HIGHER MATRIX-TREE THEOREMS

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Abstract. In this note we calculate determinants of weighted sums of reflections and of (nested) commutators of reflections. The results obtained generalize the matrix-tree theorem [3] and the results of Massbaum and Vaintrob [4].

Introduction

A famous matrix-tree theorem first proved by Kirchhoff [3] in 1847 recently attracted attention of specialists in algebraic combinatorics (see e.g. the paper [2] containing a good review of other results in the field as well, and also some generalizations in [1]). The theorem in its classical form expresses the principal minor of some $n \times n$-matrix (see L_w in Section 2.1 below) via summation over the set of trees on n numbered vertices. The matrix involved is the matrix of a weighted sum of the operators $(I - s)$ where s runs through all the reflections in the Coxeter group A_{n-1}. We prove an analog of Kirchhoff's formula for *any* system of reflections in a Euclidean space; instead of trees, our result involves summation over the set of bases made out of vectors normal to reflection hyperplanes.

In 2002 G. Massbaum and A. Vaintrob found a beautiful extension of the Kirchhoff's theorem. Their result expresses the Pfaffian of a principal minor of some skew-symmetric $(2m + 1) \times (2m + 1)$-matrix via summation over a set of 3-trees (contractible topological spaces made by gluing triangles by their vertices) with m edges. The matrix (see T in Section 2.2 below) is the weighted sum of commutants $[s_1, s_2]$ where s_1 and s_2 run through the A_{n-1} set of reflections. Again, we generalize this theorem to an arbitrary set of reflections, allowing also nested commutators of the form $[s_1, [s_2, \dots, s_k] \dots]$ for any k. This is the main result of the paper, Theorem 1.3 and its reformulation, Theorem 1.3'. The answer is given via summation over the set of "discrete one-dimensional oriented manifold with boundary", which are just directed graphs made up of chains and cycles; the weight of the graph is obtained by a sort of discrete path integration.

1. General theorems

Let V be a n-dimensional Euclidean space with an orthonormal basis $f_1, \dots, f_n$, and let $e_1, \dots, e_N \in V$ be vectors of unit length. Denote by $s_i : V \to V$ the reflection in the hyperplane normal to e_i: $s_i(v) = v - 2(e_i, v)e_i$.

Denote by $Q_e : V \to V$, where $e \stackrel{\text{def}}{=} (e_1, \dots, e_k)$, a rank 1 linear operator given by the formula

$$Q_e(v) = (v, e_1)(e_1, e_2) \dots (e_{k-1}, e_k)e_k$$

$e \mapsto Q_e$ is a $\operatorname{End}(V)$-valued quadratic function of its arguments. Fix now $e_1, \dots, e_k$, and for any permutation $\sigma \in S_k$ consider an operator $Q_{\sigma(e)} : V \to V$ where $\sigma(e) \stackrel{\text{def}}{=} (e_{\sigma(1)}, \dots, e_{\sigma(k)})$. Extend the correspondence $\sigma \mapsto Q_{\sigma(e)}$ by linearity to a quadratic map $Q : \mathbb{R}[S_k] \to \operatorname{End}(V)$ from the group algebra of the symmetric group

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S_k to the algebra of linear operators $V \to V$: if $x = \sum_{\sigma \in S_k} c_\sigma \sigma$ (where $c_\sigma \in \mathbb{R}$) then $Q(x) \stackrel{\text{def}}{=} \sum_{\sigma \in S_k} c_\sigma Q_{\sigma(e)}$.

An easy induction proves the following

Lemma 1.1. $[s_k, [s_{k-1}, [\dots, [s_2, s_1] \dots] = Q(x_k)$ *with* $x_k = -2^k (\tau_k - 1)(\tau_{k-1} - 1) \dots (\tau_2 - 1)$ *where* $\tau_\ell \stackrel{def}{=} (12 \dots \ell)$, *a permutation mapping* $1 \mapsto 2 \mapsto \dots \mapsto \ell \mapsto 1$ *and leaving all* $i > \ell$ *fixed.*

Define integers $a_k(\sigma)$ by the equation $x_k = \sum_{\sigma \in S_k} a_k(\sigma) \sigma$.

Introduce now, for all $1 \le i_1, \dots, i_k \le N$, weights $w_{i_1,\dots,i_k}$, which are elements of a commutative associative algebra. Consider the operator

$$P_w^{(k)} \stackrel{\text{def}}{=} \sum_{i_1,\dots,i_k=1}^{N} w_{i_1,\dots,i_k} [s_{i_k}, [s_{i_{k-1}}, \dots, [s_{i_2}, s_{i_1}] \dots] \tag{1}$$

By Lemma 1.1

(2)
$$P_w^{(k)} = \sum_{i_1,\dots,i_k=1}^{N} \sum_{\sigma \in S_k} a_k(\sigma) Q(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}) = \sum_{j_1,\dots,j_k=1}^{N} u_{j_1,\dots,j_k} Q(e_{j_1}, \dots, e_{j_k})$$

where $u_{j_1,\dots,j_k} \stackrel{\text{def}}{=} \sum_{\sigma \in S_k} a_k(\sigma^{-1}) w_{j_{\sigma(1)},\dots,j_{\sigma(k)}}$.

