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Classes of Graphs Critical for the Edge List-Ranking Problem

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Abstract—The edge list-ranking problem is a generalization of the classical edge coloring problem, and it is a mathematical model for some parallel processes. The computational complexity of this problem is under study for graph sets closed under isomorphism and deletion of vertices (hereditary classes). All finitely defined and minor-closed cases are described for which the problem is polynomial-time solvable (unless N=NP). We find the whole set of "critical" graph classes whose inclusion in a finitely defined class is equivalent to intractability of the edge list-ranking problem in this class (unless N=NP). It seems to be the first result on a complete description for nonartificial NP-complete graph problems. For this problem, we prove constructively that, among the inclusion minimal NP-complete hereditary cases, there are exactly five finitely defined classes and the only minor-closed class.

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INTRODUCTION

This article is a continuation of the series [1–7] and, in many respects, its terminal term. In [1–7], we study the partition of a set of the hereditary classes of graphs into "simple" and "complex" elements by the complexity of the list ranking problems. The essence of the study is to determine the "critical" classes of graphs; i.e., the classes that play a critical, exceptional role for computational complexity analysis.

A *class of graphs* is a set of simple graphs closed under isomorphism. A graph class $\mathcal{X}$ is called *hereditary* if it is closed under the operation of vertex removing . Every hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by the set of its forbidden generated subgraphs $\mathcal{S}$. In this case, by convention, we write $\mathcal{X} = \mathrm{Free}(\mathcal{S})$. There is only one minimum set $\mathcal{S}$ with this property which is denoted by $\mathrm{Forb}(\mathcal{X})$. A hereditary class $\mathcal{X}$ is called *finitely defined* if $\mathrm{Forb}(\mathcal{X})$ is finite. The family of hereditary classes is a continuum and includes such well-known subsets as the set of all monotonous and minor-closed classes of graphs. A hereditary class is called *monotonous* if it is closed under removing edges. A monotonous class is called *minor-closed* if it is closed under contracting the edges of its graphs.

The edge (vertex) ranking problem (rank coloring) for a given graph is to find the minimum number of colors (integers) for the vertices (edges) such that every path between two vertices (edges) of the same color contains a vertex (edge) with a greater color. The vertex rank coloring problem is used in the parallel computation of the Cholesky decomposition [12], VLSI design [11], whereas the edge-ranking problem is used in parallel processing of database queries [13] and assembling multi-module products [9]. The list-ranking problem is a generalization of the rank coloring problem. In this paper, we consider the edge version of the list-ranking problem. It is formulated as follows:

Let G be a graph with the set of edges E, and let $\mathfrak{L} = \{L(e) \mid e{\in}E\}$ be a set, where $L(e)$ is a finite set of integers (the colors that can be assigned to the edge e). $\mathfrak{L}$-*edge ranking* of the graph G is a coloring

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c of its edges such that (i) $c(e) \in L(e)$ for any edge e; (ii) if $c(e_1) = c(e_2)$ and $e_1 \neq e_2$ then every path between e_1 and e_2 contains an edge e_3 such that $c(e_3) > c(e_1)$.

The edge list-ranking problem (ELR problem) is to determine, using the information about G and $\mathfrak{L}$, if there is a $\mathfrak{L}$-ranking of the edges of G. Studies of the complexity of this problem may be potentially used in practice since the list-ranking models the parallelism more adequately. This task attracts our interests because all "critical" classes of some types can be completely described.

Let Π be some NP-complete task for graphs. A hereditary class of graphs is called Π-*simple* if the task Π on this class is solvable in polynomial time. Note that ELR-simple class of graphs means a hereditary class such that the ELR problem for graphs from this class is solvable in polynomial time for each set $\mathfrak{L}$. A hereditary class of graphs is called Π-*hard* if it is not Π-simple. It is assumed that P$\neq$NP, and this condition will be further omitted in the statements. A hereditary graph class $\mathcal{B}$ is called Π-*limit* if there is an infinite sequence $\mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \dots$ of Π-hard graph classes such that

$$\mathcal{B} = \bigcap_{i=1}^{\infty} \mathcal{B}_i.$$

An inclusion minimal Π-limit class is called Π-*boundary*. This notion is explained in

Theorem 1 [8]. *A finitely defined class of graphs is Π-hard if and only if it contains some Π-boundary subclass.*

It follows from Theorem 1 that the complete information about the structure of the Π-*boundary system* (i.e., the set of all Π-boundary graph classes) enables us to fully describe all finitely generated Π-simple classes. Unfortunately, prior to this work, full description of all boundary classes was not obtained for any graph problem. One of the major results of this work is such a description for the ELR problem. We demonstrate that the ELR-boundary system is formed of the 10 specific graph classes and provide the description of all ELR-simple minor closed graph classes.

So, the Π-boundary classes are "critical" classes, and all other critical classes are *minimal Π-hard classes*; i.e., the inclusion minimal Π-hard graph classes. In [2], the first examples of such classes were found and it was shown that, for some problems, there are no minimal Π-hard classes (in [8] it is proved that Π-boundary classes always exist). For all $k > 2$, for both edge and vertex variants of the k-coloring problem, there are no minimal hard classes. Currently, there are known six minimal ELR-hard cases. It is possible that there are no classes other than these.

In this paper, we prove that five specific classes form all finitely generated minimum ELR-hard classes and that some class is the unique minor closed minimum ELR-hard class.

