Finite model property of pretransitive analogs of S5*

Andrey Kudinov[†], Ilya Shapirovsky[†]

We consider propositional normal unimodal *pretransitive* logics, i.e., logics with expressible 'transitive' modality. There is a long-standing open problem about the finite model property (fmp) and decidability of pretransitive logics, in particular – for the logics $\mathrm{K}_n^m = \mathrm{K} + \Box^m p \to \Box^n p$, $n > m > 1$.

A pretransitive logic L has the fmp or is decidable, only if these properties hold for the logic L.sym*, which is the extension of L with the symmetry axiom for 'transitive' modality: like S5 can be embedded into S4, L.sym* can be embedded into L.

We show that for all $n > m \geq 1$, the logics K_n^m.sym* have the fmp.

Pretransitive logics.

Definition 1 ([2])**.** A logic L is called *pretransitive* (according to [2] – *conically expressive*), if there exists a formula $\chi(p)$ with a single variable p such that for any Kripke model M with $\mathsf{M} \vDash \mathrm{L}$ and for any w in M we have:

$$\mathsf{M}, w \vDash \chi(p) \Leftrightarrow \forall u (wR^* u \Rightarrow \mathsf{M}, u \vDash p),$$

where R^* is the transitive closure of the acceptability relation on M.

To give a syntactic description of pretransitive logics, put $\Box^{\leq n}\varphi = \bigwedge_{i=0}^{n} \Box^i \varphi$, where $\Box^0 \varphi = \varphi$, $\Box^{i+1}\varphi = \Box\Box^i\varphi$.

Lemma 2 (Shehtman, 2010)**.** L *is pretransitive iff* $\mathrm{L} \vdash \Box^{\leq m} p \to \Box^{\leq m+1} p$ *for some* $m \geq 1$.

By this lemma, for any pretransitive logic there exists the least m such that the formula $\Box^* p = \Box^{\leq m} p$ plays the role of $\chi(p)$ from Definition 1. Let $\Diamond^* \varphi = \neg\Box^*\neg\varphi$.

Consider the logics $\mathrm{K}_n^m = \mathrm{K} + \mathrm{A}_n^m$, where $\mathrm{A}_n^m = \Box^m p \to \Box^n p$, $n > m \geq 1$. For any m, n, A_n^m is a Sahlqvist formula, which corresponds to the property $R^n \subseteq R^m$; so all K_n^m are canonical, elementary and Kripke-complete pretransitive logics. If $m = 1, n = 2$, we obtain the well-known logic K4, which has the fmp. In fact, due to [1], all logics K_n^1 have the fmp. Logics with $m > 1$ were also considered (to our knowledge, K_3^2 appears already in the 1960s in papers by Segerberg and Sobociński); nevertheless, no results about the fmp or decidability for these logics are known yet.

Logics with the symmetry axiom for $\Box^*$. For a pretransitive logic L, put

$$\mathrm{L.sym}^* = \mathrm{L} + (p \to \Box^* \Diamond^* p).$$

(In [3], logics of this kind were considered in the particular case where $\mathrm{L} = \mathrm{K} + \Box^{\leq m} p \to \Box^{\leq m+1} p$.) It is well-known that for any formula φ, $\mathrm{S5} \vdash \varphi \Leftrightarrow \mathrm{S4} \vdash \Diamond\Box\varphi$ ([4]). The following is a generalization of this fact.

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[†]Institute for Information Transmission Problems, Moscow, Russia

Theorem 3. *If* L *is a pretransitive logic, then for any formula* φ *we have*

$$\mathrm{L.sym}^* \vdash \varphi \Leftrightarrow \mathrm{L} \vdash \Diamond^*\Box^*\varphi.$$

Before we prove this theorem, we formulate two simple corollaries of Lemma 2.

Proposition 4. *For a pretransitive* L *and a point generated* L*-frame* $\mathsf{F} = (W, R)$, $\mathsf{F} \vDash \mathrm{L.sym}^*$ *iff* R^* *is the universal relation on* W.

Proposition 5. *For a pretransitive* L *and a formula* φ, *let* φ^* *be the formula obtained from* φ *by replacing* $\Box$ *with* $\Box^*$ *and* $\Diamond$ *with* $\Diamond^*$. *Then for any* φ *we have:* $\mathrm{S4} \vdash \varphi \Rightarrow \mathrm{L} \vdash \varphi^*$, $\mathrm{S5} \vdash \varphi \Rightarrow \mathrm{L.sym}^* \vdash \varphi^*$.

Proof of Theorem 3. If $\mathrm{L} \vdash \Diamond^*\Box^*\varphi$, then $\mathrm{L.sym}^* \vdash \Diamond^*\Box^*\varphi$. $\mathrm{S5} \vdash (\Diamond\Box p \to p)$, so using the above proposition, we have $\mathrm{L} \vdash \varphi$.

