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Extremal values of global tolerances in combinatorial optimization with an additive objective function

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Abstract The currently adopted notion of a tolerance in combinatorial optimization is defined referring to an arbitrarily chosen optimal solution, i.e., locally. In this paper we introduce global tolerances with respect to the set of all optimal solutions, and show that the assumption of nonembededdness of the set of feasible solutions in the provided relations between the extremal values of upper and lower global tolerances can be relaxed. The equality between globally and locally defined tolerances provides a new criterion for the multiplicity (uniqueness) of the set of optimal solutions to the problem under consideration.

Keywords Combinatorial optimization problem · Additive objective function · Extremal values of tolerances

1 Introduction

Recently, tolerances have attracted more attention from the prospective to improve different algorithms for solving computationally intractable classes of combinatorial optimization problems [3,4,6,9,10]. The roots of these improvements come from, to the best of our

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knowledge, the well known Vogel's Approximation Method [17] for the Transportation Simplex Algorithm. The tolernaces have been used for a straightforward enumeration of the k-best solutions [15], for some natural k, including the Max-Regret heuristic for solving the Three-Index Assignment Problem [1]. The notion of a k-best solution contains the idea of a complete enumeration of the whole set of feasible solutions in a non-decreasing order of their objective function values. If such an enumeration has been done for a relaxed version of the original combinatorial optimization problem, then the first feasible solution to the original problem (i.e., the problem with nonrelaxed constraints) gives back an optimal solution to the original problem. The smallest value of tolerances for an optimal solution to the relaxed problem gives back the second optimal solution within the above mentioned order for the enumerated values of the objective function.

Moreover, if the smallest value among all tolerances is strictly positive, then the set of optimal solutions to the relaxed problem contains only one element, i.e., the optimal solution is unique [4]. In order to check the uniqueness of the optimal solution (see e.g., [16]) we should compute all upper and lower tolerances and choose the smallest one among all of them. If an optimal solution to a combinatorial optimization problem is defined on a graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = m$ edges, then the answers to the following questions might be useful for the reduction of the computational complexity in finding the smallest value of tolerances:

(i) is there a combinatorial optimization problem, for which the smallest value of upper tolerances and the smallest value of lower tolerances are equal;
(ii) find necessary and sufficient conditions for a general class of combinatorial optimization problems, which, if satisfied, will guarantee the equality of smallest upper and lower tolerances.

The purpose of this paper is to present a theory of global tolerances (i.e., those referring to all optimal solutions) including all consequences related to commonly known tolerances [2,4,5,7,11,12].

The paper is organized as follows. In Sect. 2 we discuss additive combinatorial optimization problems. In Sect. 3 we introduce global tolerances of the ground set elements and present their principal properties and relationships with commonly known tolerances. Section 4 is devoted to the reduction of the initial combinatorial optimization problem to an equivalent problem with canonical data. In Sect. 5 we prove three main theorems concerning the (in)equalities between the minimal global upper tolerance and the minimal global lower tolerance, which naturally includes the notion of nonembeddedness of the set of feasible solutions. The reduction to the problem with nonembedded feasible solutions is given in Sect. 6. In Sect. 7 we prove a theorem on maximal values of global upper and global lower tolerances. Finally, Sect. 8 concludes.

2 Additive combinatorial optimization problems

Let X be a finite set whose number of elements $|X| \geq 2$, called the *ground set*, and $\mathscr{S} \subset 2^X$ be a nonempty collection of nonempty subsets of X. Given a nonnegative real valued function $C : X \to \mathbb{R}$, called the *cost function* (also representing weight, distance, time, etc.), we denote by $f = f_C : \mathscr{S} \to \mathbb{R}$ the set function defined by

$$f(S) = f_C(S) = \sum_{x \in S} C(x) \quad \text{for all} \quad S \in \mathscr{S},$$

which will be called the *additive objective function*.

A *Combinatorial Optimization Problem* determined by the data $(X, C, \mathscr{S}, f_C)$, abbreviated as COP $(X, C, \mathscr{S}, f)$ or simply COP if no ambiguity arises, is to minimize or maximize the objective function f on $\mathscr{S}$. In order to be more specific, throughout the paper we concentrate on the *minimization* problem: find sets

$$S^* \in \mathscr{S} \text{ such that } f(S^*) \leq f(S) \text{ for all } S \in \mathscr{S}. \tag{2.1}$$

Any such set S^* is a ('point' of) *minimum* of f on $\mathscr{S}$ and

$$f^* = f(S^*) = \min_{\mathscr{S}} f = \min_{S \in \mathscr{S}} f(S)$$

is the *minimal value* of f on $\mathscr{S}$, which is determined uniquely. Another terminology is that S^* from (2.1) is an *optimal solution* and f^* is the *optimal value* of the COP $(X, C, \mathscr{S}, f)$. In this respect, it is convenient to say that the collection $\mathscr{S}$ is the *set of feasible solutions* of the COP and to denote by $\mathscr{S}^*$ the *set of all optimal solutions* of the COP (cf. (2.1)). Clearly, $\mathscr{S}^* \subset \mathscr{S}$ and $|\mathscr{S}^*| \geq 1$. Since $f(S^*) = f^*$ for all $S^* \in \mathscr{S}^*$, this optimal value of the COP $(X, C, \mathscr{S}, f)$ will sometimes be denoted more explicitly as

$$f(\mathscr{S}^*) = f^* = \min_{S \in \mathscr{S}} f(S). \tag{2.2}$$

Classical examples of the COPs include, among others, the following.

(A) The Traveling Salesman Problem (TSP): given $n \geq 2$ cities and distances $C(i, j)$ between i-th and j-th cities, find a closed tour of minimum length, which enters and leaves each city exactly once. Formally, let $G = (V, A)$ be a complete directed graph, where $V = \{1, 2, \ldots, n\}$ is the set of vertices and $A \subset V \times V$ is the set of arcs $(i, j) \in A$ (also called edges if the graph is undirected). The COP corresponding to the TSP is determined by the data $(X, C, \mathscr{S}, f_C)$, where $X = A$, $C : A \to \mathbb{R}$ is a nonnegative (cost, distance) function, $\mathscr{S} \subset 2^A$ is the set of all cycles S of the form $S = \{(i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n), (i_n, i_1)\} \subset A$ with all arcs (pairs) in S being pairwise different (i.e., Hamiltonian cycles) and the objective function f_C is given by $f_C(S) = \sum_{(i,j) \in S} C(i, j)$ for $S \in \mathscr{S}$.

(B) The Assignment Problem (AP): given a complete directed graph $G = (V, A)$ as above and a nonnegative cost function C on $X = A$, find a vertex permutation $\pi^* : V \to V$ such that $\sum_{i=1}^{n} C(i, \pi^*(i))$ is minimal among all such sums corresponding to all possible permutations of V. Note that a set $S \subset A$ is a feasible solution of the AP (i.e., $S \in \mathscr{S}$) if it is of the form $S = \{(1, \pi(1)), (2, \pi(2)), \ldots, (n, \pi(n))\}$ for some permutation π of V. Clearly, a Hamiltonian cycle corresponds to the cyclic permutation π of V and vice versa.

(C) The Minimal Spanning Tree Problem (MSTP): construct a net of roads of minimum cost, which connect n cities. Formally, let $G = (V, E)$ be a complete undirected graph, i.e. a graph with the set of edges E and C be a nonnegative cost function on $X = E$. A subgraph $T = (V, E')$ of G with $E' \subset E$ is a *tree* if it is connected and $|E'| = n - 1$. The MSTP is to find a tree $T^* = (V, E^*)$ such that $\sum_{x \in E^*} C(x)$ is minimal among all such sums evaluated for all trees.

The union of the collection (or a subcollection of) $\mathscr{S}$ of feasible solutions to the COP and its intersection are denoted as usual by

$$\cup\mathscr{S} = \bigcup_{S \in \mathscr{S}} S = \left\{x \in X : x \in S \text{ for some } S \in \mathscr{S}\right\}$$

and

$$\cap\mathscr{S} = \bigcap_{S\in\mathscr{S}} S = \{x \in X : x \in S \text{ for all } S \in \mathscr{S}\},$$

respectively. Since $\mathscr{S}^* \subset \mathscr{S}$, we find

$$\cup\,\mathscr{S}^* \subset \cup\mathscr{S} \quad \text{and} \quad \cap\,\mathscr{S}^* \supset \cap\mathscr{S}. \tag{2.3}$$

Informally speaking, if, for example, the TSP is under consideration, then the union $\cup\mathscr{S}$ is the collection of all arcs (edges), each of which belongs to at least one of the Hamiltonian cycles, and the intersection $\cap\mathscr{S}$ is the set of all common arcs (edges) for all Hamiltonian cycles.

The following simple observation concerning the set of feasible solutions will be used below for the study of the uniqueness of optimal solutions:

$$\cup\,\mathscr{S} = \cap\mathscr{S} \;\text{ iff }\; |\mathscr{S}| = 1, \quad \text{and} \quad \cup\,\mathscr{S} \neq \cap\mathscr{S} \;\text{ iff }\; |\mathscr{S}| \geq 2, \tag{2.4}$$

where 'iff' stands for 'if and only if'. In fact, if $|\mathscr{S}| = 1$, then $\mathscr{S} = \{S\}$ for some $S \in \mathscr{S}$, and it follows from the definitions of $\cup\mathscr{S}$ and $\cap\mathscr{S}$ that $\cup\mathscr{S} = S = \cap\mathscr{S}$. Now, if $\cup\mathscr{S} = \cap\mathscr{S}$, then we denote this common value by S and note that $S = \cup\mathscr{S}$ is nonempty (since $\mathscr{S} \neq \varnothing$ and $\varnothing \notin \mathscr{S}$). We assert that $\mathscr{S} = \{S\}$: given $T \in \mathscr{S}$, we have $T \subset \cup\mathscr{S} = S$ and $S = \cap\mathscr{S} \subset T$, and so, $T = S$. Supposing $|\mathscr{S}| \geq 2$, we find $S_1, S_2 \in \mathscr{S}$ such that $S_1 \neq S_2$, and so, $S_1 \cup S_2 \neq S_1 \cap S_2$ and $\cup\mathscr{S} \supset S_1 \cup S_2 \supset S_1 \cap S_2 \supset \cap\mathscr{S}$, i.e., $\cup\mathscr{S} \neq \cap\mathscr{S}$. Finally, if $|\mathscr{S}| < 2$, then $|\mathscr{S}| = 1$ (because $\mathscr{S} \neq \varnothing$), and so, by the above, $\cup\mathscr{S} = \cap\mathscr{S}$.

A COP is degenerated (or of no interest) if either $\cup\mathscr{S} = \cap\mathscr{S}$ (there is only one feasible solution) or $\cap\mathscr{S} \in \mathscr{S}$ (the set $\cap\mathscr{S}$ is always an optimal solution), and so, in what follows we assume that

$$|\mathscr{S}| \geq 2 \quad \text{and} \quad \cap\,\mathscr{S} \notin \mathscr{S}. \tag{2.5}$$

3 Global tolerances of the ground set elements

In this section we are interested in numerical characteristics of elements $x \in X$, which show to what extent optimal solutions of the COP (minimizing $f = f_C$) are invariant with respect to a change of the single cost $C(x)$.

Let the COP $(X, C, \mathscr{S}, f)$ be given.

By the *global upper tolerance* $u(x)$ (*global lower tolerance* $\ell(x)$) of an element $x \in X$ we mean the maximum increase (maximum decrease, respectively) of the cost $C(x)$ only, under which optimal solutions of the initial (unperturbed) COP remain optimal solutions of the perturbed COP. More formally, given $x \in X$ and $\alpha \in \mathbb{R}$, we define the perturbed cost function $C_{x,\alpha} : X \to \mathbb{R}$ as follows: $C_{x,\alpha}(y) = C(y)$ if $y \in X$ and $y \neq x$, and $C_{x,\alpha}(x) = C(x) + \alpha$. Then the global upper tolerance $u(x)$ is the least upper bound of those $\alpha \geq 0$, for which *any* optimal solution S^* of the COP $(X, C, \mathscr{S}, f)$ with $f = f_C$ is also an optimal solution of the perturbed COP $(X, C_{x,\alpha}, \mathscr{S}, f_{C_{x,\alpha}})$. The global lower tolerance $\ell(x)$ is expressed similarly if we replace the perturbed COP above by the perturbed COP $(X, C_{x,-\alpha}, \mathscr{S}, f_{C_{x,-\alpha}})$.

