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Conical Resolutions of Discriminant Varieties and Real Cohomology of the Space of Nonsingular Complex Plane Projective Quintics[1]

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We describe a general method for calculating cohomology groups of finite-dimensional spaces of nonsingular functions and calculate the real cohomology groups of the space of nonsingular curves of degree 5 in $\mathbb{CP}^2$.

1. MAIN EXAMPLE

Let Π_5 denote the space of all homogeneous polynomials $\mathbb{C}^3 \to \mathbb{C}$ of degree 5, and let P_5 be its subspace consisting of nonsingular polynomials (i.e., such that their differentials vanish nowhere except at the origin).

Theorem 1. *The Poincaré polynomial of the space* P_5 *equals* $(1 + t)(1 + t^3)(1 + t^5)$.

We denote the space $\Pi_5 \setminus P_5$ by Σ_5. According to the Alexander duality, the cohomology of P_5 is isomorphic to the Borel–Moore homology (i.e., the homology of the complex of locally finite chains) of Σ_5:

$$H^i(P_5, \mathbb{R}) = \bar{H}_{2D-1-i}(\Sigma_5, \mathbb{R}),$$

where $D = \dim_{\mathbb{C}}(\Pi_5) = 21$ and $0 < i < 2D - 1$. This reduction was used for the first time by Arnol'd in [1].

The group $\bar{H}_*(\Sigma_5, \mathbb{R})$ can be calculated with the use of the method of conical resolutions. In [4], conical resolutions of spaces of singular algebraic hypersurfaces are described and the real cohomology groups of the spaces of nonsingular plane complex curves of degree ≤ 4 are calculated. We give a more general construction of resolutions and apply it to the case of quintics (this is the first case where the application of the method from [4] involves difficulties).

The study of spaces of nonsingular hypersurfaces was motivated by problem 1970-13 from the list of [2] concerning the topology of the space of nonsingular cubics.

2. THE GENERAL METHOD OF CONICAL RESOLUTIONS

Let V be a finite-dimensional affine space of real- or complex-valued functions on a manifold $\tilde{M}$, and let $\Sigma \subset V$ be its discriminant consisting of functions having singularities of a given type. To calculate the homology of the discriminant Σ, we construct its resolution, i.e., a space σ (whose homology is easier to calculate than that of Σ) and a proper mapping $\pi: \sigma \to \Sigma$ such that the preimage of any point is contractible.

Example (see [3]). Let V be the space of polynomials of degree k on $\mathcal{F}$, where $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$, with leading coefficient 1, and let Σ consist of polynomials having a multiple root. Then a resolution σ can be constructed as follows: we take the space of pairs (f, x), where $f \in \Sigma$ and x belongs to the simplex spanned by all multiple roots of f. This simplex lives in a Euclidean space of a large dimension, into which we can embed $\mathcal{F}^1$ in such a way that the intersection of any two $\left(\left[\frac{k}{2}\right] - 1\right)$-dimensional simplices with vertices on the image of the embedding is equal to their common face (if it exists). The projection $\sigma \to \Sigma$ consists in forgetting x. The advantage of σ over Σ is that there is a natural filtration on σ: first, we take the union of all (f, x) where x belongs to the 0-dimensional simplices, then add the 1-dimensional simplices, etc. The difference of the ith and $(i - 1)$th terms of this filtration is fibered over the space of unordered collections of i points from $\mathcal{F}^1$.

Applying this idea to the case of polynomials in many variables involves some difficulties. First, the singular set of a hypersurface may be infinite, so we need to find a space in which all possible singular sets are

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properly placed. Secondly, if the singular set is infinite, then it is unclear what the simplex spanned by the points of such a set is. Thirdly, the limit of a sequence of singular sets may be not the singular set of any hypersurface in V: for example, the configuration in $\mathbb{CP}^2$ consisting of three collinear points is not the singular set of a plane cubic; however, it is the limit of generic three-point configurations. Below, we give a general construction of a resolution this construction uses the same idea as in the case of one variable. This construction applies to the case of algebraic hypersurfaces, and we hope that it might also be useful in other cases.