1. NOTATION, DEFINITIONS, AND SOME CITED RESULTS

We introduce the following notations:

kG is disconnected union of k copies of G;

P_n is a simple path with n vertices;

K_n is a complete graph with n vertices;

$K_{p,q}$ is a complete bipartite graph with p vertices in one part and q vertices in the other;

S_i is a graph that is obtained from $K_{1,i}$ by partitioning all its edges;

Comb_i is a graph that is obtained from $K_{2,i}$ by adding an edge incident to both vertices of degree i;

Cam_i is a graph obtained from S_i by connecting the edges of the vertex of degree i to all its leaves;

Com_i is a graph obtained from $K_{1,i}$ by identifying one of the endvertices of the path P_i with a vertex of degree i.

The *hereditary closure of a class $\mathcal{X}$* (denoted by $[\mathcal{X}]$) is a set of graphs generated by subgraphs of the graphs from $\mathcal{X}$. The *additive closure of a class $\mathcal{X}$* is a set of graphs whose all connected components belong to $\mathcal{X}$.

We also introduce the notations for graph classes:

$\mathcal{C}lique$ stands for the class of complete graphs;

$$\mathcal{B}at = \left[\bigcup_{i=1}^{\infty}\{K_{2,i}\}\right], \qquad \mathcal{C}omb = \left[\bigcup_{i=1}^{\infty}\{\mathrm{Comb}_i\}\right], \qquad \mathcal{S}tar = \left[\bigcup_{i=1}^{\infty}\{S_i\}\right],$$

$$\mathcal{C}amomile = \left[\bigcup_{i=1}^{\infty}\{\mathrm{Cam}_i\}\right], \qquad \mathcal{C}omet = \left[\bigcup_{i=1}^{\infty}\{\mathrm{Com}_i\}\right],$$

$\widetilde{\mathcal{T}}$ is the hereditary closure of the additive closure of a set of graphs that are obtained by adding a vertex to some path and an edge that is incident to the added vertex and to some vertex of the path;

$\widetilde{\mathcal{D}}$ is the hereditary closure of the additive closure of a set of graphs that are obtained by adding a vertex to some path and the edges that are incident to the added vertex and to some two consecutive vertices of the path;

$\widehat{\mathcal{T}}$ is the hereditary closure of the additive closure of a set of graphs that are obtained by adding a vertex to some path and the edges that are incident to the added vertex and to some two vertices of the path the distance between which is 2;

$\widehat{\mathcal{D}}$ is the hereditary closure of the additive closure of a set of graphs that are obtained by adding a vertex to some path and the edges that are incident to the added vertex and to some three consecutive vertices of the path;

In [1–7], it is proved that $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, $\mathcal{C}lique$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{D}}$ are ELR-boundary classes, the first six classes being minimal ELR-hard classes. Note that, in [7], the class $\mathcal{B}at$ was denoted by $\mathcal{B}C$; and $\mathcal{C}omb$ by $\mathcal{B}C'$; while, in [6], the class $\widetilde{\mathcal{T}}$ was denoted by $\mathcal{T}_1$.

One of the major results of this work is the statement that the set

$$\{\mathcal{B}at,\ \mathcal{S}tar,\ \mathcal{C}omet,\ \mathcal{C}omb,\ \mathcal{C}amomile,\ \mathcal{C}lique,\ \widetilde{\mathcal{T}},\ \widetilde{\mathcal{D}},\ \widehat{\mathcal{T}},\ \widehat{\mathcal{D}}\}$$

is an ELR-boundary; another result is that there are exactly five finitely generated minimal ELR-hard classes

$$\mathcal{C}lique, \quad \mathcal{B}at, \quad \mathcal{C}omb, \quad \mathcal{S}tar, \quad \mathcal{C}amomile$$

and one minor-closed minimal ELR-hard class $\mathcal{C}omet$.

2. ESTIMATES FOR THE NUMBER OF VERTICES, VERTEX DEGREES, AND DIAMETERS OF GRAPHS FROM SOME CLASSES

Recall that the set of mutually nonadjacent vertices of a graph is called *independent*, and the set of mutually adjacent vertices is called a *clique*. A *matching* in a graph is a set of mutually nonadjacent edges. A matching is called *generated* if there is no edge adjacent to any two distinct edges of the matching.

Lemma 1. *Each graph G with n vertices containing no isolated vertices and with maximum degree of Δ contains a generated matching with $\lceil n/(2\Delta^2)\rceil$ edges.*

Proof. A 2-*sphere* B_e with the center at some $e \in E(G)$ is the set of edges of G that lie in the line graph of G at the distance at most 2 from e. It is clear that, for each $e \in E(G)$, a 2-sphere B_e contains at most

$$1 + 2(\Delta - 1) + 2(\Delta - 1)^2 = \xi \stackrel{\Delta}{=} 2\Delta(\Delta - 1) + 1 < 2\Delta^2$$

elements. Therefore, each connected component H of G contains a generated matching with $\lceil |E(H)|/\xi\rceil$ edges, which can be obtained using an first-fit algorithm. If H is not a tree then $|E(H)| \geq |V(H)|$; therefore,

$$\lceil |E(H)|/\xi\rceil \geq \lceil |V(H)|/\xi\rceil \geq \lceil |V(H)|/(2\Delta^2)\rceil.$$

Let H be a tree. Let us show that

$$\lceil |E(H)|/\xi\rceil = \lceil (|V(H)| - 1)/\xi\rceil \geq \lceil |V(H)|/(2\Delta^2)\rceil.$$

Indeed, let

$$k = \lceil (|V(H)| - 1)/\xi \rceil, \qquad \lceil |V(H)|/(2\Delta^2) \rceil \geq k + 1.$$

Then $|V(H)| \geq 2\Delta^2 k + 1$ (from the last inequality); thus,

$$\lceil (|V(H)| - 1)/\xi \rceil \geq \lceil (2\Delta^2 k + 1 - 1)/\xi \rceil \geq \lceil 2\Delta^2 k/\xi \rceil.$$

Note that $\lceil 2\Delta^2 k/\xi \rceil = \lceil k + k(2\Delta - 1)/\xi \rceil \geq k + 1$ since $2\Delta > 1$ and $k \in \mathbb{N}$ (because $|V(H)| > 1$); a contradiction.

