MATHEMATICAL AND PHYSICAL BILLIARD IN PYRAMIDS

YURY KOCHETKOV AND LEV PYATKO

Abstract. In this experimental work we study billiard trajectories in triangular pyramids and try to establish conditions that guarantee the existence (or absence) of 4-cycles (there can be not more, than three of them). We formulate conjectures and prove some statements. For example, if a pyramid has two orthogonal faces, then it has not more than two 4-cycles. Also we study 4-cycles of the "physical" billiard in pyramids, i.e. in the presence of gravity. Here we present our observations for a generic case.

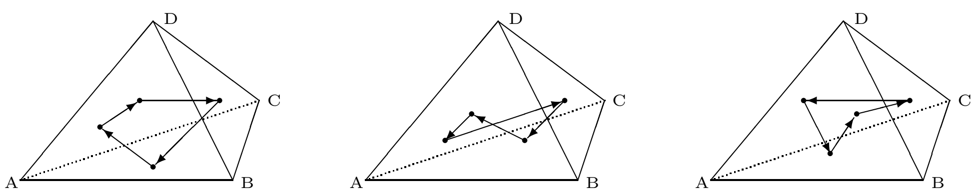
1. Introduction

The two-dimensional mathematical billiard is a vast and well known part of geometry (see [1] or [2]). However, the 3-dimensional billiard is much less known and the physical billiard (in the presence of gravity) is, probably, a novel area of investigation. We are interested in the simplest problem here: the existence of 4-cycles in triangular pyramids. Cycles in pyramids were studied in [3] and [4]. In [3] periodic trajectories in right-angled tetrahedra were constructed, but there 4-cycles do not exist. In [4] only pyramids in a neighborhood of the regular tetrahedron were considered.

Our work is organized in the following way. In the second section we explain how one can find a 4-cycle with the given order of reflections. In the third section we introduce the "map of cycles", i.e. for the given base of a pyramid and a variable altitude we study 4-cycles with the given order of reflections. In the forth section we prove statements about existence and behavior of 4-cycles in some special cases. In the fifth section we present results of our investigations in the physical case, i.e. in the presence of gravity.

2. Construction of 4-cycles

Usually we will work with pyramid $ABCD$ with base ABC in the xy plane, the apex D is in the upper half-space, A — at the origin, B — at the positive x-axis, C — in the upper half plane, DO will be the altitude and $h = |DO|$. Each 4-cycle is determined by an order of reflections. There are three possible trajectories, because each trajectory can be passed in both directions:



ABC → ABD → ACD → → BCD → ABC   ABC → ACD → ABD → → BCD → ABC   ABC → ABD → BCD → → ACD → ABC

Figure 1

Let p_0 be the starting point of our trajectory (usually it will be a point in the base ABC), $\bar{v}$ be the starting vector and $F_1F_2F_3F_4$ be the order of reflections. Each reflection defines an orthogonal operator R_i with the matrix M_i. The composition $R_1 \circ R_2 \circ R_3 \circ R_4$ is a rotation with the matrix $M = M_4M_3M_2M_1$. Thus, $\bar{v}$ is the eigenvector of M with the eigenvalue 1.

Remark 2.1. $R_1 \circ R_2$ is a rotation around an edge and $R_3 \circ R_4$ also is a rotation around another edge. These two rotations are rotations around *skew lines*, hence, the composition $R_1 \circ R_2 \circ R_3 \circ R_4$ cannot be the identity.

Example 2.1. Let $A = (0,0,0)$, $B = (4,0,0)$, $C = (2,4,0)$ and $D = (2,3,3)$, $p_0 \in ABC$ and the order of reflections be $ABD \to ACD \to BCD \to ABC$. Let M_1, M_2, M_3, M_4 be matrices of reflections with respect to planes ABD, ACD, BCD, ABC, respectively. Here

$$M_4M_3M_2M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -13/23 & -18/23 & -6/23 \\ -18/23 & 14/23 & -3/23 \\ -6/23 & -3/23 & 22/23 \end{pmatrix} \begin{pmatrix} -13/23 & 18/23 & 6/23 \\ 18/23 & 14/23 & -3/23 \\ 6/23 & -3/23 & 22/23 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{529} \begin{pmatrix} -191 & -156 & -468 \\ 468 & -216 & -119 \\ -156 & -457 & 216 \end{pmatrix} = M$$

The vector $\bar{v} = (-13, -12, 24)$ is the the eigenvector with eigenvalue 1.

