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Scenario of a mildly stable transition from codimensional one Anosov diffeomorphism to a DA-diffeomorphism

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Abstract

Smale proposed to modify the hyperbolic automorphism of the n-torus of codimension 1 in the neighbourhood of a fixed point by surgical operation to obtain the so-called DA-diffeomorphism. However, the corresponding arc of diffeomorphisms is not even mildly stable. The hypothesis of constructing a mildly stable arc between the Anosov diffeomorphism and the DA diffeomorphism was formulated by Newhouse *et al*. A detailed construction of such an arc is carried out in this paper.

Keywords: Anosov diffeomorphism, saddle-node bifurcation, mildly stable arc

Mathematics Subject Classification numbers: 37D20

1. Introduction and formulation of the results

Let $\mathrm{Diff}(M^n)$ be the space of diffeomorphisms on a closed n-dimensional manifold M^n endowed with the C^∞-topology. *A smooth arc* in the space of $\mathrm{Diff}(M^n)$ is C^∞-smooth map $\varphi : M^n \times [0,1] \to M^n$ such that for each fixed $t \in [0,1]$ the map $\varphi_t = \varphi|_{M^n \times \{t\}} \in \mathrm{Diff}(M^n)$. We say that the arc φ_t, $t \in [0,1]$, *connects diffeomorphisms* φ_0, φ_1.

Following Newhouse *et al* [10] an arc φ_t is called *mildly stable* if it is an inner point of the equivalence class with respect to the following relation: two arcs φ_t, φ'_t are called

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mildly conjugate if there are homeomorphisms $h: [0,1] \to [0,1], H_t: M \to M$ such that $H_t\varphi_t = \varphi'_{h(t)}H_t, t \in [0,1]$. If H_t depends continuously on t also then the arcs φ_t, φ'_t are called *conjugate*, and the arc φ_t is called *stable*.

The arc connecting two structurally stable diffeomorphisms φ_0, φ_1 is called *arc with a finite number of bifurcations* if there are a finite number of values $0 < t_1 < \ldots < t_q < 1$ such that the diffeomorphisms φ_{t_j} are not structurally stable and the diffeomorphisms φ_t, φ_τ are topologically conjugate if t, τ belong to the same connected component of the set $[0,1] \setminus \{t_1, \ldots, t_q\}$.

The problem of the existence of an arc with a finite number of bifurcations connecting structurally stable systems was included in the list of 50 most important problems of dynamical systems published by Palis and Pugh [16]. On manifolds of any dimension $n \geqslant 1$ there is an impressive number of counterexamples to the existence of a stable or mildly stable arc with a finite number of bifurcations between structurally stable diffeomorphisms constructed by Matsumoto [8], Blanchard [1], Grines *et al* [4, 11–14] (see also the review [9]).

Smale [21] showed that some *Anosov diffeomorphisms* (diffeomorphisms with a hyperbolic supporting manifold) on the n-torus $\mathbb{T}^n$ can be modified into DA-diffeomorphisms. DA-diffeomorphisms are called structurally stable diffeomorphisms whose basic sets consist of a $(n-1)$-dimensional expanding attractor Λ (dim $W^u_x = n-1, x \in \Lambda$) and an isolated source orbit. Williams [23] showed later that the modification can be implemented by the arc with one pitchfork bifurcation. However such an arc is neither stable nor mildly stable [10]. In the present paper we implement the transition from the Anosov diffeomorphism to the DA-diffeomorphism by the mildly stable arc with the single saddle-node bifurcation.

Let us fix our setting:

- $A \in GL(n,\mathbb{Z})$ is an integer unimodular matrix with all the eigenvalues, except for one value $\lambda \in (0,1)$, have absolute values larger than one;
- $\bar{A} : \mathbb{R}^n \to \mathbb{R}^n$ is linear map defined by the matrix A;
- $p : \mathbb{R}^n \to \mathbb{T}^n$ is the cover map given by the formula $p(x_1, \ldots, x_n) = (e^{i2\pi x_1}, \ldots, e^{i2\pi x_n})$;
- $\widehat{A} : \mathbb{T}^n \to \mathbb{T}^n$ is algebraic automorphism (Anosov diffeomorphism of codimension 1) defined by the formula $p\bar{A} = \widehat{A}p$.

The main result of the paper is the proof of the following theorem.

Theorem 1. *There exists a mildly stable arc with a single saddle-node bifurcation that connects the $\widehat{A}$ diffeomorphism with a DA-diffeomorphism.*

2. Necessary concepts and facts

Let M^n be a connected closed smooth riemannian manifold of dimension $n > 1$ with the norm $||\cdot|| : TM^n \to [0,\infty)$ and the induced metric $d : M^n \times M^n \to [0,\infty)$.

Let $f : M^n \to M^n$ be a diffeomorphism. The set $X \subset M^n$ is called *f-invariant* if $f(X) = X$.

The diffeomorphisms $f, f' : M^n \to M^n$ are called *topologically conjugate* if there exists a homeomorphism $h : M^n \to M^n$ such that $h \circ f = f' \circ h$.

f-invariant compact set $\Lambda \subset M^n$ is called *hyperbolic* if there is a continuous Df-invariant decomposition of the tangent bundle $T_\Lambda M^n$ into *stable* and *unstable* subbundles

$$E^s_\Lambda \oplus E^u_\Lambda, \ \dim E^s_x + \dim E^u_x = n, \ x \in \Lambda$$

such that for a some Riemann metric $||\cdot||$, which is called *Lyapunov*, some constants $0 < \lambda < 1 < \mu$ and any $k \in \mathbb{N}$ the following inequalities are valid:

$$\|Df^k(v)\| \leqslant \lambda^k \|v\|, \qquad \forall v \in E^s_\Lambda,$$
$$\|Df^{-k}(w)\| \leqslant \mu^{-k} \|w\|, \qquad \forall w \in E^u_\Lambda.$$

For any point x of the hyperbolic set Λ there exists an injective immersion $J^s_x : \mathbb{R}^{q_s} \to M^n$ the image $W^s_x = J^s_x(\mathbb{R}^{q_s})$ is called *a stable manifold of the point* x such that the following properties hold:

1) $T_x W^s_x = E^s_\Lambda$;
2) the points $x, y \in M^n$ belong to the same manifold W^s_x if and only if $d(f^n(x), f^n(y)) \to 0$ for $n \to \infty$;
3) $f(W^s_x) = W^s_{f(x)}$;
4) if $x, y \in \Lambda$ then either $W^s_x = W^s_y$ or $W^s_x \cap W^s_y = \emptyset$;
5) if the points $x, y \in \Lambda$ are close on M^n then W^s_x, W^s_y are C^1-close on compact sets.