So, each connected component H of G contains a generated matching with $\lceil |V(H)|/(2\Delta^2) \rceil$ edges. This implies that G itself contains a generated matching with $\lceil n/(2\Delta^2) \rceil$ edges. Let us recall that

$$\lceil x_1 \rceil + \ldots + \lceil x_k \rceil \geq \lceil x_1 + \ldots + x_k \rceil$$

for all positive numbers $x_1, x_2, \ldots, x_k$. The proof of Lemma 1 is complete. $\square$

By Ramsey's theorem, every graph with sufficiently large number of vertices contains either an independent set or a clique of a given size. Let $R(a, b)$ denote the smallest number of vertices in a graph that contains either an independent set with a vertices or a clique with b vertices.

Lemma 2. *Let $G \in \mathrm{Free}(\{K_i, K_{2,i}, \mathrm{Comb}_i, S_i, \mathrm{Cam}_i\})$ $(i \geq 2)$ and $x \in V(G)$. Then x has less than $2iR^2(i,i) + R(iR(i,i), i)$ nonleaf neighbors.*

Proof. Consider $N(x)$ and remove from it all leaves of G. Partition the remaining set into the two subsets N_1 and N_2 as follows: The subset N_1 comprises the vertices y such that $N(y) \subseteq N(x) \cup \{x\}$. The subset N_2 consists of the vertices that have at least one neighbor nonadjacent to x.

Each vertex in $N(x)$ has at most $R(i, i-1) - 1$ adjacent vertices in $N(x)$. Indeed, if there were $y \in N(x)$ with at least $R(i, i-1)$ neighbors from $N(x)$ then $N(y) \cap N(x)$ would contain either an independent set of size i or a clique with $i-1$ vertices. Then G would contain either Comb_i or K_i as a generated subgraph. Consider a subset $N_2' \subset N_2$ of the vertices adjacent to at least one vertex from N_1. The subgraph H of G generated by $N_1 \cup N_2'$ does not contain isolated vertices and, moreover, the degree of each of its vertices does not exceed $R(i, i-1) - 1$. Therefore, the set $N_1 \cup N_2'$ contains at most $2(i-1)(R(i,i-1)-1)^2 < 2iR^2(i,i)$ vertices since otherwise, by Lemma 1, there are $2i$ vertices from $N_1 \cup N_2'$ generating a matching in G (and G would contain a generated subgraph Cam_i). Therefore,

$$|N_1| < 2iR^2(i,i).$$

Consider N_2. By $N_2^{(1)}$ we will denote the largest independent subset of N_2. Put

$$N_2^{(2)} = \{z \mid \exists y \in N_2^{(1)},\ z \in N(y) \setminus (N(x) \cup \{x\})\}.$$

Consider the inclusion minimal subset

$$V = \{u_1, u_2, \ldots, u_k\} \subseteq N_2^{(2)},$$

dominating the set $N_2^{(1)}$ (i.e., $N_2^{(1)} \subseteq \bigcup_{j=1}^{k} N(u_j)$). Such a set certainly exist since each vertex of $N_2^{(1)}$ is adjacent to at least one vertex from $N_2^{(2)}$. Since V is minimal, there are vertices $v_1, v_2, \ldots, v_k \in N_2^{(1)}$ such that, for every $s \in \overline{1,k}$, v_s belongs to

$$N(u_s) \Big\backslash \bigcup_{j=1,\ j \neq s}^{k} N(u_j).$$

It is clear that every vertex of V is adjacent to at most $i-1$ vertices from $N_2^{(1)}$ (since $G \in \mathrm{Free}(\{K_{2,i}\})$); therefore, $k = |V| \geq |N_2^{(1)}|/(i-1)$. The set V contains at most $R(i,i) - 1$ elements since otherwise it would contain an independent set of size i whose vertices, together with the adjacent vertices from $\{v_1, v_2, \dots, v_k\}$ and the vertex x, would generate a subgraph S_i in G. Therefore,

$$k < R(i,i), \qquad \bigl|N_2^{(1)}\bigr| < iR(i,i).$$

It is clear that $|N_2| \le R\bigl(\bigl|N_2^{(1)}\bigr| + 1, i-1\bigr) - 1$; thus, $|N_2| < R(iR(i,i), i)$. Combining this estimate with the estimate for the size of N_1, we conclude that x has less than $2iR^2(i,i) + R(iR(i,i), i)$ nonleaf neighbors. The proof is over. $\square$

Recall that the *diameter* of a graph is the maximum distance between all pairs of vertices.

Lemma 3. *Let G be a connected graph from $\mathrm{Free}(\{\mathrm{Com}_i\})$ and $i \ge 3$. Then either the diameter of G does not exceed $2i-3$ or each vertex of G has at most $i-1$ neighboring leaves.*

Proof. Let the diameter of G be less than $2i-2$, and let some vertex x have i adjacent leaves. Let y and z be some vertices with the distance of at least $2i-2$. By the triangle inequality, there is a vertex $x' \in \{y, z\}$ such that the distance between it and x is at least $i-1$. Since G is connected, there is a generated path P between x and x' that contains at least i vertices. The path P contains no leaf neighbor of x since its length is at least 2. Some vertices in P and some leaves in G adjacent to x generate a subgraph isomorphic to Com_i; a contradiction that completes the proof. $\square$

We will say that G is an *supergraph* of a graph H if H is a generated subgraph of G. A vertex of degree 2 in a graph is called *internal* if its neighbors are not adjacent.