To prove the converse direction, we proceed by induction on a derivation of φ.

Suppose $\varphi = p \to \Box^*\Diamond^* p$. Since $\mathrm{S4} \vdash \Diamond\Box(p \to \Diamond\Box p)$, by the above proposition $\mathrm{L} \vdash \Diamond^*\Box^*\varphi$.

Suppose $\mathrm{L.sym}^* \vdash \psi_1$, $\mathrm{L.sym}^* \vdash \psi_1 \to \varphi$. By the induction hypothesis, $\mathrm{L} \vdash \Diamond^*\Box^*\psi_1$, $\mathrm{L} \vdash \Diamond^*\Box^*(\psi_1 \to \varphi)$. Then $\mathrm{L} \vdash \Box^*\Diamond^*\Box^*\psi_1$, $\mathrm{L} \vdash \Box^*\Diamond^*\Box^*(\psi_1 \to \varphi)$ (using $\Box$-rule, one can easily show that $\Box^*$-rule is admissible in L). $\mathrm{S4} \vdash \Box\Diamond\Box p \wedge \Box\Diamond\Box(p \to q) \to \Diamond\Box q$, since this formula is valid in any finite S4-frames. So using Proposition 5, we have $\Diamond^*\Box^*\varphi$.

The case when φ is obtained by the substitution rule is trivial.

Suppose $\varphi = \Box\psi$, $\mathrm{L.sym}^* \vdash \psi$. It is easy to check (e.g., using the completeness of the logics $\mathrm{K}{+}\Box^{\leq m} p \to \Box^{\leq m+1} p$) that $\mathrm{L} \vdash \Diamond^*\Box^* p \to \Diamond^*\Box^*\Box p$. By the induction hypothesis, $\mathrm{L} \vdash \Diamond^*\Box^*\psi$, so $\mathrm{L} \vdash \Diamond^*\Box^*\varphi$. □

Corollary 6. *If* L *has the fmp, then* $\mathrm{L.sym}^*$ *also has the fmp.*

Proof. If a formula φ is $\mathrm{L.sym}^*$-consistent then $\Box^*\Diamond^*\varphi$ is satisfiable in a finite L-frame (W, R). It follows that φ is satisfiable in a maximal R^*-cluster, which is an $\mathrm{L.sym}^*$-frame. □

Thus, for a pretransitive L, any negative result about decidability or the fmp for $\mathrm{L.sym}^*$ transfers to L. At the same time, the authors do not know any examples of such $\mathrm{L.sym}^*$. Moreover, next we prove that $\mathrm{K}_n^m.\mathrm{sym}^*$ have the fmp for all $n > m \geq 1$.

Finite model property. By Sahlqvist's Theorem, all logics $\mathrm{K}_n^m.\mathrm{sym}^*$ are canonical and elementary. The class of all $\mathrm{K}_n^m.\mathrm{sym}^*$-frames can be easily characterized in terms of paths and cycles. By an *R-path* Σ in (W, R) we mean a finite sequence of at least two (not necessary distinct) points $(x_0, x_1, \ldots, x_l)$, such that $x_i R x_{i+1}$ for all $i < l$; we say that Σ *connects* x_0 and x_l. l is the *length* of Σ (notation: $[\Sigma]$). If $x_l = x_0$ then Σ is an *R-cycle*.

Proposition 7. *Suppose* $n > m \geq 1$, F *is a point generated frame which is not an irreflexive singleton. Then* $\mathsf{F} \vDash \mathrm{K}_n^m.\mathrm{sym}^*$ *iff any two points in* W *belong to an R-cycle, and for any* w, u, *if* w, u *are connected by an R-path with the length* n, *then* w, u *are connected by an R-path with the length* m.

Proposition 8. *For any* $s, r \geq 0$, $\mathrm{K}_n^m \vdash \Diamond^{m+(n-m)q+r} p \to \Diamond^{m+r} p$.

Proof. By an easy induction on q. □

Proposition 9. *All logics* $\mathrm{K}_n^m.\mathrm{sym}^*$ *are different.*

Proof. Let $L_1 = \mathrm{K}_n^m.\mathrm{sym}^*$ and $L_2 = \mathrm{K}_t^s.\mathrm{sym}^*$. First, we assume that $s < m$, then we consider the following frame

$$\mathsf{F} = (W, R),\ W = \{0, 1, \ldots, m\}\,,\ xRy \Leftrightarrow y = x \text{ or } y \equiv x + 1\ (mod\ m + 1).$$

It is easy to check that $\mathsf{F} \models L_1$ and $\mathsf{F} \not\models L_2$.

