the attaching map is exactly the transition diffeomorphism.

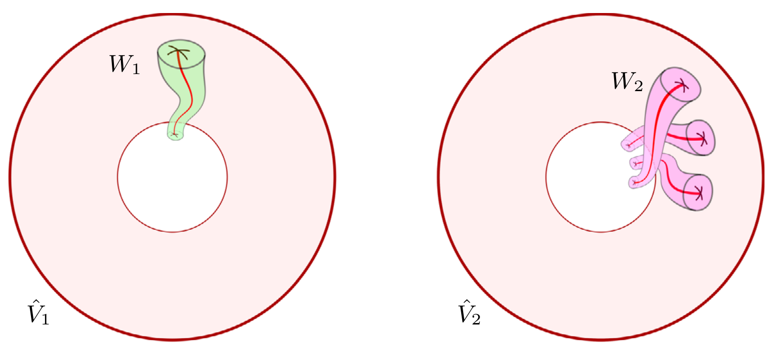

Fig. 14. The result of the surgery along T is $\hat{V}_1$ and $\hat{V}_2$.

Now we are ready to introduce the general definition of the surgery of a closed 3-manifold along a torus or Klein bottle.

Consider a closed connected smooth 3-manifold $\hat{V}$ whose fundamental group admits an epimorphism $\eta : \pi_1(\hat{V}) \to \mathbb{Z}$. Let $e^s_1 : \hat{\mathcal{N}}^s_{1,\nu_1,\mu_1} \to \hat{V}$ be a smooth embedding such that $\eta([e^s_1(c)]) > 0$ if $\eta_{\hat{\mathcal{N}}^s_{1,\nu_1,\mu_1}}([c]) = 1$. We assume

- $\hat{l}^s_1 = e^s_1(\hat{\mathcal{W}}^s_{1,\nu_1})$,
- $N(\hat{l}^s_1) = e^s_1(\hat{\mathcal{N}}^s_{1,\nu_1,\mu_1})$,
- $\bar{V}^s_1 = (\hat{V} \setminus \hat{l}^s_1) \sqcup \hat{\mathcal{N}}^u_{1,\nu_1,\mu_1}$,

- $\hat{g}_1^s = e_1^s \hat{g}_{1,\nu_1,\mu_1} : \hat{\mathcal{N}}^u_{1,\nu_1,\mu_1} \setminus \hat{\mathcal{W}}^u_{1,\nu_1} \to N(\hat{l}_1^s) \setminus \hat{l}_1^s$,
- $\hat{V}_1^s = \hat{\mathcal{N}}^u_{1,\nu_1,\mu_1} \cup_{\hat{g}_1^u} (\hat{V} \setminus \hat{l}_1^s)$

and denote by $p_1^s : \bar{V}_1^s \to \hat{V}_1^s$ the natural projection. We will call $\hat{V}_1^s$ *the manifold obtained from* $\hat{V}$ *by the surgery along the surface* $\hat{l}_1^s$ and denote it as

$$(\hat{V})_{\hat{l}_1^s}.$$

Proposition 20 ([19, Lemmas 3.4, 3.6 2)]). *Every connected component* $\hat{v}_1^s$ *of* $\hat{V}_1^s$ *is a closed smooth 3-manifold (see Fig. 15); the fundamental group of which admits the unique epimorphism* $\eta_{\hat{v}_1^s} : \pi_1(\hat{v}_1^s) \to m_{\hat{v}_1^s}\mathbb{Z}$, $m_{\hat{v}_1^s} \in \mathbb{N}$ *such that*

$$\eta_{\hat{v}_1^s}([p_1^s(c)]) = \eta([c]), \textit{ for any loop } c \subset \hat{V} \setminus \hat{l}_1^s. \tag{2.11}$$

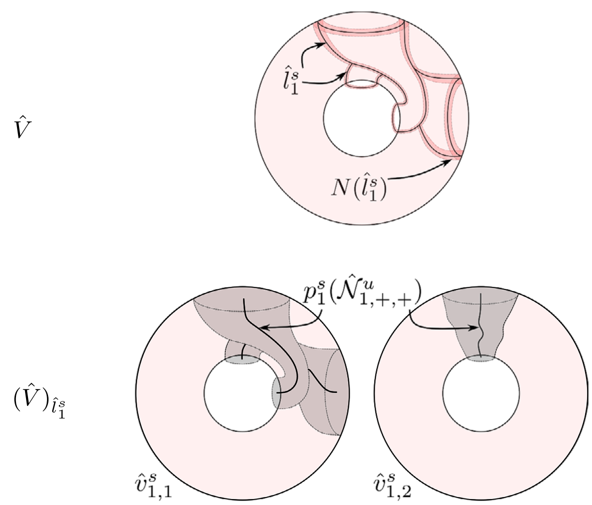

Fig. 15. The surgery of $\hat{V} = \mathbb{P}_+$ along a two-sided 2-turning torus $\hat{l}_1^s$. The manifold $(\hat{V})_{\hat{l}_1^s}$ consists of two connected components $\hat{v}^s_{1,1} \cong \hat{v}^s_{1,2} \cong \mathbb{P}_+$ such that $\eta_{\hat{v}^s_{1,1}}(\pi_1(\hat{v}^s_{1,1})) = \mathbb{Z}$, $\eta_{\hat{v}^s_{1,2}}(\pi_1(\hat{v}^s_{1,2})) = 2\mathbb{Z}$.

For $(\hat{V})_{\hat{l}_1^s}$, we denote by η_1^s the map composed of epimorphisms $\eta_{\hat{v}_1^s}$ and call it *the surgical epimorphism obtained from* η.

This notion does not cover the cases where some 2-dimensional stable separatrix takes part in heteroclinic intersections. To deal with this imperfection, we introduce the even more general notion of *a surgery along an s-lamination*.

