Journal of Nonlinear Mathematical Physics, Vol. 1, No. 1 (2009) 1–15

BILINEAR IDENTITIES ON SCHUR SYMMETRIC FUNCTIONS

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Received (04 May, 2009)
Revised (31 May, 2009)
Accepted (01 July, 2009)

A series of bilinear identities on the Schur symmetric functions is obtained with the use of Plücker relations.

Keywords: Schur symmetric function, partition, Young diagram

2000 Mathematics Subject Classification: 05E05

1. Introduction

In this note we present a wide class of bilinear identities the Schur symmetric functions satisfy. The bilinear identities are homogeneous second order polynomial relations with integer coefficients, connecting different Schur functions. For the detailed treatment of the Schur function theory, the corresponding terminology, examples etc., see the monograph [Mac]. Here we give only a short list of definitions and key examples for convenience of the reader.

A sequence of non-increasing non-negative integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i, \dots), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_i \geq \dots$$

containing only finitely many non-zero terms is called *a partition*. The total number of non-zero components, $\ell(\lambda)$, is called *the height* of a given partition λ

$$\ell(\lambda) = n \iff \lambda_n > 0, \quad \lambda_{n+1} = 0.$$

Given a partition λ with $\ell(\lambda) = n$, the Schur symmetric function (actually, it is a polynomial) $s_\lambda(t_1, \dots, t_m)$, where $m \geq \ell(\lambda)$, is an element of the ring $\mathbb{Z}[t_1, \dots, t_m]$ defined as the ratio of two determinants [Mac]

$$s_\lambda(t_1, \dots, t_m) = \left. \frac{\det \| t_i^{\lambda_j + m - j} \|}{\det \| t_i^{m-j} \|} \right|_{1 \leq i,j \leq m}.$$

The set of Schur symmetric functions $s_\lambda(t_1,\dots,t_m)$ labeled by all partitions λ with $\ell(\lambda)\le m$ forms a $\mathbb{Z}$-basis of the subring of symmetric polynomials

$$\Lambda_m=\mathbb{Z}[t_1,\dots,t_m]^{S_m}$$

where the symmetric group S_m acts on the polynomials from $\mathbb{Z}[t_1,\dots,t_m]$ by the permutating the indeterminates.

The ring Λ_m is graded

$$\Lambda_m=\bigoplus_{k\ge 0}\Lambda_m^k,$$

where Λ_m^k consists of the homogeneous symmetric polynomials of degree k. Then by a specific inverse limit (for details, see [Mac]) as $m\to\infty$ we pass from Λ_m^k to a graded ring Λ called the *ring of symmetric functions in countably many indeterminates* $\{t_i\}_{i\in\mathbb{N}}$. For each partition λ, the polynomials $s_\lambda\in\Lambda_m$, define a unique element $s_\lambda\in\Lambda$ called the *Schur symmetric function in countably many indeterminates*. Note that $s_\lambda\in\Lambda$ is no longer polynomial (as well as other elements of the ring Λ). It is a formal infinite sum of monomials, each of them being homogeneous of degree $|\lambda|=\lambda_1+\dots+\lambda_n$. The Schur symmetric functions form a $\mathbb{Z}$-basis of the ring Λ and satisfy the Littlewood-Richardson multiplication rule

$$s_\lambda s_\mu=\sum_\nu C^\nu_{\lambda\mu}s_\nu, \tag{1.1}$$

where the non-negative integers $C^\nu_{\lambda\mu}$ (the Littlewood-Richardson coefficients) are calculated by some combinatorial rule from partitions λ, μ and ν. Actually, the multiplication rule (1.1) can be taken for the formal definition of the ring Λ in the $\mathbb{Z}$-basis of Schur symmetric functions.

The bilinear identities we would like to discuss is another type of relations among the Schur functions. As was mentioned at the beginning of the section, they are of the form $p(\{s_{\lambda_i}\})=0$, where $p(\{x_i\})$ is a homogeneous second order polynomial (a bilinear form) in its indeterminates with integer coefficients. These identities follow, of course, from the multiplication rule (1.1) but we use another technique to prove them.

As the first example of such identities we mention the bilinear relations obtained in [Kir]:

$$s_{[m|n]}s_{[m|n]}=s_{[m|n-1]}s_{[m|n+1]}+s_{[m-1|n]}s_{[m+1|n]}, \tag{1.2}$$

where $[m|n]$ stands for the partition (m^n) with n components equal to m. This identity connects the characters of the irreducible representations of $SU(p+1)$, where $s_{[m|n]}$ is a character of the m-th symmetric power of the fundamental $SU(p+1)$ representation π_n corresponding to the signature $(1,1,\dots,1,0,\dots,0)$ (n units, $1\le n\le p$). The identity (1.2) played the key role in proving the completeness of the Bethe vector set for the generalized Heisenberg model. In the paper [KR], analogous bilinear identities were obtained for the characters of symmetric powers of fundamental representations of other classical Lie groups (of B, C and D series).

In the work [GPS] on quantum supermatrix algebras of $GL(m|n)$ type, we generalized the above identities to the products $s_{[a|b]}\,s_{[m|n]}$ for arbitrary integers $1\le a\le m$ and $1\le b\le n$:

$$s_{[a|b]}\,s_{[m|n]}=\sum_{\substack{k=\\ \max\{1,a+b-n\}}}^{a}(-1)^{a-k}s_{[m|n]_{a+b-k}}\,s_{[a-1|b-1]^{k-1}}+\sum_{\substack{k=\\ \max\{1,a+b-m\}}}^{b}(-1)^{b-k}s_{[m|n]^{a+b-k}}\,s_{[a-1|b-1]_{k-1}}, \tag{1.3}$$

where the symbols $[r|p]^k$ ($k\le r$) and $[r|p]_k$ ($k\le p$) denote the partitions $((p+1)^k,p^{r-k})$ and (p^r,k), respectively. These identities turned out to be useful in studying the structure of the maximal commutative subalgebras of the quantum supermatrix algebra.

In the work [Kl], identity (1.2) was generalized to the product $s_\lambda s_\lambda$ for an arbitrary partition λ. In the present paper, we give a different version of the identity for the product $s_\lambda s_\lambda$. In contrast with the result of [Kl], our formula admits the transposition of the Young diagrams which parameterize the Schur functions. In other words, given a bilinear identity for $s_\lambda s_\lambda$, we get a true identity if we change all the partitions λ by their conjugates λ' (see section 2 and [Mac]). In particular, if the Young diagram of the partition λ is symmetric under the transposition, the identity for $s_\lambda s_\lambda$ is also symmetric.

M. Fulmek and M. Kleber have found the identities for the product of two different Schur functions. Namely, in [FK], they proved that

$$s_{(\lambda_1,\dots,\lambda_n)}s_{(\lambda_2,\dots,\lambda_{n+1})} = s_{(\lambda_2,\dots,\lambda_n)}s_{(\lambda_1,\dots,\lambda_{n+1})} + s_{(\lambda_2-1,\dots,\lambda_{n+1}-1)}s_{(\lambda_1+1,\dots,\lambda_n+1)}, \tag{1.4}$$

where $(\lambda_1, \lambda_2, \dots, \lambda_{n+1})$ is a partition, $n > 0$ being an integer.

The series of the bilinear identities derived in this paper considerably generalizes the identities (1.4).