In order to be able to calculate the tolerances, we introduce some further notation. Given $x \in X$, we denote by $\chi_x : X \to \{0, 1\}$ the characteristic function of the one-point set $\{x\}$ (i.e., $\chi_x(y) = 0$ for $y \in X$, $y \neq x$, and $\chi_x(x) = 1$) and by $\delta_x : 2^X \to \{0, 1\}$ the Dirac measure (or point mass) concentrated at x (i.e., given $S \subset X$, we have: $\delta_x(S) = 1$ if $x \in S$, and $\delta_x(S) = 0$ if $x \notin S$). Clearly, $\delta_x(S) = \sum_{y\in S} \chi_x(y)$ for all $S \subset X$.

Noting that the perturbed cost function $C_{x,\alpha}$ is of the form $C_{x,\alpha} = C + \alpha\chi_x$ on X for $x \in X$ and $\alpha \in \mathbb{R}$, we find that the corresponding perturbed additive objective function $f_{C_{x,\alpha}}$ is given by:

$$\begin{aligned} f_{C_{x,\alpha}}(S) &= \sum_{y\in S} C_{x,\alpha}(y) = \sum_{y\in S} C(y) + \alpha \sum_{y\in S} \chi_x(y) \\ &= f_C(S) + \alpha\delta_x(S) = (f + \alpha\delta_x)(S) \quad \text{for all} \quad S \in \mathscr{S}, \end{aligned}$$

and so, $f_{C_{x,\alpha}} = f + \alpha\delta_x$ on $\mathscr{S}$. Now, given $x \in X$, it follows from the definitions of the global upper and lower tolerances that

$$u(x) = \sup\Big\{\alpha \geq 0 : (f + \alpha\delta_x)(S^*) = \min_{\mathscr{S}}(f + \alpha\delta_x) \text{ for all } S^* \in \mathscr{S}^*\Big\} \tag{3.1}$$

and

$$\ell(x) = \sup\Big\{\alpha \geq 0 : (f - \alpha\delta_x)(S^*) = \min_{\mathscr{S}}(f - \alpha\delta_x) \text{ for all } S^* \in \mathscr{S}^*\Big\}, \tag{3.2}$$

where $\mathscr{S}^* = \{S^* \in \mathscr{S} : f(S^*) = \min_{\mathscr{S}} f\}$. Clearly, $u(x), \ell(x) \in [0, +\infty]$, and these values are independent of a particular optimal solution $S^* \in \mathscr{S}^*$.

By virtue of (2.5), the set $(\cup\mathscr{S})\backslash(\cap\mathscr{S})$ is nonempty, and the complement of this set in X is, by deMorgan's laws, given by

$$X\backslash[(\cup\mathscr{S})\backslash(\cap\mathscr{S})] = (X\backslash(\cup\mathscr{S})) \cup (\cap\mathscr{S}),$$

and so, we have the following decomposition of the ground set X:

$$X = \big[(\cup\mathscr{S})\backslash(\cap\mathscr{S})\big] \cup \big[(X\backslash(\cup\mathscr{S})) \cup (\cap\mathscr{S})\big] \equiv X_1 \cup X_2, \tag{3.3}$$

where the sets in square brackets on the right, denoted by X_1 and X_2, are disjoint. For the TSP example, X_1 is the set of all noncommon arcs in all Hamiltonian cycles, and X_2 is the set of all arcs outside of all Hamiltonian cycles, i.e., for any $x \in X_2$ there is no Hamiltonian cycle containing x.

In a similar manner, we have two more decompositions:

$$X = \big[(\cup\mathscr{S}^*)\backslash(\cap\mathscr{S})\big] \cup \big[(X\backslash(\cup\mathscr{S}^*)) \cup (\cap\mathscr{S})\big] \equiv X_3 \cup X_4, \tag{3.4}$$

$$X = \big[(\cup\mathscr{S})\backslash(\cap\mathscr{S}^*)\big] \cup \big[(X\backslash(\cup\mathscr{S})) \cup (\cap\mathscr{S}^*)\big] \equiv X_5 \cup X_6. \tag{3.5}$$

Similarly, for the TSP, X_3 is the set of all arcs in all optimal Hamiltonian cycles without all common arcs (if any) for all not necessarily optimal Hamiltonian cycles, and X_4 is the set of all arcs outside any optimal Hamiltonian cycle.

The following notation will be needed below: given $x \in X$, we set

$$\mathscr{S}_+(x) = \{S \in \mathscr{S} : x \in S\} \quad \text{and} \quad \mathscr{S}_-(x) = \{S \in \mathscr{S} : x \notin S\}. \tag{3.6}$$

These subcollections of $\mathscr{S}$ are disjoint and their union is $\mathscr{S}$. Note that $\delta_x = 1$ on $\mathscr{S}_+(x)$ (provided $\mathscr{S}_+(x)$ is nonempty) and $\delta_x = 0$ on $\mathscr{S}_-(x)$ in any case. Note also that if $x \in (\cup\mathscr{S})\backslash(\cap\mathscr{S})$, then both these subcollections are nonempty. Moreover, if $X \neq \cup\mathscr{S}$ and $x \in X\backslash(\cup\mathscr{S})$, then $\mathscr{S}_+(x)$ is empty and $\mathscr{S}_-(x) = \mathscr{S}$, and if $\cap\mathscr{S}$ is nonempty and $x \in \cap\mathscr{S}$, then $\mathscr{S}_+(x) = \mathscr{S}$ and $\mathscr{S}_-(x)$ is empty.

The following lemma generalizes a result from [12] and shows how global upper and lower tolerances of the ground set elements can be evaluated.

Lemma 1 *Given a* COP $(X, C, \mathscr{S}, f)$ *and* $x \in X$, *we have*:

(a) $x \in (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S})$ *iff* $u(x) < +\infty$, *and, moreover,* $u(x) = \min_{\mathscr{S}_-(x)} f - f^*$;
(b) $x \in (\cup\mathscr{S})\backslash(\cap\mathscr{S}^*)$ *iff* $\ell(x) < +\infty$, *and, moreover,* $\ell(x) = \min_{\mathscr{S}_+(x)} f - f^*$.

Proof (a) ($\Rightarrow$) Let $\alpha \geq 0$ be such that any $S^* \in \mathscr{S}^*$ is a minimum of the function $f + \alpha\delta_x$ on $\mathscr{S}$ (cf. (3.1)), i.e.,

$$f^* + \alpha\delta_x(S^*) \leq f(S) + \alpha\delta_x(S) \quad \text{for all} \quad S \in \mathscr{S}. \tag{3.7}$$

If there is $S^* \in \mathscr{S}^*$ such that $x \notin S^*$ (and so, $\delta_x(S^*)$=0), or $S \in \mathscr{S}$ is such that $x \in S$ (and so, $\delta_x(S) = 1$), then, by virtue of (2.1), inequality (3.7) holds for all $\alpha \geq 0$. Now, since $x \in \cup\mathscr{S}^*$, there exists $S^* \in \mathscr{S}^*$ such that $x \in S^*$. Since $x \notin \cap\mathscr{S}$, the set $\mathscr{S}_-(x)$ of those $S \in \mathscr{S}$, for which $x \notin S$, is nonempty. It follows from (3.7) that $f^* + \alpha \leq f(S)$ for all $S \in \mathscr{S}_-(x)$, and so, $\alpha \leq \alpha_0 = \min_{\mathscr{S}_-(x)} f - f^*$. By (3.1), we have $u(x) \leq \alpha_0$. The arguments above also show that inequality (3.7) holds for $\alpha = \alpha_0$ as well, implying $u(x) = \alpha_0$ (and so, the sign sup in (3.1) can be replaced by max) and $u(x) < +\infty$.
(a) ($\Leftarrow$) On the contrary, if $x \notin (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S})$, then, by virtue of (3.4), we have either $x \in X\backslash(\cup\mathscr{S}^*)$, i.e., $x \notin S^*$ and $\delta_x(S^*) = 0$ for all $S^* \in \mathscr{S}^*$, or $x \in \cap\mathscr{S}$, and so, $x \in S$ and $\delta_x(S) = 1$ for all $S \in \mathscr{S}$. It follows from (2.1) that inequality (3.7) holds for all $\alpha \geq 0$ implying $u(x) = +\infty$, which contradicts the assumption.
(b) ($\Rightarrow$) Let $\alpha \geq 0$ be such that any $S^* \in \mathscr{S}^*$ is a minimum of the function $f - \alpha\delta_x$ on $\mathscr{S}$ (cf. (3.2)), i.e.,

$$f^* - \alpha\delta_x(S^*) \leq f(S) - \alpha\delta_x(S) \quad \text{for all} \quad S \in \mathscr{S}. \tag{3.8}$$

If there is $S^* \in \mathscr{S}^*$ such that $x \in S^*$ (and so, $\delta_x(S^*)$=1), or $S \in \mathscr{S}$ is such that $x \notin S$ (and so, $\delta_x(S) = 0$), then, by virtue of (2.1), inequality (3.8) holds for all $\alpha \geq 0$. Now, since $x \in \cup\mathscr{S}$, the set $\mathscr{S}_+(x)$ of those $S \in \mathscr{S}$, for which $x \in S$, is nonempty, and since $x \notin \cap\mathscr{S}^*$, there exists $S^* \in \mathscr{S}^*$ such that $x \notin S^*$. It follows from (3.8) that $f^* \leq f(S) - \alpha$ for all $S \in \mathscr{S}_+(x)$, and so, $\alpha \leq \alpha_1 = \min_{\mathscr{S}_+(x)} f - f^*$. By (3.2), we have $\ell(x) \leq \alpha_1$. The arguments above also show that inequality (3.8) holds for $\alpha = \alpha_1$ as well, implying $\ell(x) = \alpha_1$ (and so, the sign sup in (3.2) can be replaced by max) and $\ell(x) < +\infty$.
(b) ($\Leftarrow$) On the contrary, if $x \notin (\cup\mathscr{S})\backslash(\cap\mathscr{S}^*)$, then, by virtue of (3.5), we have either $x \in X\backslash(\cup\mathscr{S})$, i.e., $x \notin S$ and $\delta_x(S) = 0$ for all $S \in \mathscr{S}$, or $x \in \cap\mathscr{S}^*$, and so, $x \in S^*$ and $\delta_x(S^*) = 1$ for all $S^* \in \mathscr{S}^*$. It follows from (2.1) that inequality (3.8) holds for all $\alpha \geq 0$ implying $\ell(x) = +\infty$, which contradicts the assumption. □

As an immediate corollary of Lemma 1, (3.4) and (3.5), we have:

$$u(x) = +\infty \quad \text{iff} \quad x \in X_4 = (X\backslash(\cup\mathscr{S}^*)) \cup (\cap\mathscr{S}); \tag{3.9}$$
$$\ell(x) = +\infty \quad \text{iff} \quad x \in X_6 = (X\backslash(\cup\mathscr{S})) \cup (\cap\mathscr{S}^*). \tag{3.10}$$

The ground set X and the collection $\mathscr{S}$ of feasible solutions of the COP under consideration are said to be *canonical* if

$$\cup\mathscr{S} = X \quad \text{and} \quad \cap\mathscr{S} = \varnothing. \tag{3.11}$$

In this case the COP$(X, C, \mathscr{S}, f)$ will also be called *canonical*. Note that the subcollections $\mathscr{S}_+(x)$ and $\mathscr{S}_-(x)$ of $\mathscr{S}$ from (3.6) are nonempty for all $x \in X$. For $x \in X$ we denote by $[\mathscr{S}_-(x)]^*$ the set of all optimal solutions of the COP$(X, C, \mathscr{S}_-(x), f)$ and by $f([\mathscr{S}_-(x)]^*) = \min_{\mathscr{S}_-(x)} f$ its optimal value (which is coherent with notation $f(\mathscr{S}^*)$ from (2.2)); similar meanings apply to $[\mathscr{S}_+(x)]^*$ and $f([\mathscr{S}_+(x)]^*)$.

