Classification of m-spin Klein surfaces

S. M. Natanzon and A. M. Pratoussevitch

A Klein surface is a generalisation of a Riemann surface to the case of non-orientable surfaces and/or surfaces with boundary [1], [4]. A Klein surface is a quotient P/τ, where $\tau\colon P\to P$ is an anti-holomorphic involution of a Riemann surface P. The category of Klein surfaces is isomorphic to the category of real algebraic curves. A complete list of topological invariants of a connected Klein surface P/τ consists of the (algebraic) genus $g=g(P)$, the number $k=k(P/\tau)=|\partial(P/\tau)|$ of ovals, and the orientability $\varepsilon=\varepsilon(P/\tau)$. An oval is a connected component of the boundary $\partial(P/\tau)$. The orientability $\varepsilon(P/\tau)$ of the surface P/τ is $=1$ if the surface is orientable (in which case $1\leqslant k\leqslant g+1$ and $k\equiv g+1 \mod 2$) and $\varepsilon=0$ otherwise (in which case $0\leqslant k\leqslant g$). In addition to these invariants it is convenient to consider the geometric genus $\widetilde{g}=\widetilde{g}(P/\tau)$, which is equal to the number of handles (for $\varepsilon=1$) or to half the number of Möbius bands (for $\varepsilon=0$) that need to be attached to a sphere with holes in order to obtain a surface homeomorphic to P/τ. We will assume that $\widetilde{g}>1$ and hence $g>1$.

An m-spin bundle on a Riemann or Klein surface S is a complex line bundle $e\colon L\to S$ such that $e^{\otimes m}\colon L^{\otimes m}\to S$ is isomorphic to the cotangent bundle. The moduli spaces of m-spin bundles on Riemann surfaces have been described in [2], [7], [8]. These bundles and their moduli spaces appear naturally in physics [9]. The moduli space of compact Riemann surfaces P of genus $g>1$ with an m-spin bundle is connected for odd m and consists of two connected components for even m. These components are determined by the Arf invariant $\delta=\delta(P,e)\in\{0,1\}$. On a Riemann surface of genus g there are m^{2g} m-spin bundles, of which (for even m) there are $2^{-1-g}m^{2g}(2^g+1)$ bundles with $\delta=0$ and $2^{-1-g}m^{2g}(2^g-1)$ bundles with $\delta=1$. Our aim is to find topological invariants of m-spin bundles on Klein surfaces of geometric genus $\widetilde{g}>1$, to determine the number of such bundles, and to describe the moduli space of m-spin bundles on Klein surfaces. The special case $m=2$ was studied in [3] and [6].

Theorem 1. *Every connected component of the moduli space of m-spin bundles on Klein surfaces of genus g is homeomorphic to $\mathbb{R}^{3g-3}/\operatorname{Mod}$, where Mod is a discrete group.*

Theorem 2. *For odd m, there exists an m-spin bundle on a Klein surface P/τ if and only if $g(P)\equiv 1 \mod m$, in which case the number of such bundles is equal to m^g. Moreover, for odd m, the moduli space of m-spin bundles on Klein surfaces of genus g is connected.*

From now on we will assume that m is even.

Theorem 3. *For even m, there exists an m-spin bundle on a Klein surface P/τ if and only if $g(P)\equiv 1 \operatorname{mod}(m/2)$, in which case the number of such bundles with Arf invariant δ is equal to:* 1) $m^g/2$ *for* $\varepsilon=k=0$; 2) $2^{k-2}m^g$ *for* $\varepsilon=0$, $k>0$ *or* $m\equiv 0 \mod 4$, $\varepsilon=1$; 3) $(2^{k-1}+1)m^g/2$ *for* $m\equiv 2 \mod 4$, $\varepsilon=1$, $\delta=0$; 4) $(2^{k-1}-1)m^g/2$ *for* $m\equiv 2 \mod 4$, $\varepsilon=1$, $\delta=1$.

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The restriction of an m-spin bundle to an oval can be trivial or non-trivial. We denote by $k_0 = k_0(P/\tau, e)$ and $k_1 = k_1(P/\tau, e)$ the number of ovals on which the restriction of the bundle is trivial and non-trivial, respectively.

Theorem 4. *The moduli space of m-spin bundles of type $(g, \varepsilon, m, \delta, k_0, k_1)$ on Klein surfaces is non-empty if and only if $k_1 m/2 \equiv (1-g) \bmod m$. For $\varepsilon = 0$ the moduli space is connected. On any Klein surface of type $(g, k_0 + k_1, 0)$ there are $\binom{k}{k_1} m^g/2$ bundles of type $(g, 0, m, \delta, k_0, k_1)$.*

From now on we will assume that m is even, $\varepsilon = 1$, and $k_1 m/2 \equiv (1-g) \bmod m$. In this case there are additional topological invariants. One of them is the Arf invariant $\widetilde{\delta} = \widetilde{\delta}(P/\tau)$ of the Riemann surface $P/\tau \setminus \partial(P/\tau)$. Furthermore, the ovals can be divided into two classes. Two ovals c_1 and c_2 are called similar if there is a simple closed curve d that intersects c_1 and c_2 such that $\tau(d) = d$ and the restriction of the bundle to d is non-trivial [5]. Let us choose one similarity class of ovals and denote by k_0^0 and k_1^0 the number of ovals in this class on which the restriction of the bundle is trivial and non-trivial, respectively. Let $k_i^1 = k_i - k_i^0$. (The invariants $(k_0^0, k_1^0, k_0^1, k_1^1)$ are defined up to $k_i^j \mapsto k_i^{1-j}$.)

Theorem 5. *The moduli space of m-spin bundles with topological invariants*

$$(g, \widetilde{\delta}, m, k_0^0, k_1^0, k_0^1, k_1^1)$$

on Klein surfaces is connected. For $m \equiv 0 \bmod 4$ or $m \equiv 2 \bmod 4$ and $k_0 > 0$, bundles of this type exist on a Klein surface if and only if $\widetilde{\delta} = 0$, in which case their number is $2^{1-k} m^g \binom{k}{k_1}\binom{k_0}{k_0^0}\binom{k_1}{k_1^0}$. For $m \equiv 2 \bmod 4$ and $k_0 = 0$ the number of m-spin bundles of type $(g, \widetilde{\delta}, m, 0, k_1^0, 0, k_1^1)$ on a Klein surface of type $(g, k_1^0 + k_1^1, 1)$ is $(2^{-k} + 2^{-(g+k+1)/2}) m^g \binom{k}{k_1}\binom{k_0}{k_0^0}\binom{k_1}{k_1^0}$ for $\widetilde{\delta} = 0$ and $(2^{-k} - 2^{-(g+k+1)/2}) m^g \binom{k}{k_1}\binom{k_0}{k_0^0}\binom{k_1}{k_1^0}$ for $\widetilde{\delta} = 1$.

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Sergey M. Natanzon
National Research University
"Higher School of Economics" (HSE);
Institute of Theoretical and Experimental
Physics (ITEP)
E-mail: natanzons@mail.ru

Anna M. Pratoussevitch
University of Liverpool, Liverpool, UK
E-mail: annap@liv.ac.uk

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