In the next section we introduce our notation and some key operations with partitions. The third section is devoted to the derivation of bilinear identities. The main results are formulated in Proposition 3.1 and Corollary 3.7.

2. Definitions and notation

We use the terminology and definitions from the monograph [Mac].

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition of the height $\ell(\lambda) = n$, that is $\lambda_n > 0$. We omit the zero components of λ. The Schur symmetric function corresponding to the partition λ can be expressed in terms of the complete symmetric functions h_k by means of the Jacobi-Trudi relations [Mac]:

$$s_\lambda = \det \|h_{\lambda_i - i + j}\|_{1 \le i,j \le N}, \tag{2.1}$$

where the index i enumerates rows, the index j enumerates columns, and $N \ge \ell(\lambda) = n$ is an arbitrary positive integer. In the above formula it is assumed that $h_0 \equiv 1$ and $h_k \equiv 0$ if $k < 0$.

Vectors μ. As is clear from the Jacobi-Trudi determinant (2.1), any its row is completely defined by the index of the first element of the row. Therefore, the Jacobi-Trudi determinants and the corresponding Schur functions can be unambiguously parameterized by the vectors $\mu \in \mathbb{Z}^N$ of the form

$$\mu = [\mu_1, \dots, \mu_N], \qquad \mu_i := \lambda_i - i + 1 \tag{2.2}$$

that is, $\mu = \lambda - \delta^{(N)}$, $\delta^{(N)} = [0, 1, \dots, N-1]$. Unlike the partition λ, some of the components of μ can be negative. Besides, the components of μ form a *strictly* descending sequence

$$\mu_1 > \mu_2 > \dots > \mu_N.$$

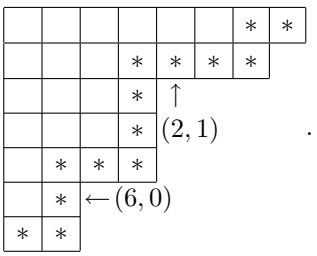
To each partition λ we assign its graphical image — the Young diagram (see [Mac]). Below we denote the Young diagram of the partition λ by the same letter (when it does not lead to a misunderstanding). Now we describe subsets of the Young diagram λ and define some operations with them; this will be used in what follows.

The complete border strip. Consider the Young diagram corresponding to a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. Let us remove $\lambda_2 - 1$ boxes from the first row of the diagram, starting from the first (the *left*-most) one. Then we extend this procedure to the other rows removing $\lambda_{k+1} - 1$ boxes from the k-th row, $1 \le k \le n-1$. We leave the last n-th row unchanged.

This procedure results in a skew-diagram which will be referred to as *the complete border strip*. Any nonempty proper subset of the complete border strip will be called a *border strip* provided this

subset can be represented as the set-theoretical difference $\lambda \setminus \nu$, where $\nu \subset \lambda$ is a Young diagram completely contained in λ.

As an example, we consider the partition $(8, 7, 4^3, 2^2)$. Its Young diagram with the complete border strip marked by star signs is depicted below:



We accept the following indexation of the boxes in the complete border strip. As follows from the definition, in the r-th row of the Young diagram λ, the boxes of the complete border strip occupy positions from the λ_{r+1}-th column till the λ_r-th one (counting from left to right). So, these boxes in the r-th row can be enumerated by the number s such that $0 \leq s \leq \lambda_r - \lambda_{r+1}$. A box of the complete border strip situated in the r-th row and in the $(\lambda_{r+1} + s)$-th column will be represented by an ordered pair of nonnegative integers (r, s). In the above example of the Young diagram, we show the coordinate pairs of two boxes in the complete border strip.

The peeling. Let us remove the complete border strip from the Young diagram λ. The new diagram thus obtained will be denoted by the symbol $\lambda\!\downarrow$. We say that $\lambda\!\downarrow$ is obtained from λ by peeling the complete border strip off. Note that the diagram $\lambda\!\downarrow$ can be the empty set if λ is a simple hook diagram:

$$(k, 1^m)\!\downarrow = \emptyset \quad \text{for all} \quad k, m \geq 0.$$

It is not difficult to see that the diagram $\lambda\!\downarrow$ can be obtained by removing the first row and the first column from λ. As a consequence, the height of $\lambda\!\downarrow$ is always less than that of λ:

$$\ell(\lambda\!\downarrow) \leq \ell(\lambda) - 1.$$

Turning to the components of the partition λ, we get the following structure of the partition $\lambda\!\downarrow$

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \quad \rightarrow \quad \lambda\!\downarrow = (\lambda_2 - 1, \lambda_3 - 1, \ldots, \lambda_n - 1, 0). \tag{2.3}$$

The corresponding μ-vectors (2.2) are connected with each other by a simple transformation

$$\mu = [\mu_1, \mu_2, \ldots, \mu_N] \quad \rightarrow \quad \mu\!\downarrow = [\mu_2, \mu_3, \ldots, \mu_N, -N + 1]. \tag{2.4}$$

In other words, the components of μ are just shifted one position to the left, the component μ_1 disappears, and on the last place we get the number $1 - N$.

Consider now the peeling a border strip off, or a *partial* peeling. In this case, we have to indicate the direction of the peeling, that is we consider a *partial up-peeling* and a *partial down-peeling*.

Let us fix a box (r, s) in the complete border strip of a Young diagram λ. Starting from the box (r, s), we remove all the boxes of the complete border strip lying to the left and down of the chosen box. That is, we remove all the boxes (r, t) with $0 \leq t \leq s$ and (p, t) with $p > r$. This procedure will be called the *partial down-peeling* from the starting box (r, s). We will only be interested in down-peelings that transform a Young diagram to a Young diagram. For this to be true, the starting box (r, s) of the partial down-peeling must be the *right-most* box in the r-th row. In other words, the number s must take the maximal possible value $s = \lambda_r - \lambda_{r+1}$. To simplify the expressions, we omit this s in notation and denote the diagram (and the partition) obtained from the diagram λ by

the partial down-peeling from the box $(r, \lambda_r - \lambda_{r+1})$ by the symbol $\lambda\downarrow^{(r)}$. The components of the partition $\lambda\downarrow^{(r)}$ read

$$\lambda\downarrow^{(r)} = (\lambda_1, \ldots, \lambda_{r-1}, \lambda_{r+1} - 1, \ldots, \lambda_n - 1, 0), \tag{2.5}$$

while for the components of the corresponding μ-vector $\mu\downarrow^{(r)}$ we obtain

$$\mu\downarrow^{(r)} = [\mu_1, \ldots, \mu_{r-1}, \mu_{r+1}, \ldots, \mu_N, -N+1]. \tag{2.6}$$

Same as the peeling the complete border strip off, the partial down-peeling decreases the height of the diagram at least by one: $\ell(\lambda\downarrow^{(r)}) \leq \ell(\lambda) - 1$.

The partial up-peeling is defined in an analogous way. We fix a starting box (r, s) in the complete border strip of a diagram λ and remove all the boxes (r, t) with $t \geq s$ and (p, t) with $p < r$. That is we remove all the boxes of the complete border strip, lying to the right and up of the chosen starting box. This procedure will be called the *partial up-peeling* from the starting box (r, s). In what follows we will be interested only in partial up-peelings that do not destroy the structure of Young diagrams. Therefore, the starting box (r, s) of the up-peeling must be chosen in such a way that there are no box of the diagram directly *under* it. This is only possible if $\lambda_r > \lambda_{r+1}$ and, besides, $s \geq 1$. The Young diagram (and the partition) obtained from the diagram λ by the partial up-peeling from the starting box (r, s) will be denoted by the symbol $\lambda\uparrow_{(r,\,s)}$.