An unstable manifold W^u_x of the point $x \in \Lambda$ is defined as a stable manifold with respect to the diffeomorphism f^{-1}. Unstable manifolds have similar properties, as the stable ones. Stable and unstable manifolds are also called *invariant manifolds*. A path connected component of the sets $W^u_x \setminus x$, $W^s_x \setminus x$ is called *separatrix*.

If the entire supporting manifold M^n of the diffeomorphism f is a hyperbolic set, then f is called to be *Anosov diffeomorphism*.

Recall that ε-*chain of length* $m \in \mathbb{N}$ joining points $x, y \in M^n$ for f is a set of points $x = x_0, \ldots, x_m = y$ such that $d(f(x_{i-1}), x_i) < \varepsilon$ for $1 \leqslant i \leqslant m$. A point $x \in M^n$ is called *chain recurrent* for f if for any $\varepsilon > 0$ there exists a natural number m depending on $\varepsilon > 0$ and ε-chain of length m joining x with itself. The set of all chain recurrent points is called *chain recurrent set* and is denoted by $\mathcal{R}_f$.

It follows from the results of [2, 15, 20, 22] that the hyperbolicity of the set $\mathcal{R}_f$ is equivalent to Ω-stability f. Recall that f is called Ω-*stable* if its C^1-small perturbations preserve the structure of a chain recurrent set up to topological conjugacy. The set $\mathcal{R}_f$ in this case consists of a finite number of pairwise disjoint subsets called *basic*, each is compact invariant and *is topologically transitive* (contains an everywhere dense orbit) [21]. If the basic set is a periodic orbit then it is called *trivial*. Otherwise, the basic set is called *nontrivial*.

A basic set Λ of Ω-stable diffeomorphism f is called an *attractor* if it has a closed trapping neighborhood $U_\Lambda \subset M^n$ such that

$$f(U_\Lambda) \subset \operatorname{int} U_\Lambda, \bigcap_{k \in \mathbb{N}} f^k(U_\Lambda) = \Lambda.$$

In this case (see, for example, [17])

$$\Lambda = \bigcup_{x \in \Lambda} W^u_x.$$

If $\dim \Lambda = \dim W^u_x$, then the attractor Λ is called *expanding*. *Repeller* is defined as an attractor for the map f^{-1}.

By theorem 3 in [17] any basic set Λ of codimension one Ω-stable diffeomorphism $f : M^n \to M^n$ is either an attractor or a repeller.

A diffeomorphism f is called *structurally stable* if there exists its neighborhood in the space Diff(M^n) with C^1-topology such that any diffeomorphism from this neighborhood is topologically conjugate to the diffeomorphism f. Due to the results of [7, 19] a diffeomorphism f is structurally stable if and only if 1) it is a Ω-stable diffeomorphism and 2) it satisfies the *strong*

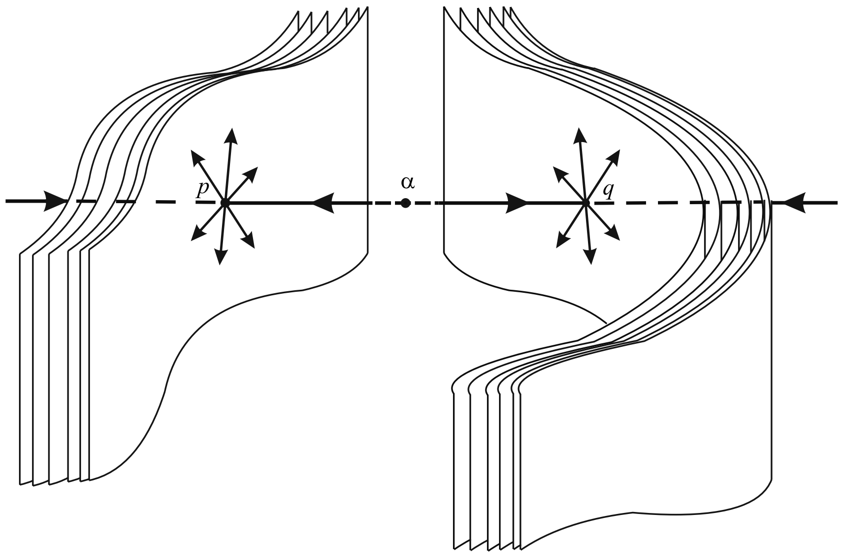

Figure 1. 2-bunch b of the two-dimensional expanding attractor Λ with boundary points p, q.

transversality condition. The latter means that $\forall x, y \in \mathcal{R}_f$ of the manifolds W^s_x and W^u_y intersect *transversely*, that is the sum of the tangent spaces to these manifolds coincides with the entire tangent space at their intersection points.

Any expanding attractor Λ of codimension 1 divides its basin W^s_Λ by a finite number of connected components. Each such a component of B defines *a bunch b* as the union of all unstable manifolds of all periodic points from Λ such that at least one of the stable separatrix of each belongs to B. The number k_b of such *boundary points* is finite and is called the *degree of the bunch b* and b is called the *k_b-bunch* with the *basin B* (see figure 1).

If $n \geqslant 3$ then according to [18, theorem 2.1] any expanding attractor of codimension 1 has only 1- or 2-bunches. In this case 1-bunches can be on non-orientable manifolds only.

A structurally stable diffeomorphism f is called a DA-diffeomorphism if its chain recurrent set consists of a single expanding attractor of codimension 1 and isolated sources.

3. Construction of the arc

Let us fix our setting for an integer unimodular matrix G:

- $\bar{G} : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map defined by G, that is $\bar{G}(x) = Gx$;
- $\widehat{G} : \mathbb{T}^n \to \mathbb{T}^n$ is algebraic automorphism defined by G, that is $\widehat{G}(x \pmod 1) = Gx \pmod 1$.