Corollary 1.2 (of Lemma 1.1)**.** $u_{j_k,\dots,j_1} = (-1)^{k-1} u_{j_1,\dots,j_k}$

Proof. According to Lemma 1.1 $u_{j_1,\dots,j_k}$ changes its sign if a permutation τ_ℓ with any $2 \le \ell \le k$ is applied to its arguments. Note now that $\tau_2 \dots \tau_k$ is a permutation of $1, \dots, k$ exchanging $1 \leftrightarrow k$, $2 \leftrightarrow k-1$, etc. □

Theorem 1.3.

$$\det P_w^{(k)} = \sum_{\substack{j^{(1)},\dots,j^{(n)}=(1,\dots,1) \\ j_k^{(1)} < \dots < j_k^{(n)}}}^{(N,\dots,N)} u_{j^{(1)}} \dots u_{j^{(n)}} \prod_{s=1}^{k-1} \prod_{t=1}^{n} (e_{j_s^{(t)}}, e_{j_{s+1}^{(t)}}) \times \operatorname{vol}(e_{j_1^{(1)}}, \dots, e_{j_1^{(n)}}) \operatorname{vol}(e_{j_k^{(1)}}, \dots, e_{j_k^{(n)}}). \tag{3}$$

Here $j^{(s)} \stackrel{def}{=} (j_1^{(s)}, \dots, j_k^{(s)})$ *is a multi-index, and* $\operatorname{vol}(b_1, \dots, b_n)$ *stands for an* n*-dimensional volume of the parallelepiped spanned by the vectors* $b_1, \dots, b_n \in V$.

Proof. Let $f_1, \dots, f_n$ be an orthonormal basis in V. Then it follows from (2) that

$$\begin{aligned}
P_w^{(k)}(f_1) \wedge \dots \wedge P_w^{(k)}(f_n) \\
&= \sum_{j^{(1)},\dots,j^{(n)}=(1,\dots,1)}^{(N,\dots,N)} u_{j^{(1)}} \dots u_{j^{(n)}} Q_{(e_{j_1^{(1)}},\dots,e_{j_k^{(1)}})}(f_1) \wedge \dots \wedge Q_{(e_{j_1^{(n)}},\dots,e_{j_k^{(n)}})}(f_n) \\
&= \sum_{j^{(1)},\dots,j^{(n)}=(1,\dots,1)}^{(N,\dots,N)} u_{j^{(1)}} \dots u_{j^{(n)}} \prod_{p=1}^{n} (e_{j_1^{(p)}}, f_p) \prod_{s=1}^{k-1} \prod_{t=1}^{n} (e_{j_s^{(t)}}, e_{j_{s+1}^{(t)}}) \\
&\qquad \times e_{j_k^{(1)}} \wedge \dots \wedge e_{j_k^{(n)}}
\end{aligned}$$

The wedge product at the end changes its sign if two multi-indices $j^{(a)}$ and $j^{(b)}$ are exchanged. Additionally, all $j_k^{(i)}$ should be distinct, else the term is zero. Thus we can restrict summation to the multi-indices such that $j_k^{(1)} < \dots < j_k^{(n)}$, and then make an additional summation over the set of permutations σ of n points:

$$\begin{aligned}
&P_w^{(k)}(f_1)\wedge\cdots\wedge P_w^{(k)}(f_n)\\
&\quad=\sum_{\substack{j^{(1)},\dots,j^{(n)}=(1,\dots,1)\\ j_k^{(1)}<\dots<j_k^{(n)}}}^{(N,\dots,N)} u_{j^{(1)}}\dots u_{j^{(n)}}\prod_{s=1}^{k-1}\prod_{t=1}^{n}(e_{j_s^{(t)}},e_{j_{s+1}^{(t)}})\\
&\quad\times\sum_{\sigma\in S_n}(-1)^\sigma\prod_{p=1}^{n}(e_{j_1^{(p)}},f_{\sigma(p)})e_{j_k^{(1)}}\wedge\cdots\wedge e_{j_k^{(n)}}\\
&\quad=\sum_{\substack{j^{(1)},\dots,j^{(n)}=(1,\dots,1)\\ j_k^{(1)}<\dots<j_k^{(n)}}}^{(N,\dots,N)} u_{j^{(1)}}\dots u_{j^{(n)}}\prod_{s=1}^{k-1}\prod_{t=1}^{n}(e_{j_s^{(t)}},e_{j_{s+1}^{(t)}})\\
&\quad\times\operatorname{vol}(e_{j_1^{(1)}},\dots,e_{j_1^{(n)}})\operatorname{vol}(e_{j_k^{(1)}},\dots,e_{j_k^{(n)}})\times f_1\wedge\cdots\wedge f_n.
\end{aligned}$$

The coefficient at $f_1\wedge\cdots\wedge f_n$ is $\det P_w^{(k)}$. □

Theorem 1.3 admits a beautiful reformulation. For every set $j^{(1)},\dots,j^{(n)}$ of multi-indices in the sum draw a graph with the vertices $1,\dots,N$ and oriented edges joining $j_1^{(i)}$ with $j_k^{(i)}$ for all $i=1,\dots,n$. The volume in the formula may be nonzero only if all its arguments are distinct; so in the graph obtained for every vertex there is at most one outgoing edge and at most one incoming edge. This means that every connected component of the graph is either an oriented cycle or an oriented chain — thus, the graph is a "discrete one-dimensional oriented manifold with boundary" (abbreviated as *DOOMB* below).