The component structure of the partition $\lambda\uparrow_{(r,\,s)}$ is as follows

$$\lambda\uparrow_{(r,\,s)} = \underset{1}{(\lambda_2 - 1}, \ldots, \underset{r-1}{\lambda_r - 1}, \underset{r}{\lambda_{r+1} + s - 1}, \underset{r+1}{\lambda_{r+1}}, \ldots, \underset{n}{\lambda_n}), \quad 1 \leq s \leq \lambda_{r-1} - \lambda_r, \tag{2.7}$$

where in the second line we have written the ordinal numbers of the corresponding components to clarify the structure. For the corresponding vector μ, we get the following expression

$$\mu\uparrow_{(r,\,s)} = \underset{1}{[\mu_2}, \ldots, \underset{r-1}{\mu_r}, \underset{r}{\mu_{r+1} + s}, \underset{r+1}{\mu_{r+1}}, \ldots, \underset{N}{\mu_N}], \quad 1 \leq s \leq \mu_{r-1} - \mu_r - 1, \tag{2.8}$$

Adding a border strip to diagram. Consider the Young diagram, corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. Choose $m \leq n - 1$ consecutive rows with numbers $r, r+1, \ldots, r+m-1$, where $2 \leq r \leq n - m + 1$. We are going to add boxes in the chosen rows in such a way that the result would be a Young diagram, and, besides, the added boxes would form a connected border strip in the new diagram. The restriction on the number of rows means that we do not add boxes into the first line of λ ($r \geq 2$) and that we do not increase the height of the diagram ($r \leq n - m + 1$). Below we use the shorthand notation $r_m := r + m - 1$.

It turns out to be convenient to treat the first (the left-most) box added into the r_m-th row as *the beginning* (or the first) box of the strip.

The last (the right-most) box added into the r-th row will be treated as *the end* (or the last) box of the strip. The beginning of the added strip can be placed in any row of λ (except for the above restriction on number) with the only requirement that the first added box must appear in the $(\lambda_{r_m} + 1)$-th column (to preserve the correct structure of the Young diagram). As for the end of the strip, it can be situated only in the row which is shorter than its preceding row: $\lambda_r < \lambda_{r-1}$.

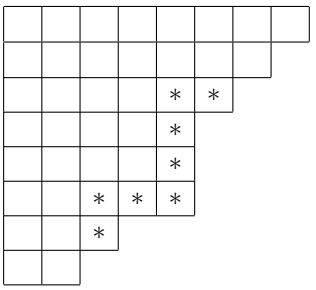
The number of boxes added into the $(r + i)$-th row reads as follows

$$p_i = \lambda_{r+i-1} - \lambda_{r+i} + 1, \quad 1 \leq i \leq m - 1. \tag{2.9}$$

Into the last, r-th, row we add $p_0 = t$ boxes, where $1 \leq t \leq \lambda_{r-1} - \lambda_r$. Therefore, the total amount of boxes added is equal to

$$p = \sum_{i=0}^{m-1} p_i = \lambda_r - \lambda_{r_m} + t + m - 1 = \mu_r - \mu_{r_m} + t.$$

Here is an example of adding a border strip for the case $\lambda = (8, 7, 4^3, 2^3)$, $r = 3$, $m = 5$ and $t = 2$:



Here stars denote the added boxes.

The symbol $\lambda^{+\,t}_{(r,m)}$ will stand for the diagram (and the partition) obtained from the diagram λ by adding a border strip of m rows from r to $r_m = r+m-1$ with t boxes in the end row r. If we add several (say k) disconnected border strips, the notation is obviously generalized to $\lambda^{+\,t_1\ \ldots\ t_k}_{(r_1,m_1)\ldots(r_k,m_k)}$.

The components of the partition $\lambda^{+\,t}_{(r,m)}$ read (recall that $1 \le t \le \lambda_{r-1} - \lambda_r$)

$$\begin{array}{l}\lambda^{+\,t}_{(r,m)} = (\lambda_1, \ldots, \lambda_{r-1},\, \lambda_r + t,\, \lambda_r + 1,\, \lambda_{r+1} + 1,\, \ldots,\, \lambda_{r_m-1} + 1,\, \lambda_{r_m+1},\, \ldots,\, \lambda_n) \\ \qquad\qquad\qquad\quad r \quad\ r+1 \quad\ r+2 \quad \ldots \quad\ r_m \quad\ r_m+1 \quad \ldots \quad n.\end{array} \tag{2.10}$$

Here in the second line we have written the ordinal numbers of the corresponding components.

The component structure of the corresponding vector $\mu^{+\,t}_{(r,m)}$ is more transparent

$$\begin{array}{l}\mu^{+\,t}_{(r,m)} = [\mu_1, \ldots, \mu_{r-1},\, \mu_r + t,\ \ \mu_r,\ \ \mu_{r+1},\, \ldots,\, \mu_{r_m-1},\, \mu_{r_m+1},\, \ldots,\, \mu_N]\,. \\ \qquad\qquad\qquad\quad\ \ r \quad\ r+1 \ \ r+2 \ \ \ldots \quad\ r_m \quad\ r_m+1 \quad \ldots \quad N\end{array} \tag{2.11}$$

As we see, the changes take place only for the components from μ_r to μ_{r_m}. Namely, the string of components $\mu_r, \ldots, \mu_{r_m-1}$ shifts one position to the right, in the r-th place (the end row of the added strip) we get the new component $\mu_r + t$ and the component μ_{r_m} (the beginning row of the strip) disappears.

3. Bilinear identities

The bilinear identities on the Schur symmetric functions follow from the Jacobi-Trudi determinant formula (2.1) and the Plücker relation on the product of two determinants (for details, see [St]). Let us formulate the corresponding statement for the reader's convenience.

Consider a pair of $p \times p$ matrices $A = \|a_{ij}\|_{i,j=1}^{p}$ and $B = \|b_{ij}\|_{i,j=1}^{p}$. Let a_{i*} denote the i-th row of the matrix A. Introduce the following notation:

$$\det A := |A|\,, \qquad A := \begin{pmatrix} a_{1*} & \ldots & a_{i*} & \ldots & a_{p*} \\ 1 & \ldots & i & \ldots & p \end{pmatrix}, \tag{3.1}$$

where the last symbol contains a detailed information on the row content of A. Namely, it says that the row a_{i*} is located in the i-th place in the matrix A (when counting from the top down).

Let us fix a set of integer data $\{k \,|\, r_1, r_2, \ldots, r_k\}$, where $1 \le k \le p$ and $1 \le r_1 < \cdots < r_k \le p$. Given these data, the Plücker relation reads

$$\begin{aligned} |A||B| = \sum_{1 \le s_1 < \cdots < s_k \le p} & \begin{vmatrix} a_{1*} & \ldots & b_{s_1*} & \ldots & b_{s_2*} & \ldots & b_{s_k*} & \ldots & a_{p*} \\ 1 & \ldots & r_1 & \ldots & r_2 & \ldots & r_k & \ldots & p \end{vmatrix} \times \\ & \begin{vmatrix} b_{1*} & \ldots & a_{r_1*} & \ldots & a_{r_2*} & \ldots & a_{r_k*} & \ldots & b_{p*} \\ 1 & \ldots & s_1 & \ldots & s_2 & \ldots & s_k & \ldots & p \end{vmatrix}, \end{aligned} \tag{3.2}$$

where the sum is taken over all possible sets $\{k \,|\, s_1, \ldots, s_k\}$.