In the important particular case of a canonical COP Lemma 1, (3.9) and (3.10) give the following easier version of Lemma 1:

Lemma 2 *Given a* COP $(X, C, \mathscr{S}, f)$ *satisfying* (3.11) *and* $x \in X$*, we have*:

(a) $x \in \cup\mathscr{S}^*$ *iff* $u(x) < +\infty$*, and* $u(x) = f([\mathscr{S}_-(x)]^*) - f(\mathscr{S}^*)$;
(a′) $x \notin \cup\mathscr{S}^*$ *iff* $u(x) = +\infty$;
(b) $x \notin \cap\mathscr{S}^*$ *iff* $\ell(x) < +\infty$*, and* $\ell(x) = f([\mathscr{S}_+(x)]^*) - f(\mathscr{S}^*)$;
(b′) $x \in \cap\mathscr{S}^*$ *iff* $\ell(x) = +\infty$.

It will be shown in the next section that any COP can be reduced to a canonical COP with the preservation of the global upper and lower tolerances. Now, under the assumption that the COP under consideration is canonical, we explore the relationship of the global tolerances, introduced above, with the well known tolerances considered for different purposes in [5,8,11,19].

Suppose that the COP $(X, C, \mathscr{S}, f)$ is canonical. Given $x \in X$, the *upper tolerance* $u_{S^*}(x)$ (*lower tolerance* $\ell_{S^*}(x)$) of $x \in X$ with respect to an optimal solution $S^* \in \mathscr{S}^*$ of the COP is the maximum increase (maximum decrease, respectively) of the cost $C(x)$, under which S^* remains an optimal solution of the perturbed COP. Following the methodology of (3.1) and (3.2), we find

$$u_{S^*}(x) = \sup\big\{\alpha \geq 0 : (f + \alpha\delta_x)(S^*) \leq (f + \alpha\delta_x)(S) \text{ for all } S \in \mathscr{S}\big\},$$
$$\ell_{S^*}(x) = \sup\big\{\alpha \geq 0 : (f - \alpha\delta_x)(S^*) \leq (f - \alpha\delta_x)(S) \text{ for all } S \in \mathscr{S}\big\}.$$

The quantities $u_{S^*}(x)$ and $\ell_{S^*}(x)$ depend on a particular optimal solution S^* of the COP. The following lemma [5,12] is a local counterpart of Lemma 2, and it is established along the same lines as Lemma 1.

Lemma 3 *Given* $S^* \in \mathscr{S}^*$ *and* $x \in X$*, we have*:

(a) $x \in S^*$ *iff* $u_{S^*}(x) < +\infty$*, and* $u_{S^*}(x) = \min_{\mathscr{S}_-(x)} f - f^*$;
(a′) $x \notin S^*$ *iff* $u_{S^*}(x) = +\infty$;
(b) $x \notin S^*$ *iff* $\ell_{S^*}(x) < +\infty$*, and* $\ell_{S^*}(x) = \min_{\mathscr{S}_+(x)} f - f^*$;
(b′) $x \in S^*$ *iff* $\ell_{S^*}(x) = +\infty$.

The global tolerances are expressed by means of tolerances as follows.

Lemma 4 *For each* $x \in X$ *we have*:

$$u(x) = \min_{S^* \in \mathscr{S}^*} u_{S^*}(x) \quad \textit{and} \quad \ell(x) = \min_{S^* \in \mathscr{S}^*} \ell_{S^*}(x).$$

Proof Since the set under the supremum sign in (3.1) is contained in the set under the supremum sign defining $u_{S^*}(x)$, and likewise for (3.2) and $\ell_{S^*}(x)$, by the definition of the supremum, we find $u(x) \leq u_{S^*}(x)$ and $\ell(x) \leq \ell_{S^*}(x)$ for all $S^* \in \mathscr{S}^*$, which proves the inequalities $\leq$ for the quantities above.

Now, we establish the inequality $\geq$ for the upper tolerances. Given $x \in X$, the following two cases are possible: (i) $x \in \cup\mathscr{S}^*$, and (ii) $x \notin \cup\mathscr{S}^*$. In the case (i) there exists $S_1^* \in \mathscr{S}^*$ such that $x \in S_1^*$, and so, by virtue of Lemmas 2(a) and 3(a), we have:

$$\min_{S^* \in \mathscr{S}^*} u_{S^*}(x) \leq u_{S_1^*}(x) = \min_{\mathscr{S}_-(x)} f - f^* = u(x).$$

In the case (ii) we find $x \notin S^*$ for all $S^* \in \mathscr{S}^*$, and so, by Lemma 3(a′), $u_{S^*}(x) = +\infty$ for all $S^* \in \mathscr{S}^*$. Now, it follows from Lemma 2(a′) that

$$\min_{S^* \in \mathscr{S}^*} u_{S^*}(x) = +\infty = u(x).$$

Let us prove the inequality $\geq$ for the lower tolerances. Given $x \in X$, we have two possibilities: (i) $x \notin \cap\mathscr{S}^*$, and (ii) $x \in \cap\mathscr{S}^*$. In the case (i) there exists $S_2^* \in \mathscr{S}^*$ such that $x \notin S_2^*$, and so, Lemmas 3(b) and 2(b) imply

$$\min_{S^* \in \mathscr{S}^*} \ell_{S^*}(x) \leq \ell_{S_2^*}(x) = \min_{\mathscr{S}_+(x)} f - f^* = \ell(x).$$

In the case (ii) we find $x \in S^*$ for all $S^* \in \mathscr{S}^*$, and so, by Lemma 3(b′), $\ell_{S^*}(x) = +\infty$ for all $S^* \in \mathscr{S}^*$. Now it follows from Lemma 2(b′) that

$$\min_{S^* \in \mathscr{S}^*} \ell_{S^*}(x) = +\infty = \ell(x). \qquad \square$$

The remaining three propositions of this section expose yet further relationships between the global tolerances and the tolerances.Note that these propositions are valid for COPs with no requirements on the set of canonical feasible solutions (e.g., relaxed nonembeddedness).

Proposition 1 *Given* $S_1^*, S_2^* \in \mathscr{S}^*$, *we have*:

(a) $S_1^* \subset S_2^*$ *iff* $u_{S_2^*}(x) \leq u_{S_1^*}(x)$ *for all* $x \in X$.
(b) $S_1^* \subset S_2^*$ *iff* $\ell_{S_1^*}(x) \leq \ell_{S_2^*}(x)$ *for all* $x \in X$.

Proof (a) ($\Rightarrow$) Since $X = S_2^* \cup (X \backslash S_2^*) = S_1^* \cup (S_2^* \backslash S_1^*) \cup (X \backslash S_2^*)$, given $x \in X$, we have the following three possibilities: (i) $x \in S_1^*$, (ii) $x \in S_2^* \backslash S_1^*$, and (iii) $x \notin S_2^*$. In case (i) we have $x \in S_1^*$ and $x \in S_2^*$, and so, by Lemma 3(a), $u_{S_1^*}(x) = \min_{\mathscr{S}_-(x)} f - f^* = u_{S_2^*}(x)$. In case (ii) we have $x \in S_2^*$ and $x \notin S_1^*$, and so, by Lemma 3 (a), (a′), $u_{S_2^*}(x) < +\infty = u_{S_1^*}(x)$. In case (iii) we have $x \notin S_2^*$ and $x \notin S_1^*$, and so, by Lemma 3(a′), $u_{S_2^*}(x) = +\infty = u_{S_1^*}(x)$.

(a) ($\Leftarrow$) Let $x \in S_1^*$. By Lemma 3(a), $u_{S_1^*}(x) < +\infty$, and since $u_{S_2^*}(x) \leq u_{S_1^*}(x)$, we have $u_{S_2^*}(x) < +\infty$, and again by Lemma 3(a), $x \in S_2^*$, which proves the desired inclusion.

(b) ($\Rightarrow$) We have the same possibilities (i), (ii) and (iii) as in (a) ($\Rightarrow$) above and we apply Lemma 3. (i) If $x \in S_1^*$ and $x \in S_2^*$, then $\ell_{S_1^*}(x) = +\infty = \ell_{S_2^*}(x)$. (ii) If $x \in S_2^*$ and $x \notin S_1^*$, then $\ell_{S_1^*}(x) < +\infty = \ell_{S_2^*}(x)$. (iii) If $x \notin S_2^*$ and $x \notin S_1^*$, then $\ell_{S_1^*}(x) = \min_{\mathscr{S}_+(x)} f - f^* = \ell_{S_2^*}(x)$.

(b) ($\Leftarrow$) If $x \in S_1^*$, then $\ell_{S_1^*}(x) = +\infty$, and since $\ell_{S_1^*}(x) \leq \ell_{S_2^*}(x)$, then $u_{S_2^*}(x) = +\infty$, and so, $x \in S_2^*$. Thus, $S_1^* \subset S_2^*$. $\square$

Proposition 2 *Given* $S^* \in \mathscr{S}^*$, *we have*:

(a) $u(x) = u_{S^*}(x)$ *for all* $x \in X$ *iff* $\cup\mathscr{S}^* = S^*$ *(i.e.,* $S \subset S^*$ *for all* $S \in \mathscr{S}^*$*).*
(b) $\ell(x) = \ell_{S^*}(x)$ *for all* $x \in X$ *iff* $\cap\mathscr{S}^* = S^*$ *(i.e.,* $S^* \subset S$ *for all* $S \in \mathscr{S}^*$*).*

Proof (a) ($\Rightarrow$) Let $x \in \cup\mathscr{S}^*$. By the assumption and Lemma 2(a), $u_{S^*}(x) = u(x) < +\infty$, and so, by Lemma 3(a), $x \in S^*$. Thus, $\cup\mathscr{S}^* \subset S^*$, and so, $\cup\mathscr{S}^* = S^*$.

(a) ($\Leftarrow$) If $\cup\mathscr{S}^* = S^*$, then $S \subset S^*$ for all $S \in \mathscr{S}^*$, and so, by Proposition 1(a), $u_{S^*}(x) \leq u_S(x)$ for all $S \in \mathscr{S}^*$ and $x \in X$. Now, the definition of a minimum and Lemma 4 imply $u_{S^*}(x) = \min_{S \in \mathscr{S}^*} u_S(x) = u(x)$ for all $x \in X$.

(b) ($\Rightarrow$) Let $x \in S^*$. By the assumption and Lemma 3(b′), $\ell(x) = \ell_{S^*}(x) = +\infty$, and so, by Lemma 2(b′), $x \in \cap\mathscr{S}^*$. Thus, $S^* \subset \cap\mathscr{S}^*$, and so, $S^* = \cap\mathscr{S}^*$.

(b) ($\Leftarrow$) If $\cap\mathscr{S}^* = S^*$, then $S^* \subset S$ for all $S \in \mathscr{S}^*$, and so, by Proposition 1(b), $\ell_{S^*}(x) \le \ell_S(x)$ for all $S \in \mathscr{S}^*$ and $x \in X$. Now, the definition of a minimum and Lemma 4 imply $\ell_{S^*}(x) = \min_{S\in\mathscr{S}^*} \ell_S(x) = \ell(x)$ for all $x \in X$. □

The following straightforward corollary of Proposition 2 concerns a characterization of the uniqueness of optimal solutions of the COP $(X, C, \mathscr{S}, f)$.

Proposition 3 *Let the* COP $(X, C, \mathscr{S}, f)$ *be canonical.*

(a) $|\mathscr{S}^*| = 1$ *iff for some* $S^* \in \mathscr{S}^*$ *we have*: $u = u_{S^*}$ *and* $\ell = \ell_{S^*}$ *on* X.
(b) $|\mathscr{S}^*| \ge 2$ *iff for all* $S^* \in \mathscr{S}^*$ *we have*: $u \ne u_{S^*}$ *or* $\ell \ne \ell_{S^*}$ *on* X.