For every edge $\varepsilon_i=(j_1^{(i)},j_k^{(i)})$ of the graph consider a path (a sequence of vertices) $j^{(i)}=(j_1^{(i)},j_2^{(i)},\dots,j_k^{(i)})$. This path has a weight $u_{j^{(i)}}(e_{j_1^{(i)}},e_{j_2^{(i)}})\dots(e_{j_{k-1}^{(i)}},e_{j_k^{(i)}})$; call the k-weight of the edge ε_i the sum of weights of all the paths of length k joining its endpoints. The k-weight of the graph Γ is the product of the k-weights of its edges. Then Theorem 1.3 is equivalent to

Theorem 1.3'. *The determinant of the operator $P_w^{(k)}$ is equal to the sum of k-weights of all the DOOMBs having n edges with the vertices $1,\dots,N$, each weight multiplied by $\operatorname{vol}(e_{p_1},\dots,e_{p_n})\operatorname{vol}(e_{q_1},\dots,e_{q_n})$ where p_i and q_i are a starting vertex and a final vertex, respectively, of the i-th edge of the graph, $1\le i\le n$.*

Remark 1.4. To formulate Theorem 1.3' it is necessary to number the edges for every DOOMB involved, but the value of the corresponding term is independent of the numbering.

Formulations of Lemma 1.1 and Theorem 1.3 assume that $k\ge 2$. The propositions remain valid, nevertheless, together with their proofs, if $k=1$, if one defines $u_i\stackrel{\text{def}}{=}w_i$ for all i and $P_w^{(1)}\stackrel{\text{def}}{=}\sum_{i=1}^N w_i(I-s_i)$ (I means the identity operator). More precisely, the following is true:

Theorem 1.3 for $k=1$.

$$\det P_w^{(1)}=2^n\sum_{1\le i_1<\dots<i_n\le N} w_{i_1}\dots w_{i_n}\operatorname{vol}^2(e_{i_1},\dots,e_{i_n}). \tag{4}$$

A reflection is an orthogonal operator, and it is an involution, so it is a symmetric operator. Therefore, a nested commutator of k reflections is symmetric for k odd and skew-symmetric for k even, and the same is true for the operator $P_w^{(k)}$. Suppose that k is even. In this case $\det P_w^{(k)}$ may be nonzero only if $n \stackrel{\text{def}}{=} \dim V$ is even, and $\det P_w^{(k)} = \mathrm{Pf}^2 P_w^{(k)}$ where Pf means the Pfaffian.

Consider now the term in Theorem 1.3' corresponding to the DOOMB Γ. Take a connected component of Γ and reverse the orientation of all the edges in it obtaining a new DOOMB Γ'. It follows from Corollary 1.2 that for k even the weight of an edge changes its sign if the orientation of the edge is reversed. Therefore the contribution to the sum of Γ' equals the contribution of Γ multiplied by $(-1)^\ell$ where ℓ is the size of the component. If ℓ is odd, the contributions of Γ and Γ' cancel — therefore, for k even only DOOMBs with connected components of even size enter the sum.

Recall that a directed graph on the vertex set $1, \dots, N$ is called a directed partial pair matching if no two its edges have a common vertex. Every DOOMB with n edges all of whose connected components have even size can be uniquely decomposed into a union of two directed partial pair matchings of $n/2$ edges each. Vice versa, a union of two directed partial pair matchings is a DOOMB with connected components of even size. Thus for k even the following statement is equivalent to Theorem 1.3:

Theorem 1.3 for k even. *If k is even then a Pfaffian of $P_w^{(k)}$ is the sum of k-weights of all the directed partial pair matchings with $n/2$ edges and vertices $1, \dots, N$, multiplied by the volume of the parallelepiped spanned by the vectors e_p where p runs through all the vertices of the pair matching.*

2. The A_n case

In this section we consider the system of reflections of the Coxeter group A_n. This means $V = \{(x_0, \dots, x_n) \mid x_1 + \dots + x_n = 0\} \subset \mathbb{R}^{n+1}$, $N = n(n+1)/2$, and the vectors e_p are

$$e_{ij} \stackrel{\text{def}}{=} (f_i - f_j)/\sqrt{2}, \quad 0 \le i < j \le n, \tag{5}$$

where $f_0, \dots, f_n$ is the standard orthonormal basis in $\mathbb{R}^{n+1}$. It will be convenient to use notation (5) also if $i > j$, so that $e_{ji} = -e_{ij}$. Consider a set of points numbered $0, \dots, n$; a vector e_{ij} will be drawn then as an arrow joining points i and j. For any set $u = \{e_{i_1 j_1}, \dots, e_{i_k j_k}\}$ we will denote by $G(u)$ a graph with vertices $0, \dots, n$ and directed edges $(i_1, j_1), \dots, (i_k, j_k)$.