Now we can obtain a bilinear identity, connecting the Schur symmetric functions labeled by a partition λ and the partition $\lambda^{+\substack{t_1 \quad \cdots \quad t_k \\ (r_1,m_1)\ldots(r_k,m_k)}}$. Here we assume that the structure of the diagram λ allows adding k border strips of the indicated size and location.

Proposition 3.1. *In the Young diagram corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, let there exist $k \geq 1$ rows with numbers $2 \leq r_1 < r_2 < \cdots < r_k \leq r_{k+1} := n$ possessing the property*

$$\lambda_{r_i} < \lambda_{r_i - 1}, \quad 1 \leq i \leq k.$$

Let the integers t_i, m_i, where $1 \leq i \leq k$, satisfy the restrictions

$$1 \leq t_i \leq \lambda_{r_i - 1} - \lambda_{r_i}, \quad 1 \leq m_i \leq r_{i+1} - r_i, \quad 1 \leq i \leq k.$$

Then the Young diagram $\lambda^{+\substack{t_1 \quad \cdots \quad t_k \\ (r_1,m_1)\ldots(r_k,m_k)}}$ can be defined and the following bilinear identity on the Schur symmetric functions holds

$$s_\lambda s_{\lambda^{+\substack{t_1 \quad \cdots \quad t_k \\ (r_1,m_1)\ldots(r_k,m_k)}\downarrow}} = s_{\lambda^{+\substack{t_1 \quad \cdots \quad t_k \\ (r_1,m_1)\ldots(r_k,m_k)}}} s_{\lambda\downarrow} + \sum_{p=1}^{k} s_{\lambda^{+\substack{t_1 \quad \cdots \quad t_k \\ (r_1,m_1)\ldots(r_k,m_k)}\downarrow^{(r_p)}}} s_{\lambda\uparrow_{(r_p\,-\,1,\,t_p)}}. \tag{3.3}$$

Proof. To prove the proposition we use the Jacobi-Trudi formulae for the Schur functions and the Plücker relation for the product of two determinants. In so doing, we shall parameterize the rows of the Jacobi-Trudi determinants in (3.2) by components of vectors μ defined in (2.2).

First of all, we inspect the structure of the Jacobi-Trudi determinants in the left-hand side of (3.3) in order to find the set of rows to be exchanged in accordance with the Plücker relation. Taking into account expression (2.11) for the μ-vector of the diagram with added border strip and expression (2.4) for the peeling the complete border strip off, we have

$$s_\lambda s_{\lambda^{+\substack{t_1 \quad \cdots \quad t_k \\ (r_1,m_1)\ldots(r_k,m_k)}\downarrow}} = \left| \begin{matrix} \mu_1 & \ldots & \mu_{r_i} & \mu_{r_i+1} & \ldots & \mu_{r_{m_i}} & \ldots & \mu_N \\ 1 & \ldots & r_i & r_i+1 & \ldots & r_{m_i} & \ldots & N \end{matrix} \right|_{\mathsf{I}} \times$$

$$\left| \begin{matrix} \mu_2 & \ldots & \mu_{r_i-1} & \mu_{r_i}+t_i & \mu_{r_i} & \mu_{r_i+1} & \ldots & \mu_{r_{m_i}-1} & \mu_{r_{m_i}+1} & \ldots & \mu_N & -N+1 \\ 1 & \ldots & r_i-2 & r_i-1 & r_i & r_i+1 & \ldots & r_{m_i}-1 & r_{m_i} & \ldots & N-1 & N \end{matrix} \right|_{\mathsf{II}},$$

where we explicitly indicated the components containing the i-th part of the added border strip. Recall that it is located in rows between r_i and $r_{m_i} = r_i + m_i - 1$. The indices I and II were introduced for convenience of references.

Let us take the data $\{k \,|\, r_{m_1}, r_{m_2}, \ldots r_{m_k}\}$ to indicate the k rows of the determinant I to be exchanged with all possible sets of k rows of the determinant II in accordance with the Plücker relation (3.2). It is not difficult to see that in the right hand side of the Plücker relation applied to the above product of the determinants I and II there are only $(k+1)$ nonzero terms. They correspond to the exchange of the rows $r_{m_1}, r_{m_2}, \ldots, r_{m_k}$ of the determinant I with rows $r_1 - 1, r_2 - 1, \ldots, r_k - 1$ and N of the determinant II. The other terms vanish since the determinants obtained in exchanging procedure possess at least two identical rows.

The nonzero terms correspond to the following ways of row exchange

$$\begin{bmatrix}\mu_{r_{m_i}}\\ r_{m_i}\end{bmatrix}_{\mathsf{I}} \longleftrightarrow \begin{bmatrix}\mu_{r_i}+t_i\\ r_i-1\end{bmatrix}_{\mathsf{II}}, \quad 1\le i\le k \qquad \text{placement A}$$

$$\left.\begin{array}{ll}\begin{bmatrix}\mu_{r_{m_i}}\\ r_{m_i}\end{bmatrix}_{\mathsf{I}} \longleftrightarrow \begin{bmatrix}\mu_{r_i}+t_i\\ r_i-1\end{bmatrix}_{\mathsf{II}}, & 1\le i\le p-1\\ \begin{bmatrix}\mu_{r_{m_j}}\\ r_{m_j}\end{bmatrix}_{\mathsf{I}} \longleftrightarrow \begin{bmatrix}\mu_{r_{j+1}}+t_{j+1}\\ r_{j+1}-1\end{bmatrix}_{\mathsf{II}}, & p\le j\le k-1\\ \begin{bmatrix}\mu_{r_{m_k}}\\ r_{m_k}\end{bmatrix}_{\mathsf{I}} \longleftrightarrow \begin{bmatrix}-N+1\\ N\end{bmatrix}_{\mathsf{II}} & \end{array}\right\} \quad \text{placements } B_p,\ 1\le p\le k$$

The row exchange in accordance with the placement A gives the first term in the right hand side of (3.3). Indeed, after such an exchange the typical part of the determinant I takes the form

$$\begin{vmatrix}\dots\ \mu_{r_i}\ \mu_{r_i+1}\ \dots\ \mu_{r_i}+t_i\ \dots\\ \dots\ r_i\ \ r_i+1\dots\ \ r_{m_i}\ \ \dots\end{vmatrix}_{\mathsf{I}}.$$

Now we have to make the cyclic permutation of rows placing the component $\mu_{r_i}+t_i$ to the r_i-th row. This gives the sign factor $(-1)^{r_{m_i}-r_i}=(-1)^{m_i-1}$ and, according to (2.11), the structure of the determinant I corresponds to the Schur function $s_{\lambda^+{}^{t_1\ \dots\ t_k}_{(r_1,m_1)\dots(r_k,m_k)}}$. As for the typical part of the second determinant, we get after the row exchange

$$\begin{vmatrix}\mu_2\ \dots\ \mu_{r_{m_i}}\ \mu_{r_i}\ \dots\ \mu_{r_{m_i}-1}\ \mu_{r_{m_i}+1}\ \dots\ \mu_N\ \ -N+1\\ 1\ \dots\ r_i-1\ \ r_i\ \ \dots\ r_{m_i}-1\ \ r_{m_i}\ \ \dots\ N-1\ \ N\end{vmatrix}_{\mathsf{II}}.$$

Here we also have to make the cyclic permutation of rows from (r_i-1) to $(r_{m_i}-1)$ placing the component $\mu_{r_{m_i}}$ to the $(r_{m_i}-1)$-th row. This generates the sign factor $(-1)^{m_i-1}$ which compensates the same factor of the determinant I. As for the structure of the determinant II, it corresponds to $s_{\lambda\downarrow}$ as directly follows from (2.4).