Proof (a) If $|\mathscr{S}^*| = 1$, then $\mathscr{S}^* = \{S^*\}$ for some $S^* \in \mathscr{S}$, and so, the equalities $u = u_{S^*}$ and $\ell = \ell_{S^*}$ follow from (3.1), (3.2) and the definitions of u_{S^*} and ℓ_{S^*}. Now, suppose that for some $S^* \in \mathscr{S}^*$ we have $u(x) = u_{S^*}(x)$ and $\ell(x) = \ell_{S^*}(x)$ for all $x \in X$. Then Proposition 2 implies $\cup\mathscr{S}^* = S^* = \cap\mathscr{S}^*$, and so, $\mathscr{S}^* = \{S^*\}$.
(b) This is the negation of the assertion in (a). □

4 Reduction to the canonical problem

The aim of this section is to show that any COP can be reduced to a canonical COP. As a motivation, we note that it follows from (3.3) to (3.5) and (2.3) that

$$\begin{aligned} X_4 \cap X_6 &= \big[(X\backslash(\cup\mathscr{S}^*)) \cup (\cap\mathscr{S})\big] \cap \big[(X\backslash(\cup\mathscr{S})) \cup (\cap\mathscr{S}^*)\big] \\ &= \big[(X\backslash(\cup\mathscr{S}^*)) \cap (X\backslash(\cup\mathscr{S}))\big] \cup \big[(X\backslash(\cup\mathscr{S}^*)) \cap (\cap\mathscr{S}^*)\big] \\ &\quad \cup [(\cap\mathscr{S}) \cap (X\backslash(\cup\mathscr{S}))] \cup \big[(\cap\mathscr{S}) \cap (\cap\mathscr{S}^*)\big] \\ &= (X\backslash(\cup\mathscr{S})) \cup (\cap\mathscr{S}) = X_2. \end{aligned}$$

Thus, if X_2 is nonempty, then (3.9) and (3.10) yield

$$u(x) = +\infty = \ell(x) \quad \text{for all} \quad x \in X_4 \cap X_6 = X_2,$$

where X_2 does not depend on the set $\mathscr{S}^*$ of optimal solutions of the initial COP. This means that elements of X_2 do not lead to any feasible solution and, hence, may waste the computational efforts within any optimization procedure. In fact, if $x \in X\backslash(\cup\mathscr{S})$, then the value $C(x)$ does not contribute to the sum $f(S) = \sum_{y\in S} C(y)$ for all $S \in \mathscr{S}$ (note also that $\delta_x(S) = 0$ for all $S \in \mathscr{S}$, cf. (3.1), (3.2)). On the other hand, if $\cap\mathscr{S}$ is nonempty, then, since $S = (\cap\mathscr{S}) \cup (S\backslash(\cap\mathscr{S}))$ for all $S \in \mathscr{S}$, we have (by the additivity of f)

$$f(S) = f(\cap\mathscr{S}) + f(S\backslash(\cap\mathscr{S})), \tag{4.1}$$

and so, $f(\cap\mathscr{S})$ is the common contribution to any sum $f(S)$, where $f(\cap\mathscr{S})$ is independent of a particular $x \in \cap\mathscr{S}$ (note that $\delta_x(S) = 1$ for all $S \in \mathscr{S}$).

We are going to show that any COP $(X, C, \mathscr{S}, f)$ satisfying (2.5) can be reduced to an equivalent COP $(X', C', \mathscr{S}', f')$ with canonical ground set X' and the set of feasible solutions $\mathscr{S}'$ (cf. (3.11)) in such a way that the upper and lower tolerances of the initial COP and the reduced COP coincide in the sense to be made precise below.

In order to do this, suppose that $|\mathscr{S}| \ge 2$, $\cap\mathscr{S} \ne \varnothing$ and $\cap\mathscr{S} \notin \mathscr{S}$.

We set $X' = X_1 = (\cup\mathscr{S})\backslash(\cap\mathscr{S})$ (cf. (3.3)), and so, by (2.5),

$$1 \le |X'| = |\cup\mathscr{S}| - |\cap\mathscr{S}| < |\cup\mathscr{S}| \le |X|.$$

It will be convenient to introduce the (so called) 'prime operation' on $\mathscr{S}$ defined as follows: given $S \in \mathscr{S}$, we set $S' = S\backslash(\cap\mathscr{S})$. We also let

$$\mathscr{S}' = \mathscr{S}\backslash(\cap\mathscr{S}) = \{S' \subset X : S' = S\backslash(\cap\mathscr{S}) \text{ for some } S \in \mathscr{S}\}. \tag{4.2}$$

Clearly, $\mathscr{S}' \subset 2^{X'}$ is a nonempty collection of nonempty subsets of X' (note that if $S \in \mathscr{S}$, then $S' = S\backslash(\cap\mathscr{S}) \subset (\cup\mathscr{S})\backslash(\cap\mathscr{S}) = X'$) and, because $\cap\mathscr{S} \notin \mathscr{S}$, we find $|\mathscr{S}'| = |\mathscr{S}|$. The 'primed' cost function $C' : X' \to \mathbb{R}$, given by $C'(x) = C(x)$ for all $x \in X'$, is the restriction of C (from X) to X', and so, we keep the notation C for it as well. Finally, the 'primed' objective function $f' = f'_C : \mathscr{S}' \to \mathbb{R}$ acts according to the rule: if $S' \in \mathscr{S}'$, then $S' = S\backslash(\cap\mathscr{S})$ for some $S \in \mathscr{S}$, and so,

$$f'(S') = f'_C(S') = \sum_{x \in S'} C(x) = f_C(S\backslash(\cap\mathscr{S})) = f(S\backslash(\cap\mathscr{S})). \tag{4.3}$$

We assert that the COP $(X', C, \mathscr{S}', f')$ is the desired reduced COP.

Clearly, X' and $\mathscr{S}'$ are canonical, i.e., $\cup\mathscr{S}' = X'$ and $\cap\mathscr{S}' = \varnothing$; in fact,

$$\cup\mathscr{S}' = \cup(\mathscr{S}\backslash(\cap\mathscr{S})) = (\cup\mathscr{S})\backslash(\cap\mathscr{S}) = X'$$

and

$$\cap\mathscr{S}' = \cap(\mathscr{S}\backslash(\cap\mathscr{S})) = (\cap\mathscr{S})\backslash(\cap\mathscr{S}) = \varnothing.$$

Now we study the relations between the optimal solutions of the COP $(X, C, \mathscr{S}, f)$ and the reduced COP $(X', C, \mathscr{S}', f')$.

We denote by $\mathscr{S}'^*$ the set of all optimal solutions of the reduced COP, i.e.,

$$\mathscr{S}'^* = \left\{S' \in \mathscr{S}' : f'^* = f'(S') = \min_{\mathscr{S}'} f'\right\},$$

where f'^* is the optimal value of the reduced COP. As a consequence of (4.1), we have

Lemma 5 (a) *If $S^* \in \mathscr{S}^*$, then $S^{*\prime} = S^*\backslash(\cap\mathscr{S}) \in \mathscr{S}'^*$ and*

$$f'^* = f'(S^{*\prime}) = f(S^*\backslash(\cap\mathscr{S})) = \min_{S \in \mathscr{S}} f(S\backslash(\cap\mathscr{S})).$$

(b) *If $S'^* \in \mathscr{S}'^*$, then $S^* = S'^* \cup (\cap\mathscr{S}) \in \mathscr{S}^*$ and*

$$f^* = f(S^*) = f(S'^* \cup (\cap\mathscr{S})) = \min_{S' \in \mathscr{S}'} f(S' \cup (\cap\mathscr{S})).$$

(c) $f^* = f(\cap\mathscr{S}) + f'^*$.

Proof (a) Suppose $S^* \in \mathscr{S}^*$, i.e., condition (2.1) is satisfied. Let $S' \in \mathscr{S}'$ be arbitrary. Then there exists $S \in \mathscr{S}$ such that $S' = S\backslash(\cap\mathscr{S})$. Inequality $f(S^*) \le f(S)$ and (4.1) imply

$$f(\cap\mathscr{S}) + f(S^*\backslash(\cap\mathscr{S})) = f(S^*) \le f(S) = f(\cap\mathscr{S}) + f(S\backslash(\cap\mathscr{S})), \tag{4.4}$$

and so, (4.3) yields

$$f'(S^{*\prime}) = f(S^*\backslash(\cap\mathscr{S})) \le f(S\backslash(\cap\mathscr{S})) = f'(S'). \tag{4.5}$$

The arbitrariness of $S' \in \mathscr{S}'$ implies $S^{*\prime} \in \mathscr{S}'^*$, i.e., by virtue of (4.3), $f'^* = f'(S^{*\prime}) = f(S^*\backslash(\cap\mathscr{S}))$. Since $S^* \in \mathscr{S}$, we have $\min_{S \in \mathscr{S}} f(S\backslash(\cap\mathscr{S})) \le f(S^*\backslash(\cap\mathscr{S}))$, and it follows from (4.5) that the reverse inequality holds as well, and so, $f(S^*\backslash(\cap\mathscr{S})) = \min_{S \in \mathscr{S}} f(S\backslash(\cap\mathscr{S}))$.

(b) Now, let $S'^* \in \mathscr{S}'^*$ and $S^* = S'^* \cup (\cap\mathscr{S})$, and let $S \in \mathscr{S}$ be arbitrary. Defining $S' \in \mathscr{S}'$ by $S' = S\backslash(\cap\mathscr{S})$ and noting that $S^{*\prime} = S^*\backslash(\cap\mathscr{S}) = S'^* \in \mathscr{S}'^*$, we find $f'(S'^*) \le f'(S')$, i.e., by virtue of (4.3),

$$f(S^*\backslash(\cap\mathscr{S})) = f'(S'^*) \le f'(S') = f(S\backslash(\cap\mathscr{S})).$$

Adding $f(\cap\mathscr{S})$ to both sides of this inequality and taking into account (4.1), we get $f(S^*) \le f(S)$. The arbitrariness of $S \in \mathscr{S}$ implies $S^* \in \mathscr{S}^*$, and so, $f^* = f(S^*) = f(S'^* \cup (\cap\mathscr{S}))$. Since $S'^* \in \mathscr{S}'$ and $S' \cup (\cap\mathscr{S}) \in \mathscr{S}$ for all $S' \in \mathscr{S}'$, we have

$$\min_{S'\in\mathscr{S}'} f(S' \cup (\cap\mathscr{S})) \le f(S'^* \cup (\cap\mathscr{S})) \le \min_{S'\in\mathscr{S}'} f(S' \cup (\cap\mathscr{S})).$$

(c) It follows from (4.4) and (4.5) that $f^* \le f(\cap\mathscr{S}) + f'(S')$ for all $S' \in \mathscr{S}'$, and so, $f^* \le f(\cap\mathscr{S}) + f'^*$. On the other hand, by virtue of (4.1), given $S \in \mathscr{S}$, we find $S\backslash(\cap\mathscr{S}) \in \mathscr{S}'$ and

$$f(\cap\mathscr{S}) + f'^* \le f(\cap\mathscr{S}) + f(S\backslash(\cap\mathscr{S})) = f(S),$$

and so, $f(\cap\mathscr{S}) + f'^* \le f^*$, which was to be proved. □

Lemma 5 can be interpreted in the following way: if we set $\mathscr{S}^{*\prime} = \mathscr{S}^*\backslash(\cap\mathscr{S})$ (similar to (4.2)), then

$$\mathscr{S}'^* = \mathscr{S}^{*\prime}. \tag{4.6}$$

In fact, if $S' \in \mathscr{S}'^*$, then, by Lemma 5(b), $S^* = S' \cup (\cap\mathscr{S}) \in \mathscr{S}^*$, and $S' = S^{*\prime} = S^*\backslash(\cap\mathscr{S})$, and so, $S' \in \mathscr{S}^{*\prime}$, which establishes the inclusion $\subset$. Now, if $S' \in \mathscr{S}^{*\prime}$, then there exists $S^* \in \mathscr{S}^*$ such that $S' = S^*\backslash(\cap\mathscr{S})$, and so, by Lemma 5, $S' \in \mathscr{S}'^*$, which proves the inclusion $\supset$.

Finally, we show that the corresponding global tolerances for the initial and reduced COPs coincide for all $x \in X' = X_1$, i.e.,

$$u'(x) = u(x) \quad \text{and} \quad \ell'(x) = \ell(x) \quad \text{for all} \quad x \in X',$$

where $u'(x)$ is the global upper tolerance and $\ell'(x)$ is the global lower tolerance of x with respect to the COP $(X', C, \mathscr{S}', f')$.