Lemma 4. *Let $H_1 \in \widetilde{\mathcal{T}}$, $H_2 \in \widetilde{\mathcal{D}}$, $H_3 \in \widehat{\mathcal{T}}$, and $H_4 \in \widehat{\mathcal{D}}$, and let G be a connected graph without internal vertices with at least three vertices from the class $\mathrm{Free}(\{H_1, H_2, H_3, H_4\})$. Then G has at most $\bigl(\Delta^{8n(n+2)} - 1\bigr)/\bigl(\Delta - 1\bigr)$ vertices, where Δ is the maximum degree of the vertices of G and*

$$n = \max\{|V(H_1)|,\ |V(H_2)|,\ |V(H_3)|,\ |V(H_4)|\}.$$

Proof. Suppose the opposite. Since G is a connected graph and contains at least three vertices, $\Delta \ge 2$. It is easy to show that a connected graph with the diameter d and with the maximum vertex degree $\Delta' > 1$ has at most

$$1 + \Delta' + \Delta'^2 + \dots + \Delta'^d = \frac{\Delta'^{d+1} - 1}{\Delta' - 1}$$

vertices. Combining this and the assumption, we obtain that the diameter of G is at least $8n(n+2)$. Consider two vertices x and y in G and a generated path between them with length equal to the diameter of G. Remove the endvertices from this path, and denote the remaining path by $P = (u_1, u_2, \dots, u_k)$. It is clear that $k \ge (8n-1)(n+2) + n + 1$. Each vertex in P is adjacent to at least one vertex from $V(G) \setminus V(P)$ (since G does not contain internal vertices and P is a generated path). Let V denote the set of the vertices from $V(G) \setminus V(P)$ that are adjacent to at least one vertex from P. Consider the set $I_v = \{i \mid (v, u_i) \in E(G)\}$ for each $v \in V$. Let us show that each I_v consists of at most three consecutive indices. Consider an arbitrary vertex u_i adjacent to v. Assume that there is a vertex u_j adjacent to v, where $j > i+2$. Then G contains a path $(x, u_1, \dots, u_i, v, u_j, \dots, u_k, y)$ of length of at most k, which is less than the diameter of G; a contradiction.

Consider the vertices $u_{n+2}, u_{3n+6}, \dots, u_{(8n-1)(n+2)}$. For $u_{(2i-1)(n+2)}$, there is an adjacent vertex $v_i \in V$. All vertices $v_1, v_2, \dots, v_{4n}$ are different (see arguments at the end of the previous paragraph). It is easy to see that all of them are mutually nonadjacent (otherwise the diameter of G would be at most k). For each $i \in \overline{1, 4n}$, the set

$$V_i = \{v_i, u_{(2i-1)(n+2)-n-1}, \dots, u_{(2i-1)(n+2)}, \dots, u_{(2i-1)(n+2)+n+1}\}$$

exists (since $k \ge (8n-1)(n+2) + n + 1$) and generates a subgraph in G which is an supergraph of every component of some graph $H^i \in \{H_1, H_2, H_3, H_4\}$. The graph G does not contain any edge that is incident to two vertices from different sets.

By Dirichlet's principle, the sequence $v_1, v_2, \dots, v_{4n}$ contains vertices $v_{i_1}, v_{i_2}, \dots, v_{i_n}$ such that the sets $V_{i_1}, V_{i_2}, \dots, V_{i_n}$ generate the graphs that are supergraphs of all components of some graph $H \in \{H_1, H_2, H_3, H_4\}$. The union of these supergraphs is an supergraph of H; therefore, G contains H as a generated subgraph; a contraction to the initial assumptions.

The proof of Lemma 4 is complete. □

We will call a graph H a *contraction* of G if G can be obtained by subdivisions of the edges of H and H contains a minimum number of vertices. It is clear that there exists a unique contraction of G. Consider a graph class $\mathcal{X}$. Let $\mathcal{X}_c$ stand for the set of contractions of the graphs from $\mathcal{X}$.

Lemma 5. *Let a hereditary class $\mathcal{X}$ contain none of the classes $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, $\widehat{\mathcal{D}}$, or $\mathcal{C}omet$. Then $[\mathcal{X}_c]$ contains none of them.*

Proof. Let us show that if $\mathcal{Y} \subseteq [\mathcal{X}_c]$ for some $\mathcal{Y}$ from $\{\widetilde{\mathcal{T}}, \widetilde{\mathcal{D}}, \widehat{\mathcal{T}}, \widehat{\mathcal{D}}$, and $\mathcal{C}omet\}$ then, for some

$$\mathcal{Y}' \in \{\widetilde{\mathcal{T}},\ \widetilde{\mathcal{D}},\ \widehat{\mathcal{T}},\ \widehat{\mathcal{D}},\ \mathcal{C}omet\},$$

we have $\mathcal{Y}' \subseteq \mathcal{X}$, which implies the claim of the lemma. A subdivision of any edges of an arbitrary graph from $\mathcal{C}omet$ (or from $\widetilde{\mathcal{T}}$) creates an supergraph of it. This and the heredity of $\mathcal{X}$ imply that the statement is true for the cases $\mathcal{Y} \in \{\mathcal{C}omet,\ \widetilde{\mathcal{T}}\}$ (here $\mathcal{Y}' = \mathcal{Y}$).

The reasoning for the remaining three cases is similar; we, therefore, provide the proof only for $\mathcal{Y} = \widetilde{\mathcal{D}}$. We will prove that if $\widetilde{\mathcal{D}} \subseteq [\mathcal{X}_c]$ then $\mathcal{X}$ contains at least one of the classes $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, and $\widehat{\mathcal{T}}$.