Turn now to a placement of B_p type for some fixed integer p such that $1\le p\le k$. We first consider the changes in the determinant I. The rows r_{m_1} to $r_{m_{p-1}}$ are exchanged in the same way as in the placement A giving rise to the following typical parts corresponding to added border strips

$$\begin{vmatrix}\mu_1\ \dots\ \mu_{r_i}+t_i\ \ \mu_{r_i}\ \ \dots\ \mu_{r_{m_i}-1}\ \dots\\ 1\ \ \dots\ \ \ r_i\ \ \ r_i+1\dots\ \ r_{m_i}\ \ \dots\end{vmatrix}_{\mathsf{I}}, \quad 1\le i\le p-1,$$

with the sign factor $(-1)^{m_i-1}$ for each strip lying in rows r_i to r_{m_i}. The remaining part of the determinant I can be transformed to

$$\begin{vmatrix}\dots\ \mu_{r_{m_p}-1}\ \mu_{r_{m_p}+1}\ \dots\ \mu_{r_j}+t_j\ \mu_{r_j}\ \dots\ \mu_{r_{m_j}-1}\ \dots\ \mu_{r_{m_k}-1}\ \mu_{r_{m_k}+1}\ \dots\ -N+1\\ \dots\ r_{m_p}-1\ \ \ r_{m_p}\ \ \ \dots\ \ r_j-1\ \ \ r_j\ \ \dots\ r_{m_j}-1\ \dots\ r_{m_k}-1\ \ \ r_{m_k}\ \ \ \dots\ \ \ N\end{vmatrix}_{\mathsf{I}}$$

with the sign factors $(-1)^{r_{j+1}-r_{m_j}-1}$ and $p\le j\le k-1$ which originate from the cyclic permutation of rows from r_{m_j} till $(r_{j+1}-1)$. This permutation results in moving the component $\mu_{r_{j+1}}+t_{j+1}$ from the r_{m_j}-th row to the $(r_{j+1}-1)$-th one. We have also a sign factor $(-1)^{N-r_{m_k}}$ since the component $(-N+1)$ moved from the r_{m_k}-th row to the last, N-th, row. Finally, taking into account the structure of the partial down-peeling (2.6), we see that, up to the above sign factors, the determinant I represents the following Schur symmetric function

$$\begin{vmatrix}\mu_1\ \dots\ \mu_{r_i}\ \mu_{r_i+1}\ \dots\ \mu_{r_{m_i}}\ \dots\ \mu_N\\ 1\ \ \dots\ \ r_i\ \ r_i+1\dots\ \ r_{m_i}\ \ \dots\ \ N\end{vmatrix}_{\mathsf{I}} \xrightarrow{B_p} s_{\lambda^+{}^{t_1\ \dots\ t_k}_{(r_1,m_1)\dots(r_k,m_k)}{}^{\downarrow(r_p)}}.$$

Consider now the changes in the determinant II under the row exchange of the same B_p type. The part of the determinant containing the rows $r_i - 1$ for $1 \le i \le p-1$ can be expressed in the following form

$$\left| \begin{matrix} \mu_2 \dots & \mu_{r_{m_i}} & \mu_{r_i} \dots & \mu_{r_{m_i}-1} & \mu_{r_{m_i}+1} \dots \\ 1 \ \dots \ r_i - 1 & r_i & \dots \ r_{m_i} - 1 & r_{m_i} & \dots \end{matrix} \right|_{\mathrm{II}}.$$

Here we have to rearrange the rows from $(r_i - 1)$ till $(r_{m_i} - 1)$ by cyclic permutation in order to move the component $\mu_{r_{m_i}}$ to the $(r_{m_i} - 1)$-th row. This gives rise to the sign factor $(-1)^{r_i - r_{m_i}} = (-1)^{m_i - 1}$ for each $1 \le i \le p-1$. The sign factors compensate the analogous sign factors originated from the determinant I.

The rest part of the determinant II reads $(p \le j \le k-1)$

$$\left| \begin{matrix} \dots \ \mu_{r_p} + t_p & \mu_{r_p} \dots & \mu_{r_{m_j}} & \mu_{r_{j+1}+1} \dots & \mu_{r_{m_{j+1}}-1} & \mu_{r_{m_{j+1}}+1} \dots & \mu_{r_{m_k}} \\ \dots \ r_p - 1 & r_p \ \dots \ r_{j+1} - 1 & r_{j+1} & \dots \ r_{m_{j+1}} - 1 & r_{m_{j+1}} & \dots & N \end{matrix} \right|_{\mathrm{II}}.$$

On moving the component $\mu_{r_{m_j}}$ from the $(r_{j+1} - 1)$-th row to the r_{m_j}-th one we get the sign factor $(-1)^{r_{j+1} - r_{m_j} - 1}$ for each $p \le j \le k-1$. Also we get the factor $(-1)^{N - r_{m_k}}$ since the component $\mu_{r_{m_k}}$ moved from the last, N-th, row to the row r_{m_k}. All these sign factors exactly compensate the corresponding sign factors appearing in the determinant I. The final structure of the determinant II is as follows:

$$\left| \begin{matrix} \mu_2 \dots & \mu_{r_i} & \dots & \mu_{r_{p-1}} & \mu_{r_p} + t_p & \mu_{r_p} \dots & \mu_{r_j} \dots & \mu_N \\ 1 \ \dots \ r_i - 1 & \dots & r_p - 2 & r_p - 1 & r_p & \dots & r_j & \dots & N \end{matrix} \right|_{\mathrm{II}}.$$

On comparing the above determinant with (2.8), we conclude that under the row exchange of the B_p type the determinant II transforms to the Schur function $s_{\lambda \uparrow_{(r_p - 1,\, t_p)}}$ (up to the sign factors compensated by the corresponding factors of the determinant I).

At last, summing over all placements of the B_p type and adding the result of the placement A we get the final formula (3.3). ■

Consider now some important corollaries of Proposition 3.1.

Corollary 3.2. *The identity (3.3) is preserved under the simultaneous transposition of all the Young diagrams parameterizing the Schur functions in (3.3).*

Proof. Recall (see [Mac]) that the partition λ' is said to be the *conjugate* of a given partition λ if the Young diagram λ' is obtained from the Young diagram λ by the transposition with respect to the main diagonal. In other words,

$$\lambda'_i = \#(j \mid \lambda_j \ge i).$$

The key point in the proof of the Corollary 3.2 is the following Jacobi-Trudi determinant formula for the Schur symmetric function s_λ

$$s_\lambda = \det \| e_{\lambda'_i - i + j} \|_{1 \le i,j, \le M}, \tag{3.4}$$

where e_k is the k-th elementary symmetric function, and $M \ge \ell(\lambda') = \lambda_1$ is an arbitrary positive integer. Here, as well as in relation (2.1), we set: $e_k \equiv 0$ for $k < 0$ and $e_0 \equiv 1$.