Let us continue our construction and consider a smooth embedding[2)] $e_2^s : \hat{\mathcal{N}}^s_{1,\nu_2,\mu_2} \to \hat{V}_1^s$ such that

1) $\eta_1^s([e_2^s(c)]) > 0$ if $\eta_{\hat{\mathcal{N}}^s_{1,\nu_2,\mu_2}}([c]) = 1$,

2) each leaf of the foliation $e_2^s(\hat{\mathcal{F}}^{ss}_{1,\nu_2,\mu_2}) \cap p_1^s(\hat{\mathcal{N}}^u_{1,\nu_1,\mu_1})$ is a subset of some leaf of the foliation $p_1^s(\hat{\mathcal{F}}^{su}_{1,\nu_2,\mu_2})$.

Let $\bar{l}_2^s = e_2^s(\hat{\mathcal{W}}^s_{1,\nu_2})$, $N(\bar{l}_2^s) = e_2^s(\hat{\mathcal{N}}^s_{1,\nu_2,\mu_2})$ and $\hat{l}_2^s = (p_1^s|_{\hat{V} \setminus \hat{l}_1^s})^{-1}(\bar{l}_2^s)$. We will say (see Fig. 16) that *the manifold* $\hat{V}_2^s = (\hat{V}_1^s)_{\bar{l}_2^s}$ *is obtained from* $\hat{V}$ *by a surgery along the surfaces* $\hat{l}_1^s \cup \hat{l}_2^s$ and denote it as

$$(\hat{V})_{\hat{l}_1^s \cup \hat{l}_2^s}.$$

[2)]The possibility of such an embedding follows from [5].

Due to Proposition 20, each connected component $\hat{v}_2^s$ of $\hat{V}_2^s$ is a closed smooth 3-manifold (see Fig. 16), the fundamental domain of which admits the unique epimorphism $\eta_{\hat{v}_2^s} : \pi_1(\hat{v}_2^s) \to m_{\hat{v}_2^s}\mathbb{Z}$, $m_{\hat{v}_2^s} \in \mathbb{N}$ such that

$$\eta_{\hat{v}_2^s}([p_2^s(c)]) = \eta_{\hat{v}_1^s}([c]), \text{ for any loop } c \subset \hat{V}_2^s \setminus \hat{l}_2^s.$$

These epimorphisms construct the map η_2^s (from the Cartesian product of fundamental groups to the Cartesian product of the corresponding subgroups of $\mathbb{Z}$), which we will also call *the surgical epimorphism obtained from* η.

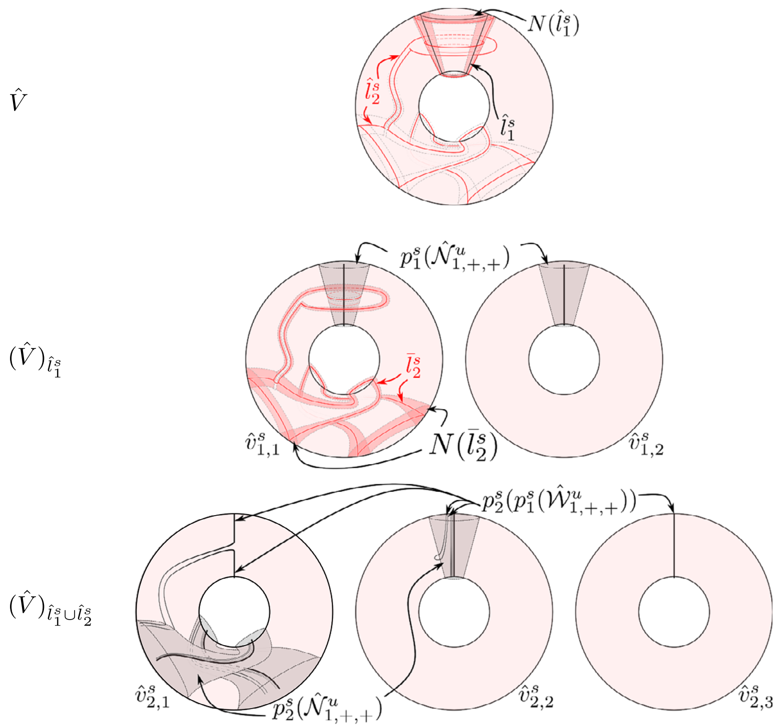

Fig. 16. The surgery of $\hat{V} = \mathbb{P}_+$ along two-sided tori: 1-turning torus $\hat{l}_1^s$ and 2-turning punctured torus $\hat{l}_2^s$. The manifold $(\hat{V})_{\hat{l}_1^s}$ consists of two connected components $\hat{v}^s_{1,1} \cong \hat{v}^s_{1,2} \cong \mathbb{P}_+$ such that $\eta_{\hat{v}^s_{1,1}}(\pi_1(\hat{v}^s_{1,1})) = \eta_{\hat{v}^s_{1,2}}(\pi_1(\hat{v}^s_{1,2})) = \mathbb{Z}$. The manifold $(\hat{V})_{\hat{l}_1^s \cup \hat{l}_2^s}$ consists of $\hat{v}^s_{2,1} \cong \hat{v}^s_{2,2} \cong \hat{v}^s_{2,3} \cong \mathbb{P}_+$ such that $\eta_{\hat{v}^s_{2,1}}(\pi_1(\hat{v}^s_{2,1})) = \eta_{\hat{v}^s_{2,2}}(\pi_1(\hat{v}^s_{2,2})) = \mathbb{Z}$, $\eta_{\hat{v}^s_{2,3}}(\pi_1(\hat{v}^s_{2,3})) = 2\mathbb{Z}$.

Iterating this process a finite number n_s of times, we construct a closed 3-manifold

$$(\hat{V})_{\hat{l}_1^s \cup \dots \cup \hat{l}_{n_s}^s}$$

called *the manifold obtained from* $\hat{V}$ *by a surgery along the* s*-lamination* $\hat{L}^s = \hat{l}_1^s \cup \dots \cup \hat{l}_{n_s}^s$. We denote it by

$$(\hat{V})_{\hat{L}^s}$$

and by $\eta^s_{\hat{L}^s}$ we denote the surgical epimorphism obtained from η.

Similarly, we can perform a surgery along a u*-lamination* $\hat{L}^u = \hat{l}_1^u \cup \dots \cup \hat{l}_{n_u}^u$ on the manifold $\hat{V}$, thereby obtaining a closed 3-manifold

$$(\hat{V})_{\hat{L}^u}$$

and a surgical epimorphism $\eta^u_{\hat{L}^u}$ obtained from η.

