Modelling dependence between Lévy processes

Vladimir Panov
Laboratory for Structural Methods of Data Analysis in Predictive Modeling (MIPT), University Duisburg-Essen
vladimir.panov@uni-due.de

Abstract
In this paper, we introduce a principally new method for modelling the dependence structure between two Lévy processes. The proposed method is based on some special properties of the time-changed Lévy processes and can be viewed as a reasonable alternative to the copula approach.

1. Introduction

Copula is with no doubt the most popular tool for modelling the dependence structure between two random variables. The theoretical base for this is provided by Sklar theorem, which states that for any two random variables \( Y_1 \) and \( Y_2 \) there exists a copula \( C \) (a two-dimensional distribution function with domain \( [0, 1]^2 \) and uniform margins) such that

\[
P\{Y_1 \leq u_1, Y_2 \leq u_2\} = C(P\{Y_1 \leq u_1\}, P\{Y_2 \leq u_2\}),
\]

for any \( u_1, u_2 \geq 0 \). We refer to [14], [18] for a comprehensive overview of the copula theory.

Nevertheless, the direct application of this approach to stochastic processes meets several problems. For example, consider a two-dimensional Lévy process \( \tilde{X}(t) = (X_1(t), X_2(t)) \) and assume that for a fixed time \( t \) the dependence between components is described by some copula \( C_t \), i.e.,

\[
P\{X_1(t) \leq u_1, X_2(t) \leq u_2\} = C_t(P\{X_1(t) \leq u_1\}, P\{X_2(t) \leq u_2\}),
\]

for any \( u_1, u_2 \geq 0 \). Such description of the dependence has at least one important drawback - the Lévy copula \( C_t \) typically depends on \( t \), see [24] for examples and [13] for the most recent results. Moreover, the class \( C_t \) depends on the class of marginal laws \( X_i(t) \) since the distribution of \( \tilde{X}(t) \) is infinitely divisible. This leads to the ambitious aim, which is formulated in [15] as follows:

“To characterize the dependence structure of a Lévy process in a time-independent fashion.”

In this paper, we discuss different ways of such characterization. The first way - introducing the so-called Lévy copulas is quite popular but the implementation of this approach is challenging. We discuss this issue in the next section. Afterwards we present the notion of stochastic time change in Lévy processes and expand the methodology to the two-dimensional case. Finally, we describe the algorithm for modeling dependence between Lévy processes and discuss some issues concerning this new approach.

2. Lévy copulas

The first way of solving the difficulties mentioned in the introduction is to define the so-called Lévy copula. The two-dimensional Lévy copula is defined as a function from \( \mathbb{R}^2 \) to \( \mathbb{R} \) such that

1. \( F(u_1, u_2) = 0 \) if \( u_i = 0 \) for at least one \( i = 1, 2 \).
2. \( F(u_1, u_2) \neq \infty \) for \( (u_1, u_2) \neq (\infty, \infty) \).
3. \( F \) is 2-increasing.
4. \( F(u, \infty) = F(\infty, u) = u \).

This notion was introduced in [23], and then studied in [2], [11], [15]. Similar to probabilistic copulas, a Lévy copula describes the dependence structure of a multivariate Lévy process.
It is worth mentioning that the Lévy copula can be easily written in the case when the one-dimensional Lévy measures are infinite and have no atoms. In this case, the construction for the Lévy copula uses the so-called tail integrals, which are defined for a one-dimensional Lévy process \( L \) with Lévy measure \( \nu \) as

\[
U_Z(x) := (-1)\text{sign}(x) \nu_Z(I(x)),
\]

where \( I(x) = (x, +\infty) \) if \( x > 0 \) and \( I(x) = (-\infty, x) \) if \( x < 0 \). The reason for this definition is that in the case of infinite measure \( \nu_Z \), \( U_Z(A) = \infty \) for any set \( A \) which contains 0. Analogously, for a two-dimensional process \( \tilde{Z} \) with Lévy measure \( \nu_{\tilde{Z}} \),

\[
U_{\tilde{Z}}(x_1, x_2) := (-1)^{\text{sign}(x_1) + \text{sign}(x_2)} \nu_{\tilde{Z}}(I(x_1) \times I(x_2)),
\]

and this definition is also correct for any real \( x_1 \) and \( x_2 \).