The proof of Proposition 3.1 is based on formula (2.1), which contains the complete symmetric functions h_k. But we do not use any specific properties of these functions in course of the proof. The functions h_k are just the matrix elements of determinants in the Plücker relation. If we change all the complete symmetric functions $h_{\lambda_i - i + j}$ for $e_{\lambda_i - i + j}$ the identity (3.3) still remains true *determinant* identity. The interpretation of the determinants involved will, however, be different. As can be

seen from (3.4), the determinants will now parameterize the Schur functions corresponding to the conjugate partitions λ'. ■

Another useful consequence of the proof of Proposition 3.1 is a possibility to remove the first line or the first column of some partitions and get a new identity. Indeed, as can be easily seen from the proof, the first row of the Jacobi-Trudi determinant corresponding to the Schur function s_λ (the component μ_1) does not play an active role in the calculations. In principle, it can be changed for an arbitrary row and identity (3.3) will be still valid as the *determinant* identity (though the interpretation of the corresponding determinants as Schur functions will be lost in general). But if we change the row μ_1 by the N-dimensional row $(1, 0, \dots, 0)$, the determinants s_λ, s_{λ^+} and $s_{\lambda^+\downarrow^{(r)}}$ can be interpreted as the Schur functions corresponding to the partition with the first component removed. Here is an example of the procedure:

$$s_{(\lambda_1,\dots,\lambda_n)} = \det\|h_{\lambda_i-i+j}\| \quad \longrightarrow \quad \begin{vmatrix} 1 & 0 & \dots & 0 \\ h_{\lambda_2-1} & h_{\lambda_2} & \dots & h_{\lambda_2+n-2} \\ \vdots & \dots & \vdots & \vdots \\ h_{\lambda_n-n+1} & h_{\lambda_n-n+2} & \dots & h_{\lambda_n} \end{vmatrix} = s_{(\lambda_2,\dots,\lambda_n)}.$$

Due to Corollary 3.2 the same is true for removing the first column in the diagram λ. Therefore, the following corollary holds true.

Corollary 3.3. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition satisfying the conditions of Proposition 3.1. Denote by $\bar\lambda$ the partition obtained from λ by removing the first line or the first column from the Young diagram λ, that is*

$$\bar\lambda = (\lambda_2, \dots, \lambda_n) \quad \text{or} \quad \bar\lambda = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_n - 1).$$

Then identity (3.3) implies that

$$s_{\bar\lambda} s_{\lambda^+{}^{t_1\ \dots\ t_k}_{(r_1,m_1)\dots(r_k,m_k)}\downarrow} = s_{\overline{\lambda^+}\,{}^{t_1\ \dots\ t_k}_{(r_1,m_1)\dots(r_k,m_k)}} s_{\lambda\downarrow} + \sum_{p=1}^{k} s_{\overline{\lambda^+}\,{}^{t_1\ \dots\ t_k}_{(r_1,m_1)\dots(r_k,m_k)}\downarrow^{(r_p)}} s_{\lambda\uparrow_{(r_p\,-\,1,\,t_p)}}. \tag{3.5}$$

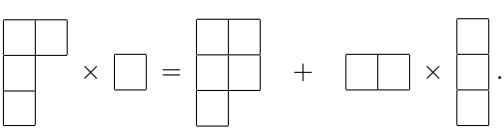
Here $\overline{\lambda^+}$ and $\overline{\lambda^+}\downarrow^{(r_p)}$ are the Young diagrams obtained from the diagrams λ^+ and $\lambda^+\downarrow^{(r_p)}$ by removing the first row (column).

We give two examples illustrating the above formulae.

Example 3.4. Let $\lambda = (2, 1, 1)$, $k = 1$, $r_1 = 2$, $m_1 = 1$ in accordance with the notation of Proposition 3.1. That is we add a single box in the second row of the Young diagram λ. Then the main identity (3.3) reads:

$$s_{(2,1,1)} s_{(1)} = s_{(2,2,1)} + s_{(2)} s_{(1,1,1)}$$

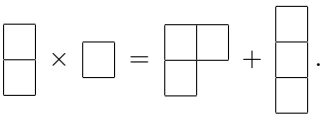
or, loosely denoting the Schur functions s_λ by the corresponding Young diagrams λ (for more visual clarity)



On removing the first row ($\lambda \to \bar\lambda = (1, 1)$), we get

$$s_{(1,1)} s_{(1)} = s_{(2,1)} + s_{(1,1,1)},$$

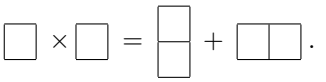
or, in the graphic form



On removing the first column, we find

$$s_{(1)}s_{(1)} = s_{(1,1)} + s_{(2)},$$

or, in the graphic form

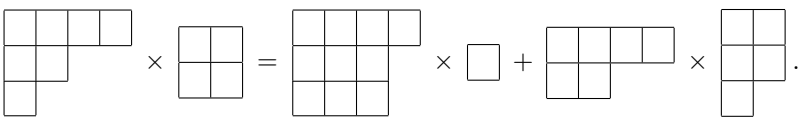


Evidently, these are nothing but the well known Littlewood-Richardson relations on the Schur functions.

Example 3.5. Take $\lambda = (4, 2, 1)$, $k = 1$, $r_1 = 2$, $m_1 = 2$, that is we add a border strip in the second and the third rows of λ. The main identity takes the form

$$s_{(4,2,1)}s_{(2,2)} = s_{(4,3,3)}s_{(1)} + s_{(4,2)}s_{(2,2,1)}.$$

In the graphic form it reads



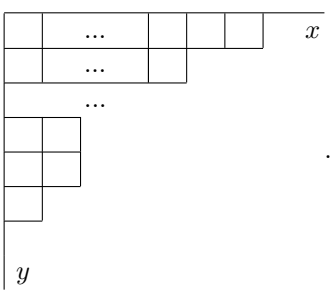
Removing the first row or the first column gives rise to a pair of new identities

$$s_{(2,1)}s_{(2,2)} = s_{(3,3)}s_{(1)} + s_{(2)}s_{(2,2,1)}$$

and

$$s_{(3,1)}s_{(2,2)} = s_{(3,2,2)}s_{(1)} + s_{(4,2)}s_{(1,1)}.$$

Before proving the next corollary, we should introduce new notation. With a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_n)$ we associate a coordinate system with x and y axes directed as shown in the picture below



The size of each box is accepted to be 1×1, being measured in the units of the x and y axes.