The scalar product in $\mathbb{R}^n$ is standard, so the scalar product of e_{ij} is given by

$$(e_{i_1,j_1}, e_{i_2,j_2}) = \begin{cases} 1, & \text{if } ; \\ \text{-1}, & \text{if } ; \\ 1/2, & \text{if } \quad \text{or} \quad ; \\ \text{-1/2}, & \text{if } ; \\ 0, & \text{in all the other cases, that is, if } \end{cases} \tag{6}$$

Lemma 2.1. *The determinant* $\mathrm{vol}(e_{i_1,j_1}, \dots, e_{i_n,j_n})$ *is zero unless the graph* $G(e_{i_1,j_1}, \dots, e_{i_n,j_n})$ *is a tree. If it is a tree then the volume is equal to* $\pm\sqrt{(n+1)/2^n}$.

Proof. Denote $u = (e_{i_1,j_1}, \dots, e_{i_n,j_n})$. If $G(u)$ is not a tree, then it contains at least one cycle. Without loss of generality the cycle is formed by the vertices $i_1, \dots, i_k$ and the edges $(i_1, i_2), \dots, (i_k, i_1)$. Then $e_{i_1 i_2} + \dots + e_{i_k i_1} = 0$, and u is linearly dependent, so that the volume is zero.

Let now $G(u)$ be a tree; the vertex 0 will serve as its root. Changing a direction of an edge $(i,j) \in G(u)$ means replacement $e_{ij} \mapsto -e_{ij}$. So, preserving $\mathrm{vol}(u)$ up to a sign one may assume that all the edges of $G(u)$ are directed from the root outwards.

Let p be a hanging vertex of $G(u)$ with the parent q and the grandparent r. Replacement of the edge (q,p) by (r,p) means replacement of the vector $\pm e_{qp} \in u$ by $e_{rp} = e_{qp} + e_{rq}$, obtaining a new set u'. Since $e_{rq} \in u$, one has $\mathrm{vol}(u') = \mathrm{vol}(u)$. Doing like this several times one obtains a tree where the root 0 is connected with the n other vertices $1, \dots, n$. The square $\gamma_n \stackrel{\mathrm{def}}{=} \mathrm{vol}^2(u)$ is the Gram determinant; by (6) for such tree γ_n is the determinant of the $n \times n$-matrix with 1s in the main diagonal, 1/2's one line above and one line below it, and zeros in the other places. Expanding the determinant along the first row one obtains $\gamma_n = \gamma_{n-1} - \frac{1}{4} \cdot \gamma_{n-2}$, hence $\gamma_n = (n+1)/2^n$ by induction. □

Remark. The procedure described in the proof allows to determine the sign of the volume as well. Namely, take the vertex number 0 as a root of the tree and relate to every edge of the tree its outer endpoint. This puts the edges into a one-to-one correspondence with the vertices $1, \dots, n$. The ordering of the vectors $e_{i_1 j_1}, \dots, e_{i_n j_n}$ determines the ordering of the vertices $1, \dots, n$, that is, a permutation of $1, \dots, n$. A direct computation shows that the sign of $\mathrm{vol}(u)$ is equal to $\varepsilon_1 \varepsilon_2$ where ε_1 is the sign of this permutation and ε_2 is the parity of the number of edges in $G(u)$ looking *towards* the root.

Here are two particular cases of Theorem 1.3 for the A_n system of roots:

2.1. $k = 1$**: a matrix-tree theorem.** In view of Lemma 2.1, Theorem 1.3' (for $k = 1$) gives for the case considered: $\det P_w^{(1)} = (n+1) \sum w_{i_1 j_1} \dots w_{i_n j_n}$ where the sum is taken over all sets $\{(i_1, j_1), \dots, (i_n, j_n)\}$ of pairs such that the corresponding graph is a tree.

Kirchhoff's (or Laplacian) matrix L_w is defined as the $(n+1) \times (n+1)$-matrix with the entries $(L_w)_{ij} = -w_{ij}$ when $i \neq j$, and $(L_w)_{ii} = \sum_{j \neq i} w_{ij}$. L_w is the matrix of the operator $-P_w^{(1)}$ in the standard basis $f_0, \dots, f_n \in \mathbb{R}^{n+1}$. The matrix L_w is degenerate: $L_w(f_0 + \dots + f_n) = 0$; therefore $L_w = L_w R$ where R is the orthogonal projection to the subspace $V \subset \mathbb{R}^{n+1}$; explicitly $R f_i = f_i - \frac{1}{n+1}(f_0 + \dots + f_n)$.

For any $i = 0, \dots, n$ define the operator M_i by $M_i(f_i) = 0$ and $M_i(f_j) = f_j$ for $j \neq i$. M_i is the orthogonal projection to the subspace $W_i \stackrel{\mathrm{def}}{=} \langle f_0, \dots, \widehat{f_i}, \dots, f_n \rangle \subset \mathbb{R}^{n+1}$. Then $M_i|_{V \to M_i} P_w^{(1)} R|_{W_i \to V}$ is the operator $W_i \to W_i$ whose matrix in the basis $f_0, \dots, \widehat{f_i}, \dots, f_n$ is the $n \times n$-submatrix of L_w obtained by deletion of the i-th row and the i-th column. Hence, the determinant of the submatrix (the principal minor of L_w) is equal to $\det M_k R|_{W_k} \cdot \det P_w^{(1)}$. The matrix of the operator $M_k R|_{W_k}$ in the same basis is $I - \frac{1}{n+1} U$ where the matrix U is defined as $U_{ij} = 1$ for all i, j. So, the rank of U is 1, and $Uf = nf$ where $f = f_0 + \dots + \widehat{f_k} + \dots + f_n$. Therefore, the characteristic polynomial of $\frac{1}{n+1} U$ is $\chi(t) = t^{n-1}(t - \frac{n}{n+1})$, and hence $\det(I - \frac{1}{n+1} U) = \chi(1) = 1/(n+1)$. This proves the classical *matrix-tree theorem* (see [3] and also [2] for another proof):