Returning to Lévy copulas, note that

\[
F(u_1, u_2) = U\left(U_1^{-1}(u_1), U_2^{-1}(u_2)\right), \tag{3}
\]

where \( U, U_1 \) and \( U_2 \) are the tail integrals of the Lévy measures of the processes \( \tilde{X}(t), X_1(t), X_2(t) \) resp. [15].

Despite the compactness of the formula (3), the practical implementation in the general case is quite intricate. The point is that the formula (3) holds only in the case when the Lévy measures are infinite. In this case, Lévy measure is infinite in the vicinity of zero, and the tail integrals \( U, U_1 \) and \( U_2 \) should be estimated in each quadrant separately. Therefore, estimation of the Lévy copula requires solving four separate tasks.

The last observation motivates the new method for describing the dependence between two Lévy processes, which is the objective of this research. This method is based on the so-called time-changed Lévy models, which we introduce in the next section.

3. Time-changed Lévy models

In a one-dimensional case, the time-changed Lévy process is defined as

\[
Y_s = L(\mathcal{T}(s)), \tag{4}
\]

where \( L \) is a Lévy process, and \( \mathcal{T}(s) \) - a non-negative, non-decreasing stochastic process with \( \mathcal{T}(0) = 0 \). The economical interpretation of the time change is based on the idea that the “business” time \( \mathcal{T}(s) \) may run faster than the physical time in some periods, for instance, when the amount of transactions is high, see [1] and [25].

On the other side, this class of models has mathematical background based on the so-called Monroe theorem [17], which stands that any semimartingale can be represented as a time-changed Brownian motion and vice versa, any time-changed Brownian motion is a semimartingale. Various aspects of this theory are discussed in [4] and [9]. Nevertheless, the first part of the Monroe theorem doesn’t hold if one introduces any of the following additional assumptions:

1. Processes \( W \) and \( \mathcal{T} \) are independent. This assumption is widely used in the statistical literature and is quite convenient for both theoretical and practical purposes.

2. Time change process \( \mathcal{T} \) is itself a Lévy process; such processes are known as subordinators. In this case any resulting process \( Y_s \) is also a Lévy process, which is usually called a subordinated process.

These drawbacks of the time-changed Brownian motion lead to the idea of considering more general model (4) with any Lévy process instead of the Brownian motion and introducing the assumption that the processes \( \mathcal{T} \) and \( L \) are independent. This model has been attracting attention of many researches, see, e.g., [6], [7], [8], [10], [21].

Nevertheless, there is no clear understanding in the literature how to extend this model to a two-dimensional case. The most popular construction is to consider the model (4) with a two-dimensional Lévy process \( \tilde{L} \) and to provide a time change in each component with the same time change process \( \mathcal{T} \) [20].

Interestingly enough, in the case when \( \tilde{L} \) is a Brownian motion, the correlation coefficient between subordinated processes is bounded by the correlation coefficient between the components of the Brownian motion, see [12] (moreover, these coefficients coincides in some cases, see [11]). Therefore, the question that naturally arises in this context is how to determine the model, which is able to reproduce not only the correlation but also the dependence between components. The answer is given in the next section.
4. Two-dimensional subordinated processes

In the sequel we will be mainly interested in the following two-dimensional generalization of the model (4). Introduce a two-dimensional subordinator \( \tilde{T}(s) = (T_1(s), T_2(s)) \) - a Lévy process such that each coordinate of \( \tilde{T}(s) \) is a subordinator. An interesting question that immediately arises is how to construct such processes. This issue is quite sophisticated and merits separate publication; at the moment we only mention that the process in the form \( (Z_1(s) + \alpha_1 Z_3(s), Z_2(s) + \alpha_2 Z_3(s)) \) with independent one-dimensional subordinators \( Z_1, Z_2, Z_3 \) and positive \( \alpha_1, \alpha_2 \) is a two-dimensional subordinator, see [22].

Consider now a two-dimensional Lévy process \( \tilde{L}(t) = (L_1(t), L_2(t)) \) with independent components and a two-dimensional subordinator \( \mathcal{F}(s) = (\mathcal{R}_1(s), \mathcal{R}_2(s)) \) such that \( \mathcal{R}_i(s) \) is independent of \( L_i(s) \) for any \( i = 1, 2 \). Define the subordinated process by composition

\[
\tilde{X}(s) = (X_1(s), X_2(s)) := \left( L_1(\mathcal{R}_1(s)), L_2(\mathcal{R}_2(s)) \right).
\]

This construction, known as multivariate subordination, was introduced by Ole E. Barndorff-Nielsen, Jan Pederson and Ken-Iti Sato in [3]. The next theorem sheds some light to the characteristics of such processes.

