Material deformation tensor in time-reversal symmetry breaking turbulence

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The properties of material deformation tensor in time irreversible turbulence are determined. It is shown that time irreversibility in the Lagrangian framework is connected with energy flux of the turbulence.

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1. Introduction

The Lagrangian evolution of material elements has been extensively studied in various works [1–3]. These studies were connected with different problems: such as Lagrangian turbulence [5] or passive scalar decay [2]. One of the most important results of these studies is the proof of the intermittency in structure functions. This intermittency seems closely connected with the intermittency of the developed turbulence. The results of these studies were summarized in [2]. But the most of theoretical papers consider hydrodynamic turbulence as a Gaussian noise. In the case of a Gaussian stochastic process there is time symmetry $t$ to $-t$.

In real uniform and isotropic turbulence energy flows from the scale at which it is injected, $L$, to the scale where it is dissipated, $\eta$. For intense three-dimensional turbulence, $L \gg \eta$, and the energy flows from large to small scales [1]. As a result, time symmetry is broken, since the time reversal $t$ to $-t$ would also reverse the direction of the energy flux. Exploring the implications of this time asymmetry on the relative motion between fluid particles is of great interest.

The simplest problem in this context concerns the dispersion of two particles whose positions, $r_1(t)$ and $r_2(t)$, are separated by $|r_2(t) - r_1(t)|$. The growth of the mean $n$-th power of the separation, forward ($t > 0$) and backward in time ($t < 0$) is a fundamental question in turbulence research [4].

The aim of this paper is an introduction of a model which takes into account time irreversibility and an investigation of the influence of this time asymmetry on material deformation tensor. Note that the time evolution of this tensor is a basis for the theories discussed above.

2. The statement of the problem

A description of material finite deformation in a continuum is done using the deformation tensor defined as

$$D_{ij}(X, t) = \frac{\partial x_i}{\partial X_j},$$

where $x_i(X, t)$ denotes the position at time $t$ of a fluid particle that was at the position $x_i(X, t_0) = X_i$ at the initial time $t_0$. $D_{ij}$ thus describes the variation of the position of a particle at the current time when one slightly changes the initial position. The fluid particle obeys $dx_i/dt = u_i(x, t)$, and differentiating this expression with respect to $X_j$, one obtains the evolution equation for $D_{ij}$:

$$\frac{dD_{ij}}{dt} = A_{ik}D_{kj}$$

(1)

The value $A_{ij}$ is called velocity gradient tensor. Since $u_i(x, t)$ is the turbulent velocity $A_{ij}$ is a stochastic process. Many important properties of the developed turbulence such as geometric and statistical information, the alignment of vorticity with respect to the strain-rate eigenvectors, rate of deformation and shapes of fluid material volumes, non-Gaussian statistics, and intermittency, encode in the tensor $A_{ij}$. In the inertial range of turbulence, similar properties can be described using the coarse-grained or filtered velocity gradient tensor. Strictly speaking to get $A_{ij}(t)$ you can differentiate coarse-grained velocity only. But according to experiments [6,7] and numerical simulations [8,9], the trajectory of a Lagrangian particle in the developed turbulence flow consists of two parts: regular one and trapped one into vortex filaments. But vortex filament moves also as a set of Lagrangian particles. Thus if you consider the distance between two particles which one of them in a filament and another one in a regular, smooth flow (or in another filament) you can at first approximation to consider smooth
part flow \( u \) only. Actually, small oscillations of one particle around vortex center do not affect on global distance growth between two particles.

Below we will discuss some general properties of the tensor \( D_{ij} \) proposing arbitrary stochastic process \( A_{ik}(t) \).

3. Properties of \( A_{ik} \) in stationary uniform and isotropic turbulence

It is naturally to suppose that stationary, uniform and isotropic turbulence flow is a general stochastic process defined by probability distribution functional \( P(u(r,t)) \). The conditions of stationarity, uniformity and isotropy take the form:

\[
P\left[u(r,t)\right] = P\left[u(r, t + t_0)\right], \quad P\left[u(r, t)\right] = P\left[u(r + r_0, t)\right], \quad P\left[u(r, t)\right] = P\left[Ru(Rr, t)\right]
\]

(2)

These relations must be valid for any moment \( t_0 \), position \( r_0 \) and rotation matrix \( R \).

Analogous one can define probability functional \( P_A[A(t)] \), for any matrix \( A_{ij}(r_0, t) \), here \( r_0 \) is some point fixed in space. The uniformity means that \( P_A[A(t)] \) does not depend on \( r_0 \). Any mean value of functional \( F[A(t)] \) is defined by functional integral

\[
\langle F[A(t)] \rangle = \int F[A(t)] P_A[A(t)]
\]

The stationarity and isotropy of the flow means that

\[
P_A[A(t)] = P_A[A(t + t_0)], \quad P_A[A(t)] = P_A[R^{-1}A(t)R]
\]

Let us suppose below that stochastic process \( A(t) \) is statistically independent at different moments of time:

\[
P_A[A(t)] = \Pi_t p(A(t))
\]

here \( p(A(t)) \) is some universal function of the traceless matrix \( A(t) \).