The collection $S = (\hat{V}, \eta, \hat{L}^s, \hat{L}^u)$ is *an abstract scheme* if:

- $\hat{V}$ is a connected closed smooth 3-manifold with a fundamental group admitting some epimorphism $\eta : \pi_1(\hat{V}) \to \mathbb{Z}$;
- $\hat{L}^s$, $\hat{L}^u \subset \hat{V}$ are transversally intersecting s- and u-laminations, respectively;
- each connected component $(\hat{V})_{\hat{L}^s}$ and $(\hat{V})_{\hat{L}^u}$ is diffeomorphic to $\mathbb{P}_{\pm}$.

Proposition 21 ([8, Theorem 1]). *The scheme of any $f \in MS(M^3)$ is an abstract scheme. Conversely, for any abstract scheme S there is a diffeomorphism $f \in MS(M^3)$ such that the scheme S_f is equivalent to S (see Fig. 17).*

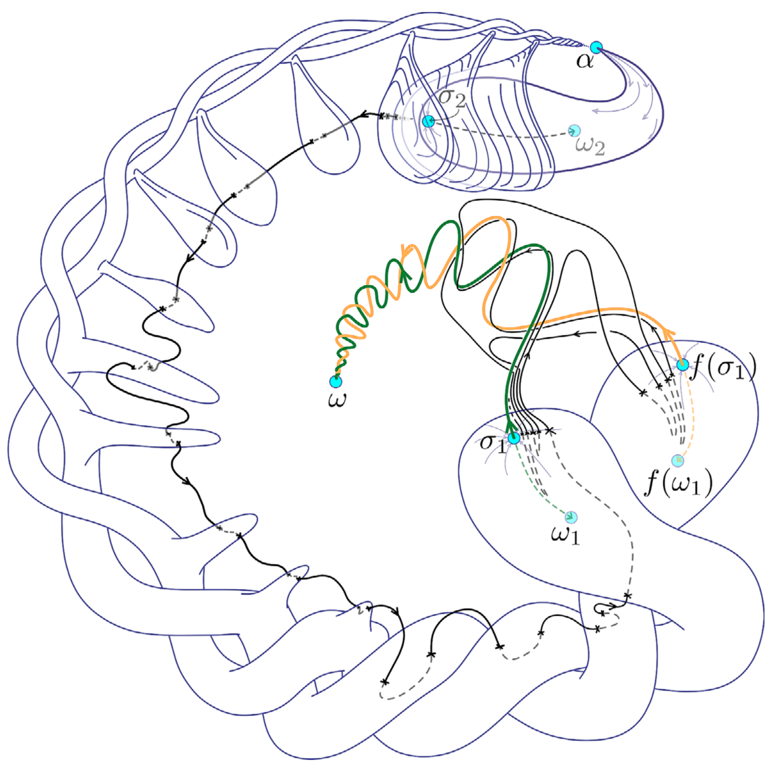

Fig. 17. The phase portrait of a diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ with a scheme equivalent to that in Fig. 16.

2.5.3. Interrelation between the periodic data of the diffeomorphism $f \in MS(M^3)$ and its scheme

The purpose of this subsection is to establish the connections between the scheme S_f and the periodic data of the diffeomorphism $f \in MS(M^3)$.

Due to Proposition 9, for any $f \in MS(M^3)$ there is such a numbering of saddle orbits $\mathcal{O}_1, \ldots, \mathcal{O}_{n_s}$ with Morse index 1 that

$$\operatorname{cl} W^u_{\mathcal{O}_i} \subset \left(\bigcup_{j=i+1}^{n_s} W^u_{\mathcal{O}_j} \cup \Omega_0 \right).$$

Let

$$W^u_i = W^u_{\mathcal{O}_i},\ W^s_i = W^s_{\mathcal{O}_i},\ A_i = \bigcup_{j=i}^{n_s} W^u_{\mathcal{O}_j} \cup \Omega_0,\ i \in \{1, \ldots, n_s\}.$$

Then

$$A_f = A_1 \supset A_2 \supset \cdots \supset A_{n_s} \supset \Omega_0 = A_{n_s+1}.$$

For $i \in \{1, \ldots, n_s + 1\}$, let

$$V_i^s = W^s_{A_i \cap \Omega_f} \setminus A_i.$$

Proposition 22 ([15, Theorem 1], [19, Proposition 3.1]). *Let $f \in MS(M^3)$. Then for any $i \in \{1, \ldots, n_s + 1\}$*

- *the set A_i is an attractor of f;*
- *f acts freely and discontinuously on V_i^s;*
- *$\hat{V}_i^s = V_i^s/f$ is a closed 3-manifold;*
- *$\hat{V}_{i+1}^s$ can be obtained from $\hat{V}_i^s$ by a surgery along $\bar{l}_i^s = p_{\hat{V}_i^s}(W_i^s \setminus \mathcal{O}_i)$;*
- *map $\eta_{\hat{V}_i^s}$ induced by projection $p_{\hat{V}_i^s} : V_i^s \to \hat{V}_i^s$ coincides with the surgical epimorphism obtained from η_f.*

As an immediate consequence of Propositions 22 and 15 we have the following relations between manifolds $\hat{V}_i^s$ and the periodic data of a diffeomorphism f.

Proposition 23. *Let $S_f = (\hat{V}_f, \eta_f, \hat{L}_f^s, \hat{L}_f^u)$ be the scheme of $f \in MS(M^3)$. Then*

- *$\hat{L}_f^s = \hat{l}_1^s \sqcup \cdots \sqcup \hat{l}_{n_s}^s$, here $\hat{l}_i^s = p_f(W_i^s \setminus A_f)$ and consequently the number of path-connected components of lamination $\hat{L}_f^s$ coincides with the number of saddle periodic points of f with Morse index 1;*
- *$\eta_{\hat{V}_i^s}\Big(j_{\bar{l}_i^s *}(\pi_1(\bar{l}_i^s))\Big) = m_i^s\mathbb{Z}$, here $j_{\bar{l}_i^s} : \bar{l}_i^s \to \hat{V}_i^s$ is the inclusion map and m_i^s is the period of orbit $\mathcal{O}_i$;*
- *$(\hat{V})_{\hat{L}^s} = \hat{W}^s_{\Omega_0}$, which implies that the number of connected components of $(\hat{V})_{\hat{L}^s}$ coincides with the number of sink periodic orbits;*
- *for a surgical epimorphism $\eta_{\hat{v}} : \pi_1(\hat{v}) \to m_{\hat{v}}\mathbb{Z}$ (here $\hat{v}$ is some connected component of $(\hat{V})_{\hat{L}^s}$) obtained from η_f, the number $m_{\hat{v}}$ coincides with a period of the corresponding sink orbit.*

Similarly, we can construct manifolds $\hat{V}_i^u$, $i \in \{1, \ldots, n_u + 1\}$ by ordering the set of periodic saddle orbits with the Morse index 2 and then establish their relation with the periodic data of f.