Now assume that $s = m$ and $t < n$. Put $k = n - m$,

$$\mathsf{F}' = (W', R'),\ W' = \{0, 1, \ldots, k - 1\}\,,\ xR'y \Leftrightarrow y \equiv x + 1\ (mod\ k).$$

It is also easy to see that $\mathsf{F}' \models L_1$ and $\mathsf{F}' \not\models L_2$. □

Theorem 10. *The logics* $\mathrm{K}_n^m.\mathrm{sym}^*$ *have the fmp for all* $n > m \geq 1$.

If $m = 1$, the statement of the theorem immediately follows from [1] and Corollary 6. Also, for the case $m = n+1$, this theorem can be easily proved by the straightforward filtration argument (the same reasoning works if we consider $\mathrm{K} + \Box^{\leq m}p \to \Box p^{\leq m+1}$ instead of K_{m+1}^m, [3]). Nevertheless, the standard filtration argument does not work for the arbitrary case: to preserve validity of A_n^m, we have to construct a countermodel in a more subtle way. First, we need the following slightly modified version of filtration.

Definition 11. Let $\mathsf{M} = (W, R, \theta)$ be a model, φ be a formula, $\sim$ be an equivalence relation on W. For $u, v \in W$, we define

$$u \sim^\varphi v \text{ iff } u \sim x \text{ and } M, u \vDash \psi \Leftrightarrow M, v \vDash \psi \text{ for every subformula } \psi \text{ of } \varphi.$$

Let $\overline{W} = W/\sim^\varphi$, $\overline{u}\overline{R}\overline{v} \Leftrightarrow \exists u' \in \overline{u}\ \exists v' \in \overline{v}(u'Rv')$, $\overline{\theta}(p) = \{\overline{u} \mid u \in \theta(p)\}$ for all variables of φ (and put $\overline{\theta}(p) = \varnothing$ for other variables). The model $(\overline{W}, \overline{R}, \overline{\theta})$ is called *the (minimal)* $\sim$*-filtration of* M *through* φ.

Note that in the case when $\sim$ is the universal relation, the $\sim$-filtration is the standard *minimal filtration.* Clearly, $\sim$-filtrations preserve truth of subformulas of φ. Also, if $W/\sim$ is finite, then $W/\sim^\varphi$ is finite too.

Proposition 12. *Let* $(\overline{W}, \overline{R}, \overline{\theta})$ *be a* $\sim$*-filtration of* (W, R, θ).

- *For any* $l > 0$, $xR^l y$ *implies* $\overline{x}\,\overline{R}^l\,\overline{y}$.
- *If* R^* *is universal on* W, *then* $\overline{R}^*$ *is universal on* $\overline{W}$.

The proof of the above proposition is straightforward. The main difficulty in the proof of the theorem is to find an appropriate equivalence relation to make sure that A_n^m is valid in the resulting frame.

For a set of integers I, let $gcd(I)$ denotes its greatest common devisor.

Proof of Theorem 10. Let $\mathrm{L} = \mathrm{K}_n^m.\mathrm{sym}^*$, $k = m - n$. Consider an infinite rooted L-frame $\mathsf{F} = (W, R)$, and suppose that $\mathsf{M} = (W, R, \theta), x \vDash \varphi$. We construct a finite L-frame $\overline{\mathsf{F}} = (\overline{W}, \overline{R})$ where φ is satisfiable.

For a positive integer d, consider the relation $\sim_d$ on W: $u \sim_d w$ iff there exists an R-path Γ from u to w such that d divides $[\Gamma]$.

Claim 1. If d divides the length of any R-cycle in F, then $\sim_d$ is an equivalence relation and $W/{\sim_d}$ is finite.

Clearly, $\sim_d$ is transitive. $\sim_d$ is reflexive, since for any $w \in W$ there exists an R-path from w to w. If $u \sim_d w$, then d divides $[\Gamma^\uparrow]$ for some R-path $\Gamma^\uparrow$ from u to w. Let $\Gamma^\downarrow$ be an R-path from w to u. Then d divides $[\Gamma^\uparrow] + [\Gamma^\downarrow]$, so d divides $[\Gamma^\downarrow]$, and $w \sim_d u$.

To show that $W/{\sim_d}$ is finite, take points $w_1 R w_2 R \ldots R w_d$ (we can choose these points because F is serial). If $u \in W$, then some Γ connects w_d and u. Then $w_{d-r} \sim_d u$, where r is the remainder of the division $[\Gamma]$ by d.

To illustrate the following construction, first we consider the simplest case when k is a prime number or $k = 1$. In this case, we have two possibilities:
(a) there exists an R-cycle Γ_0 such that $gcd([\Gamma_0], k) = 1$;
(b) k divides the length of any R-cycle in F.