Suppose the opposite. Let $G_k^{(1)}$ be the result of adding a vertex to the path P_{2k+1} and an edge incident to the added vertex and the middle vertex of the path. Let $G_k^{(2)}$ be the result of identifying two ends of two paths P_{k+1} with two distinct vertices of the triangle. Let $G_k^{(3)}$ be the result of identifying two ends of the paths P_{k+1} with two nonadjacent vertices of a cycle of length 4. Note that, for every k,

$$G_k^{(1)} \in \widetilde{\mathcal{T}}, \qquad G_k^{(2)} \in \widetilde{\mathcal{D}}, \qquad G_k^{(3)} \in \widehat{\mathcal{T}}.$$

There is k' such that

$$\mathcal{X} \subseteq \mathrm{Free}\big(\{k'G_{k'}^{(1)}, k'G_{k'}^{(2)}, k'G_{k'}^{(3)}\}\big).$$

By definition of the class $\mathcal{X}_c$ and since $\mathcal{X}$ is hereditary, for every k there is a graph $H_k \in \mathcal{X}$ with $3k$ connected components that can be transformed into $3kG_k^{(2)}$ through a number of contractions. Note that each connected component of H_k is an supergraph of either $G_k^{(1)}$, or $G_k^{(2)}$, or $G_k^{(3)}$. Therefore, there is $i_k \in \overline{1,3}$ such that $kG_k^{(i_k)}$ is a generated subgraph of H_k. Let $k > k'$. Therefore, $kG_k^{(i_k)} \in \mathcal{X}$ and $k'G_{k'}^{(i_k)} \in \mathcal{X}$ (since $\mathcal{X}$ is hereditary); this contradiction completes the proof of Lemma 5. □

3. POLYNOMIAL CASES OF THE EDGE-LIST RANKING PROBLEM

Theorem 2 [10]. *For an arbitrary fixed C, the* ELR *problem is solvable in polynomial time in the class of graphs with at most C nonleaf vertices.*

In [7], a notion of a polyclass was introduced: We say that a graph class is a *polyclass* if each of its connected graphs G with n vertices contains at most $p(n)$ subsets of $V(G)$ each of which generates a connected subgraph in G, where $p(n)$ is some polynomial in n.

The following reveals our interest in the notion of polyclass:

Theorem 3 [7]. *The* ELR *problem is solvable in polynomial time for all graphs from each of the polyclasses.*

In [7], it is shown that some specific graph classes are polyclasses.

Lemma 6. *For every fixed d and k, the class of graphs whose all vertices have degrees at most d and having at most k vertices with degree greater than 2 is a polyclass.*

4. STRUCTURE OF ALL FINITELY DEFINED AND MINOR CLOSED EASY-SOLVABLE CASES FOR THE ELR PROBLEM AND ELR-"CRITICAL" GRAPH CLASSES

A graph H is called a *minor* of G if H can be obtained from a subgraph of G by contracting its edges. The class of graphs is called *minor-closed* if every minor of each graph from this class is also in this class. Every minor-closed class $\mathcal{X}$ can be defined by the set of its forbidden minors $\mathcal{S}$, which can be written as

$$\mathcal{X} = \mathrm{Free}_m(\mathcal{S}).$$

By the well-known Robertson-Seymour theorem, the minimal set (by graph minor relationship) of forbidden minors is finite for every minor-closed class. For example, for the class of planar graphs, this set coincides with $\{K_{3,3}, K_5\}$ by the Pontryagin–Kuratowski criterion.

The statement and proof of Lemma 4 for minor-closed classes are somewhat easier:

Lemma 7. *Let G be a connected graph from* $\mathrm{Free}_m(\{\mathrm{Com}_i\})$ $(i \geq 2)$ *with at least three vertices, and let Δ stand for the largest vertex degree in G. Then the contraction of G has at most* $\big(\Delta^{4i+1} - 1\big)/\big(\Delta - 1\big)$ *vertices.*

Proof. Suppose the opposite. It is clear that $\Delta > 1$. Let H be the contraction of G. If G is a simple path then $H = K_2$. The equality holds for such G.

Let G not be a simple path. It is clear that H does not contain internal vertices, the degrees of all vertices in H do not exceed Δ, and some vertex in H has the degree exactly Δ. Consider a generated path P in H of length equal to the diameter of H. This path contains at least $4i + 2$ vertices (recall that a connected graph with diameter d and maximum vertex degree $\Delta' > 1$ has at most $(\Delta'^{d+1} - 1)/(\Delta' - 1)$ vertices). Each nonendvertex in P is adjacent to at least one vertex from $V(H) \setminus V(P)$, and each vertex in $V(H) \setminus V(P)$ is adjacent to at most three vertices from P. Consider the set V_1 consisting of $3i$ nonendvertices from P. There exist some set V_2 that consists of i vertices from $V(H) \setminus V(P)$, each of which is adjacent to at least one vertex from V_1.

Consider a subgraph H' of H consisting of all nonendedges of P and i arbitrary edges each of which is incident to one vertex from V_2 and one vertex from V_1, wherein all these i edges are incident to all vertices from V_2. It is easy that we can obtain Com_i from H' by contracting some of its edges. Therefore, H contains Com_i as a minor; a contradiction. □

A class of graphs $\mathcal{X}$ is called a *minor of some class* $\mathcal{Y}$ if, for each graph $H \in \mathcal{X}$, there is $G \in \mathcal{Y}$ such that H is a minor of G. The class $\mathcal{X}$ is called a *strong minor of the class* $\mathcal{Y}$, if, for every graph from $H \in \mathcal{X}$, there is $G \in \mathcal{Y}$ such that H is a minor of G, wherein the number of vertices in G is upper-bounded by some polynomial in the number of vertices of H.