It is convenient to accept a different notation for components of a given partition λ. Namely, we denote by ξ_i, where $1 \le i \le k \le n$, all *distinct* components of the partition λ. That is $\lambda = (\xi_1^{m_1}, \xi_2^{m_2}, \ldots, \xi_k^{m_k})$ with some integers $m_i \ge 1$, $m_1 + \cdots + m_k = n$. Note, that by definition $\xi_i > \xi_j$ if $i < j$. Besides, it is convenient to set $\xi_{k+1} = 0$. We also introduce a set of integers y_i, where $0 \le i \le k$, by the rule

$$y_0 = 0, \quad y_i = m_i + y_{i-1}, \quad 1 \le i \le k.$$

An *inner corner* of a diagram λ is a point with coordinates (ξ_i, y_{i-1}) with respect to the above coordinate system. The collection of all inner corners will be called the *inner corner set* of the diagram λ. So, the inner corner set $\mathfrak{C}_\lambda$ of the Young diagram $\lambda = (\xi_1^{m_1}, \dots, \xi_k^{m_k})$ consists of the following $k+1$ points α_i

$$\mathfrak{C}_\lambda = \{\alpha_i = (\xi_i, y_{i-1}) \mid 1 \le i \le k+1\}. \tag{3.6}$$

For example, the inner corner set of the Young diagram $(6, 5, 2^2, 1)$ includes five elements: $(6, 0)$, $(5, 1)$, $(2, 2)$, $(1, 4)$ and $(0, 5)$.

By the above definition, the inner corner set of any non-empty Young diagram λ is a non-empty set, containing at least two elements — the points $(\xi_1, 0)$ and $(0, \ell(\lambda))$. Note that knowing the inner corner set of a diagram allows one to restore the diagram itself.

Introduce now the vertical and horizontal shifts of inner corners. Let $\alpha_i = (\xi_i, y_{i-1})$, where $\xi_i \neq 0$, be an inner corner of a partition $\lambda = (\xi_1^{m_1}, \dots, \xi_k^{m_k})$. The *horizontal shift* $h_i^{\pm}$ of the corner α_i by ± 1 means increasing or decreasing the component ξ_i by 1. If $\xi_i + 1 = \xi_{i-1}$ or $\xi_i - 1 = \xi_{i+1}$, then the corresponding rows of the diagram are united:

$$\lambda = (\dots, \xi_{i-1}^{m_{i-1}}, \xi_i^{m_i}, \dots) \xrightarrow{h_i^+} \begin{cases} (\dots, \xi_{i-1}^{m_{i-1}}, (\xi_i+1)^{m_i}, \dots) & \text{if} \quad \xi_{i-1} - \xi_i \ge 2 \\ (\dots, \xi_{i-1}^{m_{i-1}+m_i}, \dots) & \text{if} \quad \xi_{i-1} - \xi_i = 1, \end{cases}$$

$$\lambda = (\dots, \xi_i^{m_i}, \xi_{i+1}^{m_{i+1}}, \dots) \xrightarrow{h_i^-} \begin{cases} (\dots, (\xi_i-1)^{m_i}, \xi_{i+1}^{m_{i+1}}, \dots) & \text{if} \quad \xi_i - \xi_{i+1} \ge 2 \\ (\dots, \xi_{i+1}^{m_i+m_{i+1}}, \dots) & \text{if} \quad \xi_i - \xi_{i+1} = 1. \end{cases}$$

The other components of λ preserve their values.

Similarly, the *vertical shift* $v_i^{\pm}$ of the corner $\alpha_i = (\xi_i, y_{i-1})$, where $y_{i-1} \neq 0$, by ± 1 affects the exponents m_i and m_{i-1} in the following way

$$(\dots, \xi_{i-1}^{m_{i-1}}, \xi_i^{m_i}, \dots) \xrightarrow{v_i^-} \begin{cases} (\dots, \xi_{i-1}^{m_{i-1}-1}, \xi_i^{m_i+1}, \dots) & \text{if} \quad m_{i-1} \ge 2 \\ (\dots, \xi_{i-2}^{m_{i-2}}, \xi_i^{m_i+1}, \dots) & \text{if} \quad m_{i-1} = 1, \end{cases}$$

$$(\dots, \xi_{i-1}^{m_{i-1}}, \xi_i^{m_i}, \dots) \xrightarrow{v_i^+} \begin{cases} (\dots, \xi_{i-1}^{m_{i-1}+1}, \xi_i^{m_i-1}, \dots) & \text{if} \quad m_i \ge 2 \\ (\dots, \xi_{i-1}^{m_{i-1}+1}, \xi_{i+1}^{m_{i+1}}, \dots) & \text{if} \quad m_i = 1. \end{cases}$$

The other components of λ remain unchanged.

Note that we do not define the horizontal shifts for the corner $(0, \ell(\lambda))$ and vertical shifts for the corner $(\xi_1, 0)$.

For example, for partition $\lambda = (6, 5, 2^2, 1)$, the horizontal shift of the corner $\alpha_3 = (2, 2)$ by $+1$ and the vertical shift of the corner $\alpha_2 = (5, 1)$ by -1 lead to the following transformations:

$$\lambda \xrightarrow{h_3^+} (6, 5, 3^2, 1), \qquad \lambda \xrightarrow{v_2^-} (5^2, 2^2, 1).$$

Define now two transformations of any partition λ generated by shifts of the inner corners of the corresponding Young diagram.

Definition 3.6. Let λ be a partition and $\alpha_i = (\xi_i, y_{i-1})$ an inner corner of the Young diagram λ. Make the *horizontal* shift by $+1$ of all the inner corners situated *above* α_i in the diagram λ (that is the corners (ξ_j, y_{j-1}) with $j < i$). Besides, make the *vertical* shift by -1 of all the inner corners situated *below* α_i (that is, the corners (ξ_j, y_{j-1}) with $j > i$). The corner α_i keeps its position unchanged. The Young diagram thus obtained will be denoted $\lambda_-^+(\alpha_i)$. In a similar way, shifting the corners above

α_i by -1 in the horizontal direction and those below α_i by $+1$ in the vertical direction, we get the diagram $\lambda^-_+(\alpha_i)$.

Here is an example of the above procedures for the partition $\lambda = (6, 5, 2^2, 1)$ and the inner corner $\alpha_3 = (2, 2)$:

$$\lambda = (6, 5, 2^2, 1) \quad \Rightarrow \quad \lambda^+_-(\alpha_3) = (7, 6, 2, 1), \quad \lambda^-_+(\alpha_3) = (5, 4, 2^3, 1).$$

Corollary 3.7. *Let $\lambda = (\xi_1^{m_1}, \ldots, \xi_k^{m_k})$ be an arbitrary partition and let $\mathfrak{C}_\lambda$ be the inner corner set of the Young diagram λ. Then the following identity holds true*

$$s_\lambda s_\lambda = \sum_{\alpha \in \mathfrak{C}_\lambda} s_{\lambda^+_-(\alpha)} s_{\lambda^-_+(\alpha)}. \tag{3.7}$$

This identity generalizes (1.2) to the case of an arbitrary partition.