Let A be an integer unimodular matrix with all the eigenvalues, except for one value $\lambda \in (0,1)$, have absolute values larger than one. According to the Jordan normal form theorem for the matrix A there exists a square non-singular matrix C such that given by the matrix $J = C^{-1}AC$ linear map $\bar{J}$ has the form

$$\bar{J}(x,y) = (\lambda x, \bar{B}(y)) \quad (1)$$

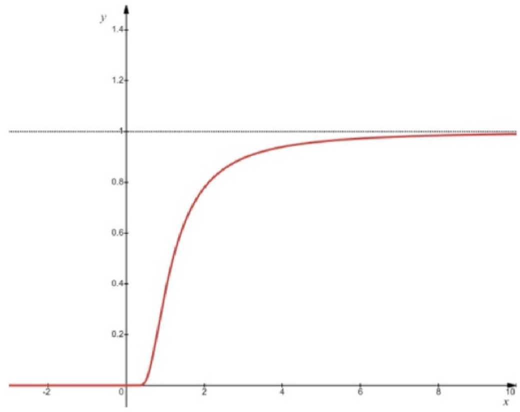

Figure 2. Graph of the function $g(x)$.

in coordinates $x \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$. Here $\bar{B} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the linear map defined by the matrix B. Absolute values of all eigenvalues of B are greater than one and there is $\mu > 1$ such that

$$||B^{-k}y|| < \mu^{-k}||y||,\ \forall y \in \mathbb{R}^{n-1}. \tag{2}$$

Let

$$V = \left\{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 \leqslant x \leqslant r,\ ||y|| \leqslant \delta\right\}, \tag{3}$$

for any $r > 0$, $\delta > 0$ where the constants r, δ are chosen so that the cover p is a homeomorphism on the set $\bar{V} = \bar{C}(V)$.

Next we describe how to construct the function $\varphi(x)$ so that the diffeomorphism $\lambda x + \varphi(x)$ coincides with the linear contraction outside the segment $[0, r]$ and has two hyperbolic fixed points on the interval $(0, r)$ which are a source and a sink. In addition this function requires described below in the lemma 3.1 properties allowing to construct the desired arc based on it. To do this we define C^∞-smooth function $g : \mathbb{R} \to [0, 1]$ by the formula (see figure 2)

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ \mathrm{e}^{-\frac{1}{x^2}}, & x > 0. \end{cases}$$

We define C^∞-smooth function $\sigma : \mathbb{R} \to [0, 1]$ by the formula (see figure 3)

$$\sigma(x) = \frac{g(x+1)}{g(x+1) + g(1-x)}.$$

It is directly verified that the function $\sigma(x)$ increases monotonously from 0 to 1 on the interval $(-1, 1)$, it is constant outside $(-1, 1)$ and (see figure 4)

$$\sigma'(x) < \frac{9}{8}, x \in \mathbb{R}. \tag{4}$$

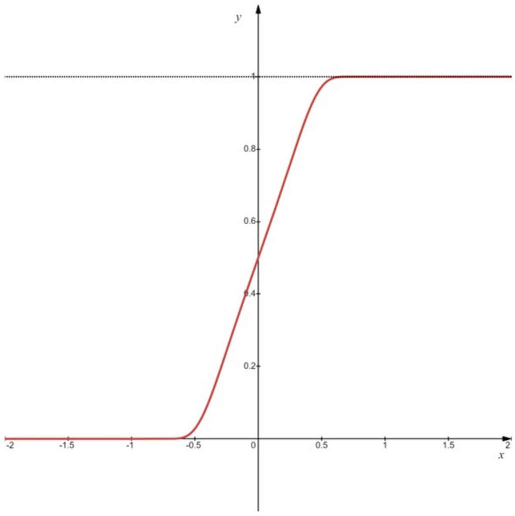

Figure 3. Graph of the function $\sigma(x)$.

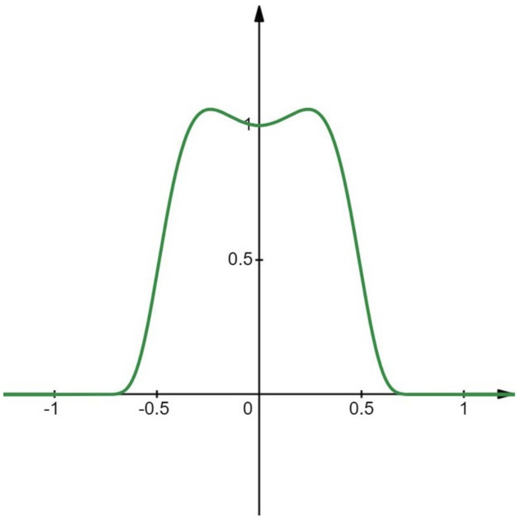

Figure 4. Graph of the function $\sigma'(x)$.

Let $h_-, h_+ : \mathbb{R} \to \mathbb{R}$ are C^∞-smooth functions and let $h_-(x_0) = h_+(x_0)$ and the function $h : \mathbb{R} \to \mathbb{R}$ is defined by the formula

$$h(x) = \begin{cases} h_-(x), & x \leqslant x_0, \\ h_+(x), & x > x_0. \end{cases}$$

Let the function $h(x)$ is not smooth at the point x_0. We say C^∞-smooth function

$$\tilde{h}(x) = \left(1 - \sigma\left(\frac{x - x_0}{\varepsilon}\right)\right) h_-(x) + \sigma\left(\frac{x - x_0}{\varepsilon}\right) h_+(x)$$

to be *smoothing of the function h(x) at the point* x_0 for any $\varepsilon > 0$ (see figure 5).

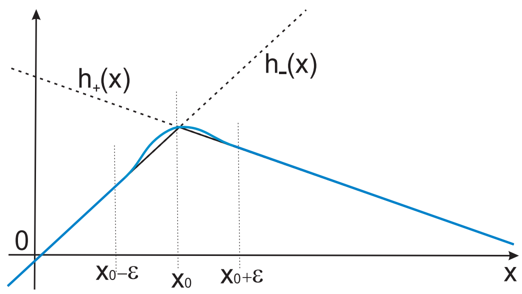

Figure 5. Graph of the smoothing $\tilde{h}(x)$ of the function $h(x)$ at the point x_0.

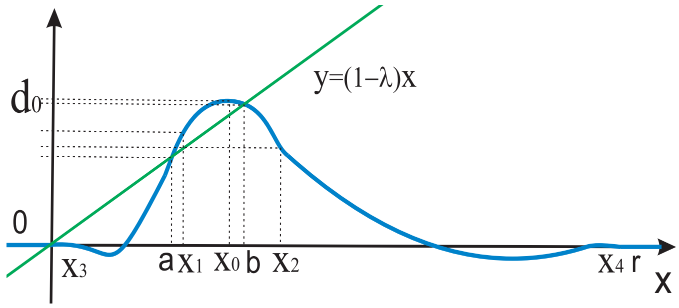

Figure 6. Graph of the function $\varphi(x)$.

By construction $\tilde{h}(x)$ differs from $h(x)$ only in ε-neighborhood of the point x_0 and

$$\lim_{\varepsilon\to 0}\left(\max_{x\in[x_0-\varepsilon,x_0+\varepsilon]}|\tilde{h}(x)-h(x)|\right)=0. \tag{5}$$

Concept of smoothing is naturally generalized to continuous functions composed by a finite number of smooth parts.