3. Periodic Data of Diffeomorphisms from Class MS_κ

This section is devoted to the proof of some auxiliary statements which give a general picture of MS_κ-diffeomorphisms and which we will use in the following.

Recall that by MS_κ we denote a subclass of MS-diffeomorphisms $f : M^3 \to M^3$ such that the elements of κ are exactly the periods of sink orbits of f, where

$$\kappa = (k_1, \ldots, k_r), \quad k_1 \leqslant \cdots \leqslant k_r, \quad k_1 + \cdots + k_r = k.$$

Let us also recall that the set Ω_q, $q \in \{0, \ldots, n\}$, is defined to be the subset of Ω_f consisting of points with the Morse index q; $C_q = |\Omega_q|$.

Lemma 1. *For any $f \in MS_\kappa$ there exists an integer $n_f \geqslant 0$ such that*

$$|\Omega_f| = 2(k + n_f).$$

Proof. It follows from the definition of class MS_κ that $|\Omega_0| = k$. We know from Proposition 11 that $|\Omega_f| \geqslant 2|\Omega_0| = 2k$, with number $|\Omega_f|$ being even. Then let us choose as n_f the number $\frac{|\Omega_f|-2k}{2}$, from the above reasoning we conclude that it is a nonnegative integer. □

For any $n \in \mathbb{N} \cup \{0\}$ we denoted by $MS_{\kappa,n}$ the subset (possibly empty) of MS_κ which is of diffeomorphisms f with $|\Omega_f| = 2(k+n)$.

Lemma 2. *A diffeomorphism $f \in MS_\kappa$ belongs to $MS_{\kappa,n}$ iff $C_2 = n$.*

Proof. $\Rightarrow$ Suppose $f \in MS_{\kappa,n}$, i.e., $|\Omega_f| = 2|\Omega_0| + 2n = 2k + 2n$, where k is the sum of all k_i in κ. Then

$$k + C_1 + C_2 + C_3 = 2k + 2n. \tag{3.1}$$

Using the Lefschetz–Hopf formula from Proposition 10, we find that

$$k - C_1 + C_2 - C_3 = 0. \tag{3.2}$$

Adding equalities (3.1) and (3.2), we get $C_2 = n$.

$\Leftarrow$ Suppose $f \in MS_\kappa$ and $C_2 = n$. Due to Lemma 1, there is $n_f \in \mathbb{N} \cup \{0\}$ such that $|\Omega_f| = 2(k + n_f)$. Then, again using the Lefschetz–Hopf formula, we obtain

$$k + C_1 + C_2 + C_3 = 2k + 2n_f. \tag{3.3}$$

Adding (3.2) to (3.3), given that $C_2 = n$, we get $n_f = n$. □

Lemma 3. *Let $f : M^3 \to M^3$ belong to $MS_{\kappa,0}$. Then*

1) $C_3 = 1$, $C_1 = k - 1$;

2) M^3 *is a* 3*-sphere* $\mathbb{S}^3$.

Proof. Suppose $f \in MS_{\kappa,0}$. We know from the previous lemma that $C_2 = 0$. With an estimate $C_3 \geqslant 1$ on the number of sources (see Proposition 11), we obtain $C_1 \leqslant k - 1$ from the equality (3.1). The opposite inequality $C_1 \geqslant k - 1$ follows from the Morse inequality $C_1 - C_0 \geqslant \beta_1 - \beta_0$ and the relations $\beta_1 \geqslant 0$, $\beta_0 = 1$, which are true for a connected (by definition) M^3. Hence, the first statement of this lemma is proved by solving Eq. (3.2).

Since $\Omega_2 = \varnothing$, we conclude from Proposition 12 that $M^3 \cong \mathbb{S}^3$. □

Lemma 4. *Let $f : M^3 \to M^3$ be in $MS_{\kappa,1}$. Then the periodic data of f has one of the following forms:*

1) $C_0 = k$, $C_1 = k$, $C_2 = 1$, $C_3 = 1$;

2) $C_0 = k$, $C_1 = k - 1$, $C_2 = 1$, $C_3 = 2$.

Proof. Suppose $f \in MS_{\kappa,1}$. It follows from Lemma 2 that $C_2 = 1$. With the upper-bound estimate $C_3 \geqslant 1$ on the number of sources (see Proposition 11), we get another estimate $C_1 \leqslant k$ from equality (3.1). Then the estimate $C_1 \geqslant k - 1$ follows from the second Morse inequality $C_1 - C_0 \geqslant \beta_1 - \beta_0$ and relations $\beta_1 \geqslant 0$, $\beta_0 = 1$, which are true for a connected (by definition) M^3. Thus, $|C_1|$ equals either k or $k - 1$, therefore, the points of this lemma are proved by solving Eq. (3.2). □

4. CHARACTERIZATION OF AN r-TUPLE κ FOR THE CLASS $MS_{\kappa,0}$

In the present section we prove Theorem 1:

$MS_{\kappa,0}$ is non-empty iff tuple κ has the following property:

$$\exists\, i \in \{1, \ldots, r\},\ m_i \in \mathbb{N} \cup \{0\} :\ k_i = 2^{m_i}. \tag{1.2}$$

Moreover, for any κ with property (1.2), the set $MS_{\kappa,0}$ contains both orientation-preserving and orientation-reversing diffeomorphisms.