Proof. Let $\lambda = (\xi_1^{m_1}, \ldots, \xi_k^{m_k})$ be an arbitrary partition of height $\ell(\lambda) = n$. We introduce an auxiliary partition ν with $n + 1$ components

$$\nu = (\xi_1 + 1, \xi_1^{m_1}, \xi_2^{m_2}, \ldots, \xi_k^{m_k}).$$

On adding to the diagram ν all possible *strictly vertical* border strips, we get the partition

$$\nu^+ = ((\xi_1 + 1)^{m_1+1}, (\xi_2 + 1)^{m_2}, \ldots, (\xi_k + 1)^{m_k}).$$

The inner corner sets of the new partitions are

$$\mathfrak{C}_\nu = (\xi_1 + 1, 0) \cup \{(\xi_i, y_{i-1} + 1)\,,\ 1 \le i \le k + 1\}$$

$$\mathfrak{C}_{\nu^+} = \{(\xi_1 + 1, 0), (0, y_k + 1)\} \cup \{(\xi_i + 1, y_{i-1} + 1)\,,\ 2 \le i \le k\}$$

Now we apply identity (3.3) of Proposition 3.1 to the product of the Schur functions $s_\nu s_{\nu^+\downarrow}$ and then we use Corollary 3.3 in order to remove the first line of length $\xi_1 + 1$ from the diagram ν:

$$\nu \mapsto \bar{\nu} = (\xi_1^{m_1}, \ldots, \xi_k^{m_k}) = \lambda.$$

Besides, as follows from (2.3), $\nu^+\downarrow = \lambda$. So, in our case, the left hand side of identity (3.5) in Corollary 3.3 reads $s_{\bar{\nu}} s_{\nu^+\downarrow} = s_\lambda s_\lambda$. We consider the right hand side of (3.5) and verify that it coincides with that of (3.7).

The first term in the right hand side of (3.5) in our case has the form $s_{\overline{\nu^+}} s_{\nu\downarrow}$. Recall that the bar over the symbol of partition means removing the first row of the corresponding Young diagram. The inner corner sets of the diagrams $\overline{\nu^+}$ and $\nu\downarrow$ are as follows

$$\mathfrak{C}_{\overline{\nu^+}} = \{(\xi_i + 1, y_{i-1})\,,\ 1 \le i \le k\} \cup (0, y_k)$$

$$\mathfrak{C}_{\nu\downarrow} = \{(\xi_i - 1, y_{i-1})\,,\ 1 \le i \le k\} \cup (0, y_k),$$

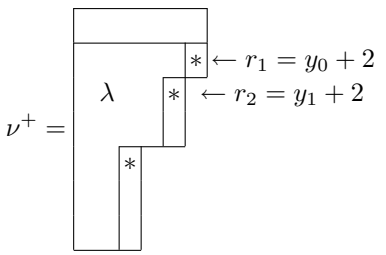
and therefore, as follows from the structure of the inner corner set $\mathfrak{C}_\lambda$ (3.6) and Definition 3.6,

$$\overline{\nu^+} = \lambda^+_-(\alpha_{k+1}), \quad \nu\downarrow = \lambda^-_+(\alpha_{k+1}), \qquad \alpha_{k+1} = (0, y_k).$$

Consider now the sum over the partial peelings in (3.5). In our case, this sum takes the form

$$\sum_{p=1}^{k} s_{\overline{\nu^+}\downarrow(r_p)} s_{\nu\uparrow_{(r_p-1,1)}}.$$

The starting points r_p of partial peelings in the diagram ν^+ are the end points of the vertical border strips added to the diagram ν. The numbers $\{r_p\}$ are expressed in terms of $\{y_p\}$ by the relation $r_p = y_{p-1} + 2$ as illustrated in the diagram below



Here the star signs mark the end points of the added border strips — the starting points r_p of the partial down-peelings. As is not difficult to see, the inner corner set of the diagram $\overline{\nu^+}\downarrow^{(y_{p-1}+2)}$ has the following structure

$$\mathfrak{C}_{\overline{\nu^+}\downarrow^{(y_{p-1}+2)}} = \{(\xi_i + 1, y_{i-1}) \mid 1 \le i \le p-1\} \cup (\xi_p, y_{p-1}) \cup \{(\xi_j, y_{j-1} - 1) \mid p+1 \le j \le k+1\}.$$

By Definition 3.6 this means that

$$\overline{\nu^+}\downarrow^{(y_{p-1}+2)} = \lambda^+_-(\alpha_p), \quad \alpha_p = (\xi_p, y_{p-1}).$$

In analogous way we find that $\nu\uparrow_{(y_{p-1}+1,1)} = \lambda^-_+(\alpha_p)$. Lastly, summation over p gives the final result (3.7). ■

As an example we write down the bilinear relation for the square $s^2_{(3,2,1)}$:

$$s_{(3,2,1)}s_{(3,2,1)} = s_{(4,3,2)}s_{(2,1)} + s_{(4,3)}s_{(2,1^3)} + s_{(4,1)}s_{(2^3,1)} + s_{(3^2,2,1)}s_{(2,1)}.$$

In what follows, we give a simple proof of the result (1.4) [FK].

Corollary 3.8. [***FK***] *Let $(\lambda_1, \lambda_2, \dots, \lambda_{n+1})$ be a partition with an integer $n > 0$. Then the following identity holds true*

$$s_{(\lambda_2,\dots,\lambda_{n+1})}s_{(\lambda_1,\dots,\lambda_n)} = s_{(\lambda_1+1,\dots,\lambda_n+1)}s_{(\lambda_2-1,\dots,\lambda_{n+1}-1)} + s_{(\lambda_2,\dots,\lambda_n)}s_{(\lambda_1,\dots,\lambda_{n+1})}. \tag{3.8}$$

Proof. The result is based on identity (3.3) and the following steps.

(1) Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n+1})$, we construct an auxiliary partition

$$\hat{\lambda} = (\lambda_1 + 1, \lambda_2, \dots, \lambda_{n+1})$$

and take it as the initial partition for Proposition 3.1.

(2) Then we add to $\hat{\lambda}$ the connected border strip from the second row till the last one ($k = 1$, $r_1 = 2$, $m = n$) and get the partition (see (2.10))

$$\hat{\lambda}^+ = (\lambda_1 + 1, \lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1).$$

(3) Peeling the complete border strip off and partial peelings from the end point of the added strip result in the following partitions (see (2.3), (2.5) and (2.7)):

$$\hat{\lambda}^+\downarrow = (\lambda_1, \lambda_2, \dots, \lambda_n)$$
$$\hat{\lambda}\downarrow = (\lambda_2 - 1, \dots, \lambda_{n+1} - 1)$$
$$\hat{\lambda}^+\downarrow^{(2)} = (\lambda_1 + 1, \lambda_2, \dots, \lambda_n)$$
$$\hat{\lambda}\uparrow_{(1,\lambda_1-\lambda_2+1)} = (\lambda_1, \lambda_2, \dots, \lambda_{n+1}).$$

(4) Lastly, the identity (3.3) for the above Schur functions gives

$$s_{(\lambda_1+1,\lambda_2,\ldots,\lambda_{n+1})}s_{(\lambda_1,\ldots,\lambda_n)} = s_{(\lambda_1+1,\lambda_1+1,\lambda_2+1,\ldots,\lambda_n+1)}s_{(\lambda_2-1,\ldots,\lambda_{n+1}-1)} + s_{(\lambda_1+1,\lambda_2,\ldots,\lambda_n)}s_{(\lambda_1,\lambda_2,\ldots,\lambda_{n+1})}.$$

Removing from the above identity the first row $(\lambda_1 + 1)$ in accordance with Corollary 3.3, we come to the result desired (3.8). ■

Note added in proof. After this paper had been accepted for publication, M. Fulmek communicated to us that identity (3.3) can be proved in another way, as a corollary of Lemma 16 in [FK] (for details, see [F]).

Acknowledgement

This work was jointly supported by RFBR and CNRS, grants 09-01-93107-NCNIL-a and GDRI-471. The work of P.P. and P.S. was also supported by the RFBR grant 08-01-00392-a. The work of P.P. was supported by the grant SU-HSE 09-01-0026.

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