In what follows, we assume that V is a vector space (as in our main example). All arguments can be easily transferred to the affine case. Suppose that each function $f \in \Sigma$ is assigned a nonempty compact subset K_f of a certain compact manifold M. For instance, if $\tilde{M} = \mathbb{C}^{n+1} \backslash \{0\}$, V is the space of homogeneous polynomials of a given degree, and Σ consists of polynomials whose gradient vanishes at some point of $\tilde{M}$, then it is natural to set M equal to $\mathbb{CP}^n$ and K_f equal to the image of the set of singular points of f under the obvious projection $\tilde{M} \to \mathbb{CP}^n$.

In the general case, we assume that the following three conditions are satisfied: (i) if $K_f \cap K_g \neq \emptyset$, then $f + g \in \Sigma$ and $K_f \cap K_g \subset K_{f+g}$; (ii) $K_{\lambda f} = K_f$ if $\lambda \neq 0$; and (iii) $K_0 = M$, where 0 is the zero function. For any $K \subset M$, let $L(K) \subset V$ be the vector space consisting of all f such that $K \subset K_f$. We assume also that the dimension of $L(\{x\})$ is the same for all $x \in M$ (say equals d) and the mapping $x \mapsto L(\{x\})$ from M to $G_d(V)$ (the corresponding Grassmannian) is continuous.

By a configuration in M, we mean an arbitrary nonempty compact subset $K \subset M$. We denote the space of all configurations in M by 2^M. Suppose that the topology on M is induced by a metric ρ. We introduce the Hausdorff metric on 2^M in the usual way as

$$\tilde{\rho}(K, L) = \max_{x \in K} \rho(x, L) + \max_{x \in L} \rho(x, K).$$

It is easy to check that, if M is compact, then the space 2^M with the metric $\tilde{\rho}$ is also compact. For any subset $A \subset 2^M$, we denote the closure of A in 2^M by $\bar{A}$. For instance, if $A = B(M, k)$ is the space of all k-element subsets of M, then $\bar{A} = \bigcup_{j \le k} B(M, j)$.

Let $X_1, X_2, \ldots, X_N$ be subspaces of 2^M satisfying the following conditions.

(i) For any function $f \in \Sigma$, the set K_f belongs to some X_i;

(ii) If $K \in X_i$, $L \in X_j$, and $K \subsetneq L$, then $i < j$;

(iii) For any i and all configurations $K \subset X_i$, the dimensions of the corresponding spaces $L(K)$ (consisting of functions f such that $K \subset K_f$) coincide;

(iv) $X_i \cap X_j = \emptyset$ if $i \neq j$;

(v) Any $K \in \bar{X}_i \backslash X_i$ belongs to some X_j, where $j < i$.

Under these assumptions, we construct a filtered resolution σ.

By condition (iii), for any $i = 1, 2, \ldots, N$, there exists an obvious mapping $K \mapsto L(K)$ from X_i into the Grassmann manifold $G_{d_i}(V)$. It is continuous, because the mapping $x \mapsto L(\{x\})$ is.

Consider the space $Y = \bigcup_{i=1}^{N} \bar{X}_i = \bigcup_{i=1}^{N} X_i$. We assume that Y can be embedded continuously into some finite-dimensional Euclidean space. Let X denote the space Y^{*N}, i.e., the Nth self-join of Y.[2] The spaces Y and Y^{*N} are compact. A simplex $\Delta \subset X$ is called coherent if the configurations corresponding to its vertices form an increasing chain. By condition (ii), the vertices of such a simplex belong to distinct X_i. The principal vertex of a coherent simplex is the vertex corresponding to the configuration maximal among all configurations corresponding to the vertices of the simplex. Let Λ denote the union of all coherent simplices. For any $K \in X_i$, we denote the union of all coherent simplices whose principal vertices are equal to K by $\Lambda(K)$. Any set $\Lambda(K)$ is contractible.

Let $\partial\Lambda(K)$ denote the union $\bigcup_{\kappa} \Lambda(K)$ over all maximal $\kappa \subsetneq K$. The space $\Lambda(K)$ is the union of all segments connecting points of $\partial\Lambda(K)$ with K; hence, it is homeomorphic to the cone over $\partial\Lambda(K)$. Let us introduce a filtration $\emptyset \subset \Phi_1 \subset \Phi_2 \subset \ldots \subset \Phi_N = \Lambda$ in Λ by setting $\Phi_i = \bigcup_{j \le i} \bigcup_{K \in X_j} \Lambda(K)$. It is easy to verify that all the spaces Λ, $\Lambda(K)$, and Φ_i are compact.