Now we must find the starting point. We construct the pyramid ABC_1D, where C and C_1 are symmetric with respect to the plane ABD. Then we construct the pyramid AB_1C_1D, where B and B_1 are symmetric with respect to the plane AC_1D. And then we construct the pyramid $A_1B_1C_1D$, where A and A_1 are symmetric with respect to the plane B_1C_1D. Here $C_1 = (2,0,4)$, $B_1 = \left(-\frac{52}{23}, \frac{24}{23}, \frac{72}{23}\right)$, $A_1 = \left(-\frac{432}{529}, \frac{1176}{529}, \frac{3528}{529}\right)$. We replace a polygonal trajectory of our 4-cycle by a line that connects a point $F \in \Delta ABC$ with barycentric coordinates (x, y, z) and a point $F_1 \in \Delta A_1B_1C_1$ with the same barycentric coordinates. If F is the starting point of the 4-cycle, then the vector $\overline{FF_1}$ is collinear to the vector $\bar{v}$. This condition defines x, y and z:

$$x = \frac{115}{1778}\,,\ y = \frac{583}{1778}\,,\ z = \frac{540}{889} \Rightarrow F = \left(\frac{2246}{889}, \frac{2160}{889}, 0\right).$$

It remains to check that the line FF_1 intersects planes B_1C_1D, AC_1D and ABD *inside* triangles ΔB_1C_1D, ΔAC_1D and ΔABD, respectively.

Proposition 2.1. *For a given order of reflections we have either one 4-cycle, or none.*

Proof. After constructing pyramids ABC_1D, AB_1C_1D and $A_1B_1C_1D$, we solve a linear system to find a starting point F. But a linear system has either a unique solution, or infinitely many (a line), or none. We must demonstrate that infinite case is impossible.

Let us assume that points of a line $\ell \subset ABC$ are solutions of our system. Points with the same barycentric coordinates constitute a line $\ell_1 \subset A_1B_1C_1$. Let points $P \in \ell$ and $P_1 \in \ell_1$ have the same coordinates and points $Q \in \ell$ and $Q_1 \in \ell_1$ also. Then lines PP_1 and QQ_1 are parallel. As $|PQ| = |P_1Q_1|$, then PQQ_1P_1 is a parallelogram, i.e. $\ell \parallel \ell_1$.

Let R be the composition of reflections with respect to planes ABD, AC_1D, B_1C_1D. Then R is a rotation ρ around some line L and a reflection π with respect to the plane Π, orthogonal to L. As R maps ℓ into ℓ_1, then ℓ is parallel to L. But the reflection π changes the barycentric order of points in ℓ into opposite. □

3. Computations and conjectures

Let $ABCD$ be a pyramid, where ΔABC is a fixed acute triangle in the plane xy. The point O — the base of the altitude DO is fixed, but the height $h = |DO|$ is variable. Let $ABC \to ACD \to ABD \to BCD \to ABC$ be the order of reflections. Then a 4-cycle with this order of reflections either

(1) exists for all $h > a \geqslant 0$;
(2) or exists for $0 \leqslant a < h < b < \infty$;
(3) or does not exist for all $h > 0$.

Definition 3.1. Points O for which the first case is realized constitute the α-set. Points O for which the second case is realized constitute the β-set. Points O for which the third case is realized constitute the γ-set. The arrangement of these three sets in the plane will be called the *the map of cycles*.

Remark 3.1. α-set for the cycle $ABC \to ACD \to ABD \to BCD \to ABC$ and α-set for the cycle $ABC \to ABD \to ACD \to BCD \to ABC$ are, of course, different sets.

How one can describe these sets? We can give only a partial answer. Let ABC be a fixed acute triangle: A at origin, B at the positive x-axis, C at the upper half-plane. We will describe the map of cycles for the order $ABC \to ACD \to ABD \to BCD \to ABC$. The construction of the map is performed in the following steps (Figure 2): a) we rotate ΔABC on π around the center of AB and obtain the triangle ABC'; b) we draw altitudes AG, AG', BF, BF'; c) we draw the lines CC', FF' and GG' (the last two lines are parallel).

