WEAK REGULARITY OF GAUSS MASS TRANSPORT

ALEXANDER V. KOLESNIKOV

ABSTRACT. Given two probability measures μ and ν we consider a mass transportation mapping T satisfying 1) T sends μ to ν, 2) T has the form $T = \varphi \frac{\nabla \varphi}{|\nabla \varphi|}$, where φ is a function with convex sublevel sets. We prove a change of variables formula for T. We also establish Sobolev estimates for φ, and a new form of the parabolic maximum principle. In addition, we discuss relations to the Monge–Kantorovich problem, curvature flows theory, and parabolic nonlinear PDE's.

Keywords: optimal transportation, Monge–Kantorovich problem, Gauss curvature flows, parabolic Monge–Ampère equation, Alexandrov maximum principle, parabolic maximum principle, Sobolev and Hölder a priori estimates.

1. Introduction

In this paper we study a class of mass transportation mappings having the form

$$T = \varphi \frac{\nabla \varphi}{|\nabla \varphi|}$$

with some potential φ. The mappings of this type have been introduced in [9], [8]. Assume we are given a couple of probability measures $\mu = \rho_0 \ dx$ and $\nu = \rho_1 \ dx$. It has been shown that, under general assumptions, there exists a unique φ with convex sublevel sets $A_t = \{x \colon \varphi(x) \le t\}$ such that

$$T \colon x \to \varphi(x) \cdot \mathrm{n}(x),$$

where $\mathrm{n}(x)$ is the normal vector to ∂A_t at x with $t = \varphi(x)$ and T satisfies the equality $\nu = \mu \circ T^{-1}$. We point out that the restriction of T to every level set ∂A_t coincides (up to the factor t) with the Gauss map of ∂A_t. In what follows we use the name "Gauss mass transport" for T.

Mappings of this kind are closely related to several areas of research. They can be considered as "parabolic" analogs of optimal transportation mappings, which attract attention of researchers from the most diverse fields, including probability, partial differential equations, geometry, and infinite-dimensional analysis (see [36], [37], and [7]). In addition, they arise naturally in the Gauss curvature flow theory. Concerning transformations of measures of other related types, see [5], [6], [10].

The main goal of this paper is to establish some regularity properties of the mapping T. More precisely, we prove that T satisfies a change of variables formula, which can be considered as the weakest regularity property of T.

The corresponding result in the elliptic case (optimal mappings between measures with densities always satisfy a change of variables formula) belongs to McCann [29]. This result turns out to be quite useful for different applications. Applications of the change of variables formula include, for instance, the so-called above-tangent formalism which is a crucial technique in variational problems, PDE's, and probability (see [37], [2], [4]).

The paper is organized as follows.

In Section 2, we briefly describe the main results of [9] that are used throughout. These are the results on existence and uniqueness of Gauss maps, a description of an important scaling procedure, and certain duality relations. In addition, we describe the relations to curvature flows and the parabolic Monge–Ampère equation.

Our main result is proved in Section 3. We show that T satisfies the following change of variables formula:

$$\rho_1 \circ T \cdot \mathcal{J} = \rho_0 \quad \text{with} \quad \mathcal{J} = \det D_a T,$$

where $D_a T$ can be understood as the absolutely continuous part of the distributional derivative of T. One has

$$\mathcal{J} = \varphi^{d-1} |D_a \varphi| K,$$

where $|D_a \varphi|$ is the absolutely continuous part of the full variation of the vector-valued measure $\nabla \varphi$ and K is the Gauss curvature of $\partial A_{\varphi(x)}$.