It turned out that, using the notion of a strong minor of a class of graphs, we can fully describe all polynomial cases of ELR problem for some family of graph classes that contains all finitely defined and minor-closed classes. It seems, this is the first result that provides full description for nonartificial NP-complete problems on graphs.

Theorem 4. *Let $\mathcal{X}$ be a class of graphs for which the ELR problem is not solvable in polynomial time. Then each class for which $\mathcal{X}$ is a strong minor also has this property.*

Proof. Let $\mathcal{X}$ be a strong minor of the class $\mathcal{Y}$. Let us show that the ELR problem for the graphs from $\mathcal{X}$ can be reduced in polynomial time to the same problem for the graphs from $\mathcal{Y}$. This will prove the theorem.

Let $H \in \mathcal{X}$ and $\mathfrak{L}$ be the input data for the ELR problem, and let C denote the maximum color in the sets from $\mathfrak{L}$. Let G be the graph that exists according to the definition of a strong minor of a class. The graph H can be obtained from G by removing vertices and edges and then by contracting some edges of G. There is a set of edges of G that become the edges of H. Let E be the set of such edges in G, let E_1 denote the set of edges in G that are contracted while constructing H, and let $E_2 = E(G) \setminus (E \cup E_1)$. The sets E, E_1, and E_2 are computed in polynomial time.

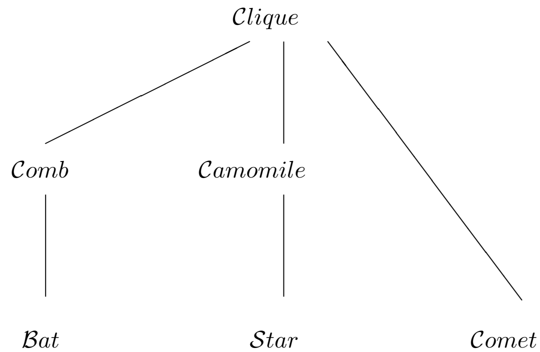

Fig. 1.

Construct the set $\mathfrak{L}'$ of assignments of acceptable colors to edges of G as follows: If $e \in E$ then put $L'(e)$ equal to the union of colors from $L(e)$ increased by $|E_1|$. Enumerate all edges in E_1 from 1 to $|E_1|$. For the ith edge $e \in E_1$, put $L'(e) = \{i\}$. Enumerate all edges in E_2 from 1 to $|E_2|$. For the ith edge $e \in E_2$, put $L'(e) = \{C + |E_1| + i\}$. It is apparent that $\mathfrak{L}'$-ranking of edges of G exists if and only if there is $\mathfrak{L}$-ranking of edges of H. At the same time, the length of input data for $(G, \mathfrak{L}')$ is bounded above by some polynomial in the length of input data for $(H, \mathfrak{L})$. This allows for the earlier identified polynomial reduction, and Theorem 4 is proved. $\square$

Theorem 4 allows for construction of the new ELR-hard cases from the already known ELR-hard cases. This is especially useful for detection of new minimal ELR-hard classes. Thus, relying on ELR-hard classes $\mathcal{B}at$, $\mathcal{S}tar$, and $\mathcal{C}omet$, it was proved in [5, 7] in a similar way that the classes $\mathcal{C}omb$, $\mathcal{C}amomile$, and $\mathcal{C}lique$ are ELR-hard.

We can introduce the graph strong minor relationship on the set of all minimal ELR-hard classes; this relationship is necessarily a quasi-order (it is reflexive and transitive). It can be checked that this relationship is an order on the set

$$\{\mathcal{B}at,\ \mathcal{S}tar,\ \mathcal{C}omet,\ \mathcal{C}omb,\ \mathcal{C}amomile,\ \mathcal{C}lique\}$$

The Hasse diagram for this order is shown in Fig. 1.

It is shown below that the classes $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omb$, $\mathcal{C}amomile$, and $\mathcal{C}lique$ form a full set of finitely defined minimal ELR-hard classes. Therefore, the given diagram (without $\mathcal{C}omet$) is complete for these classes.

Consider the family of graph classes $\mathcal{M}$. A hereditary graph class $\mathcal{X}$ belongs to $\mathcal{M}$ if one of the following conditions is true:

(i) none of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$ is a minor of $\mathcal{X}$;

(ii) if at least one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$ is a minor of $\mathcal{X}$ then at least one of them is also a strong minor of $\mathcal{X}$.

It is clear that all minor-closed classes belong to $\mathcal{M}$. It is proved below that all finitely defined classes belong to $\mathcal{M}$.

Lemma 8. *Every finitely defined class belongs to $\mathcal{M}$.*

Proof. Let $\mathcal{X}$ be a finitely defined class such that at least one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$ is its minor. We can assume that $\mathcal{X}$ contains none of the six classes $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, and $\mathcal{C}lique$ since otherwise one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, and $\mathcal{C}omet$ is a strong minor of $\mathcal{X}$.