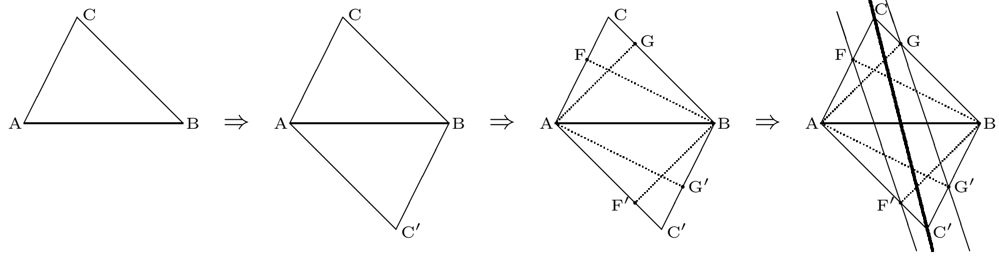


Figure 2

Conjecture 3.1. The map of cycles in the upper half-plane. *(See Figure 3). The infinite polygonal domain $QNMR$ minus the segment $[G, K]$ and the ray FR is the α-set. The union of open triangles ΔAFM and ΔBGN and the open infinite sector with vertex K, bounded by rays KP and KQ, is the β-set. All other points of the upper half-plane belong to the γ-set.*

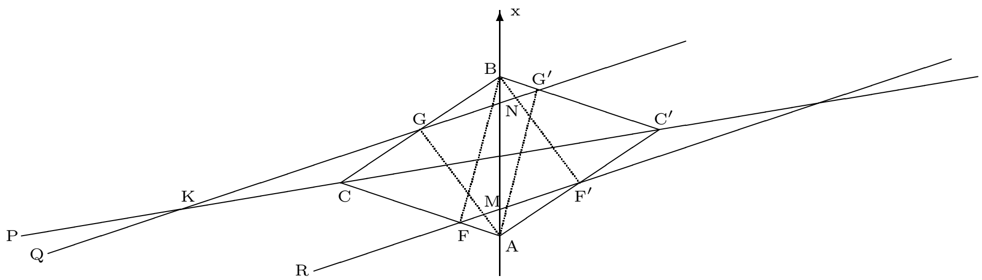


Figure 3

Remark 3.2. Situation in the lower half-plane is much more complicated. In particular, there are points *inside* $\Delta ABC'$ that belong to the γ-set.

Remark 3.3. Cases of a right or an obtuse triangles ABC are also much more complicated.

Example 3.1. Let us consider the acute triangle ΔABC, $A = (0, 0)$, $B = (15, 0)$, $C = (5, 10)$. The line $l : 3x + y = \frac{45}{2}$ is the central line of the α-set in the upper half-plane. Here is the plot of the function $a(y)$, $(x, y) \in l$, $0 \leqslant y < \infty$:

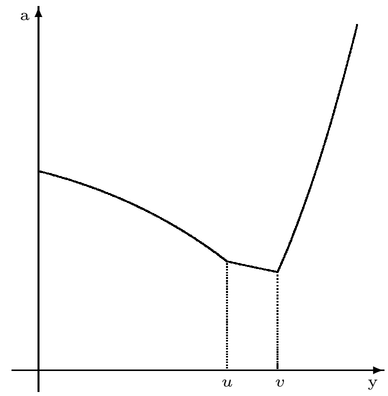


Figure 4

Here $a(0) = 7.5$; $a(u) \approx 4.1$, where $u \approx 7.1$ is the y-coordinate of the intersection point of l and the circle $x^2 + y^2 - 45x = 0$; $a(v) \approx 3.7$,where $v = 9$ is the y-coordinate of the intersection point of l and AC.

4. Theorems

Here we will prove three results about 4-cycles in special cases.

Proposition 4.1. *There are no 4-cycles in a right pyramid, i.e. in a section of the first octant.*

Proof. Let $P = ABCO$ be a pyramid, where A is a point in the positive x-axis, B — a point in the positive y — axis, C — a point in the positive z-axis and O is the origin. A billiard trajectory begins at a point $S \in \Delta ABC$ with a vector $\bar{v}$. After three reflections from planes xy, xz and yz it returns to ΔABC with the directing vector $-\bar{v}$. The reflection from ΔABC must transforms $-\bar{v}$ into $\bar{v}$, hence, the vector $\bar{v}$ is orthogonal to ABC. But in this case the returning point cannot be S. □

Proposition 4.2. *If a pyramid has the right dihedral angle, then it has not more, than two 4-cycles.*

Proof. Let $ABCD$ be a pyramid, where $A = (0, 0, 0)$, $B = (a, 0, b)$, $C = (0, c, d)$ and $D = (0, 0, e)$, i.e. the dihedral angle at the edge AD is $\frac{\pi}{2}$. We will consider 4-cycles

$$ABC \to ABD \to ACD \to BCD \to ABD \text{ and } ABC \to ACD \to ABD \to BCD \to ABC$$

and will prove that both two cannot exist.