Further, we separately prove the necessity of property (1.2). After that, for any κ with this property we construct two abstract schemes, realized by the required orientation-preserving and orientation-reversing diffeomorphisms of $MS_{\kappa,0}$. This construction implies the sufficiency of property (1.2).

4.1. Necessity of Property (1.2)

First of all, let us prove a lemma that allows us to see how exactly the epimorphism $\eta : \mathbb{P}_\pm \to n\mathbb{Z}$ changes under the surgery along a torus or Klein bottle.

Lemma 6. *Suppose that*

- *$\mathbb{P}_\pm$ is equipped with a nontrivial homomorphism $\eta : \pi_1(\mathbb{P}_\pm) \to \mathbb{Z}$ such that $\eta\big(\pi_1(\mathbb{P}_\pm)\big) = n\mathbb{Z}$;*
- *$T_\nu \subset \mathbb{P}_\pm$ is an m-turning surface;*
- *$\bar{V} = (\mathbb{P}_\pm)_{T_\nu}$;*
- *$\bar{\eta}$ is the surgical homomorphism obtained from η.*

Then

1) *if $T_+ \subset \mathbb{P}_+$, then $\bar{V}$ consists of two connected components $\bar{V}_1 \cong \bar{V}_2 \cong \mathbb{P}_+$ and $\bar{\eta}\big(\pi_1(\bar{V}_1)\big) = n\mathbb{Z}$, $\bar{\eta}\big(\pi_1(\bar{V}_2)\big) = mn\mathbb{Z}$;*

2) *if $T_- \subset \mathbb{P}_+$, then $\bar{V} \cong \mathbb{P}_+$ and $\bar{\eta}\big(\pi_1(\bar{V})\big) = 2n\mathbb{Z}$;*

3) *if $T_+ \subset \mathbb{P}_-$ is a one-sided torus, then $\bar{V} \cong \mathbb{P}_+$ and $\bar{\eta}\big(\pi_1(\bar{V})\big) = 2n\mathbb{Z}$;*

4) *if $T_+ \subset \mathbb{P}_-$ is a two-sided torus, then $\bar{V}$ consists of two connected components $\bar{V}_1 \cong \mathbb{P}_-$, $\bar{V}_2 \cong \mathbb{P}_+$ and $\bar{\eta}\big(\pi_1(\bar{V}_1)\big) = n\mathbb{Z}$, $\bar{\eta}\big(\pi_1(\bar{V}_2)\big) = mn\mathbb{Z}$;*

5) *if $T_- \subset \mathbb{P}_-$, then $\bar{V}$ consists of two connected components $\bar{V}_1 \cong \bar{V}_2 \cong \mathbb{P}_-$ and $\bar{\eta}\big(\pi_1(\bar{V}_1)\big) = n\mathbb{Z}$, $\bar{\eta}\big(\pi_1(\bar{V}_2)\big) = mn\mathbb{Z}$.*

Proof. We consider only cases 1) and 2), since the proof is similar in other cases.

1) By Proposition 8, the tubular neighborhood $T_+ \subset \mathbb{P}_+$ is homeomorphic to $N_+(\mathbb{T}_+)$ and $\mathbb{P}_+ \setminus \operatorname{int} N(T_+)$ consists of two connected components, at least one of which, Y, is an m-turning solid torus, i. e., $j_{Y*}\big(\pi_1(Y)\big) = m\mathbb{Z}$. We denote by Y' its second connected component. In this case, Y' must satisfy the property $j_{Y'*}\big(\pi_1(Y')\big) = \mathbb{Z}$. It follows from the definition of a surgery of $\mathbb{P}_+$ along a homotopically nontrivial torus T_+ that manifold $\bar{V}$ consists of two connected components $\bar{V}_1 \cong Y' \cup_{g_1} \mathbb{W}_+$ and $\bar{V}_2 \cong Y \cup_{g_2} \mathbb{W}_+$, where $g_1, g_2 : \partial Y \to \partial \mathbb{W}_+$ are homeomorphisms mapping a meridian of Y into some meridian of $\mathbb{W}_+$. Therefore, by Proposition 4, $\bar{V}_1 \cong \bar{V}_2 \cong \mathbb{P}_+$. Since $\eta\big(\pi_1(\mathbb{P}_+)\big) = n\mathbb{Z}$, the surgical homomorphism $\bar{\eta}$ satisfies $\bar{\eta}\big(\pi_1(\bar{V}_1)\big) = n\mathbb{Z}$ and $\bar{\eta}\big(\pi_1(\bar{V}_2)\big) = mn\mathbb{Z}$.

2) By Proposition 8, a tubular neighborhood of a Klein bottle $T_- \subset \mathbb{P}_+$ is homeomorphic to $N_-(\mathbb{T}_-)$ and $Y = \mathbb{P}_+ \setminus \operatorname{int} N(T_-)$ is a 2-turning solid torus, i. e., $j_{Y*}\big(\pi_1(Y)\big) = 2\mathbb{Z}$. It follows

from the definition of a surgery of $\mathbb{P}_+$ along a compact homotopically nontrivial Klein bottle T_- that $\bar{V} = Y \cup_g \mathbb{W}_+$, here $g : \partial Y \to \partial \mathbb{W}_+$ is a homeomorphism mapping a meridian of Y into some meridian of $\mathbb{W}_+$. Then, by Proposition 4, $\bar{V} \cong \mathbb{P}_+$. Since $\eta\big(\pi_1(\mathbb{P}_+)\big) = n\mathbb{Z}$, the surgical homomorphism $\bar{\eta}$ has to satisfy $\bar{\eta}\big(\pi_1(\bar{V})\big) = 2n\mathbb{Z}$.