Let N denote the sum of numbers of vertices in the graphs from $\mathrm{Forb}(\mathcal{X})$. Consider the graph $G_{i,j}$, where $i \in \overline{1,4}$ and $j \in \mathbb{N}$, that is obtained as follows: Take a path with $(j+1)N + j\delta_i$ vertices, where $\delta_i = i$ for $i \in \overline{1,3}$ and $\delta_4 = 3$. Number the vertices of this path from 1 to $(j+1)N + j\delta_i$ from one end to the other and add j new vertices to it also numbered from 1 to j. For every $k \in \overline{1,j}$, consider the added vertex with number k. Connect it with the vertex with the number $kN + (k-1) + 1$ of the path if $i = 1$, or with the vertices with the numbers

$$kN + 2(k-1) + 1 \text{ and } kN + 2(k-1) + 2, \quad \text{if } i = 2,$$

$$kN + 3(k-1) + 1,\ kN + 3(k-1) + 2 \text{ and } kN + 3(k-1) + 3, \quad \text{if } i = 3,$$

$$kN + 3(k-1) + 1 \text{ and } kN + 3(k-1) + 3, \quad \text{if } i = 4.$$

Let $\mathcal{Z}_i = \left[\bigcup\limits_{j=1}^{\infty}\{G_{i,j}\}\right]$. It is clear that, for every $i \in \overline{1,4}$, $\mathcal{C}omet$ is a strong minor of the class $\mathcal{Z}_i$. Therefore, we can assume that each of these four classes is not in $\mathcal{X}$. Note that, for every $i \in \overline{1,4}$, each graph with N vertices from $\mathcal{Z}_i$ belongs either to $\widetilde{\mathcal{T}}$ (if $i = 1$), or $\widetilde{\mathcal{D}}$ (if $i = 2$), or $\widehat{\mathcal{D}}$ (if $i = 3$), or $\widehat{\mathcal{T}}$ (if $i = 4$). Therefore, in $\mathrm{Forb}(\mathcal{X})$, there are graphs that belong to each of the sets $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{D}}$. By assumption, this holds also for the classes $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, and $\mathcal{C}lique$.

By Lemmas 2 and 3, there are constants C_1 and C_2 (depending on the graphs from $\mathrm{Forb}(\mathcal{X})$ that belong to $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, and $\mathcal{C}lique$) such that, for every connected graph $G \in \mathcal{X}$, either the degrees of all vertices are at most C_1, or the diameter of G is at most C_2. Let $\mathcal{X}_1$ be the set of connected graphs from $\mathcal{X}$ with the degrees of all vertices of at most C_1. Let $\mathcal{X}_2$ be the set of connected graphs from $\mathcal{X}$ with diameter at most C_2. A contraction of a graph does not contain internal vertices. Therefore, by Lemmas 4 and 5, there is a constant C_3 (it also depends on the graphs from $\mathrm{Forb}(\mathcal{X})$ that belongs to $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, $\mathcal{C}lique$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{D}}$), such that a contraction of every graph from $\mathcal{X}_1$ has at most C_3 vertices. By Lemma 2, there is a constant C_4 (depending on the graphs from $\mathrm{Forb}(\mathcal{X})$ that belongs to $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, and $\mathcal{C}lique$) such that every graph from $\mathcal{X}_2$ has at most C_4 nonleaf vertices.

It is obvious that there are finitely many graphs from

$$\{K_{2,i} \mid i \in \mathbb{N}\} \cup \{S_i \mid i \in \mathbb{N}\} \cup \{\mathrm{Com}_i \mid i \in \mathbb{N}\}$$

that are minors of at least one graph from $\mathcal{X}_1$ (otherwise the sequence of the numbers of vertices of contractions of the graphs from $\mathcal{X}_1$ is not bounded). This also holds for the class $\mathcal{X}_2$ since each graph, for which at least one of $K_{2,i}$, S_i, and Com_i is a minor, contains at least $i-1$ nonleaf vertices. Therefore, none of the classes $\mathcal{B}at$, $\mathcal{S}tar$, and $\mathcal{C}omet$ can be a minor of $\mathcal{X}$; a contradiction. The proof of Lemma 8 is complete. □

We are ready now to prove the criterion for effective solvability of the ELR problem in the family $\mathcal{M}$:

Theorem 5. *The* ELR *problem for the class $\mathcal{X} \in \mathcal{M}$ is solvable in polynomial time if and only if none of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$ is a minor of $\mathcal{X}$.*

Proof. Recall that all three classes $\mathcal{B}at$, $\mathcal{S}tar$, and $\mathcal{C}omet$ are ELR-hard. This, together with Theorem 4, implies that if at least one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$ was a strong minor of $\mathcal{X}$ then the ELR problem would not be solvable in polynomial time in $\mathcal{X}$. Let none of the three classes be a strong minor of $\mathcal{X}$. Then, by definition of the family $\mathcal{M}$, none of them is a minor of $\mathcal{X}$. Therefore, there are graphs $H_1 \in \mathcal{B}at$, $H_2 \in \mathcal{S}tar$, and $H_3 \in \mathcal{C}omet$ such that none of the three graphs is a minor of any of the graphs from $\mathcal{X}$. Thus, $\mathcal{X} \subseteq \mathrm{Free}_m(\{H_1, H_2, H_3\})$. There is $i = i(\mathcal{X})$ such that

$$\mathcal{X} \subseteq \mathrm{Free}(\{K_i,\ K_{2,i},\ \mathrm{Comb}_i,\ S_i,\ \mathrm{Cam}_i,\ \mathrm{Com}_i\}).$$

Indeed, if such i did not exist then the class $\mathcal{X}$ would include one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, or $\mathcal{C}lique$ (since $\mathcal{X}$ is hereditary); therefore, one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$ is a strong minor of $\mathcal{X}$.

By analogy to the second part of the proof of Lemma 8 (using Lemmas 2, 3 and 7), it can be shown that there is a constant C (depending on i and H_3) such that each connected graph from $\mathcal{X}$ either has at most C nonleaf vertices or its contraction contains at most C vertices.