As faces ABD and ACD are orthogonal, then reflection operators with respect to these faces commute. But then these cycles have the same starting vector and, thus, the same starting point. □

Let a pyramid $ABCD$ be symmetric with respect to the plane CDE, where E is the middle point of the edge AB. Then the altitude DO belongs to CDE, lines CE and AB are perpendicular and lines DE and AB are also perpendicular. Let us consider the cycle $\mathcal{C} = ABC \to ACD \to ABD \to BCD \to ABC$. Let K, L, M, N be points of $\mathcal{C}$ that belong to ΔABC, ΔACD, ΔABD and ΔBCD, respectively. The cycle $ABC \to BCD \to ABD \to ACD \to ABC$ is the same cycle, passed in the reversed direction. As the symmetry with respect to the plane CDE maps the first cycle into the second, then the cycle $\mathcal{C}$ is symmetric: $K \in [C, E]$, $M \in [D, E]$ and points L and N are symmetric. As the plane LMN contains the normal vector to the plane ABD, then $LM \perp DE$.

Let us consider the pyramid $ACDB'$ which is symmetric to $ABCD$ with respect to the plane ACD, and let points E' and M' be symmetric to E and M, respectively. Points K, L, M' are collinear (because KLM is the billiard trajectory) and $LM' \perp DE'$ (because $LM \perp DE$). Thus, the starting vector $\overline{KL}$ is orthogonal both to CE and DE'.

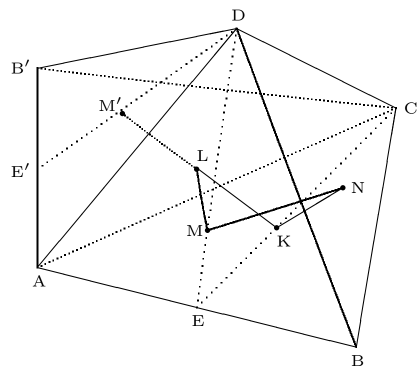


Figure 5

Now we can describe the geometry of our 4-cycle.

Proposition 4.3. *The staring point and the starting vector belongs to the common perpendicular of two skew lines CE and DE'.*

Conjecture 4.1. *If a pyramid has k obtuse dihedral angles, then the number of its 4-cycles is not more, than $3-k$.*

5. Physical billiard. Computations

In this section we will study the movement of a mass point in the presence of a gravity field with the constant g. A point moves inside a triangular pyramid with elastic reflections (i.e. without loss of the energy), thus, trajectories of the point between reflections are segments of parabolas.

It must be noted that each billiard trajectory can be passed in both directions. Thus, we must consider three different orders of reflections. As opposed to the mathematical case, each order of reflections can admit infinitely many trajectories. Computations in the physical case are much more complex, than in the mathematical one, because we have to work with *nonlinear* systems. We present here only results of computations and don't formulate statements or conjectures.

The above assumptions about our pyramid are preserved: the base ABC is in the xy plane: A at the origin, B at the positive x-axis, C in the upper half-plane. The apex D is in the upper half-space. The gravity force with the constant g is directed downwards.

Example 5.1. Let $A = (0, 0, 0)$, $B = (4, 0, 0)$, $C = (3, 3, 0)$, $D = (2, 1, 3)$. Here

$$ABD : 3y - z = 0,\ ACD : 3x - 3y - z = 0,\ BCD : -9x - 3x - 5z + 36 = 0$$

— are equations of planes and

$$\bar{n}_1 = (0, 3, -1),\ \bar{n}_2 = (3, -3, -1),\ \bar{n}_3 = (-9, -3, -5)$$

— are corresponding normal vector (all of them are directed inside the pyramid). Let the order of reflections be $ABD \to ACD \to BCD \to ABC$, the starting point be $p_1 = (a, b, 0)$, the starting vector be $\bar{v}_1 = (k, l, m)$, $m > 0$.

We introduce new variables t_1 — the time interval from the start to the encounter with the ABD plane, t_2 — the time interval between encounters with planes ABD and ACD, t_3 — the time interval between encounters with planes ACD and BCD and t_4 — the time interval between the encounter with the plane BCD and the return to the starting point.