It is common knowledge that any traceless matrix \( A(t) \) could be split into symmetric and antisymmetric parts:

\[
A = B + \Omega, \quad B = \frac{1}{2}(A + A^T), \quad \Omega = \frac{1}{2}(A - A^T)
\]


\[
B^T = B, \quad \Omega^T = -\Omega, \quad \Omega_{ij} = \epsilon_{ijk}\omega_k
\]

here \( \omega_k \) is a polar vector.

Because of SO(3) symmetry function \( p(A) \) depends on rotational invariants only. For traceless matrix \( A \), probability \( p(A) \) depends on 3 invariants. Let us restrict our consideration by analytic functions \( p(A) \). In this case you can choose the following invariants:

\[
\begin{align*}
\text{tr}B^2 & \quad \text{tr}B^3 \quad \omega^2 \\
\text{tr}B\Omega^2 & \quad \text{tr}B^2\Omega^2
\end{align*}
\]

It is important that \( \omega \) in these invariants is a quadratic form only.

In this case

\[
P(A^T) = P(A)
\]

4. Lyapunov exponents

For further consideration let us introduce matrix

\[
\frac{dQ}{dt} = Q A, \quad Q(t_0) = I
\]

(4)

here \( I \) is a unit matrix. One can see from [1] that matrix \( Q \) is connected with \( D \) by simple relation: \( D = Q^T(A^T) \). The matrix \( Q \) is more convenient object, so we will discuss below its properties.

To examine the solution of (4), we proceed to a discrete approximation. Consider a discrete sequence of moments separated by \( \Delta t \) and let \( A_{ij}(t) = (A_{ij})_0 \) be constant inside each small (n-th) interval. Then, for each \( \Delta t \), the solution to Eq. (4) is described by an exponent and we get

\[
Q_n = Q_{n-1}e^{A_{ij}\Delta t}
\]

The matrix \( Q \) is a multiplication of \( N \) random real unimodular matrices with the same distribution (discrete \( t \)-exponent). The asymptotic behavior of this object has been studied carefully and a number of important results have been obtained. (For a short summation of them, see [10].) In particular, the following theorems have been proved for reasonable conditions.

Let us consider the Iwasawa decomposition of the matrix \( Q \):

\[
Q = z(Q) d(Q) s(Q),
\]

where \( z \) is an upper triangular matrix with diagonal elements equal to 1, \( d \) is a diagonal matrix with positive eigenvalues \( d = |d_1, d_2, d_3| \), and \( s \) is an orthogonal matrix.

Theorem 1. From [11] we have that with probability 1, there exists the limit \( \lim_{N \to \infty} \frac{1}{N} \text{tr} d(Q) = \lambda_i \), where \( \lambda_i \) are not random, i.e., do not depend on the realization of the process \( A_{ij}(t) \) but only on the statistical properties of the process, and \( \lambda_1 < \lambda_2 < \lambda_3 \), with the ordering due to the triangular matrix, which provides the inequality of the axes.

Theorem 2. From [12,13] we have that the distribution of \( \xi_i \) is \( \xi_i = \frac{1}{\sqrt{N}} \sum_{t_1}^N d_i(Q(N)) = \lambda_i \), is not random, i.e., do not depend on the realization of the process \( A_{ij}(t) \) but only on the statistical properties of the process, and \( \lambda_1 < \lambda_2 < \lambda_3 \), with the ordering due to the triangular matrix, which provides the inequality of the axes.

Theorem 3. From [14] we have that with probability 1, \( z(Q) \) converges as \( N \to \infty \); contrary to \( \lambda_i \), the values \( z_\infty = z(Q_\infty) \) are different in different realizations of \( A_{ij}(t) \).

Theorem 4. From [15] we have that the values \( \xi_i(Q) \) and \( z(Q) \) are asymptotically independent.