For any simplex $\Delta \subset X$, we use $\mathring{\Delta}$ to denote its interior, i.e., the union of all points that do not belong to faces of smaller dimensions. For any point $x \in X$, there is a unique simplex Δ such that $x \in \mathring{\Delta}$.

We define the conical resolution σ as the subspace of $V \times \Lambda$ consisting of pairs (f, x) such that $f \in \Sigma$ and $x \in \Lambda(K_f)$. There are obvious projections $\pi: \sigma \to \Sigma$ and $p: \sigma \to \Lambda$. We introduce a filtration in σ by setting $F_i = p^{-1}(\Phi_i)$. The mapping π is proper, since the preimage of any compact set $C \subset \Sigma$ is closed in the compact space $C \times \Lambda$.

Below, we describe two additional technical conditions ensuring that the difference of the ith and the

[2] For any finite-dimensional space Y, its kth self-join Y^{*k} can be defined as follows. We take a generic embedding $i: Y \to \mathbb{R}^{\Omega}$ with some large but finite Ω and define Y^{*k} as the union of all $(k-1)$-dimensional simplices with vertices in $i(Y)$ ("generic" means that any two such simplices can intersect only in a common face).

$(i-1)$th terms of the filtration is the space of a certain fiber bundle over X_i.

(vi) For each i, let $\mathcal{T}_i$ denote the space consisting of pairs (x, K) such that $x \in K$ and $K \in X_i$; let $\mathrm{pr}_i: \mathcal{T}_i \to X_i$ be the projection to the second factor. The triple $(\mathcal{T}_i, X_i, \mathrm{pr}_i)$ should be a locally trivial fiber bundle. We call this fiber bundle the tautological[3] bundle over X_i.

(vii) Note that any local trivialization of $\mathcal{T}_i$ has the form $(x, K') \mapsto (t(x, K'), K')$, where x is a point in some $K \in X_i$, K' belongs to some neighborhood $U \ni K$ in X_i, and $t: K \times U \to M$ is a continuous mapping such that, fixing a $K' \in U$, we obtain a homeomorphism $t_{K'}: K' \to K$. We require that, for any $K \in X_i$, there exist a neighborhood $U \ni K$ and a local trivialization of $\mathcal{T}_i$ over U such that any mapping $t_{K'}: K \to K'$ induces a one-to-one correspondence between the subsets of K and K' belonging to $\bigcup_{j \le i} X_i$.

Theorem 2. Let $X_1, X_2, \ldots, X_N$ *be subspaces of* 2^M *satisfying conditions* (i)–(v). *Then*,

(a) π *induces an isomorphism of Borel–Moore homology groups*;

(b) *each space* $F_i \backslash F_{i-1}$ *is the space of a vector bundle of dimension* $\dim(L(K))$, *where* $K \in X_i$, *over* $\Phi_i \backslash \Phi_{i-1}$;

(c) *if, in addition*, $X_1, X_2, \ldots, \ldots, X_N$ *satisfy conditions* (vi) *and* (vii), *then the space* $\Phi_i \backslash \Phi_{i-1}$ *is fibered over* X_i, *and the fiber is homeomorphic to* $\Lambda(K) \backslash \partial\Lambda(K)$, *where* $K \in X_i$.

Assertion (a) is obvious. The mapping $F_i \backslash F_{i-1} \to \Phi_i \backslash \Phi_{i-1}$ from (b) is p. There exists a mapping $f_i: \Phi_i \backslash \Phi_{i-1} \to X_i$ taking x to the principal vertex of the coherent simplex whose interior contains x. This is the projection of the fibration from (c). Note that the fiber bundle $p: F_i \backslash F_{i-1} \to \Phi_i \backslash \Phi_{i-1}$ is the inverse image of the tautological bundle over $G_{d_i}(V)$ under the mapping $x \overset{f_i}{\mapsto} K \mapsto L(K)$, and $f_i^{-1}(K) = \Lambda(K) \backslash \partial\Lambda(K)$. A complete proof of the theorem is given in [5].

In the case of plane quintics, the majority of configurations of singular points are discrete (consist of finitely many points). To make the fiber of the bundle $\Phi_i \backslash \Phi_{i-1} \to X_i$ as simple as possible, we introduce yet another condition.

(viii) If K is a finite configuration of X_i, then any subset $L \subset K$ is contained in some X_j with $j < i$.