Let

$$\theta=\frac{\mu-1}{1-\lambda}>0, \tag{6}$$

$$c_0=\left(1+\frac{\theta}{4}\right)(1-\lambda). \tag{7}$$

Lemma 3.1. *There is a C^∞-smooth function $\varphi(x)$ (see figure 6), having the following properties for some values*

$$0<x_3<a<x_1<x_0<b<x_2<x_4<r:$$

1) $\varphi(x)\equiv 0$ outside the segment $[x_3,x_4]$;
2) $|\varphi(x)|\leqslant\varphi(x_0)=d_0$ and $x_0(1-\lambda)<d_0<x_0c_0$;
3) $-\frac{\lambda}{2}<\varphi'(x)<\left(1+\frac{\theta}{2}\right)(1-\lambda)$ and $\varphi'(x_1)>\frac{\varphi(x_1)}{x_1}$;
4) $\varphi(x)\equiv -d_1(x-x_0)^2+d_0$ on the segment $[x_1,x_2]$ where $d_1>0$;
5) The equation $\varphi(x)=(1-\lambda)x$ for $x>0$ has exactly two solutions $x=a$, $x=b$ such that $\varphi'(a)>1-\lambda$, $\varphi'(b)<0$.

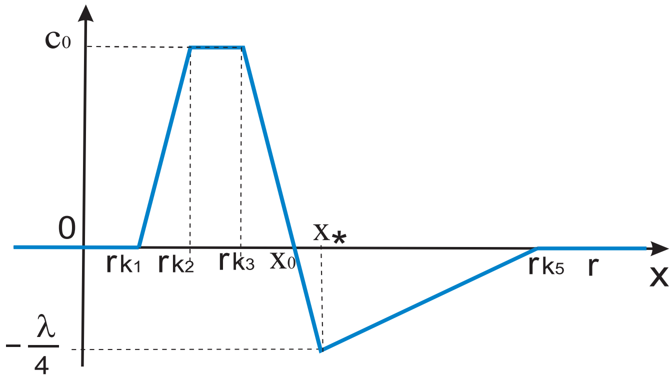

Figure 7. Graph of the function $\psi(x)$.

Proof. We will find a function $\varphi(x)$, satisfying the conditions 1)-5), in the form

$$\varphi(x) = \int_{-\infty}^{x} \tilde{\psi}(s)\,\mathrm{d}s,$$

where $\tilde{\psi}$ is a C^∞-smoothing of a piecewise linear function ψ. For this aim let us define a piecewise linear function $\psi : \mathbb{R} \to \mathbb{R}$, depending on the parameters $0 < k_1 < k_2 < k_3 < k_4 < k_5 < 1$, by its graph on figure 7(here $x_0 = rk_4$, $x_* = \psi^{-1}\left(-\frac{\lambda}{4}\right)$). Let

$$\bar{\varphi}(x) = \int_{-\infty}^{x} \psi(s)\,\mathrm{d}s.$$

To prove the present lemma it is enough to show that there are such constants k_i, $i \in \{1, \ldots, 5\}$ that the function $\bar{\varphi}(x)$ has the following properties:

$\bar{1}$) $\bar{\varphi}(x) \equiv 0$ outside the segment $[rk_1, rk_5]$;
$\bar{2}$) $|\bar{\varphi}(x)| \leqslant \bar{\varphi}(x_0) = \bar{d}_0$ and $x_0(1-\lambda) < \bar{d}_0 < x_0 c_0$;
$\bar{3}$) $-\frac{\lambda}{2} < \bar{\varphi}'(x) < \left(1 + \frac{\theta}{2}\right)(1-\lambda)$ and $\bar{\varphi}'(rk_3) > \frac{\bar{\varphi}(rk_3)}{rk_3}$;
$\bar{4}$) $\bar{\varphi}(x) \equiv -\bar{d}_1 (x - x_0)^2 + \bar{d}_0$ on the segment $[rk_3, x_*]$ where $\bar{d}_1 > 0$;
$\bar{5}$) the equation $\bar{\varphi}(x) = (1-\lambda)x$ for $x > 0$ has exactly two solutions $x = \bar{a}$, $x = \bar{b}$ where $rk_2 < \bar{a} < x_0 < \bar{b} < rk_5$ at the same time $\bar{\varphi}'(\bar{a}) > 1 - \lambda$, $\bar{\varphi}'(\bar{b}) < 0$.

$\bar{1}$) The condition $\bar{\varphi}(x) \equiv 0$ outside the segment $[rk_1, rk_5]$ is equivalent to the equality of the areas of the trapezoid above the Ox axis and the triangle under the Ox axis, which is expressed by the following equality

$$(k_4 - k_1 + k_3 - k_2)\left(1 + \frac{\theta}{4}\right)(1-\lambda) = (k_5 - k_4)\frac{\lambda}{4}. \tag{8}$$

$\bar{2}$) The property $|\bar{\varphi}(x)| \leqslant \bar{\varphi}(x_0)$ is an immediate consequence of the fact that the function $\psi(x)$ changes the sign from $+$ to $-$ at the point x_0. Note that $\bar{d}_0 = \bar{\varphi}(x_0)$ is equal to the area of the trapezoid located above the Ox axis so $\bar{d}_0 < x_0 c_0$. Fulfilling the condition $\bar{d}_0 > x_0(1-\lambda)$ is equivalent to inequality

$$(k_4 - k_1 + k_3 - k_2)\left(1 + \frac{\theta}{4}\right) > 2k_4. \tag{9}$$

$\bar{3}$) The property $-\frac{\lambda}{2} < \bar{\varphi}'(x) < \left(1+\frac{\theta}{2}\right)(1-\lambda)$ follows directly from the inequality

$$-\frac{\lambda}{4} \leqslant \psi(x) \leqslant \left(1+\frac{\theta}{4}\right)(1-\lambda).$$

Property $\bar{\varphi}'(rk_3) > \frac{\bar{\varphi}(rk_3)}{rk_3}$ is executed because $\varphi(rk_3)$ is the area of a trapezoid located above the Ox axis and having a lower base $[rk_1, rk_3]$, $\bar{\varphi}'(rk_3) = \psi(rk_3) = c_0$ and therefore

$$\bar{\varphi}(rk_3) = r(2k_3 - (k_1 + k_2))\frac{c_0}{2} < c_0 rk_3.$$

$\bar{4}$) Since the function $\psi(x)$ on the segment $[rk_3, x_*]$ is a straight line with a negative slope then the function $\bar{\varphi}(x)$ on this segment is a quadratic function with a negative coefficient at x^2. It follows from point $\bar{2}$) that the vertex of the parabola is at the point $(x_0, \bar{d}_0)$ and, therefore,

$$\bar{\varphi}(x) \equiv -\bar{d}_1(x - x_0)^2 + \bar{d}_0$$

on the segment $[rk_3, x_*]$ where $\bar{d}_1 > 0$.