Theorem 2.2 (matrix-tree theorem, [3])**.** *A principal n-minor of the Kirchhoff's matrix is equal to the sum $\sum w_{i_1 j_1} \dots w_{i_n j_n}$ taken over all sets $\{(i_1, j_1), \dots, (i_n, j_n)\}$ representing edges of a tree with vertices $0, \dots, n$.*

2.2. $k = 2$**: Massbaum–Vaintrob theorem.** Consider now the operator $P_w^{(2)}$ for the A_n system of roots $e_{ij} = (f_i - f_j)/\sqrt{2}$. The Pfaffian of $P_w^{(2)}$ may be nonzero only if $\dim V = n$ is even; hence $n \stackrel{\mathrm{def}}{=} 2m$.

Let s_{pq} be a reflection in the hyperplane normal to e_{pq}, $p \neq q$; thus $s_{qp} = s_{pq}$. One has $[s_{ij}, s_{kl}] = 0$ if the $\{i,j\} \cap \{k,l\} = \emptyset$ or $\{i,j\} = \{k,l\}$, so the only nonzero terms in (1) are $w_{ij,jk}[s_{ij}, s_{jk}] \stackrel{\text{def}}{=} w_{ijk}M_{ijk}$. An immediate computation shows that $M_{ijk} = \mu_{ijk} - \mu_{ikj}$ where $\mu_{ijk} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a 3-cyclic permutation of the coordinates: it maps $f_i \mapsto f_j \mapsto f_k \mapsto f_i$ and leaves all the other f_p invariant. Hence, the operators M_{ijk} are fully skew-symmetric (change the sign when any two indices are exchanged), and one has $P_w^{(2)} = \sum_{1\le i<j<k\le N} \lambda_{ijk}M_{ijk}$ where λ_{ijk} is the result of alternation of the w_{ijk}: $\lambda_{ijk} = w_{ijk} - w_{ikj} - w_{jik} - w_{kji} + w_{jki} + w_{kij}$. The coefficient λ_{ijk} is fully skew-symmetric, too.

The matrix $T = (t_{pq})$, $0 \le p, q \le n$, of the operator $P_w^{(2)}$ in the standard basis $f_0, \dots, f_n$ is $t_{pq} = \sum_{r=1}^N \lambda_{pqr}$. The matrix is degenerate because it is skew-symmetric and $n+1 = 2m+1$ is odd; reasoning like in Section 2.1 one proves that the Pfaffian of any principal n-minor of T is equal to $\sqrt{n+1}$ times the Pfaffian of the operator $P_w^{(2)}$ acting in the subspace $V \subset \mathbb{R}^{n+1}$.

The right-hand side of (3) contains the coefficients $u_{ij,kl} = w_{ij,kl} - w_{kl,ij}$, which are all zeros except, possibly, $u_{ij,jk} = w_{ij,jk} - w_{jk,ij} = w_{ijk} - w_{kji} \stackrel{\text{def}}{=} u_{ijk}$; this implies $\lambda_{ijk} = u_{ijk} + u_{jki} + u_{kij}$.

Theorem 1.3 for $k = 2$ and the A_n system of roots looks as follows:

$$\operatorname{Pf} P_w^{(2)} = \sum \prod_{p=1}^m u_{i_pj_pk_p}(e_{i_pj_p}, e_{j_pk_p}) \operatorname{vol}(e_{i_1j_1}, e_{j_1k_1}, \dots, e_{i_mj_m}, e_{j_mk_m});$$

the sum is taken over the set of all m-edge partial pair matchings $\big((i_1, j_1), (j_1, k_1)\big)$, $\dots, \big((i_m, j_m), (j_m, k_m)\big)$ over the set of vertices (i, j), $0 \le i < j \le n$. By Lemma 2.1 the volume term is nonzero if and only if $(i_1, j_1), (j_1, k_1), \dots, (i_m, j_m), (j_m, k_m)$ are edges of a tree; in particular, the pairs $(i_1, j_1), (j_1, k_1), \dots, (i_m, j_m), (j_m, k_m)$ are all distinct and therefore $\big((i_1, j_1), (j_1, k_1)\big), \dots, \big((i_m, j_m), (j_m, k_m)\big)$ is indeed a partial pair matching.

By Remark 1.4 and skew symmetry of u_{ijk} one can ensure that $i_s < j_s < k_s$ for every $s = 1, \dots, m$. According to (6) the scalar products in the formula are then equal to $-1/2$, and the volume, according to Lemma 2.1, is $\pm\sqrt{(n+1/2^n}$; thus every term is equal to $(-1)^m\sqrt{n+1}/2^n\varepsilon(i_1, \dots, k_m) \prod_{s=1}^m u_{i_sj_sk_s}$, where $\varepsilon(i_1, \dots, k_m) \stackrel{\text{def}}{=} \operatorname{sgn}\operatorname{vol}(e_{i_1j_1}, e_{j_1k_1}, \dots, e_{i_mj_m}, e_{j_mk_m}) = \pm 1$ is the sign of the volume.