Suppose (a). Let us show that $wR^l u$ for any $l \geq m$, $w, u \in W$. Let v be the starting point of Γ_0, Γ_1 be an R-path from w to v, and Γ_2 be an R-path from v to u. For some $r < k$ we have $l + [\Gamma_1] + [\Gamma_2] \equiv r \ (mod\ k)$. Consider the path $\Gamma = \Gamma_1 \Gamma_0^{l+k-r} \Gamma_2$ (that is, Γ goes along Γ_1 then $l + k - r$ times along Γ_0 and then along Γ_2). Thus Γ connects w and u, and $[\Gamma] = l + qk$ for some $q > 0$. By Proposition 8, $wR^l u$.

Let $(\overline{\mathsf{F}}, \overline{\theta})$ be the minimal filtration of M through φ. By Proposition 12, between any two point in $\overline{W}$ there exists an $\overline{R}$-path with the length m, so $\overline{\mathsf{F}} \vDash \mathrm{L}$.

Suppose (b). In this case, $\sim_k$ is an equivalence relation on W. Let $(\overline{W}, \overline{R}, \overline{\theta})$ be the $\sim_k$-filtration of M through φ. Let us show that $(\overline{W}, \overline{R}) \vDash \mathrm{A}_n^m$. Suppose that $\overline{x}\overline{R}^n\overline{y}$. It means that we have for some $x_0, x_0', \ldots, x_n, x_n'$: $x_0 = x$, $x_n' = y$, and $x_i \sim_d x_i'$ & $x_i' R x_{i+1}$ for all $i < n$. Now, since $x_i \sim_d x_i'$ implies $x_i R^{q_i k} x_i'$ (for some q_i), there is an R-path Γ from x to y with $[\Gamma] = n + qk$, $q = \sum q_i$. Thus, $xR^{m+(q+1)k}y$, and $xR^m y$ (Proposition 8), and so $\overline{x}\overline{R}^m\overline{y}$ (Proposition 12). Hence $\overline{\mathsf{F}} \vDash \mathrm{L}$.

Now we extend the above construction for arbitrary k. In this case, we need a combination of reasonings from (a) and (b).

Let $D = \{gcd([\Gamma], k) \mid \Gamma \text{ is an } R\text{-cycle in } W\}$, and let d be the greatest common devisor of D. Let us assume that $D = \{d_1, \ldots, d_s\}$.

Claim 2. There exists positive integers $a_1, \ldots, a_s$ and R-cycles $\Gamma_1, \ldots, \Gamma_s$ such that

$$a_1[\Gamma_1] + \cdots + a_s[\Gamma_s] \equiv d \ (mod\ k).$$

To prove this claim, note that for every d_i there exists an R-cycle Γ_i and a positive integer l_i, such that

$$[\Gamma_i] = l_i d_i \text{ and } l_i \equiv 1 \ (mod\ k).$$

By the Euclidean algorithm, we have $\sum_{i=1}^s b_i d_i = d$ for some integers b_i, therefore $\sum_{i=1}^s a_i d_i \equiv d \ (mod\ k)$ for some $a_i > 0$. Since $l_i \equiv 1 \ (mod\ d)$, $\sum_{i=1}^s a_i l_i d_i \equiv d \ (mod\ k)$, which proves the claim.

By Claim 1, $\sim_d$ is an equivalence on W. Let $(\overline{W}, \overline{R}, \overline{\theta})$ be the $\sim_d$-filtration of M through φ. Similarly to the case (b), we obtain that if $\overline{u}\overline{R}^n\overline{w}$, then $u \in R^{n+dr}w$ for some $r \geq 0$. By Proposition 8, we may assume that $r < k$.

Let v_i denote the starting point of Γ_i, $\Delta_i^{\uparrow}$ be an R-path from w to v_i, $\Delta_i^{\downarrow}$ – from v_i to w. Let $\Sigma_i = {\Delta_i^{\uparrow}}^{k-1} {\Delta_i^{\downarrow}}^{k-1} \Delta_i^{\uparrow} \Gamma_i^{(k-r)a_i} \Delta_i^{\downarrow}$. So Σ_i is an R-path from w to w and $[\Sigma_i] \equiv (k - r)a_i[\Gamma_i] \ (mod\ k)$. Let $\Gamma = \Sigma_0 \Sigma_1 \ldots \Sigma_s$, where Σ_0 is an R-path from u to w with the length $n + dr$. By Claim 2, $[\Gamma] \equiv m \ (mod\ k)$. Thus, $uR^m w$, $\overline{u}\overline{R}^m\overline{w}$ and $\overline{\mathsf{F}} \vDash \mathrm{A}_n^m$. □

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