$\bar{5}$) The function $\bar{\varphi}(x)$ on the segment $[rk_3, x_*]$ coincides with the parabola $y = -\bar{d}_1(x - x_0)^2 + \bar{d}_0$ whose vertex lies above the straight line $y = (1-\lambda)x$ and coincides with a straight line with a slope c_0 on the segment $[rk_2, rk_3]$ by construction. Then to prove the property of $\bar{5}$) it is enough to achieve the condition

$$\bar{\varphi}(rk_2) < (1-\lambda)\, rk_2. \tag{10}$$

Note that $\bar{\varphi}(rk_2)$ is equal to the area of a located above the Ox axis and having a base of $[rk_1, rk_2]$ triangle. Therefore the condition (10) is equivalent to inequality

$$(k_2 - k_1)\left(1 + \frac{\theta}{4}\right) < 2k_2. \tag{11}$$

Let us show how to choose constants k_i satisfying all the described conditions. Indeed, let

$$\ell = \frac{1}{1 + \frac{\theta}{4}}$$

and represent the constants $k_i - k_{i-1}$, $i \in \{1, \ldots, 5\}$, $k_0 = 0$ as

$$k_i - k_{i-1} = \ell_i k_4.$$

Then the inequalities (9) and (11) have the following forms

$$(2\ell_3 + \ell_2 + \ell_4) > 2\ell, \tag{12}$$
$$\ell_2 < 2(\ell_1 + \ell_2)\,\ell. \tag{13}$$

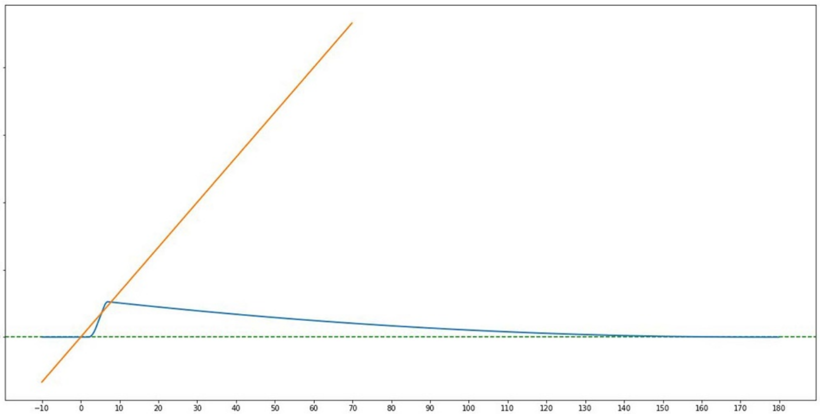

Figure 8. Graph of the function $\bar{\varphi}(x)$ for $\theta = 4$, $\lambda = \frac{1}{4}$, $r = 10$.

To achieve the inequality (12) let us put

$$\ell_3 = \ell,$$

and for the inequality (13) let

$$\ell_1 = \frac{\ell(1-\ell)}{1+\ell}, \ell_2 = \ell_1 \ell.$$

The constant ℓ_4 is calculated from the condition

$$\ell_1 + \ell_2 + \ell_3 + \ell_4 = 1$$

and therefore

$$\ell_4 = (1-\ell)^2.$$

The equality (8) in the entered variables has the following form

$$4(2\ell_3 + \ell_2 + \ell_4)(1-\lambda) = \ell_5 \lambda \ell. \tag{14}$$

We find ℓ_5 from (14). Since $k_5 = (1+\ell_5)k_4 < 1$ then k_4 can be taken as any value satisfying the inequality

$$k_4 < \frac{1}{1+\ell_5}.$$

As illustration we calculate all values ℓ_i, k_i for $\theta = 4$, $\lambda = \frac{1}{4}$, $r = 10$ and plot the graph of the function $\bar{\varphi}(x)$ using a computer (see figure 8). □

We define a C^∞-smooth function $\varphi_t : \mathbb{R} \to \mathbb{R}$ for $t \in [0,1]$ by the formula

$$\varphi_t(x) = \sigma(2t-1)\varphi(x). \tag{15}$$

Lemma 3.2. *The functions $\varphi_t(x)$ for $x > 0$ have the following properties:*

1) *there is a unique value $t_0 \in (0,1)$ such that $\varphi_{t_0}(x) < (1-\lambda)x$ for any $x > 0$, excepted a unique point $x = q_0 \in (x_1, x_0)$, where $\varphi_{t_0}(q_0) = (1-\lambda)q_0$ é $\varphi'_{t_0}(q_0) = (1-\lambda)$;*
2) *$\varphi_t(x) < (1-\lambda)x$ for any $t \in [0, t_0)$;*
3) *for any $t \in (t_0, 1]$ the equation $\varphi_t(x) = (1-\lambda)x$ has exactly two solutions $x = a_t, x = b_t$ where $x_1 < a_t < b_t < r$ and $\varphi'_t(a_t) > 1-\lambda$, $\varphi'_t(b_t) < 1-\lambda$.*

Proof. It follows from the definition of the function σ that $\varphi_0(x) \equiv 0$ and $\varphi_1(x) \equiv \varphi(x)$. It is obviously that the function $\varphi_1(x)$ has property 3) by lemma 3.1. The intersection points of the graph of the function $\varphi_t(x)$ with the line $y = (1-\lambda)x$ for any $t \in (0,1)$ are exactly the intersection points of the graph of the function $\varphi(x)$ with the line $y = \nu_t x$ where

$$\nu_t = \frac{1-\lambda}{\sigma(2t-1)}.$$

It follows from item 5) of lemma 3.1 that if such points exist then they belong to the segment $[a,b]$. Since ν_t accepts any values greater than $1-\lambda$, there exists a value $t_* \in (0,1)$ such that $\nu_{t_*} = \frac{\varphi(x_1)}{x_1}$. According to item 4) of lemma 3.1 the function $\varphi(x)$ coincides with the parabola $y = -d_1(x-x_0)^2 + d_0$ on the segment $[x_1, x_2]$ therefore the desired intersection points are exactly solutions of the equation

$$-d_1(x-x_0)^2 + d_0 = \nu_t x \tag{16}$$

for any $t \leqslant t_*$. Let the discriminant of the quadratic equation (16) equals 0 at $t = t_0$. Then this equation has exactly two solutions a_t, b_t for $t_0 < t \leqslant t_*$, one solution q_0 for $t = t_0$ and has no any solutions for $t < t_0$. The point q_0 is the tangent point of the graph of the function $\varphi_{t_0}(x)$ and the straight line $y = (1-\lambda)x$, hence $\varphi'_{t_0}(q_0) = (1-\lambda)$. The graph of the function $\varphi_{t_0}(x)$ intersects transversely the line $y = (1-\lambda)x$ at the point a_t with a slope $\varphi'_t(a_t) > 1-\lambda$ and at the point b_t with a slope $\varphi'_t(b_t) < 1-\lambda$.