Observe that, by virtue of (4.6),

$$\cup\mathscr{S}'^* = \cup\mathscr{S}^{*\prime} = (\cup\mathscr{S}^*)' = (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S})$$

and, similarly, $\cap\mathscr{S}'^* = (\cap\mathscr{S}^*)\backslash(\cap\mathscr{S})$. Taking into account that $\cap\mathscr{S}' = \varnothing$, $\cup\mathscr{S}' = X' = (\cup\mathscr{S})\backslash(\cap\mathscr{S})$ and $\cap\mathscr{S} \subset \cap\mathscr{S}^*$, it follows that (cf. (3.4) and (3.5))

$$X'_3 = (\cup\mathscr{S}'^*)\backslash(\cap\mathscr{S}') = (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S}) = X_3,$$

$$\begin{aligned} X'_5 &= (\cup\mathscr{S}')\backslash(\cap\mathscr{S}'^*) = \big[(\cup\mathscr{S})\backslash(\cap\mathscr{S})\big]\backslash\big[(\cap\mathscr{S}^*)\backslash(\cap\mathscr{S})\big] \\ &= (\cup\mathscr{S})\backslash((\cap\mathscr{S}) \cup (\cap\mathscr{S}^*)) = (\cup\mathscr{S})\backslash(\cap\mathscr{S}^*) = X_5 \end{aligned}$$

and

$$(\cup\mathscr{S}'^*)\backslash(\cap\mathscr{S}'^*) = \big[(\cup\mathscr{S}^*)\backslash(\cap\mathscr{S})\big]\backslash\big[(\cap\mathscr{S}^*)\backslash(\cap\mathscr{S})\big] = (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S}^*),$$

where we note also that $(\cup\mathscr{S}^*)\backslash(\cap\mathscr{S}^*) = X_3 \cap X_5$.

By virtue of (3.4) and (3.9), it suffices to prove that $u'(x) = u(x)$ only for elements $x \in X_3' = X_3$, and by virtue of (3.5) and (3.10), it suffices to show that $\ell'(x) = \ell(x)$ for all $x \in X_5' = X_5$. Applying Lemma 1(a),(b), we have

$$u'(x) = \min_{(\mathscr{S}')_-(x)} f' - f'^* \quad \text{and} \quad u(x) = \min_{\mathscr{S}_-(x)} f - f^*, \qquad x \in X_3' = X_3,$$

and

$$\ell'(x) = \min_{(\mathscr{S}')_+(x)} f' - f'^* \quad \text{and} \quad \ell(x) = \min_{\mathscr{S}_+(x)} f - f^*, \qquad x \in X_5' = X_5.$$

Now the desired equalities of tolerances will readily follow from Lemma 5(c) if we show that (see below)

$$\min_{\mathscr{S}_-(x)} f = f(\cap\mathscr{S}) + \min_{(\mathscr{S}')_-(x)} f', \qquad x \in X_3' = X_3, \tag{4.7}$$

and

$$\min_{\mathscr{S}_+(x)} f = f(\cap\mathscr{S}) + \min_{(\mathscr{S}')_+(x)} f', \qquad x \in X_5' = X_5. \tag{4.8}$$

In fact, it follows from (4.7) and Lemma 5(c) that

$$u'(x) = \min_{(\mathscr{S}')_-(x)} f' - f'^* = \min_{\mathscr{S}_-(x)} f - f(\cap\mathscr{S}) - f'^* = \min_{\mathscr{S}_-(x)} f - f^* = u(x),$$

and similar arguments apply to show that $\ell'(x) = \ell(x)$.

Proof of (4.7). First, we prove the inequality $\geq$. If $S \in \mathscr{S}_-(x)$, then $S \in \mathscr{S}$ and $x \notin S$, and so, $S' = S\backslash(\cap\mathscr{S}) \in \mathscr{S}'$ and $x \notin S'$, i.e., $S' \in (\mathscr{S}')_-(x)$. By virtue of (4.1), this implies

$$f(S) = f(\cap\mathscr{S}) + f'(S') \geq f(\cap\mathscr{S}) + \min_{(\mathscr{S}')_-(x)} f',$$

and the desired inequality follows from the arbitrariness of S. In order to establish inequality $\leq$ in (4.7), we let $S' \in (\mathscr{S}')_-(x)$. Then $S' \in \mathscr{S}'$ and $x \notin S'$, and so, there exists $S \in \mathscr{S}$ such that $S' = S\backslash(\cap\mathscr{S})$. We assert that $x \notin S$, so that $S \in \mathscr{S}_-(x)$; in fact, $x \notin S' = S\backslash(\cap\mathscr{S})$ implies $x \notin S$ or $x \in \cap\mathscr{S}$, and the latter inclusion is impossible because $x \in X_3 = (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S})$. Now, it follows from (4.1) that

$$\min_{\mathscr{S}_-(x)} f \leq f(S) = f(\cap\mathscr{S}) + f'(S'),$$

and it remains to take into account the arbitrariness of S'. □

Proof of (4.8). ($\geq$) If $S \in \mathscr{S}_+(x)$, then $S \in \mathscr{S}$ and $x \in S$, and so, $S' = S\backslash(\cap\mathscr{S}) \in \mathscr{S}'$. Since $x \in X_5$, we have $x \notin \cap\mathscr{S}^*$, but (cf. (2.3)) $\cap\mathscr{S}^* \supset \cap\mathscr{S}$, and so, $x \notin \cap\mathscr{S}$, which implies $x \in \mathscr{S}'$, i.e., $S' \in (\mathscr{S}')_+(x)$. By (4.1), we find

$$f(S) = f(\cap\mathscr{S}) + f'(S') \geq f(\cap\mathscr{S}) + \min_{(\mathscr{S}')_+(x)} f',$$

and it remains to take the minimum over all $S \in \mathscr{S}_+(x)$.

($\leq$) If $S' \in (\mathscr{S}')_+(x)$, then $S' \in \mathscr{S}'$ and $x \in S'$, and so, there exists $S \in \mathscr{S}$ such that $S' = S\backslash(\cap\mathscr{S})$. It follows that $x \in S$, $S \in \mathscr{S}_+(x)$ and, by (4.1),

$$\min_{\mathscr{S}_+(x)} f \leq f(S) = f(\cap\mathscr{S}) + f'(S'),$$

and it remains to take into account the arbitrariness of $S' \in (\mathscr{S}')_+(x)$.

□

5 Minimal values of upper and lower tolerances

Throughout the rest of the paper we assume that the COP $(X, C, \mathscr{S}, f)$ is canonical (cf. (3.11)), $|X| \geq 2$ and $|\mathscr{S}| \geq 2$.

The next lemma is a consequence of Lemma 2 and is preparatory for our first main result (Theorem 1).

Lemma 6 *Given a canonical* COP $(X, C, \mathscr{S}, f)$ *and* $x \in X$*, we have* :

(a) $u(x) = 0$ *iff* $x \in (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S}^*)$ *iff* $\ell(x) = 0$ *(in this case* $|\mathscr{S}^*| \geq 2$*);*
(b) $x \in \cap\mathscr{S}^*$ *iff* $0 < u(x) < +\infty$;
(c) $x \notin \cup\mathscr{S}^*$ *iff* $0 < \ell(x) < +\infty$.

As for the uniqueness of optimal solutions of the COP*, we have*:

(d) $|\mathscr{S}^*| = 1$ *iff* $0 < u(x) < +\infty$ *for all* $x \in \cup\mathscr{S}^*$;
(e) $|\mathscr{S}^*| = 1$ *iff* $0 < \ell(x) < +\infty$ *for all* $x \in X\backslash(\cap\mathscr{S}^*)$.

Proof (a) Let $x \in (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S}^*)$. Since $x \notin \cap\mathscr{S}^*$, there exists $S_1^* \in \mathscr{S}^*$ such that $x \notin S_1^*$, and so, $S_1^* \in \mathscr{S}_-(x)$, and since $x \in \cup\mathscr{S}^*$, Lemma 2(a) implies

$$0 \leq u(x) = \min_{\mathscr{S}_-(x)} f - f^* \leq f(S_1^*) - f^* = 0.$$

Similarly, $x \in \cup\mathscr{S}^*$ implies the existence of $S_2^* \in \mathscr{S}^*$ such that $x \in S_2^*$, so that we find $S_2^* \in \mathscr{S}_+(x)$. Now, condition $x \notin \cap\mathscr{S}^*$ and Lemma 2(b) yield

$$0 \leq \ell(x) = \min_{\mathscr{S}_+(x)} f - f^* \leq f(S_2^*) - f^* = 0.$$

Suppose $u(x) = 0$. The finiteness of $u(x)$ and Lemma 2(a) imply $x \in \cup\mathscr{S}^*$ and $\min_{\mathscr{S}_-(x)} f = f^*$, and so, $f(S_1) = f^*$ for some $S_1 \in \mathscr{S}_-(x)$. It follows that $S_1 \in \mathscr{S}^*$ and $x \notin S_1$, i.e., $x \notin \cap\mathscr{S}^*$. Thus, $x \in (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S}^*)$. Now, assume that $\ell(x) = 0$. By Lemma 2(b), we have $x \in X\backslash(\cap\mathscr{S}^*)$ and $\min_{\mathscr{S}_+(x)} f = f^*$, and so, $f(S_2) = f^*$ for some $S_2 \in \mathscr{S}_+(x)$. Therefore, $S_2 \in \mathscr{S}^*$ and $x \in S_2$, and so, $x \in \cup\mathscr{S}^*$. This again implies $x \in (\cup\mathscr{S}^*)\backslash(\cap\mathscr{S}^*)$.

(b) ($\Rightarrow$) By Lemma 2(a), we find $u(x) < +\infty$, and item (a) of this lemma gives (by contradiction) $u(x) > 0$.

(b) ($\Leftarrow$) Condition $u(x) < +\infty$ and Lemma 2(a) yield $x \in \cup\mathscr{S}^*$. Since $u(x) > 0$, item (a) of this lemma implies (by contradiction) $x \in \cap\mathscr{S}^*$.

(c) ($\Rightarrow$) Since $x \notin \cup\mathscr{S}^*$, we have $x \notin \cap\mathscr{S}^*$, and so, by Lemma 2(b), $\ell(x) < +\infty$. It follows from item (a) above (by contradiction) that $\ell(x) > 0$.

(c) ($\Leftarrow$) Since $\ell(x) < +\infty$, Lemma 2(b) implies $x \notin \cap\mathscr{S}^*$. Now, condition $\ell(x) > 0$ and item (a) above give (by contradiction) $x \notin \cup\mathscr{S}^*$.

(d), (e) ($\Rightarrow$) If $|\mathscr{S}^*| = 1$, then $\mathscr{S}^* = \{S^*\}$ for some $S^* \in \mathscr{S}$, and so (cf. (2.4)), $\cup\mathscr{S}^* = S^* = \cap\mathscr{S}^*$. If $x \in \cup\mathscr{S}^*$, then, by item (b) above, $0 < u(x) < +\infty$; and if $x \notin \cap\mathscr{S}^*$, then, by item (c) of this lemma, $0 < \ell(x) < +\infty$.

(d) ($\Leftarrow$) For any $x \in \cup\mathscr{S}^*$ we have $0 < u(x) < +\infty$, and so, item (b) of this lemma implies $x \in \cap\mathscr{S}^*$. Thus, $\cup\mathscr{S}^* = \cap\mathscr{S}^*$ (cf. (2.4)), and so, $\mathscr{S}^* = \{S^*\}$ with $S^* = \cap\mathscr{S}^*$.