Let us draw for every term $\prod_{s=1}^m u_{i_sj_sk_s}$ a 3-graph with vertices $0, \dots, n$ and 3-edges $(i_sj_sk_s).1 \le s \le m$. The sides (i_s, j_s) and (j_s, k_s) of all the edges (recall that $i_s < j_s < k_s$) form a tree, which is contractible (homotopy equivalent to a point). Since a triangle can be retracted onto a union of its two sides, the 3-graph obtained is contractible, too, and therefore is a 3-tree.

Theorem 2.3 ([4]). *The Pfaffian of the principal $n \times n$-submatrix of the matrix T is equal to the sum of the terms $\delta(i_1, \dots, k_m) \prod_{s=1}^m u_{i_sj_sk_s}$ where the sign $\delta(i_1, \dots, k_m) = \pm 1$ is defined as follows. Number the 3-edges from 1 to m and consider a product of the 3-cycles $(i_1j_1k_1) \dots (i_mj_mk_m)$. This product is a cyclic permutation $(a_0 \dots a_n)$. The order of the vertices $a_0 \dots a_n$ inside the cycle defines a permutation σ of $0, \dots, n$; then $\delta(i_1, \dots, k_m)$ is the parity of this permutation.*

Another definition of $\delta(i_1, \dots, k_m)$ (also found in [4]) is the following. Number, again, the 3-edges; so, for every vertex the 3-edges containing it are linearly ordered. This defines an embedding, up to a homotopy, of the 3-graph into a disk such that the vertices lie in its boundary, and for every vertex the linear ordering of the 3-edges containing it corresponds to their ordering "left to right". Let $\sigma(i_1, \dots, k_m) = (c_0, \dots, c_n)$ be the vertices of the graph listed counterclockwise. Then σ is a permutation of $0, \dots, n$, and $\delta(i_1, \dots, k_m) = \operatorname{sgn}\sigma(i_1, \dots, k_m)$.

To derive Theorem 2.3 from Theorem 1.3 one should prove

Lemma 2.4. $\delta(i_1, \dots, k_m) = \varepsilon(i_1, \dots, k_m)$

Proof. First, the lemma is true for a simplest 3-tree with the edges $(123), (145), \dots, (1, n-1, n)$ — both signs are $+1$.

Consider now an arbitrary sequence $i_1, \dots, k_m$, that is, an arbitrary 3-tree with numbered edges. Prove first that both $\varepsilon(i_1, \dots, k_m)$ and $\delta(i_1, \dots, k_m)$ do not depend on the choice of the numbering of the edges. It suffices to show that ε and δ stay the same when one exchanges the numbers of the two edges, the s-th and the $(s+1)$-th. So, in $\operatorname{vol}(e_{i_1 j_1}, \dots, e_{j_m, k_m})$ one exchanges the vectors $e_{i_s, j_s}, e_{j_s, k_s}$ with $e_{i_{s+1}, j_{s+1}}, e_{j_{s+1}, k_{s+1}}$. Such an exchange is an even permutation, so $\varepsilon(i_1, \dots, k_m) = \operatorname{sgn} \operatorname{vol}(e_{i_1 j_1}, \dots, e_{j_m, k_m})$ remains the same. Consider now $\delta(i_1, \dots, k_m)$. If the s-th and the $(s+1)$-th edge have no common vertices, the embedding of the 3-tree described above does not change, and neither does δ. If the edges have a common vertex r, the subtrees rooted at r and containing the edges are swapped. Denote by $a_1 \dots, a_p$ the vertices of the first subtree (other than the root) and by $b_1, \dots, b_q$, the vertices of the second subtree. Then the fragments $a_1 \dots, a_p$ and $b_1, \dots, b_q$ in the permutation $\sigma(i_1, \dots, k_m)$ exchange their places. Every 3-tree contains an odd number of vertices, so both p and q are even here, and the exchange is an even permutation, so that $\delta(i_1, \dots, k_m) = \operatorname{sgn} \sigma(i_1, \dots, k_m)$ is preserved.

Let now $(i_s j_s k_s)$, $i_s < j_s < k_s$, be the s-th edge of the 3-tree. Shift the numbers of the vertices cyclically: $i_s \mapsto j_s \mapsto k_s \mapsto i_s$ and prove that this will not influence ε and δ. In $\operatorname{vol}(e_{i_1 j_1}, \dots, e_{j_m k_m})$ one replaces $e_{i_s j_s}, e_{j_s k_s}$ by $e_{j_s, k_s}, e_{k_s i_s} = -e_{i_s j_s} - e_{j_s k_s}$. So the volume stays the same, and so does its sign, $\varepsilon(i_1, \dots, k_m)$. The permutation $\sigma(i_1, \dots, k_m)$ is multiplied by the 3-cycle $(i_s j_s k_s)$, which is even, and therefore $\delta(i_1, \dots, k_m)$ would not change either.