As $\varphi'_t(x) = \sigma(2t-1)\varphi'(x)$ then its values on the segment $[a, x_1]$ are not less than $\sigma(2t-1)\varphi'(x_1)$ for $t \in (t_*, 1)$. Therefore the smallest value of the derivative is $\sigma(2t_*-1)\varphi'(x_1)$. Since t_* is a solution of the equation

$$\sigma(2t_*-1)\varphi(x_1) = (1-\lambda)x_1,$$

and, by item 2) of lemma 3.1,

$$\varphi(x_1) < x_1\varphi'(x_1),$$

then

$$\sigma(2t_*-1)\varphi(x_1) > 1-\lambda.$$

Since the function $\varphi_t(x)$ is monotonously decreasing on the segment $[a, x_1]$ then its graph has a unique intersection point a_t with the line $y = (1-\lambda)x$ on this segment and $\varphi'_t(a_t) > 1-\lambda$. Since the function $\varphi_t(x)$ coincides on the segment $[x_1, x_2]$ with a parabola whose branches are directed downward and the vertex lies to the right of the point x_1 then there is exactly one more intersection point $b_t > x_1$ of the graph of the function $\varphi_t(x)$ with a straight line $y = (1-\lambda)x$ and $\varphi'_t(b_t) < 1-\lambda$. □

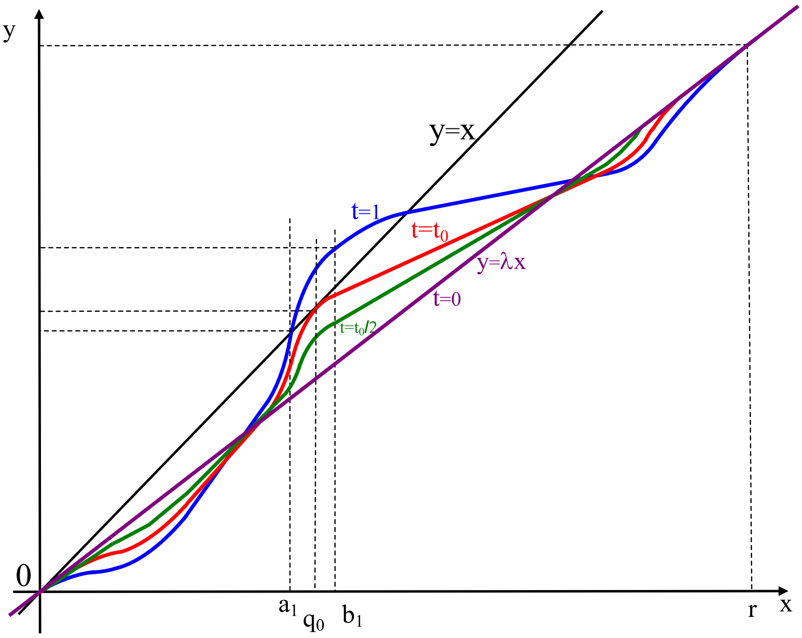

Figure 9. Graphs of the funcions $\gamma_t(x)$ for $t = 0$, $t = \frac{t_0}{2}$, $t = t_0$, $t = 1$.

Note that by lemma 3.2, the graphs of the functions $\gamma_t(x) = \lambda x + \varphi_t(x)$ have the form shown in figure 9.

Let

$$\vartheta(y) = \left(1 - \sigma\left(\frac{4||y|| - 2\delta}{\delta}\right)\right) \tag{17}$$

for $y \in \mathbb{R}^{n-1}$, where $||y|| < \delta$. It follows directly from the formula (4) that

$$|\vartheta'(y)| < \frac{9}{2\delta}. \tag{18}$$

We define C^∞-smooth map $\phi_t : V \to \mathbb{R}$ by the formula

$$\phi_t(x,y) = \vartheta(y)\varphi_t(x), \tag{19}$$

and define C^∞-smooth map $\Phi_t : V \to \mathbb{R} \times \mathbb{R}^{n-1}$ by the formula

$$\Phi_t(x,y) = (\phi_t(x,y), 0). \tag{20}$$

Notice, that by construction $\Phi_t(x,y) = (0,0)$ if $(x,y) \in \partial V$. Let

$$\bar{\Phi}_t = \bar{C}\Phi_t\bar{C}^{-1} : \bar{V} \to \mathbb{R}^n.$$

Then the map $\bar{\Phi}_t$ can be continued on $\mathbb{R}^n$ so that the following properties are met in coordinates $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with the origin $\bar{0} = (0, \ldots, 0)$

- $\bar{\Phi}_t(\bar{x}) = \bar{0}$ outside of all integer shifts of the set $\bar{V}$;
- $\bar{\Phi}_t(\bar{x})$ is 1-periodical function on the vector argument $\bar{x}$.

We define a smooth arc $\bar{f}_t : \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$\bar{f}_t(\bar{x}) = \bar{A}(\bar{x}) + \bar{\Phi}_t(\bar{x})$$

and a smooth arc $f_t : \mathbb{T}^n \to \mathbb{T}^n$ from the formula

$$p\bar{f}_t = f_t p. \tag{21}$$

The above construction holds for any $r > 0$, $\delta > 0$ and admits an obvious generalization to the case of modification of the Anosov diffeomorphism in the neighbourhood of any number of periodic orbits. For simplisity we prove theorem 1 for the constructed arc f_t in the next section.