(e) ($\Leftarrow$) For any $x \notin \cap\mathscr{S}^*$ we have $0 < \ell(x) < +\infty$, and so, by item (c) above, $x \notin \cup\mathscr{S}^*$. It follows that $\cap\mathscr{S}^* \supset \cup\mathscr{S}^*$, and so, $\cap\mathscr{S}^* = \cup\mathscr{S}^*$. Thus, $\mathscr{S}^* = \{S^*\}$ with $S^* = \cup\mathscr{S}^*$. □

Our first main result is the following

Theorem 1 *Given a canonical* COP $(X, C, \mathscr{S}, f)$ *and* $S^* \in \mathscr{S}^*$, *if feasible solutions from* $\mathscr{S}$ *are nonembedded (i.e.,* $S_1 \backslash S_2 \neq \varnothing$ *for all* $S_1, S_2 \in \mathscr{S},\ S_1 \neq S_2$*), then*

$$\min_{y \in X \backslash S^*} \ell(y) = \min_{y \in X \backslash (\cap \mathscr{S}^*)} \ell(y) = \min_{x \in \cup \mathscr{S}^*} u(x) = \min_{x \in S^*} u(x). \tag{5.1}$$

Proof First, we show that

$$\min_{y \in X \backslash S^*} \ell(y) \leq \min_{x \in \cup \mathscr{S}^*} u(x). \tag{5.2}$$

For this, it suffices to verify that for each $x \in \cup \mathscr{S}^*$ there exists $y \in X \backslash S^*$ such that $\ell(y) \leq u(x)$. Let $x \in \cup \mathscr{S}^*$. By Lemma 2(a), there exists $S_1 \in \mathscr{S}_-(x)$ such that $u(x) = f(S_1) - f^*$. By the nonembeddedness, $S_1 \backslash S^* \neq \varnothing$. Choose a $y \in S_1 \backslash S^*$. Then $y \in S_1$ and $y \notin S^*$, and so, $S_1 \in \mathscr{S}_+(y)$ and $y \notin \cap \mathscr{S}^*$. It follows from Lemma 2(b) that

$$\ell(y) = \min_{\mathscr{S}_+(y)} f - f^* \leq f(S_1) - f^* = u(x).$$

Now, we prove that

$$\min_{x \in S^*} u(x) \leq \min_{y \in X \backslash (\cap \mathscr{S}^*)} \ell(y). \tag{5.3}$$

It suffices to show that for each $y \in X \backslash (\cap \mathscr{S}^*)$ there exists $x \in S^*$ such that inequality $u(x) \leq \ell(y)$ holds. If $y \in X \backslash (\cap \mathscr{S}^*)$, then, by virtue of Lemma 2(b), we have $\ell(y) = f(S_2) - f^*$ for some $S_2 \in \mathscr{S}_+(y)$. By the nonembeddedness, $S^* \backslash S_2 \neq \varnothing$. Fix an $x \in S^* \backslash S_2$. Then $x \in S^*$ and $x \notin S_2$, and so, $x \in \cup \mathscr{S}^*$ and $S_2 \in \mathscr{S}_-(x)$. It follows from Lemma 2(a) that

$$u(x) = \min_{\mathscr{S}_-(x)} f - f^* \leq f(S_2) - f^* = \ell(y).$$

The desired equality (5.1) now follows from (5.2) and (5.3) if we take into account that

$$\min_{x \in \cup \mathscr{S}^*} u(x) \leq \min_{x \in S^*} u(x) \quad \text{and} \quad \min_{y \in X \backslash (\cap \mathscr{S}^*)} \ell(y) \leq \min_{y \in X \backslash S^*} \ell(y). \qquad \square$$

Remark 1 If $|\mathscr{S}^*| \geq 2$, then, by (2.4), $\cup \mathscr{S}^* \neq \cap \mathscr{S}^*$, and so, $(\cup \mathscr{S}^*) \backslash (\cap \mathscr{S}^*)$ contains an element x_0, for which, by Lemma 6(a), we have $\ell(x_0) = 0 = u(x_0)$. Thus, all values in (5.1) are equal to zero. In particular, it is interesting to note that

$$\min_{y \in X \backslash S^*} \ell(y) = 0 = \min_{x \in S^*} u(x) \quad \text{for all} \quad S^* \in \mathscr{S}^*.$$

Now we study the case when feasible solutions from $\mathscr{S}$ are not necessarily nonembedded (see also Sect. 6).

Theorem 2 *Given a canonical* COP $(X, C, \mathscr{S}, f)$, *we have*:

$$\min_{y \in X \backslash (\cap \mathscr{S}^*)} \ell(y) \leq \min_{x \in \cup \mathscr{S}^*} u(x) \leq \min_{y \in X \backslash [(\cap \mathscr{S}^*) \cup (\cup \mathscr{S}_0)]} \ell(y),$$

where $\mathscr{S}_0 = \{S_0 \in \mathscr{S} : \cup \mathscr{S}^* \subset S_0\}$ $(\min \varnothing = +\infty)$. *In particular, if feasibile solutions* $\mathscr{S}$ *are nonembedded, then* $\mathscr{S}_0 = \varnothing$ *and* $\min_{y \in X \backslash (\cap \mathscr{S}^*)} \ell(y) = \min_{x \in \cup \mathscr{S}^*} u(x)$.

Proof In order to prove the left hand side inequality, it suffices to show that for each $x \in \cup \mathscr{S}^*$ there exists $y \in X \backslash (\cap \mathscr{S}^*)$ such that $\ell(y) \leq u(x)$.

Let $x \in \cup \mathscr{S}^*$ be arbitrarily fixed. By Lemma 2(a), there exists a set $S_1 \in \mathscr{S}_-(x)$, i.e., $S_1 \in \mathscr{S}$ with $x \notin S_1$, such that $u(x) = f(S_1) - f^*$. We have two possibilities: either $|\mathscr{S}^*| = 1$ or $|\mathscr{S}^*| \geq 2$.

First, suppose $|\mathscr{S}^*| = 1$, and so, $\mathscr{S}^* = \{S^*\}$ for some $S^* \in \mathscr{S}$. Noting that $\cap\mathscr{S}^* = S^*$, we assert that

$$S_1 \backslash (\cap\mathscr{S}^*) = S_1 \backslash S^* \neq \varnothing. \tag{5.4}$$

In fact, on the contrary assume that $S_1 \backslash S^*$ is empty, and so, $S_1 \subset S^*$. Since $S^* \in \mathscr{S}^*$, we have $f(S^*) \leq f(S_1)$, and since the cost function C is nonnegative and $S_1 \subset S^*$, we find

$$f(S_1) = \sum_{z \in S_1} C(z) \leq \sum_{z \in S^*} C(z) = f(S^*).$$

Thus, $f(S_1) = f(S^*) = f^*$, and so, $S_1 \in \mathscr{S}^*$. It follows from $x \in \cup\mathscr{S}^* = S^*$ and $x \notin S_1$ that $S_1 \neq S^*$, which contradicts the uniqueness of the optimal solution S^*. The inequality $\ell(y) \leq u(x)$ for some $y \in X \backslash (\cap\mathscr{S}^*)$ will be established in a more general case below (see (ii)).

Now, suppose $|\mathscr{S}^*| \geq 2$. Two cases are possible: either (i) $S_1 \backslash (\cap\mathscr{S}^*) = \varnothing$, or (ii) $S_1 \backslash (\cap\mathscr{S}^*) \neq \varnothing$ (this includes (5.4) as a particular case).

(i) In this case $S_1 \subset \cap\mathscr{S}^*$, and so, $S_1 \subset S^*$ for all $S^* \in \mathscr{S}^*$. As above, we have $f(S^*) \leq f(S_1) \leq f(S^*)$, implying $f(S_1) = f(S^*) = f^*$ and $S_1 \in \mathscr{S}^*$. Since $x \notin S_1$, then $x \notin \cap\mathscr{S}^*$, but $x \in \cup\mathscr{S}^*$, and so, $x \in (\cup\mathscr{S}^*) \backslash (\cap\mathscr{S}^*)$. Setting $y = x$, by Lemma 6(a), we find

$$\ell(y) = \ell(x) = 0 = u(x).$$

(ii) If $S_1 \backslash (\cap\mathscr{S}^*) \neq \varnothing$, we fix a $y \in S_1 \backslash (\cap\mathscr{S}^*)$. Then $y \in S_1$ and $y \notin \cap\mathscr{S}^*$, and so, $S_1 \in \mathscr{S}_+(y)$. By Lemma 2(b) we conclude that

$$\ell(y) = \min_{\mathscr{S}_+(y)} f - f^* \leq f(S_1) - f^* = u(x).$$

This completes the proof of the left hand side inequality.

Now we prove the right hand side inequality. It suffices to show that for each $y \in X \backslash [(\cap\mathscr{S}^*) \cup (\cup\mathscr{S}_0)]$ there exists $x \in \cup\mathscr{S}^*$ such that $u(x) \leq \ell(y)$.

Let y exposed above be arbitrarily fixed. Since $y \notin \cap\mathscr{S}^*$, by virtue of Lemma 2(b), we have $\ell(y) = f(S_2) - f^*$ for some $S_2 \in \mathscr{S}_+(y)$, i.e., $S_2 \in \mathscr{S}$ and $y \in S_2$. Now, since $y \notin \cup\mathscr{S}_0$, we have $y \notin S_0$ for all $S_0 \in \mathscr{S}_0$. Taking into account that $y \in S_2$, we find that $S_2 \notin \mathscr{S}_0$, and so, $(\cup\mathscr{S}^*) \backslash S_2 \neq \varnothing$ and it contains an element x. Since $x \notin S_2$, we have $S_2 \in \mathscr{S}_-(x)$, and since $x \in \cup\mathscr{S}^*$, by Lemma 2(a), we get

$$u(x) = \min_{\mathscr{S}_-(x)} f - f^* \leq f(S_2) - f^* = \ell(y).$$

Finally, suppose that feasible solutions from $\mathscr{S}$ are nonembedded. Then for any $S \in \mathscr{S}$ we find $(\cup\mathscr{S}^*) \backslash S = \bigcup_{S^* \in \mathscr{S}^*} (S^* \backslash S) \neq \varnothing$, and so, $\mathscr{S}_0$ is empty. □

Remark 2 As it was already mentioned in Remark 1, if $|\mathscr{S}^*| \geq 2$, then

$$\min_{y \in X \backslash (\cap\mathscr{S}^*)} \ell(y) = 0 = \min_{x \in \cup\mathscr{S}^*} u(x);$$

however, the right hand side inequality in Theorem 2 may be strict as Example 2 below shows. Thus, Theorem 2 is of main interest when we have only one optimal solution: if $\mathscr{S}^* = \{S^*\}$, then

$$\min_{y \in X \backslash S^*} \ell(y) \leq \min_{x \in S^*} u(x) \leq \min_{y \in X \backslash [S^* \cup (\cup\mathscr{S}_0)]} \ell(y), \tag{5.5}$$

where $\mathscr{S}_0 = \mathscr{S}_0(S^*) = \{S_0 \in \mathscr{S} : S^* \subset S_0\}$.

Inequalities in Theorem 2 may be strict as the following examples show. In Examples 1 and 2 below we set $X = \{x_1, x_2, x_3, x_4\}$.

Example 1 Let $C(x_1) = C(x_2) = 1$, $C(x_3) = a > 1$, $C(x_4) = b > 2$ and $\mathscr{S} = \{S_1, S_2, S_3, S_4\}$, where $S_1 = \{x_1\}$, $S_2 = \{x_1, x_2\}$, $S_3 = \{x_1, x_3\}$ and $S_4 = \{x_4\}$. Clearly, the COP $(X, C, \mathscr{S}, f)$ is canonical having the unique optimal solution $S^* = S_1$ with the optimal value $f^* = f(S_1) = 1$. By virtue of Lemmas 2 and 6, the global upper and lower tolerances of elements from X are given as follows: $u(x_1) = f(S_4) - 1 = b - 1$, $u(x_2) = u(x_3) = u(x_4) = +\infty$, $\ell(x_1) = +\infty$, $\ell(x_2) = f(S_2) - 1 = 1$, $\ell(x_3) = f(S_3) - 1 = a$ and $\ell(x_4) = f(S_4) - 1 = b - 1$. It follows that $\cap\mathscr{S}^* = S^* = \cup\mathscr{S}^*$ and

$$\min_{y\in X\setminus S^*} \ell(y) = \min_{i=2,3,4} \ell(x_i) = \ell(x_2) = 1 < b - 1 = u(x_1) = \min_{x\in S^*} u(x),$$

and so, the left hand side inequality in Theorem 2 is strict. Note that $\mathscr{S}_0 = \{S_1, S_2, S_3\}$ and $\cup\mathscr{S}_0 = \{x_1, x_2, x_3\}$, which shows that on the right hand side we have the equality:

$$\min_{x\in S^*} u(x) = u(x_1) = b - 1 = \ell(x_4) = \min_{y\in X\setminus(\cup\mathscr{S}_0)} \ell(y).$$

Moreover, since $S_1 \subset S_2 \cap S_3$, some feasible solutions from $\mathscr{S}$ are embedded into each other.