The other cases are proved similarly with the only correction: to determine the topology of the complement $\mathbb{P}_- \setminus N(T_\nu)$, we use Lemma 5; to determine the topology of connected components of the manifold $\bar{V}$, we need to use the properties of Klein bottle homeomorphisms from Proposition 5. □

Lemma 7. *Let $f \in MS_{\kappa,0}$. Then*

$$\exists\, i \in \{1, \ldots, r\},\ m_i \in \mathbb{N} \cup \{0\} :\ k_i = 2^{m_i}. \tag{1.2}$$

Proof. Suppose $f \in MS_{\kappa,0}$. Then, due to Lemma 3, the nonwandering set of any such f contains a unique source point. It follows from Lemmas 2 and 3 that $\hat{L}^u_f = \varnothing$ and $\hat{V}_f \cong \mathbb{P}_\pm$. If $\hat{L}^s_f = \varnothing$, then, according to [22], f is a "source-sink" system, thus property (1.2) obviously holds. Otherwise, due to Proposition 23, the orbit space of sinks' attracting basins can be obtained from the manifold $\hat{V}_f$ by a surgery along s-lamination $\hat{L}^s_f$, while sinks' periods are determined by the surgical homomorphism obtained from η_f. The surgery along an s-lamination $\hat{L}^s_f$ is just a step-by-step sequence of surgeries along compact tori or Klein bottles. By Lemma 6, on each such step we have components with all previous epimorphisms or (if the surface is the Klein bottles) one of them will be doubled. Since $\eta_f\big(\pi_1(\hat{V}_f)\big) = \mathbb{Z}$, at least one component $\hat{v}$ of $(\hat{V})_{\hat{L}^s}$ is equipped with the surgical epimorphism $\eta_{\hat{v}} : \pi_1(\hat{v}) \to 2^m\mathbb{Z}$, $m \geqslant 0$. Then Proposition 23 implies that there is some sink orbit with period $k_i = 2^{m_i}$, where $m_i = m$. This concludes the proof of the lemma. □

4.2. Construction of an Abstract Scheme by a Given κ

Let $\kappa = (k_1, \ldots, k_r)$ be an r-tuple of natural numbers such that

$$1 \leqslant k_1 \leqslant \cdots \leqslant k_i = 2^{m_i} \leqslant \cdots \leqslant k_r.$$

This section describes the construction of abstract schemes $\mathcal{S}^+_{\kappa,0}$ and $\mathcal{S}^-_{\kappa,0}$, realized (see Propositions 23, 21) by an orientation-preserving and an orientation-reversing diffeomorphism from $MS_{\kappa,0}$ respectively.

For an orientation-preserving one, we consider the scheme $\mathcal{S}^+_{\kappa,0} = (\hat{V}, \eta, \hat{L}^s, \hat{L}^u)$ (see Fig. 18), where:

1) $\hat{V} = \mathbb{P}_+$;
2) $\eta : \pi_1(\mathbb{P}_+) \to \mathbb{Z}$ is an isomorphism;
3) $\hat{L}^u = \varnothing$;
4) $\hat{L}^s = \hat{l}^s_1 \cup \cdots \cup \hat{l}^s_{r+m_i-1}$, where
   - $\hat{l}^s_1, \ldots, \hat{l}^s_{r-1}$ are k_1-, …, k_{i-1}-, k_{i+1}-,…, k_r-turning tori, respectively;
   - $\hat{l}^s_r, \ldots, \hat{l}^s_{r+m_i-1}$ are 1-,2-, …, 2^{m_i-1}-turning punctured Klein bottles, respectively.

An orientation-reversing diffeomorphism realizes, for example, the scheme $\mathcal{S}^-_{\kappa,0} = (\hat{V}, \eta, \hat{L}^s, \hat{L}^u)$, where:

1) $\hat{V} = \mathbb{P}_-$;
2) $\eta : \pi_1(\mathbb{P}_-) \to \mathbb{Z}$ is an isomorphism;

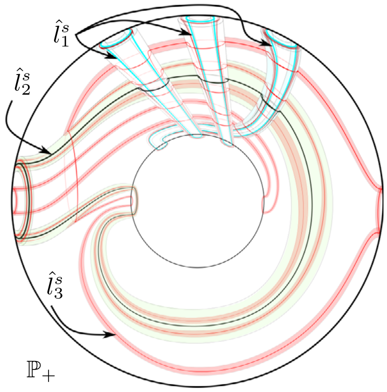

Fig. 18. Scheme $\mathcal{S}_{\kappa,0}$, where $\kappa=(3,4)$.

3) $\hat{L}^u=\varnothing$;

4) $\hat{L}^s=\hat{l}^s_1\cup\dots\cup\hat{l}^s_{r+m_i-1}$, where

- $\hat{l}^s_1,\dots,\hat{l}^s_{r-1}$ are one-sided k_1-, …, k_{i-1}-, k_{i+1}-, …, k_r-turning tori, respectively;
- $\hat{l}^s_r$ is a 1-turning punctured one-sided torus;
- $\hat{l}^s_{r+1},\dots,\hat{l}^s_{r+m_i-1}$ are 1-,2-, …, 2^{m_i-1}-turning punctured Klein bottles, respectively.

Next, we will prove the following remarkable fact.

Theorem 2. *Let κ satisfy* (1.2). *Then every diffeomorphism $f\in MS_{\kappa,0}$ has the following properties:*

a) if $k_1\geqslant 2$, then Ω_f contains a fixed saddle point σ such that $f|_{W^u_\sigma}$ is orientation-reversing;

b) if $k_1\geqslant 3$, then the wandering set of f contains heteroclinic points.

Proof. *a*) If $k_1\geqslant 2$, then due to property (1.2) of κ, the inequality $m_i\geqslant 1$ holds. Since $\hat{V}_f$ is equipped with the epimorphism η_f to group $\mathbb{Z}$, it follows that (see Propositions 23, 8 and Lemma 5) the obtained inequality $m_i\geqslant 1$ implies that the lamination $\hat{L}^s_f$ must contain a Klein bottle (a one-sided torus) $\hat{l}^s_{i_0}$, $i_0\in\{1,\dots,r\}$ if $\hat{V}_f\cong\mathbb{P}_+$ ($\hat{V}_f\cong\mathbb{P}_-$). Moreover, $\bar{l}^s_{i_0}\subset\hat{v}^s_{i_0-1}$, $\hat{v}^s_{i_0-1}\cong\hat{V}_f$ and $\eta_{\hat{v}^s_{i_0-1}}\Big(j_{\bar{l}^s_{i_0}*}\big(\pi_1(\bar{l}^s_{i_0})\big)\Big)=\mathbb{Z}$. After the surgery along $\bar{l}^s_{i_0}$, the 2-turning torus $\partial N(\bar{l}^s_{i_0})$ bounds a solid torus $N(\hat{l}^u_{i_0})$ in the manifold $\hat{v}^s_{i_0}\cong\mathbb{P}_+$ (here $\eta_{\hat{v}^s_{i_0}}\big(\pi_1(\hat{v}^s_{i_0})\big)=2\mathbb{Z}$), being a tubular neighborhood of a trivial knot $\hat{l}^u_{i_0}$. Due to Propositions 23, 15, the knot $\hat{l}^u_{i_0}$ is a projection of the unstable separatrices of some saddle point $\sigma\in\Omega_f$ such that $f|_{W^u_\sigma}$ reverses orientation.