The coordinates of the velocity vector $\bar{v}_2$ in the moment of the encounter with the ABD-plane are $\bar{v}_2 = (k, l, m - g \cdot t_1)$. At this moment the position p_2 of our mass point has coordinates

$$p_2 = \left(a + k \cdot t_1, b + l \cdot t_1, m \cdot t_1 - g \cdot t_1^2/2\right) .$$

Thus, we have the first equation:

$$3 \cdot (b + l \cdot t_1) - \big(m \cdot t_1 - g \cdot t_1^2/2\big) = 0 \tag{Eq.1}$$

Let s be the dot product $s = (\bar{v}_2, \bar{n}_1)$, then

$$\bar{v}_3 = \left(k, l - \frac{6s}{10}, m - g \cdot t_1 + \frac{2s}{10}\right)$$

is the velocity vector after the reflection from the ABD-plane.

The next step: we find the velocity vector $\bar{v}_4$ in the moment $t_1 + t_2$, i.e. in the moment of the contact with the plane ACD:

$$\bar{v}_4 = (\bar{v}_3[1], \bar{v}_3[2], \bar{v}_3[3] - g \cdot t_2)$$

and the position p_3 of our mass point at this moment:

$$p_3 = (p_2[1] + \bar{v}_3[1] \cdot t_2, p_2[2] + \bar{v}_3[2] \cdot t_2, p_2[3] + \bar{v}_3[3] \cdot t_2 - g \cdot t_2^2/2).$$

This gives us the second equation:

$$3 \cdot p_3[1] - 3 \cdot p_3[2] - p_3[3] = 0. \tag{Eq.2}$$

In the same manner we obtain the third equation.

The contact of the mass point with the base ABC gives us 6 equations: we must come to the starting point with the velocity vector $\bar{v}_8 = (k, l, -m)$. Thus, we have a nonlinear system of nine equations and ten variables $\{a, b, k, l, m, g, t_1, t_2, t_3, t_4\}$.

Given such system we find the Groebner basis for the lexicographic order $\{a, b, k, l, m, g, t_4, t_3, t_2, t_1\}$ and get the following results (in generic case).

(1) We have two independent variables t_2 and t_3 and all other variables are their rational functions.
(2) Numerators and denominators of these fractions are homogeneous polynomials in t_2 and t_2:
  (a) the degrees of numerators and denominators of t_1 and t_4 are two and one, respectively;
  (b) the degrees of the numerator and the denominator of g are three and five, respectively;
  (c) the degrees of numerators and denominators of k, l and m are four and five respectively;
  (d) the degrees of numerators and denominators of a and b are five and five.
(3) The geometry of a trajectory depends only on the ratio $t = t_3/t_2$. The parameter t_2 defines the duration of the cycle passage.

The continuation of Example 5.1. Demands on the positivity of g, t_1, t_2, t_3, t_4 and demands on the positions of reflection points (inside faces) define an admissible interval for t. In our case $0 < t < 0.48$. Admissible starting points constitute a curve $s(t)$ inside ΔABC, where $G = s(0) \approx (2, 1)$ and $H = s(0.48) \approx (2.6, 0.8)$ (Figure 6).

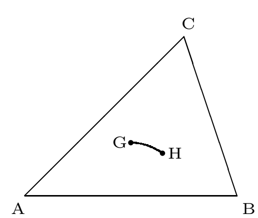


Figure 6

Remark 5.1. For the above pyramid there exist three families of billiard trajectories for each order of reflections.

Remark 5.2. The pyramid $ABCD$: $A = (0, 0, 0)$, $B = (4, 0, 0)$, $C = (3, 3, 0)$, $D = (3, 2, 1)$ has three obtuse dihedral angles and there are no 4-cycles in it.

Remark 5.3. The pyramid $ABCD$: $A = (0, 0, 0)$, $B = (9, 0, 0)$, $C = (6, 3, 0)$, $D = (6, 2, 4)$ has obtuse angle ACB in the base ΔABC and only one order of reflections: $\Delta ABC \to \Delta ACD \to \Delta ABD \to \Delta BCD \to \Delta ABC$ produces a family of 4-cycles.

Outside the scope of generic cases, symmetric cases are the most interesting.

Example 5.2. Let us consider the symmetric pyramid $ABCD$: $A = (0, 0, 0)$, $B = (6, 0, 0)$, $C = (3, 4, 0)$, $D = (3, 2, 4)$. Let CH be the altitude of ΔABC. There exists a family of 4-cycles for the reflection order $\Delta ABC \to \Delta ACD \to \Delta ABD \to \Delta BCD \to \Delta ABC$. The starting point always belongs to CH, the starting vector is orthogonal to the y-axis. A trajectory meets ΔABD at a point with x-coordinate 3.

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Email address: `yukochetkov@hse.ru, lapyatko@edu.hse.ru`