Example 2 Let $C(x_1) = C(x_2) = C(x_3) = 1$, $C(x_4) = b > 2$ and $\mathscr{S} = \{S_1, S_2, S_3, S_4\}$, where $S_1 = \{x_1, x_2\}$, $S_2 = \{x_2, x_3\}$, $S_3 = \{x_1, x_2, x_3\}$ and $S_4 = \{x_4\}$. The corresponding COP is canonical with the set of optimal solutions $\mathscr{S}^* = \{S_1, S_2\}$ and the optimal value $f^* = f(S_1) = f(S_2) = 2$. Again by Lemmas 2 and 6, we have: $u(x_1) = 0$, $u(x_2) = f(S_4) - 2 = b - 2$, $u(x_3) = 0$, $u(x_4) = +\infty$, $\ell(x_1) = 0$, $\ell(x_2) = +\infty$, $\ell(x_3) = 0$ and $\ell(x_4) = f(S_4) - 2 = b - 2$. We find $S_1 \cup S_2 = S_3$, $S_1 \cap S_2 = \{x_2\}$, $\mathscr{S}_0 = \{S_3\}$, $\cup\mathscr{S}_0 = S_3$, and so,

$$\min_{x\in S_1\cup S_2} u(x) = \min_{i=1,2,3} u(x_i) = \min\{0, b-2, 0\} = 0 < b - 2 = \ell(x_4) = \min_{y\in X\setminus S_3} \ell(y),$$

implying that the right hand side inequality in Theorem 2 is strict. It turns out that the left hand side inequality is the equality:

$$\min_{y\in X\setminus(S_1\cap S_2)} \ell(y) = \min_{i=1,3,4} \ell(x_i) = \min\{0, 0, b-2\} = 0 = \min_{x\in S_1\cup S_2} u(x),$$

and that $\mathscr{S}$ does not satisfy the nonembeddedness condition.

Our next example shows that in the case when feasible solutions of the COP are not necessarily nonembedded the minimal values of *global* upper and lower tolerances may still be equal while minimal values of ordinary upper and lower tolerances are not.

Example 3 In this example we set $X = \{x_1, x_2, x_3\}$, $C(x_1) = 0$, $C(x_2) = 1$, $C(x_3) = 2$ and $\mathscr{S} = \{S_1, S_2, S_3\}$, where $S_1 = \{x_1, x_2\}$, $S_2 = \{x_2\}$ and $S_3 = \{x_3\}$. The corresponding COP $(X, C, \mathscr{S}, f)$ is canonical, $\mathscr{S}^* = \{S_1^*, S_2^*\}$ with $S_1^* = S_1$ and $S_2^* = S_2$ is the set of all optimal solutions of the COP and $f^* = f(S_1^*) = f(S_2^*) = 1$ is the optimal value of the COP.

First, we calculate the global tolerances. Since $\cup\mathscr{S}^* = S_1$ and $\cap\mathscr{S}^* = S_2$, Lemmas 2 and 6 imply the following values of global tolerances of elements from X: $u(x_1) = 0$, $u(x_2) = f(S_3) - 1 = 1$, $u(x_3) = +\infty$, $\ell(x_1) = 0$, $\ell(x_2) = +\infty$ and $\ell(x_3) = f(S_3) - 1 = 1$. Thus,

$$\min_{y\in X\setminus(\cap\mathscr{S}^*)} \ell(y) = \min\{\ell(x_1), \ell(x_3)\} = 0 = \min\{u(x_1), u(x_2)\} = \min_{x\in\cup\mathscr{S}^*} u(x).$$

Now, let us evaluate the tolerances with respect to the optimal solutions S_1^* and S_2^*. By Lemma 3, we have: $u_{S_1^*}(x_1) = \min\{f(S_2^*), f(S_3)\} - 1 = 0$, $u_{S_1^*}(x_2) = f(S_3) - 1 = 1$, $u_{S_1^*}(x_3) = +\infty$, $\ell_{S_1^*}(x_1) = +\infty$, $\ell_{S_1^*}(x_2) = +\infty$, $\ell_{S_1^*}(x_3) = f(S_3) - 1 = 1$, and $u_{S_2^*}(x_1) = +\infty$, $u_{S_2^*}(x_2) = f(S_3) - 1 = 1$, $u_{S_2^*}(x_3) = +\infty$, $\ell_{S_2^*}(x_1) = f(S_1^*) - 1 = 0$, $\ell_{S_2^*}(x_2) = +\infty$ and $\ell_{S_2^*}(x_3) = f(S_3) - 1 = 1$. In particular, the equalities in Lemma 4 are clearly seen.

Also, we have:

$$\min_{y \in X \setminus S_1^*} \ell_{S_1^*}(y) = \ell_{S_1^*}(x_3) = 1 > 0 = \min\{u_{S_1^*}(x_1), u_{S_1^*}(x_2)\} = \min_{x \in S_1^*} u_{S_1^*}(x)$$

and

$$\min_{y \in X \setminus S_2^*} \ell_{S_2^*}(y) = \min\{\ell_{S_2^*}(x_1), \ell_{S_2^*}(x_3)\} = 0 < 1 = u_{S_2^*}(x_2) = \min_{x \in S_2^*} u_{S_2^*}(x).$$

By virtue of Proposition 3(b), the inequalities above imply the existence of more than one optimal solution to the COP under consideration.

The last theorem of this section addresses the case when the cost function of the COP under consideration is strictly positive.

Theorem 3 *Given a canonical* COP $(X, C, \mathscr{S}, f)$ *and* $S^* \in \mathscr{S}^*$*, if the cost function* C *is strictly positive on* X*, then*

$$\min_{y \in X \setminus S^*} \ell(y) \le \min_{x \in \cup \mathscr{S}^*} u(x) \le \min_{x \in S^*} u(x) \le \min_{y \in X \setminus [(\cap \mathscr{S}^*) \cup (\cup \mathscr{S}_0)]} \ell(y)$$
$$\le \min_{y \in X \setminus [S^* \cup (\cup \mathscr{S}_0)]} \ell(y).$$

where $\mathscr{S}_0 = \mathscr{S}_0(S^*) = \{S_0 \in \mathscr{S} : S^* \subset S_0\}$.

Proof In order to prove the first inequality, we show that for each $x \in \cup \mathscr{S}^*$ there exists $y \in X \setminus S^*$ such that $\ell(y) \le u(x)$. Let $x \in \cup \mathscr{S}^*$. By Lemma 2(a), there exists $S_1 \in \mathscr{S}_-(x)$, i.e., $S_1 \in \mathscr{S}$ and $x \notin S_1$, such that $u(x) = f(S_1) - f^*$. From the strict positivity of C it follows that $S_1 = S^*$ or $S_1 \setminus S^* \neq \varnothing$ (in fact, if $S_1 \neq S^*$ and $S_1 \subset S^*$, then $f(S_1) < f(S^*)$, which contradicts the optimality of S^*). If $S_1 = S^*$, we set $y = x$, and so, condition $x \notin S_1$ implies $y = x \in (\cup \mathscr{S}^*) \setminus (\cap \mathscr{S}^*)$ and, by Lemma 6(a), $\ell(y) = \ell(x) = 0 = u(x)$. If $S_1 \setminus S^* \neq \varnothing$, we choose a $y \in S_1 \setminus S^*$, so that $y \in S_1$ and $y \notin S^*$, i.e., $S_1 \in \mathscr{S}_+(y)$ and $y \notin \cap \mathscr{S}^*$. It follows from Lemma 2(b) that

$$\ell(y) = \min_{\mathscr{S}_+(y)} f - f^* \le f(S_1) - f^* = u(x).$$

The second inequality follows from the inclusion $S^* \subset \cup \mathscr{S}^*$.

In order to prove the third inequality, we show that for each element y from the set $X \setminus [(\cap \mathscr{S}^*) \cup (\cup \mathscr{S}_0)]$ there exists $x \in S^*$ such that $u(x) \le \ell(y)$. Let y be as in the previous sentence. Since $y \notin \cap \mathscr{S}^*$, Lemma 2(b) implies $\ell(y) = f(S_2) - f^*$ for some $S_2 \in \mathscr{S}_+(y)$, i.e., $S_2 \in \mathscr{S}$ and $y \in S_2$. On the other hand, since $y \notin \cup \mathscr{S}_0$, then $y \notin S_0$ for all $S_0 \in \mathscr{S}_0$. Thus, the inclusion $y \in S_2$ implies $S_2 \notin \mathscr{S}_0$, and so, $S^* \setminus S_2 \neq \varnothing$. Choosing an $x \in S^* \setminus S_2$, we find $x \in S^*$ and $x \notin S_2$, i.e., $x \in \cup \mathscr{S}^*$ and $S_2 \in \mathscr{S}_-(x)$. Now, Lemma 2(a) yields

$$u(x) = \min_{\mathscr{S}_-(x)} f - f^* \le f(S_2) - f^* = \ell(y).$$

The last inequality is a consequence of the inclusion $\cap \mathscr{S}^* \subset S^*$. □

6 'Reduction' to nonembedded feasible solutions

In this section we are going to show that any canonical COP $(X, C, \mathscr{S}, f)$ can be treated as an 'almost equivalent' COP $(\widetilde{X}, \widetilde{C}, \widetilde{\mathscr{S}}, \widetilde{f})$, whose feasible solutions from $\widetilde{\mathscr{S}} \subset \mathscr{S}$ are nonembedded. This is particularly important when the cost function C is strictly positive. We will show that for the 'tilde' COP the global upper and lower tolerances are not less than those for the original COP.

We denote by $\mathscr{S}_c$ the set of those elements from $\mathscr{S}$, which cover some other elements from $\mathscr{S}$ (the subscript 'c' stands for 'cover'):

$$\mathscr{S}_c = \big\{S_c \in \mathscr{S} : S_c \supset S \text{ for some } S \in \mathscr{S},\ S \neq S_c\big\},$$

and we assume that $\mathscr{S}_c$ is nonempty. The complement of $\mathscr{S}_c$ in $\mathscr{S}$ is denoted by $\widetilde{\mathscr{S}}$:

$$\widetilde{\mathscr{S}} = \mathscr{S} \backslash \mathscr{S}_c = \big\{\widetilde{S} \in \mathscr{S} : S \backslash \widetilde{S} \neq \varnothing \text{ for all } S \in \mathscr{S},\ S \neq \widetilde{S}\big\};$$

note that sets from $\widetilde{\mathscr{S}}$ do not cover any other sets from $\mathscr{S}$.

Clearly, $\mathscr{S}_c \cup \widetilde{\mathscr{S}} = \mathscr{S}$ and $\mathscr{S}_c \cap \widetilde{\mathscr{S}} = \varnothing$.

We set $\widetilde{X} = \cup \widetilde{\mathscr{S}}$, and so, $\widetilde{X} \subset X$. Thus, $\widetilde{\mathscr{S}} \subset 2^{\widetilde{X}}$ is a nonempty collection of nonempty subsets of $\widetilde{X}$. Since $\mathscr{S}_c \neq \varnothing$ and $|\mathscr{S}| = |\mathscr{S}_c| + |\widetilde{\mathscr{S}}|$, we have $|\widetilde{\mathscr{S}}| < |\mathscr{S}|$. The 'tilde' cost function $\widetilde{C} : \widetilde{X} \to \mathbb{R}$, given by $\widetilde{C}(x) = C(x)$ for all $x \in \widetilde{X}$, is the restriction of C from X to $\widetilde{X}$, and so, we keep the notation C for it as well. Also, the 'tilde' objective function $\widetilde{f} = \widetilde{f}_C : \widetilde{\mathscr{S}} \to \mathbb{R}$, given by $\widetilde{f}(\widetilde{S}) = f(\widetilde{S})$ for all $\widetilde{S} \in \widetilde{\mathscr{S}}$, is the restriction of f from $\mathscr{S}$ to $\widetilde{\mathscr{S}}$, and so, we keep the notation f for it as well.

Now, we study the properties of the 'tilde' COP $(\widetilde{X}, C, \widetilde{\mathscr{S}}, f)$.