4. Mild stability of the arc f_t

Lemma 4.1. *There are parameters $r > 0$, $\delta > 0$ such that the arc f_t, defined by the formula (21), is mildly stable and joints the diffeomorphism $f_0 = \widehat{A}$ with a DA-diffeomorphism f_1.*

Proof.
I. Mild stability of the arc. For any point $x \in \mathbb{T}^n$ we denote by $\mathcal{F}^s{}_x$ and $\mathcal{F}^u{}_x$ the leaves of stable $\mathcal{F}^s$, and unstable $\mathcal{F}^u$ foliations of the Anosov diffeomorphism $\widehat{A}$ passing through the point x respectively. By construction, the diffeomorphism f_t for any $t \in [0,1]$ preserves the invariant foliation $\mathcal{F}^s$. Note that in a general case the foliation $\mathcal{F}^u$ is not invariant with respect to the diffeomorphism f_t. However, every diffeomorphism f_t has a fixed saddle point $O = p(\bar{0})$. Moreover the diffeomorphism f_t coincides with the diffeomorphism $\widehat{A}$ in some neighborhood of the point O.

To prove the mild stability of the arc f_t, by [10], it is enough to achieve the following its properties:

1) any diffeomorphism f_t, $t \in [0, t_0)$ is an Anosov diffeomorphism;
2) the arc f_t *unfolds generically* at the saddle-node point (q, t_0), where $q = p(\bar{C}(q_0, 0))$, that is the arc f_t undergoes a bifurcation at the saddle-node, whose strongly unstable foliation is transversal to the foliation $\mathcal{F}^s$;
3) any diffeomorphism f_t, $t \in (t_0, 1]$ is a structurally stable diffeomorphism whose chain recurrent set consists of a $(n-1)$-dimensional expanding attractor Λ_t with two boundary points $\beta_0 = p(\bar{C}(0,0))$, $\beta_t = p(\bar{C}(b_t, 0))$ and an isolated source $\alpha_t = p(\bar{C}(a_t, 0))$.

Recall that an arc f_t *undergoes a saddle-node bifurcation* at a point (q, t_0), if the arc f_t is conjugate to the arc

$$\tilde{f}_{\tilde{t}}(x_1, x_2, \dots, x_n) = \left(x_1 - \frac{x_1^2}{2} + \tilde{t}, \pm 2x_2, 2x_3, \dots, 2x_n\right), \tag{22}$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\tilde{t}| < 1/10$ in some neighborhood of this point (see figure 10).

In local coordinates $(x_1, \dots, x_n, \tilde{t})$ the bifurcation occurs at the time $\tilde{t} = 0$, the coordinates origin $O \in \mathbb{R}^n$ being the saddle-node. Here

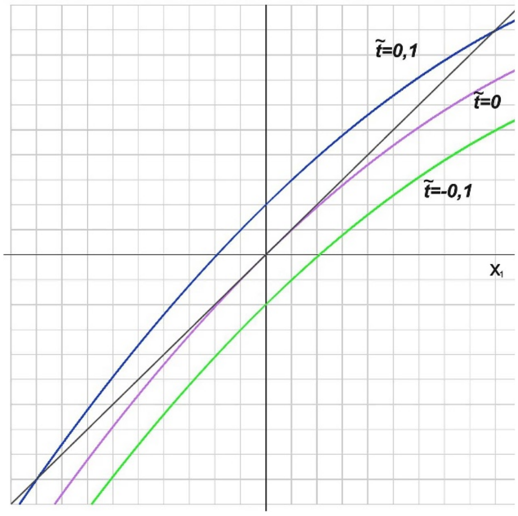

Figure 10. Graphs of the map $x_1 - \frac{x_1^2}{2} + \tilde{t}$ for $\tilde{t} = -0,1; \tilde{t} = 0$ and $\tilde{t} = 0,1$.

- Ox_1—the *central manifold*;
- $W^s_O = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \geqslant 0, x_2 = \cdots = x_n = 0\}$—the *stable manifold* of O;
- $W^u_O = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \leqslant 0\}$—the *unstable manifold* of O;
- $\mathcal{F}^{uu}_O = \bigcup\limits_{c>0} \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 = c\}$—the *strongly unstable foliation*.

If q is the saddle-node of the diffeomorphism f_{t_0} then, by [5], there is a unique f_{t_0}-invariant *strongly unstable foliation* $\mathcal{F}^{uu}_q$ with smooth leaves (see Fig 11) such that ∂W^u_q is a leaf of this foliation.

II. The existence of cones. Let

$$F_t = \bar{C}^{-1} \bar{f}_t \bar{C} : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}.$$

Then the diffeomorphism $F_t|_V$ has a form

$$F_t(x,y) = (\Gamma_t(x,y), \bar{B}(y)), \tag{23}$$

where $\Gamma_t : V \to \mathbb{R}$ is the map defined by the formula

$$\Gamma_t(x,y) = \lambda x + \Phi_t(x,y). \tag{24}$$

It follows from formulas (19) and (20) that the Jacobi matrix $Y_{F_t}(x,y)$ of the map $F_t(x,y)$ has the form

$$Y_{F_t}|_{(x,y)} = \begin{pmatrix} \lambda + L_t(x,y) & M_t(x,y) \\ 0 & B \end{pmatrix}, \tag{25}$$

where

$$L_t(x,y) = \vartheta(y)\varphi_t'(x),$$
$$M_t(x,y) = \vartheta'(y)\varphi_t(x).$$

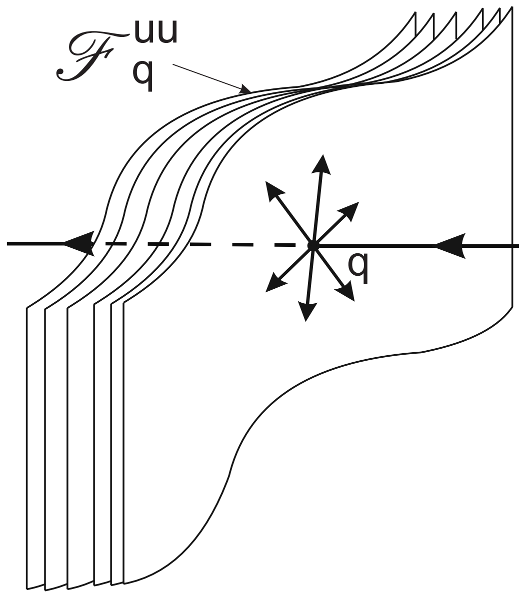

Figure 11. Strongly unstable lamination of the saddle node q.