Let $\mathcal{X}_1$ be the set of connected graphs from $\mathcal{X}$ with at most C nonleaf vertices, and let $\mathcal{X}_2$ be a set of connected graphs from $\mathcal{X}$ such that their contractions contain at most C vertices. It follows from

Theorem 2 that the ELR problem is solvable in polynomial time for the graphs from $\mathcal{X}_1$. By Lemma 6 and Theorem 3, the ELR problem is solvable in polynomial time in $\mathcal{X}_2$. Ranking of the edges of a disconnected graph is ranking the edges of each of its connected components. Whether a graph from $\mathcal{X}$ belongs to each of the classes $\mathcal{X}_1$ and $\mathcal{X}_2$ can be tested in polynomial time in the number of vertices. Therefore, the ELR problem is solvable in polynomial time in $\mathcal{X}$.

The proof of Theorem 5 is complete. □

Theorem 5 is useful when there is a rule for determining whether at least one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$ is a minor of a given class from $\mathcal{M}$. This can be easily done for minor-closed classes since it is only necessary to check if a finite set of forbidden minors contains graphs from $\mathcal{B}at$, $\mathcal{S}tar$, and $\mathcal{C}omet$. At the same time, Theorem 5 is difficult to apply to finitely defined classes. However, ELR-boundary classes described below can be helpful here.

In [3] the following is proved:

Theorem 6. *The* Π-*boundary classes* $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ *form a* Π-*boundary system if and only if the class* $\mathrm{Free}(\{G_1, G_2, \dots, G_k\})$ *is* Π-*simple for all*

$$G_1 \in \mathcal{B}_1,\ G_2 \in \mathcal{B}_2,\ \dots,\ G_k \in \mathcal{B}_k.$$

Theorem 7. *The* ELR-*boundary system coincides with*

$$\{\mathcal{B}at,\ \mathcal{S}tar,\ \mathcal{C}omet,\ \mathcal{C}omb,\ \mathcal{C}amomile,\ \mathcal{C}lique,\ \widetilde{\mathcal{T}},\ \widetilde{\mathcal{D}},\ \widehat{\mathcal{T}},\ \widehat{\mathcal{D}}\}.$$

Proof. Recall that each of the ten classes $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, $\mathcal{C}lique$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{D}}$ is ELR-boundary [1–7].

Let us apply Theorem 6. Let $\mathcal{X}$ be an arbitrary hereditary class of graphs that do not contain any of the ten given boundary classes. Let us show that $\mathcal{X}$ is ELR-simple. By Lemma 5, a contraction of every graph from $\mathcal{X}$ belongs to $\mathrm{Free}(\{H_1, H_2, H_3, H_4, H_5\})$ for some $H_1 \in \widetilde{\mathcal{T}}$, $H_2 \in \widetilde{\mathcal{D}}$, $H_3 \in \widehat{\mathcal{T}}$, $H_4 \in \widehat{\mathcal{D}}$, and $H_5 \in \mathcal{C}omet$.

Further proof of the ELR-simplicity of $\mathcal{X}$ is similar to the proof of Theorem 5 and is based on Lemmas 2–4, 6 together with Theorems 2 and 3.

The proof of Theorem 7 is complete. □

The criteria are related of effective solvability of the ELR problem for minor-closed classes and finitely defined classes. Namely, at least one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$ is a (strong) minor of some finitely defined class of graphs if and only if it includes at least one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}omet$, $\mathcal{C}omb$, $\mathcal{C}amomile$, $\mathcal{C}lique$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{D}}$. This follows from Theorems 5 and 7, Lemma 8, and definition of $\mathcal{M}$.

Using Theorems 5 and 7, we can easily enumerate all finitely defined minimal ELR-hard classes and all minor-closed minimal ELR-hard classes:

Theorem 8. *There are exactly five finitely defined minimal* ELR-*hard classes*: *$\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}lique$, $\mathcal{C}omb$, and $\mathcal{C}amomile$. The only minor-closed minimal ELR-hard class is $\mathcal{C}omet$.*

Proof. The classes $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}lique$, $\mathcal{C}omb$, and $\mathcal{C}amomile$ are finitely defined. The corresponding sets of forbidden generated subgraphs are described in [3–7]. It is easy to see that $\mathcal{C}omet$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{D}}$ are infinitely defined (for each of these classes, all simple cycles of length 5 and up belong to the minimal set of forbidden generated subgraphs).

Let $\mathcal{X}$ be an ELR-hard finitely defined class that does not include $\mathcal{B}at$, $\mathcal{S}tar$, $\mathcal{C}lique$, $\mathcal{C}omb$, and $\mathcal{C}amomile$. By Theorems 1 and 7, $\mathcal{X}$ includes at least on of the classes $\mathcal{C}omet$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{D}}$. Since they are not finitely defined, while $\mathcal{X}$ is; therefore, $\mathcal{X}$ coincides with none of them. Therefore, $\mathcal{X}$ contains a graph G that does not belong to at least one of the classes $\mathcal{C}omet$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{D}}$, $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{D}}$. The class $\mathcal{X} \cap \mathrm{Free}(\{G\})$ is, then, ELR-hard by Theorem 1; therefore, $\mathcal{X}$ is not a minimal ELR-hard class.

By Theorem 5, each minor-closed ELR-hard class necessarily includes at least one of the classes $\mathcal{B}at$, $\mathcal{S}tar$, or $\mathcal{C}omet$. Only $\mathcal{C}omet$ among them is minor-closed. Therefore, it is the only minor-closed minimal ELR-hard class. This completes the proof. □

Probably, the set of all minimal ELR-hard classes is described by the classes $\mathcal{C}lique$, $\mathcal{B}at$, $\mathcal{C}omb$, $\mathcal{S}tar$, $\mathcal{C}amomile$, and $\mathcal{C}omet$.

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