The key observation in establishing that the 'tilde' COP is canonical (and some other properties) is the following

Lemma 7 *For each $S_c \in \mathscr{S}_c$ there exists $\widetilde{S} \in \widetilde{\mathscr{S}}$ such that $\widetilde{S} \subset S_c$.*

Proof Let $S_c \in \mathscr{S}_c$. By the definition of $\mathscr{S}_c$, there exists $S_1 \in \mathscr{S}$ such that $S_1 \subset S_c$ and $S_1 \neq S_c$. If $S_1 \in \widetilde{\mathscr{S}}$, we are through. Now, let $S_1 \notin \widetilde{\mathscr{S}}$. Then $S_1 \in \mathscr{S}_c$, and so, as above, there exists $S_2 \in \mathscr{S}$ such that $S_2 \subset S_1$ and $S_2 \neq S_1$. Again, if $S_2 \in \widetilde{\mathscr{S}}$, then we are through, and if $S_2 \notin \widetilde{\mathscr{S}}$, then $S_2 \in \mathscr{S}_c$, and so, there exists $S_3 \in \mathscr{S}$ such that $S_3 \subset S_2$ and $S_3 \neq S_2$. We proceed this way further on constructing decreasing sets from $\mathscr{S}$. If for some step we have found $\widetilde{S} \in \widetilde{\mathscr{S}}$ such that $\widetilde{S} \subset \cdots \subset S_2 \subset S_1 \subset S_c$, then we are through. Otherwise, since $\mathscr{S}_c$ is a finite set, on a k-th step we end up with a one-point set $S_k = \{x\} \in \mathscr{S}$ such that $S_k \subset S_{k-1} \subset \cdots \subset S_1 \subset S_c$. It is clear that $S_k \in \widetilde{\mathscr{S}}$, which completes the proof. □

We assert that the COP $(\widetilde{X}, C, \widetilde{\mathscr{S}}, f)$ is canonical: in fact, $\cup \widetilde{\mathscr{S}} = \widetilde{X}$ (by the definition); also, by Lemma 7, for each $S_c \in \mathscr{S}_c$ there exists $\widetilde{S} \in \widetilde{\mathscr{S}}$ such that $\widetilde{S} \subset S_c$, and so, $\cap \widetilde{\mathscr{S}} \subset \widetilde{S} \subset S_c$, implying $\cap \widetilde{\mathscr{S}} \subset \cap \mathscr{S}_c$ and

$$\cap \widetilde{\mathscr{S}} = (\cap \mathscr{S}_c) \cap (\cap \widetilde{\mathscr{S}}) = \cap \mathscr{S} = \varnothing.$$

Clearly, feasible solutions from $\widetilde{\mathscr{S}}$ are nonembedded.

We denote by $(\widetilde{\mathscr{S}})^*$ the set of all optimal solutions of the 'tilde' COP, i.e.,

$$(\widetilde{\mathscr{S}})^* = \left\{\widetilde{S} \in \widetilde{\mathscr{S}} : \widetilde{f}^* = f(\widetilde{S}) = \min_{\widetilde{\mathscr{S}}} f\right\},$$

where $\widetilde{f}^*$ is the optimal value of the 'tilde' COP. The interrelations of optimal solutions are contained in the following

Lemma 8 (a) *If* $S^* \in \mathscr{S}^*$ *is unique, then* $S^* \in (\widetilde{\mathscr{S}})^*$.
(b) $(\widetilde{\mathscr{S}})^* = \mathscr{S}^* \cap \widetilde{\mathscr{S}}$.
(c) $\widetilde{f}^* = f^*$.

Proof (a) First, we show that $S^* \in \widetilde{\mathscr{S}}$. On the contrary, assume that $S^* \in \mathscr{S}_c$. Then there exists $S \in \mathscr{S}$ such that $S \subset S^*$ and $S \neq S^*$. Since $C \geq 0$, then (as in the proof of Theorem 2) $S \in \mathscr{S}^*$, which contradicts the uniqueness of S^*. Now, for any $\widetilde{S} \in \widetilde{\mathscr{S}}$ we have $\widetilde{S} \in \mathscr{S}$, and so, by (2.1), $f(S^*) \leq f(\widetilde{S})$, i.e., $S^* \in (\widetilde{\mathscr{S}})^*$.
(b) ($\subset$) Let $\widetilde{S}^* \in (\widetilde{\mathscr{S}})^*$, i.e., $\widetilde{S}^* \in \widetilde{\mathscr{S}}$ and $f(\widetilde{S}^*) \leq f(\widetilde{S})$ for all $\widetilde{S} \in \widetilde{\mathscr{S}}$. Now, let $S \in \mathscr{S} = \widetilde{\mathscr{S}} \cup \mathscr{S}_c$ be arbitrary. It suffices to verify the inequality $f(\widetilde{S}^*) \leq f(S)$ only for $S \in \mathscr{S}_c$. For such an S, by Lemma 7, there exists $\widetilde{S} \in \widetilde{\mathscr{S}}$ such that $\widetilde{S} \subset S$, and so, $f(\widetilde{S}^*) \leq f(\widetilde{S}) \leq f(S)$. Thus, $\widetilde{S}^* \in \mathscr{S}^*$, implying $\widetilde{S}^* \in \mathscr{S}^* \cap \widetilde{\mathscr{S}}$.
(b) ($\supset$) If $S^* \in \mathscr{S}^* \cap \widetilde{\mathscr{S}}$, then $S^* \in \mathscr{S}^*$ and $S^* \in \widetilde{\mathscr{S}}$, and so, for any $\widetilde{S} \in \widetilde{\mathscr{S}}$ we have $\widetilde{S} \in \mathscr{S}$ and $f(S^*) \leq f(\widetilde{S})$, i.e., $S^* \in (\widetilde{\mathscr{S}})^*$.
(c) Since $\widetilde{f}^* = \min_{\widetilde{\mathscr{S}}} f$, $f^* = \min_{\mathscr{S}} f$ and $\widetilde{\mathscr{S}} \subset \mathscr{S}$, then $f^* \leq \widetilde{f}^*$. On the other hand, the arguments in (b) ($\subset$) show that if $\widetilde{S}^* \in (\widetilde{\mathscr{S}})^*$, i.e., $\widetilde{f}^* = f(\widetilde{S}^*)$, then $f(\widetilde{S}^*) \leq f(S)$ for all $S \in \mathscr{S}$, and so, $\widetilde{f}^* \leq f^*$. □

Finally, we establish the relationships between the global tolerances of the original COP and the'tilde' COP.

By Lemma 8(b), we find

$$\cup(\widetilde{\mathscr{S}})^* = \cup(\mathscr{S}^* \cap \widetilde{\mathscr{S}}) \subset \cup\mathscr{S}^* \quad \text{and} \quad \cap(\widetilde{\mathscr{S}})^* = \cap(\mathscr{S}^* \cap \widetilde{\mathscr{S}}) \supset \cap\mathscr{S}^*.$$

If $x \in \cup(\mathscr{S}^* \cap \widetilde{\mathscr{S}})$, then, taking into account that $(\widetilde{\mathscr{S}})_-(x) = \mathscr{S}_-(x) \cap \widetilde{\mathscr{S}}$, by Lemmas 2(a) and 8(c), we have for the global upper tolerance $\widetilde{u}(x)$ with respect to the COP $(\widetilde{X}, C, \widetilde{\mathscr{S}}, f)$:

$$\widetilde{u}(x) = \min_{(\widetilde{\mathscr{S}})_-(x)} \widetilde{f} - \widetilde{f}^* = \min_{\mathscr{S}_-(x) \cap \widetilde{\mathscr{S}}} f - f^* \quad \text{and} \quad u(x) = \min_{\mathscr{S}_-(x)} f - f^*.$$

In a similar manner, if $x \in \widetilde{X} \backslash (\cap(\mathscr{S}^* \cap \widetilde{\mathscr{S}}))$, then $(\widetilde{\mathscr{S}})_+(x) = \mathscr{S}_+(x) \cap \widetilde{\mathscr{S}}$ and

$$\widetilde{\ell}(x) = \min_{(\widetilde{\mathscr{S}})_+(x)} \widetilde{f} - \widetilde{f}^* = \min_{\mathscr{S}_+(x) \cap \widetilde{\mathscr{S}}} f - f^* \quad \text{and} \quad \ell(x) = \min_{\mathscr{S}_+(x)} f - f^*.$$

Thus, we have shown that

$$u(x) \leq \widetilde{u}(x) \quad \text{and} \quad \ell(x) \leq \widetilde{\ell}(x) \quad \text{for all} \quad x \in \widetilde{X}.$$

A simple remark is that if $x \in \cup(\widetilde{\mathscr{S}})^*$ is such that $\mathscr{S}_-(x) \subset \widetilde{\mathscr{S}}$, then $u(x) = \widetilde{u}(x)$, and similarly for $\ell(x) = \widetilde{\ell}(x)$ if $\mathscr{S}_+(x) \subset \widetilde{\mathscr{S}}$.

In a particular case when $C(x) > 0$ for all $x \in X$, we have $\mathscr{S}^* \subset \widetilde{\mathscr{S}}$ (and so, $(\widetilde{\mathscr{S}})^* = \mathscr{S}^*$). In fact, if (on the contrary) $S^* \in \mathscr{S}_c$ for some $S^* \in \mathscr{S}^*$, then there exists $S \in \mathscr{S}$, $S \neq S^*$, such that $S \subset S^*$, and so, $f(S) < f(S^*)$, which contradicts the optimality of S^*.

7 Maximal values of upper and lower tolerances

The case of maximal values of global upper and lower tolerances is more intricate and involved. However, we have the following partial result (cf. Theorem 4). In the next theorem we employ notations introduced before Lemma 2.

Theorem 4 *Let the* COP $(X, C, \mathscr{S}, f)$ *be canonical such that its feasible solutions from* $\mathscr{S}$ *are nonembedded, and let* $S^* \in \mathscr{S}^*$ *be the unique optimal solution of the* COP. *We have*:

(a) *if* $\left(\bigcup_{x\in S^*}\left(\cup[\mathscr{S}_-(x)]^*\right)\right)\backslash S^* = X\backslash S^*$, *then* $\max_{y\in X\backslash S^*} \ell(y) \le \max_{x\in S^*} u(x)$;

(b) *if* $S^* \subset \bigcup_{y\in X\backslash S^*}\left(X\backslash(\cap[\mathscr{S}_+(y)]^*)\right)$, *then* $\max_{x\in S^*} u(x) \le \max_{y\in X\backslash S^*} \ell(y)$.

Proof (a) It suffices to show that for each element $y \in X\backslash S^*$ there exists $x \in S^*$ such that $\ell(y) \le u(x)$. Let $y \in X\backslash S^*$. By the assumption, there exists $x \in S^*$ such that $y \in \left(\cup[\mathscr{S}_-(x)]^*\right)\backslash S^*$ (the set on the right is nonempty due to the nonembeddedness of feasible solutions $\mathscr{S}$), and so, there is $S_1 \in [\mathscr{S}_-(x)]^*$ such that $y \in S_1$ and $y \notin S^*$. Since $S_1 \in \mathscr{S}_+(y)$, it follows from Lemma 2(a), (b) that

$$
\begin{aligned}
u(x) &= f([\mathscr{S}_-(x)]^*) - f(\mathscr{S}^*) = \min_{\mathscr{S}_-(x)} f - f^* = f(S_1) - f^* \\
&\ge \min_{\mathscr{S}_+(y)} f - f^* = f([\mathscr{S}_+(y)]^*) - f(\mathscr{S}^*) = \ell(y).
\end{aligned}
$$

(b) It suffices to establish that for each element $x \in S^*$ there exists $y \in X\backslash S^*$ such that $u(x) \le \ell(y)$. Let $x \in S^*$. By the assumption, there exists $y \in X\backslash S^*$ such that $x \notin \cap[\mathscr{S}_+(y)]^*$, and so, there is $S_2 \in [\mathscr{S}_+(y)]^*$ such that $x \notin S_2$, i.e., $S_2 \in \mathscr{S}_-(x)$. Now, it follows from Lemma 2(a), (b) that

$$
\begin{aligned}
\ell(y) &= f([\mathscr{S}_+(y)]^*) - f(\mathscr{S}^*) = \min_{\mathscr{S}_+(y)} f - f^* = f(S_2) - f^* \\
&\ge \min_{\mathscr{S}_-(x)} f - f^* = f([\mathscr{S}_-(x)]^*) - f(\mathscr{S}^*) = u(x).
\end{aligned}
$$

This completes the proof of Theorem 4. □

8 Conclusion

In this paper the theory of *global* tolerances of the ground set elements is developed and the relationship with the commonly known tolerances is clarified. We show that it suffices to study the global tolerances only for canonical combinatorial optimization problems. For such problems (in)equalities for the minimal and maximal values of global upper tolerances and global lower tolerances are established under the assumption of nonembeddedness of feasible solutions as well as without it. We prove a new criterion characterizing the uniqueness of optimal solution of the combinatorial optimization problem under consideration: the optimal solution is unique if and only if for some optimal solution the upper and lower tolerances are global ones. Examples are presented illustrating the sharpness of our results.

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