If condition (viii) holds, then the fiber of the bundle $(\Phi_i \backslash \Phi_{i-1}) \to X_i$ over a point $K \in X_i$ is an open simplex whose vertices correspond to the points of the configuration K. (This can be proved by induction on the number of points in the configuration $K \in X_i$.) In this situation, the space $\Lambda(K)$ is piecewise linear isomorphic to the first barycentric subdivision of the simplex Δ spanned by the points of K. The complex $\partial\Lambda(K)$ is isomorphic to the first barycentric subdivision of $\partial\Delta$. Let Λ^{fin} denote the union of spaces $\Lambda(K)$ over all finite K.

Lemma. *If condition* (viii) *holds, then there exists a mapping* $C: \Lambda^{\mathrm{fin}} \to M^{*N}$ *taking* K *to some interior point of the simplex* Δ *spanned by the points of* K. *This mapping is a homeomorphism onto its image and homeomorphically maps* $\Lambda(K)$ [*respectively*, $\partial\Lambda(K)$] *onto* Δ (*respectively, onto* $\partial\Delta$).

This lemma implies that, for any i such that X_i consists of finite configurations, the fiber bundle $\Phi_i \backslash \Phi_{i-1}$ is isomorphic to the restriction to X_i of the natural bundle $M^{*k} \backslash M^{*(k-1)} \to B(M, k)$, where k is the number of points in a configuration from X_i. Thus, we have

$$\bar{H}_*(\Phi_i \backslash \Phi_{i-1}, \mathbb{R}) = \bar{H}_{*-(k-1)}(X_i, \pm\mathbb{R}),$$

where $\pm\mathbb{R}$ is the alternating local system with fiber $\mathbb{R}$ (changing the sign every time two points of a configuration are transposed). Suppose that V consists of complex-valued functions. Then, by assertion (ii) of Theorem 2, we have

$$\bar{H}_*(F_i \backslash F_{i-1}) = \bar{H}_{*-2d_i}(\Phi_i \backslash \Phi_{i-1}) = \bar{H}_{*-2d_i-(k-1)}(X_i, \pm\mathbb{R}),$$

where $d_i = \dim_{\mathbb{C}} L(K)$ for $K \in X_i$.

3. SPECTRAL SEQUENCE OF A CONICAL RESOLUTION OF THE SPACE Σ_5

Let us apply the method described in the preceding section to the case of plane complex quintics. Let us specify a system of spaces X_i satisfying conditions (i)–(viii). For this purpose, we consider the subspace of $2^{\mathbb{CP}^n}$ consisting of all K_f over all singular $f \in \Sigma_5$, take the closure of this space, and stratify it by the dimensions of the spaces $L(K)$ in such a way that all sets in the same stratum are at least homeomorphic to each other and the boundary of every stratum lies in the union of the preceding strata. As a result, we obtain a list of 42 items (see [5]). Applying Theorem 2, we obtain a conical resolution of the space Σ_5 and a 42-term filtration on it.

Theorem 3. *The spectral sequence for the Borel–Moore homology of the resolved space* Σ_5 *is determined by the following conditions*.

(i) *The group* $E^1_{p,q}$ *is isomorphic to* $\mathbb{R}$ *if* $(p, q) = (1, 35)$, $(1, 37)$, $(1, 39)$, $(2, 31)$, $(2, 33)$, $(2, 35)$, or $(3, 29)$; *otherwise*. $E^1_{p,q} = 0$.

(ii) *The spectral sequence stabilizes at the term* $E^1 \equiv E^\infty$.

It follows from the computations performed in [4, 3] that almost all columns are zero: it is shown in [4, 3] that the groups $\bar{H}_*(B(\mathbb{C}^n, k), \pm\mathbb{R})$ and $\bar{H}_*(B(\mathbb{CP}^n, k),$

[3] For instance, if $M = \mathbb{CP}^n$ and X_i consists of projective subspaces M of the same dimension, then this bundle is the restriction of the projectivization of the usual tautological fiber bundle over some Grassmannian manifold.

$\pm\mathbb{R}$) are zero for any $k > 1$ and any $k > n + 1$, respectively. This is sufficient for calculating the majority of columns in the spectral sequence. However, there are two columns that should be considered in more detail. They correspond, roughly speaking, to the cases of the intersection points of three lines and a quadric in general position and the intersection points of two quadrics and a line in general position. "General position" means that the number of intersection points is maximal, i.e., it equals 9 in the former case and 8 in the latter. Complete proofs are given in [5].

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