To calculate the Lyapunov exponents let us introduce matrix

\[
\Gamma = Q^T Q
\]

according to (5).

\[
\Gamma = z d^2 z^T
\]

Taking into account that \( z \) is upper triangle matrix one can get:

\[
d_3^2 = \Gamma_{33}, \quad d_2^2 = (\Gamma_{11}^{-1})_{11}^{-1}, \quad d_1^2 = (\Gamma_{11}^{-1})_{11}^{-1}
\]

Basing on these relations and theorems it is easy to get Lyapunov’s exponents:

\[
\lambda_3 = \lim_{t \to \infty} \frac{\ln \Gamma_{33}}{2t}, \quad \lambda_1 = -\lim_{t \to \infty} \frac{\ln (\Gamma_{11}^{-1})_{11}}{2t},
\]

\[
\lambda_2 = -\lambda_1 - \lambda_3
\]

and expressions for the Gaussian noise \( \xi(t) \):

\[
e^{2\int \xi_t dt}d_3^2 = e^{-2\lambda_3 t} \Gamma_{33} \quad e^{2\int \xi_t dt}d_2^2 = e^{-2\lambda_2 t} (\Gamma_{33}^{-1})_{11}^{-1}
\]

These relations are exact. It is important that these limits exist with probability 1. That is why it is possible to calculate these limits by averaging over process \( A \) with probability \( P_A \):
\[ \lambda_3 = \lim_{t \to \infty} \frac{\ln(\Gamma_{33})}{2t}, \quad \lambda_1 = -\lim_{t \to \infty} \frac{\ln(\Gamma^{-1}_{11})}{2t} \]
\[ \langle e^{2/\xi_1(t)}d\xi_1(t) \rangle = e^{2\lambda_1 t} \langle (\Gamma^{-1}_{11}) \rangle \]
\[ \langle e^{-2/\xi_1(t)}d\xi_1(t) \rangle = e^{2\lambda_1 t} \langle (\Gamma^{-1}_{11}) \rangle \]

The expression for \( \lambda_1 \) could be rewritten in another form if one introduce the transformation \( N \to NA(t) = -A^T(t) \)

In this case from (6)
\[ \Gamma^{-1}(t)[A] = \Gamma^{-1}[NA] \]

So the expression for \( \lambda_1 \) could be presented in the form:
\[ \lambda_1 = -\lim_{t \to \infty} \frac{\ln(\Gamma^{-1}[A])}{2t} \]
\[ \langle e^{2/\xi_1(t)}d\xi_1(t) \rangle = e^{2\lambda_1 t} \langle (\Gamma^{-1}[A]) \rangle \]

Let us consider now an evolution of the vector \( r(t) \) defined by relation
\[ r(t) = r(0)Q_{ij} \]

One can see that evolution of this vector is determine by equation
\[ \frac{dr}{dt} = r(t)A_{ij}(t) \]

Hence
\[ r^2(t) = r(0)Q_{ij}Q_{jk}r(0) \]

Let us choose \( r(0) = (0, 0, 1) \) in this case \( r^2(t) = \Gamma_{33}(t) \).

If we choose \( r(0) = (1, 0, 0) \) one can get \( r^2(t) = \Gamma_{11}(t) \).

Basing on these relations we can determine Lyapunov exponents via evolution of \( r(t) \):
\[ \lambda_3 = \lim_{t \to \infty} \frac{\ln(r^2[t, A])}{2t}, \quad r(0) = (0, 0, 1) \]
\[ \lambda_1 = -\lim_{t \to \infty} \frac{\ln(r^2[t, -A^T])}{2t}, \quad r(0) = (1, 0, 0) \]

and
\[ \langle e^{2/\xi_1(t)}d\xi_1(t) \rangle = e^{-2\lambda_1 t} \langle r^2(t, A) \rangle \]
\[ \langle e^{-2/\xi_1(t)}d\xi_1(t) \rangle = e^{2\lambda_1 t} \langle r^2(t, -A^T) \rangle \]

5. Isotropic and time-reversible process

Let us consider transformation \( T \) which acts on the flow \( u(r, t) \) defining as:
\[ T(u(r, t)) = -u(r, t) \]

this transformation is the time reverse. The matrix of deformations in this case changes analogously:
\[ TA(t) = -A(t) \]

Now let us prove a statement:

If the process \( A(t) \) is an isotropic one \( P[A(t)] = P[R^{-1}(t) \times A(t)R(t)] \) and invariant to the time reverse, \( P[A(t)] = P[-A(t)] \),

then \( \lambda_1 = -\lambda_3, \lambda_2 = 0 \) for any process \( P[A(t)] \). Previously the proof of this statement was presented for the Gaussian process only [16].