Take now a dangling edge (ijk), $i < j < k$, of the 3-tree. As we proved, one may suppose that this edge has the number m and hangs on the vertex i. Delete the edge, obtaining a new tree $(i_1, \dots, k_{m-1})$ with $m-1$ edges and $n-2 = 2m-1$ vertices. In $\operatorname{vol}(e_{i_1 j_1}, \dots, e_{j_m k_m})$ one deletes the two last vectors; indices of the other vectors do not contain j_m and k_m. In the corresponding matrix (ε is the sign of its determinant) the two last rows and the j_m-th and the k_m-th column are deleted; so $\varepsilon(i_1, \dots, k_{m-1}) = (-1)^{j_m + k_m - 1} \varepsilon(i_1, \dots, k_m)$. On the other hand, permutation $\sigma(i_1, \dots, k_{m-1})$ is obtained from $\sigma(i_1, \dots, k_m)$ by deletion of the numbers j_m and k_m; so the number inversions is decreased by $j_m + k_m - 1$, and $\delta(i_1, \dots, k_{m-1}) = (-1)^{j_m + k_m - 1} \delta(i_1, \dots, k_m)$.

Thus, the equality $\varepsilon(i_1, \dots, k_m) = \delta(i_1, \dots, k_m)$ is proved by induction on m. □

3. The D_n and B_n cases

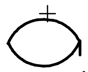
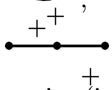
The D_n root system in the space $V = \mathbb{R}^n$ contains vectors $e_{ij}^+ = (f_i + f_j)/\sqrt{2}$ and $e_{ij}^- = (f_i - f_j)/\sqrt{2}$, for all $1 \le i < j \le n$; here $f_1, \dots, f_n \in \mathbb{R}^n$ is the standard orthonormal basis. The B_n root system contains the same vectors and also the vectors f_i for all $i = 1, \dots, n$. Denote the corresponding reflections by s_{ij}^+, s_{ij}^- and s_i, respectively. The D_n root system is a subset of the B_n root system, hence it is enough to consider the B_n.

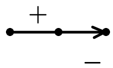
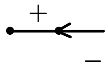
To any set $u = \{u_1, \dots, u_k\}$ of B_n-roots we associate a graph $G(u)$ with the vertices $1, \dots, n$ and the following edges: for any $e_{pq}^- \in u$ the graph $G(u)$ has an edge marked "$-$" and directed from p to q; for any $e_{pq}^+ \in u$ it contains an undirected edge joining p and q and marked "$+$"; for any $f_p \in u$ it contains a loop attached to the vertex p.

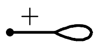
All the vectors e^σ_{pq} and f_p have unit length; the other scalar products are:

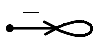
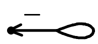
(7)

$(e^-_{i_1j_1}, e^-_{i_2j_2})$ is like in (6);

$(e^+_{i_1j_1}, e^+_{i_2j_2}) =$
- $1,$ if ,
- $1/2,$ if ,
- $0,$ otherwise (i.e. if the edges have no common vertices).

$(e^+_{i_1j_1}, e^-_{i_2j_2}) =$
- $1/2,$ if , but the final point does not,
- $-1/2,$ if ,
- $0,$ otherwise.

$(e^+_{ij}, f_p) =$
- $1/\sqrt{2}$ if 
- 0 otherwise

$(e^-_{ij}, f_p) =$
- $-1/\sqrt{2}$ if 
- $1/\sqrt{2}$ if 
- 0 otherwise

Consider a graph with n vertices and n edges, some of them directed (the "$-$"-edges), some not (the "+"-edges), loops allowed (and not directed, indeed). Such a graph will be called B-basic if any connected component of it contains exactly one cycle, and this cycle either is a loop or has an odd number of "+"-edges in it. The following lemma explains the term:

Lemma 3.1. *Let $u = \{u_1, \dots, u_n\} \in \mathbb{R}^n$ be a system of B_n-roots. The volume $\mathrm{vol}(u_1, \dots, u_n)$ is nonzero (that is, u is a basis in $\mathbb{R}^n$) if and only if the graph $G(u)$ is B-basic. If $G(u)$ is B-basic, then the volume is equal to $\pm 2^{d-\ell/2-n/2}$ where d is the number of connected components in $G(u)$ and ℓ is the number of loops in it.*

Proof. Suppose first that $G(u)$ contains no loops (that is, u is a system of D_n-roots).

Let the edges $(i_1, i_2), \dots, (i_m, i_1)$, marked $\sigma_1, \dots, \sigma_m$ (where $\sigma_i = +$ or $-$), form a cycle. Denote by τ_p the number of +s among $\sigma_1, \dots, \sigma_p$. Then $e^{\sigma_1}_{i_1i_2} + (-1)^{\tau_1} e^{\sigma_2}_{i_2i_3} + \dots + (-1)^{\tau_{m-1}} e^{\sigma_m}_{i_mi_1} = f_{i_1}(1 + (-1)^{\tau_m+1})$. If τ_m is even (that is, the cycle contains an even number of "+"-edges), then the sum is zero and the vectors are linearly dependent, hence $\mathrm{vol}(u) = 0$.