**Theorem 4.1.** Let \( B_i, i = 1, 2 \) be two independent Brownian motions and let \( \mathcal{F}_i, i = 1, 2 \) be two (dependent) subordinators.

- Then the process

\[
\tilde{X}(s) := \left( B_1(\mathcal{R}_1(s)), B_2(\mathcal{R}_2(s)) \right)
\]

is a two-dimensional Lévy process.

- Denote the Lévy triplets of the process \( T_i \) by \( (\rho_i, 0, \lambda_i) \), where \( \rho_i \) is a drift and \( \lambda_i \) - a Lévy measure of the process \( T_i \). In this notation, the Lévy triplet of the process \( \tilde{X} \) is equal to

\[
\left( 0, \text{diag}(\rho_1, \rho_2), v \right),
\]

where \( \text{diag}(\rho_1, \rho_2) \) is a diagonal matrix with the values \( \rho_1 \) and \( \rho_2 \) on the diagonal, and the measure \( v \) is defined as

\[
v(B) := \int_{\mathbb{R}^2} Q_\text{diag}(\rho_1, \rho_2) (B) v(dy_1, dy_2), \quad B \subset \mathbb{R}^2,
\]

where by \( Q_A \) with a matrix \( A \) we denote the normal distribution with mean \( 0 \) and covariance matrix \( A \).

5. Reproducing the dependence structure

Assume that the observations of the process \( X_i \) and the time points \( \Delta, 2\Delta, \ldots, n\Delta \) are given for some fixed \( \Delta > 0 \) (low-frequency setup, see [19]). The aim is to construct a two-dimensional subordinator \( \mathcal{F}(s) \) such that (6) holds with this \( \mathcal{F}(s) \) and some independent Brownian motions \( B_i(t), i = 1, 2 \).

The algorithm consists in the following steps:

1. Estimation of the tail integrals \( U, U_1 \) and \( U_2 \) of the processes \( \tilde{X}, X_1 \) and \( X_2 \) resp. by reconstruction the Lévy measure from the low-frequency observations, see [5].

2. Estimation of the tail integrals \( U^*, U_1^* \) and \( U_2^* \) of the processes \( \tilde{F}, \mathcal{F}_1 \) and \( \mathcal{F}_2 \) resp. by reconstruction the Lévy triplet of the subordinator from the Lévy triplet of the subordinated process, see Theorem 4.1 and [9].

3. Estimation of the Lévy copula \( F^* \) of the process \( \mathcal{F} \) by the formula (3).

6. Discussion

The main idea of the proposed algorithm is to use the construction (6) to reproduce the dependence structure of the Lévy process \( \tilde{X} \). More precisely, we study the relationship between the Lévy copula \( F \) of the subordinated process \( \tilde{X} \) and the Lévy copula \( F^* \) of the subordinator \( \mathcal{F}(s) \).

The trick presented above allows to avoid the difficulties arising by estimation of the general Lévy copula - in fact, only estimation (and simulation) of the positive copula is used. Moreover, the proposed method corresponds to the general philosophy of the time-changed models.

Returning now to the representation (3), we arrive at an interesting observation that the Lévy copula of the time-changed process \( \tilde{X} \) can be represented in this form if the process \( \tilde{L} \) is a Brownian motion and the Lévy measures of the one-dimensional subordinators \( \mathcal{R}(s) \) are infinite for \( i = 1, 2 \). This fact allows to estimate the Lévy copulas of the subordinated processes for a large class of subordinators.
To the best of our knowledge, there were only few efforts to combine the idea of stochastic time change with the notion of Lévy copulas. For instance, Paul Kettler [16] provides some simulations for the Ornstein-Uhlenbeck processes. In this respect, this research is the first contribution to this theory.

The comprehensive theoretical study of the proposed algorithm (in particular, comparison of the copulas $F$ and $F^*$) is the topic of the future research.

7. Acknowledgments

The author is partially supported by Laboratory for Structural Methods of Data Analysis in Predictive Modeling, MIPT, RF government grant, ag. 11.G34.31.0073 and by the Deutsche Forschungsgemeinschaft through the SFB 823 “Statistical modelling of nonlinear dynamic processes”.

References