b) If $k_1\geqslant 3$, then due to property (1.2) of κ, it is true that $m_i\geqslant 2$, therefore, in addition to the surface $\hat{l}^s_{i_0}$ the lamination $\hat{L}^s_f$ must contain a Klein bottle $\hat{l}^s_{i_1}$, $i_0<i_1\leqslant r$, such that $\bar{l}^s_{i_1}\subset\hat{v}^s_{i_1-1}\cong\hat{v}^s_{i_0}$. Assuming that the diffeomorphism f has no heteroclinic points, we find that the 1-turning knot $\hat{l}^u_{i0}$ does not intersect the Klein bottle $\bar{l}^s_{i_1}$ inside the manifold homeomorphic to $\mathbb{P}_+$, which contradicts Proposition 7. □

5. CHARACTERIZATION OF THE CLASS $MS_{\kappa,1}$

In this last section, for any κ we prove the non-emptyness of the class $MS_{\kappa,1}$ (we prove Theorem 3) and determine the topology of manifolds admitting such diffeomorphisms (we prove Theorem 4).

Theorem 3. *For any κ, the set $MS_{\kappa,1}$ is never empty and contains both types of diffeomorphisms: given on orientable and non-orientable* 3*-manifolds.*

Proof. Let us describe abstract schemes $\mathcal{S}^+_{\kappa,1}$ and $\mathcal{S}^-_{\kappa,1}$, realizable (see Propositions 23, 21) by diffeomorphisms from $MS_{\kappa,1}$ defined on the manifolds $\mathbb{P}_+$ and $\mathbb{P}_-$, respectively.

Let us define the scheme $\mathcal{S}^+_{\kappa,1}$ to be $(\hat V, \eta, \hat L^s, \hat L^u)$ (see Fig. 19), where:

1) $\hat V = \mathbb{T}_+ \times \mathbb{S}^1 = \mathbb{T}^3$ and $q_+ : \mathbb{R}^3 \to \mathbb{T}^3$ is a natural projection;

2) $\eta : \pi_1(\hat V) \to \mathbb{Z}$ is an epimorphism defined by $\eta([q_+(\upsilon)]) = 1$, where $\upsilon = \{(0,0,z),\ z \in [0,1]\}$;

3) $\hat L^u = \hat l^u_1 = q_+(Oxz)$;

4) $\hat L^s = \hat l^s_1 \cup \dots \cup \hat l^s_r$, where:

- $\hat l^s_1, \dots, \hat l^s_{r-1}$ are pairwise disjoint two-sided tori each of which, $\hat l^s_i$, is a tubular neighborhood of the knot $c_i = q_+(C_i)$ with $C_i = \left\{(x,y,z) \in \mathbb{R}^3 : z = k_i x, y = \frac{i}{r}\right\}$;
- $\bar l^s_r = q_+(P)$ is a two-sided torus, where $P = \left\{(x,y,z) \in \mathbb{R}^3 : z = -k_r y + 1\right\}$.

The scheme $\mathcal{S}^-_{\kappa,1} = (\hat V, \eta, \hat L^s, \hat L^u)$ is constructed similarly to the scheme $\mathcal{S}^+_{\kappa,1}$, but it should be done on the manifold $\hat V = \mathbb{T}_- \times \mathbb{S}^1$, using natural projection $q_- : \mathbb{R}^3 \to \hat V$ (see Fig. 19). ☐

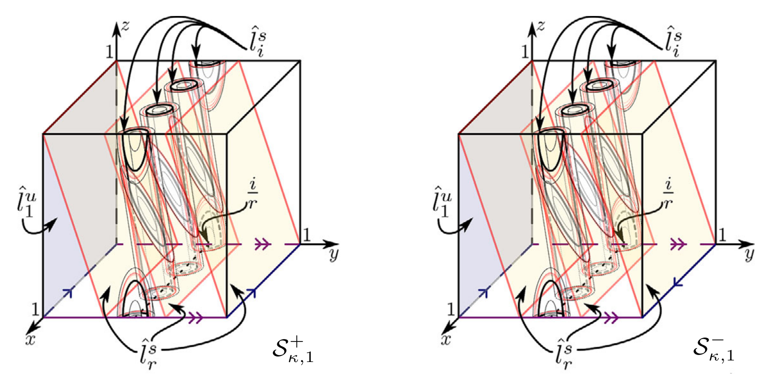

Fig. 19. Abstract schemes $\mathcal{S}^\pm_{\kappa,1}$.

Theorem 4. *Let a diffeomorphism $f : M^3 \to M^3$ be in $MS_{\kappa,1}$ and $\mathrm{cl}(W^s_\sigma)$, $\sigma \in \Omega_2$, be a submanifold of M^3. Then M^3 is homeomorphic either to a lens space or to the skew product $\mathbb{S}^2 \tilde\times \mathbb{S}^1$.*

Proof. To determine the topology of the ambient manifold M^3, without loss of generality we can suppose that all points in Ω_f are fixed for $f : M^3 \to M^3$, $f \in MS_{\kappa,1}$. It follows from the definition of the class $MS_{\kappa,1}$ that the repeller R_f is of the form

$$R_f = \mathrm{cl}(W^s_\sigma) = W^s_\sigma \cup \Omega_3.$$

In addition, we will assume that all saddle points of Ω_1 are ordered $\sigma_1, \dots, \sigma_r$ such that $p_f(W^u_{\sigma_i} \cap V_f) = \hat l^u_i$, $i \in \{1, \dots, r\}$. According to Lemma 4, we break up the following proof into two separate cases: 1) $|C_3| = 1$ and 2) $|C_3| = 2$.