To prove it let present a rotation matrix \( R_\alpha \) which has elements \( R_{13} = -1, R_{22} = R_{12} = 1 \). This matrix transforms element 11 into 33. One can see that \( A_{11} = (R_\alpha A R_\alpha^{-1})_{33} \) and correspondingly \( \Gamma_{11} = (R_\alpha \Gamma R_\alpha^{-1})_{33} \). Then from (5)

\[ \lambda_1 = -\lim_{t \to \infty} \frac{\ln(\Gamma_{11}[A])}{t} \]

and from (3)
\[ \lambda_3 = -\lim_{t \to \infty} \frac{\ln(\Gamma_{11}[A])}{t} \]

and according to (7), (8) we have:
\[ \lambda_1 = -\lim_{t \to \infty} \frac{\ln(\Gamma_{11}[A])}{t} \]
\[ = -\lim_{t \to \infty} \frac{\ln(\Gamma_{33}[R_c(-A)R_c^{-1}])}{t} \]
\[ = -\lim_{t \to \infty} \frac{\ln(\Gamma_{33}[A])}{t} = -\lambda_3 \]

Thus we see that for any isotropic and time invariant process minimal and maximal Lyapunov exponents have equal modulus but opposite sign. Basing on relations (7), (8), (9) one can see also that
\[ \langle e^{2/\xi_1}d\xi_1 \rangle = e^{-2\lambda_1} \langle r^2(t) \rangle, \quad \langle e^{-2/\xi_1}d\xi_1 \rangle = e^{2\lambda_1} \langle r^2(t) \rangle \]

Since in the time-reversible process \( \lambda_1 = -\lambda_3 \), from (11) follows that dispersions of Gaussian processes \( \xi_1(t) \) and \( \xi_3(t) \) are equal too.
\[ \langle \xi_1(t) \xi_1(t') \rangle = \langle \xi_3(t) \xi_3(t') \rangle \]

6. Isotropic, but time-irreversible process

As it was mentioned above, a real turbulent flow is time irrelevent due to dissipation. Energy flows from large scale into small. Now let us discuss the properties of matrix \( A(t) \) which is a coarse grained velocity deformation tensor.

Irreversibility of the process \( A(t) \) means that \( P[A(t)] \neq P[-A(t)] \). It is actually possible if \( P[A(t)] \) contains not only quadratic invariants but also \( tr B^2 \) and \( tr \Omega \Omega^T \). Namely these components are responsible for energy flux. Indeed, the contribution of the large-scale velocity component to the average energy flux \( \Phi \) through any sphere of intermediate radius \( r < L \) is
\[ \langle \Phi \rangle = \left\langle \int U^2 U d\Omega \right\rangle \]
\[ = \left\langle A_{ij} A_{kp} \int \int_{r \Omega \Omega^T} \int \int \Omega A^3 \right\rangle \]

The flux must flow inside the volume. Thus condition \( \langle \Phi \rangle < 0 \) gives \( tr A^3 < 0 \). In general case all Lyapunov exponents are different. This difference is closely connected with irreversibility of the turbulence.

7. Distance between Lagrangian particles

On the basis of general solutions (9), (6) one can easy calculate the distance \( r \) between two Lagrangian particles
\[ r^2 = \alpha_1 d_1^2 + \alpha_2 d_2^2 + \alpha_3 d_3^2 \]
where \( \alpha_1, \alpha_2, \alpha_3 \) became a constants at \( t \to \infty \). According to Theorems 1 and 2, the value \( d_i \) takes a form:
\[ d_1 = e^{\lambda_1 t + f_i d_i} \]
\[ d_3 = e^{\lambda_3 t + f_i d_i} \]

and
\[ d_2 = e^{\lambda_2 t + f_i d_i} \]

Now let us determine the asymptotic behavior at \( t \to +\infty \) and at
$t \to -\infty$ of the value $\langle r^n \rangle$. One can see that if $t \to +\infty$, asymptotically $r(t) = d_3(t)$ and if $t \to -\infty$, $r(-t) = d_1(-t)$.

To find the behavior of $\langle r^n(t) \rangle$ (for the case $t \to +\infty$) it is necessary to calculate functional integral:

$$\langle r^n(t) \rangle = \int e^{-\frac{1}{2} n \int s^2(t') dt' - 2 n^2 \int s(t') dt'} D[\xi(t)]$$

Since this integral is a Gaussian one it is easy to write the answer

$$\langle r^n(t) \rangle = e^{\gamma_n(t - t)}$$

To calculation of the asymptotic behavior $\langle r^n(-t) \rangle$ is quite analogous. The result is:

$$\langle r^n(-t) \rangle = e^{\gamma_n(-t - t)}$$

Thus we see that the distance between two Lagrangian particles in turbulent flow always grows exponentially and does not depend on time direction. The discussed process is reversible because in accordance with relations (10), $\lambda_1 = -\lambda_3$ and $D_1 = D_3$.

But in the case of time non-invariant turbulence the trajectory divergence is defined by different exponents. This difference in Lagrangian exponents is determined by even invariants $tr(B^3)$ and $tr(\Omega B\Omega)$ which are closely connected with the energy flux in the system. The difference in the $\gamma_n(\cdot)$ and $\gamma_n(-\cdot)$ is the indication of time irreversibility in Lagrangian framework and very important in calculations of turbulent scaling exponents [17].

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References