Then the differential DF_t of the map F_t in coordinates $v \in E_v \cong \mathbb{R}$, $w \in E_w \cong \mathbb{R}^{n-1}$ of the tangent bundle has the form

$$\mathrm{DF}_t(v,w) = ((\lambda + L_t)v + M_t w, Bw) = (v', w'). \tag{26}$$

Directly from lemmas 3.1, 3.2 and formulas (18), (17) we obtain the following estimates

$$\frac{\lambda}{2} < \lambda + L_t < \frac{\mu+1}{2}. \tag{27}$$

$$|M_t| < \frac{9rc_0}{2\delta}. \tag{28}$$

We choose the values of r, δ so that

$$|M_t| < \frac{\mu-1}{4}\sqrt{\mu^2-1} = m_0. \tag{29}$$

Let

$$\gamma = \frac{\sqrt{\mu^2-1}}{2} \tag{30}$$

and

$$K_\gamma = \{(v,w) : ||v|| \leqslant \gamma ||w||\}.$$

We show that for any $(v,w) \in K_\gamma$ the inequalities

$$||v'|| < \gamma ||w'||, \tag{31}$$

$$v'^2 + w'^2 > v^2 + w^2 \tag{32}$$

are valid. Indeed, we get (31) from the following chain of inequalities

$$||v'|| = ||(\lambda + L_t)v + M_t w|| \leqslant ||(\lambda + L_t)v|| + ||M_t w|| <$$
$$< \frac{\mu+1}{2}||v|| + m_0||w|| \leqslant \left(\frac{\mu+1}{2}\gamma + m_0\right)\frac{||w'||}{\mu} <$$
$$< \frac{\sqrt{\mu^2-1}}{2}||w'|| = \gamma||w'||.$$

We get (32) from the following chain of inequalities

$$v'^2 + w'^2 = ||(\lambda + L_t)v + M_t w||^2 + ||w'||^2 \geqslant \mu^2||w||$$
$$\geqslant v^2 + w^2 + \left(\frac{\mu^2-1}{\gamma^2} - 1\right)v^2 > v^2 + w^2.$$

III. Fulfilling the properties 1)-3) of a mildly stable arc. Due to the criterion of cones (see, for example, [6, Corollary 6.4.8]), constructed in the previous section the cone K_γ guarantees the existence of a continuous DF_t-invariant decomposition of the tangent bundle into subbundles

$$E_v \oplus E^u,\ \dim E^u = n-1 \tag{33}$$

such that for some constant $\mu_t > 1$ and any $k \in \mathbb{N}$ the inequality holds

$$||\mathrm{DF}_t^{-k}(u)|| \leqslant \mu_t^{-k}||u||, \quad \forall u \in E^u.$$

1) If $t \in [0, t_0)$ then according to lemma 3.2

$$0 < \lambda + L_t < \lambda_t < 1,$$

and for any $k \in \mathbb{N}$ the inequality

$$||\mathrm{DF}_t^{k}(v)|| \leqslant \lambda_t^{k}||v||, \quad \forall v \in E_v$$

is valid. Thus f_t is an Anosov diffeomorphism.

2) It follows from lemma 3.2 that the arc $\gamma_t(x) = \lambda x + \varphi_t(x)$ is conjugate to the arc $x_1 - \frac{x_1^2}{2} + \tilde{t}$ in a neighborhood of the point (q_0, t_0). Since the map $\bar{B}$ is a linear hyperbolic extension then (see, for example, [15, theorem 5.5]) the arc f_t is conjugate to the arc $\tilde{f}_{\tilde{t}}$ in some neighborhood of the point $(q, t_0) \in \mathbb{T}^n \times [0, 1]$. Thus, the saddle-node bifurcation unfolds generically on the arc f_t at the point (q, t_0) where $q = p(\bar{C}(q_0, 0))$.

3) For any $t \in (t_0, 1]$, the diffeomorphism $\gamma_t(x) = \lambda x + \varphi_t(x)$ has exactly three fixed points: hyperbolic sinks $x = 0$, $x = a_t$ and hyperbolic source $x = b_t$. In this case the point a_t is an isolated source of the diffeomorphism F_t and the point $\alpha_t = p(\bar{C}(a_t, 0))$ is an isolated source of the diffeomorphism f_t.

We show that $\mathbb{T}^n \setminus W^u_{\alpha_t}$ is a hyperbolic set. To do this it is enough to show that the set $U = \{(x,y) \in V : \lambda + L_t(x,y) \geqslant 1\}$ is a subset of the unstable manifold $W^u_{a_t}$.

Note that the diffeomorphism $F_t(x,y)$ has the form

$$F_t(x,y) = (\lambda + \varphi_t(x,y), \bar{B}(x,y)).$$

on the set $G = [0, b_t] \times \left[-\frac{\delta}{4}, \frac{\delta}{4}\right]$. It follows from the properties of the diffeomorphism φ_t that $(U \cap G) \subset W^u_{a_t}$ and therefore $F_t^{-1}(U \cap G) \subset \mathrm{int}(U \cap G)$. Let $U_y = U \cap \mathbb{R} \times \{y\}$. Since $\vartheta(y)$ decreases monotonously from 1 to 0 with respect to the variable $||y||$ then $F_t^{-1}(U_y) \subset U_{\bar{B}^{-1}(y)}$ and therefore $U \subset W^u_{a_t}$.

Let $\Lambda_t = \mathcal{R}_{f_t} \setminus a_t$. We prove that Λ_t is a $(n-1)$-dimensional expanding attractor.

To do this we first show that $W^u_O \subset \Lambda_t$. Since $f_t = \widehat{A}$ in some neighborhood of the point O then $O \notin W^u_{a_t}$ and therefore $W^u_O \subset \Lambda_t$. In addition $\mathrm{cl}(W^s_O \cap W^u_O) = W^u_O$, that directly implies that all points of W^u_O are chain recurrent. Since $\Lambda_t \subset (\mathbb{T}^n \setminus W^u_{\alpha_t})$ then Λ_t is a hyperbolic set and therefore periodic points are dense in it.

Let us show that $\mathrm{cl}\, W^u_O = \Lambda_t$.

Indeed there is a periodic point p of the period m_p in any neighborhood of U_a of any point $a \in \Lambda_t$. Then $f_t^{m_p}(\mathcal{F}^s{}_p) = \mathcal{F}^s{}_p$. Since the leaf $\mathcal{F}^s{}_p$ is everywhere dense on the torus $\mathbb{T}^n$, then there is a point $q \in (\mathcal{F}^s{}_p \cap W^u_O)$. Then $\lim\limits_{k\to\infty} d(f_t^{m_p k}(q), p) = 0$ and therefore $U_a \cap W^u_O \neq \emptyset$.

Thus Λ_t is a basic set. Its topological dimension is less than n since otherwise it would coincide with the entire torus $\mathbb{T}^n$ (see, for example, [3, Lemma 8.1]). On the other hand $\dim W^u_x = n-1$, $x \in \Lambda_t$ which means $\dim \Lambda_t = n-1$ and therefore Λ_t is a $(n-1)$-dimensional expanding attractor. □

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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