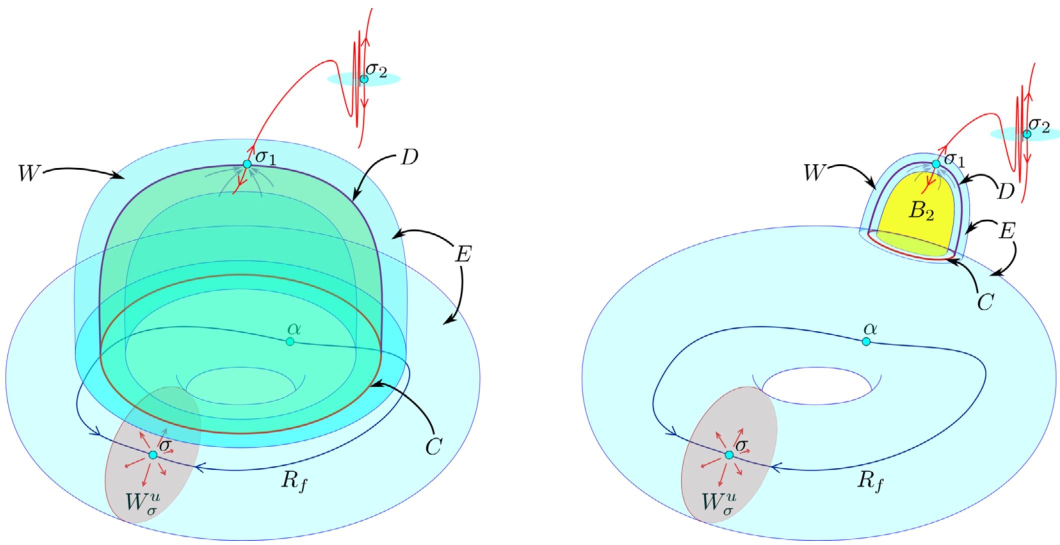

Fig. 20. Case 1a): C is noncontractible on the left and contractible on the right.

In case 1), R_f is a knot in M^3. Since R_f is a submanifold, a trapping neighborhood W of R_f is diffeomorphic to either a solid torus or solid Klein bottle. Let us look at these two possibilities separately: $1a$) $W \cong \mathbb{W}_+$; $1b$) $W \cong \mathbb{W}_-$.

$1a$) Using the same technique as in the proof of the lemmas [20, Lemmas 3.1, 3.2], we can choose the solid torus W so that the stable manifold $W^s_{\sigma_1}$ transversally intersects the torus ∂W along the unique knot $C = W^s_{\sigma_1} \cap \partial W$. Let us denote by $D \subset W^s_{\sigma_1}$ the two-dimensional disk bounded by the knot C and denote by $N(D) \subset M^3$ its tubular neighborhood transversally intersecting the torus ∂W along a tubular neighborhood $N(C) \subset \partial W$. Let us put $E = N(D) \cup W$ and $\Sigma = \partial E$ (see Fig. 20).

If C is a noncontractible knot on the torus ∂W, then Σ is a 2-dimensional sphere (see Fig. 20). Let $\tilde{M}^3 = (M^3 \setminus \operatorname{int} E) \cup_g \mathbb{D}^3$, where $g : \Sigma \to \mathbb{S}^2$ is some diffeomorphism. Then $\tilde{M}^3$ admits a Morse–Smale diffeomorphism $\tilde{f}$ topologically conjugated to f outside the ball $\mathbb{D}^3$ and having the unique nonwandering point — a source — inside $\mathbb{D}^3$ (see, for example, [18, Lemmas 4, 5]). Since all saddle points of $\tilde{f}$ have the Morse index 1, it follows, due to Lemmas 2 and 3, that $\tilde{M}^3$ is homeomorphic to the 3-sphere $\mathbb{S}^3$. An immediate consequence of this fact is that $M^3 \setminus \operatorname{int} E$ is homeomorphic to the 3-ball, therefore $M^3 \setminus \operatorname{int} W$ is homeomorphic to the solid torus, and thus M^3 is a lens space.

If C is a contractible knot on the torus ∂W, then Σ consists of two connected components, one of which, Σ_1, is homeomorphic to a 2-torus, while the other Σ_2 is homeomorphic to a 2-sphere. Using a method similar to that above, we can prove that Σ_2 bounds a 3-ball B_2. Then $E_2 = E \cup B_2$ is a solid torus. Let us repeat our reasoning for saddle points $\sigma_2, \ldots$ until the trace of a stable manifold of some saddle point is an essential curve on the boundary of the solid torus. Such saddle point must exist, since otherwise, having exhausted all saddle points, we obtain the fundamental domain of the sink point basin, homeomorphic to $\mathbb{T}_+ \times [0, 1]$, which is impossible by virtue of Proposition 15.

$1b$) In view of the above considerations, without loss of generality, we can assume that C is a nontrivial knot on the Klein bottle ∂W. As $\hat{V}_f$ is homeomorphic to the direct product of the Klein bottle with $\mathbb{S}^1$ and $p_f(W^s_{\sigma_1} \cap V_f)$ is a-turning torus there, C is a meridian of W. Similarly to case 1a), this implies that $M^3 \setminus \operatorname{int} W$ is homeomorphic to the solid Klein bottle, and thus, due to Proposition 5, $M^3 \cong \mathbb{P}_-$.

In case 2), R_f is a closed arc in M^3. Since R_f is a submanifold, its tubular neighborhood is a 3-ball B. Then M^3 admits a Morse–Smale diffeomorphism $\tilde{f}$, coinciding with f outside B and having a unique nonwandering point (namely, source) inside B (see, for example, [18, Lemmas 4, 5]).

Since all saddle points of $\tilde{f}$ have the Morse index 1, it follows, due to Lemmas 2 and 3, that the manifold M^3 is homeomorphic to the 3-sphere $\mathbb{S}^3$, which is a lens space $L_{1,0}$. □

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CONFLICT OF INTEREST

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