Let a connected component of $G(u)$ contain two cycles, $(i_1, \dots, i_p)$ and $(j_1, \dots, j_q)$ joined by a path $k_1, k_2, \dots, k_r$ where $k_1 = i_1$, $k_r = j_1$. If either cycle contains an even number of "+"-edges, u is proved to be lineary dependent. Suppose that each cycle contain an odd number of "+"-edges. Then the linear hull of u contains vectors $2f_{i_1}$ and $2f_{j_1}$, as well as the vector $e^{\sigma_1}_{k_1k_2} + (-1)^{\tau_1} e^{\sigma_2}_{k_2k_3} + \dots + (-1)^{\tau_{r-2}} e^{\sigma_r}_{k_{r-1}k_r} = f_{i_1} + (-1)^{\tau_{r-1}} f_{j_1}$ (notation as above) — hence, u is linearly dependent, too. So, if a connected component of a $G(u)$ contains two (or more) cycles, u is linearly dependent and $\mathrm{vol}(u) = 0$.

The graph $G(u)$ has n vetices and n edges. So if every connected component of $G(u)$ contains not more than one cycle, it contains exactly one. Thus, the first part of the lemma is proved for the D_n case (no loops).

If $f_p \in u$ then $G(u)$ contains a loop. Add then a new dimension, and replace f_p by $e^-_{p,n+1}$ and $e^+_{p,n+1}$, obtaining a new system u'. Since $f_p = \frac{1}{\sqrt{2}}(e^-_{p,n+1} + e^+_{p,n+1})$, the system u' is linearly dependent if and only if u is. Thus, the first part of the lemma for the B_n case follows from the same statement for the D_n.

Suppose now that $G(u)$ is connected and B-basic. Let (ij) and (jk) be two edges of $G(u)$ sharing a vertex j. Changing the signs of the vectors if necessary (this will affect the sign of $\operatorname{vol}(u)$ but not its absolute value) we may suppose that the vectors corresponding to these edges are $a = e_{ij}^{\sigma_1} \in u$ and $b = e_{jk}^{\sigma_2} \in u$. Replace then the vector b by $c = \pm b \pm a$ (the signs depend on σ_1 and σ_2 and are easily determined) one arrives to a system u' containing a and $c = e_{ik}^{\sigma}$ (σ also depends on σ_1 and σ_2); it is clear that $\operatorname{vol}(u') = \pm\operatorname{vol}(u)$ and that $G(u')$ is still connected and B-basic. Applying this transformation several times one can replace $G(u)$ with a connected B-basic graph containing either a loop f_i or a pair of edges $e_{ij_1}^{+}, e_{ij_1}^{-}$ (the exponents must be different because the graph is B-basic), and some edges $e_{ij_p}^{\sigma_p}$, $p = 2, \dots, n$, all j_p being distinct. In the first case the volume is

$$\det \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 1 & 1 & \dots \\ 0 & \pm 1 & 0 & \dots \\ 0 & 0 & \pm 1 & \dots \end{pmatrix} = \pm 2^{1-1/2-n/2},$$

and in the second case it is

$$\det \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & \pm 1 & 0 & \dots \\ 0 & 0 & 0 & \pm 1 & \dots \end{pmatrix} = \pm 2^{1-n/2},$$

(easily checked using column expansion).

In the general case a B-basic graph $G(u)$ is not connected but every its connected component contains exactly one cycle. If two edges have no common vertices, then the corresponding vectors are orthogonal (see (7)), so the volume is the product of volumes (of appropriate dimension) corresponding to connected components; thus the volume is $2^{d-\ell/2-n/2}$ where d is the number of connected components of $G(u)$ and ℓ, the number of loops in it. $\square$

Apply now Theorem 1.3 for $k = 1$ to the B_n case. Denote by T_w the matrix of the operator $P_w^{(1)} = \sum_{1 \le i < j \le n} (w_{ij}^{+}(I - s_{ij}^{+}) + w_{ij}^{-}(I - s_{ij}^{-})) + \sum_{1 \le i \le n} w_i (I - s_i)$ in the standard basis $f_1, \dots, f_n$. The matrix T_w is symmetric; one has $(T_w)_{ij} = -w_{ij}^{-} + w_{ij}^{+}$ for $i \ne j$, and its (i,i)-th entry is $(T_w)_{ii} = \sum_{j \ne i} (w_{ij}^{+} + w_{ij}^{-}) + 2w_i$; here we assume that $w_{ij}^{\sigma} = w_{ji}^{\sigma}$ where $\sigma = +$ or $-$. Combining this with Theorem 1.3 for $k = 1$ and Lemma 3.1 we obtain the following B-version of the matrix-tree theorem:

Theorem 3.2. *The determinant of the matrix T_w described above is equal to the sum of weights of all the B-basic graphs on the vertex set $1, \dots, n$, the weight of each graph multiplied by 2^{2d} where d is the number of its connected components. A weight of a B-basic graph is defined as the product of all w_{ij}^{+} where (ij) is a "+"-edge, times the product of all w_{ij}^{-} where (ij) is a "−"-edge, times the product of all the w_i where i is a vertex to which a loop is attached.*

Proof. Follows immediately from Theorem 1.3 and Lemma 3.1; note that the factor 2 at the w_i compensates for $2^{-\ell/2}$ term in Lemma 3.1). $\square$

An analog of the Massbaum–Vaintrob theorem for the B_n case can also be proved using Theorem 1.3 for $k = 2$. The formula obtained, though, involves some messy summation over the set of the 3-graph with possibly singular 3-edges and with additional structure on it; we do not formulate the result here.

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