In Section 4 we establish some natural Sobolev a-priori estimates for φ. We emphasize that φ is not Sobolev but only BV in general. Under assumption that $\rho_1 = \frac{C}{r^{d-1}}$ we show that for every $p > 0$

$$C_{p,R} \int_A |\nabla \varphi|^{p+1} \ d\mu \le \int_A \Bigl| \frac{\nabla \rho_0}{\rho_0} \Bigr|^{p+1} \ d\mu + \int_{\partial A} K^{-p} \rho_0^{p+1} \ d\mathcal{H}^{d-1}.$$

Another natural question arising in the study of the Gauss mass transport is the validity of some parabolic analogs of the maximum principle. Applying the mass transportation arguments one can establish (see Section 5) the following form of the parabolic maximum principle: every smooth function f on a convex set A satisfies the inequality

$$\sup_A f \le \sup_{\partial A} f + \ C(d) \int_{\mathcal{C}_{-f,l}} |\nabla f| K \ dx,$$

where

$$\mathcal{C}_{-f,l} = \{x \colon x \in A_t \cap \partial \operatorname{conv}(A_t)\}, \quad A_t = \{-f \le t\}$$

is the set of contact points for the sublevel sets of $-f$, $\operatorname{conv}(A_t)$ is the convex envelope of A_t, and K is the corresponding Gauss curvature. This estimate is naturally related to the Gauss mass transport and the second-order nonlinear parabolic differential operator $f \mapsto |\nabla f| K$ (similarly to the Monge–Ampère operator in the classical maximum principle). The inverse mapping $S = T^{-1}$ is associated with another parabolic differential operator:

$$f \mapsto \frac{f_r \cdot \det\bigl(f \cdot \mathrm{Id} + D^2_\theta f\bigr)}{r^{d-1}},$$

where $D^2_\theta f$ is the Hessian on S^{d-1}. The corresponding maximum principle is proved.

In Section 7, we are concerned with the regularity of the parabolic Monge–Ampère equation. In particular, we briefly explain how the arguments employed in [34] can be extended to our situation to prove Hölder's regularity of φ. Thus we establish Hölder's continuity of φ assuming that $\rho_1, \rho_2 \in C^{2,\alpha}(A)$ and ∂A is smooth and uniformly convex.

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2. Existence and basic properties

In what follows we denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure on $\mathbb{R}^d$, $m \le d$. For Lebesgue measure we also use common notation λ. We denote by S^{d-1} the unit sphere in $\mathbb{R}^d$ (and by S^{d-1}_+ its upper-half). We also use the symbols D_θ, D^2_θ for the gradient and the Hessian on S^{d-1}.

It will be assumed throughout the paper that

A1) the measure μ is supported on a compact convex set A

A2) the measure ν is supported on $B_R = \{x\colon |x| \le R\}$ for some $R > 0$

A3) the measure μ is absolutely continuous with respect to $\lambda|_A$ and ν is absolutely continuous with respect to $\lambda|_{B_R}$.

We start with a brief outline of two areas of research closely related to the Gauss mass transport.

1) Optimal transportation.

Optimal transportation can be described as a problem of optimization of a certain functional associated with a pair of measures. The quadratic transportation cost $W_2^2(\mu,\nu)$ between two probability measures μ,ν on $\mathbb{R}^d$ is defined as the minimum of the Kantorovich functional:

$$m \mapsto \int_{\mathbb{R}^d\times\mathbb{R}^d} |x_1 - x_2|^2 \, dm(x_1,x_2), \quad m \in \mathcal{P}(\mu,\nu), \tag{1}$$

where $\mathcal{P}(\mu,\nu)$ is the set of all probability measures on $\mathbb{R}^d\times\mathbb{R}^d$ with the marginals μ and ν; here $|v|$ denotes the Euclidean norm of $v\in\mathbb{R}^d$. The problem of minimizing (1) is called the mass transportation problem. In many cases there exists a mapping $T\colon \mathbb{R}^d \to \mathbb{R}^d$, called the optimal transport between μ and ν, such that $\nu = \mu\circ T^{-1}$ and

$$W_2^2(\mu,\nu) = \int_{\mathbb{R}^d} |x - T(x)|^2 \, \mu(dx).$$

If μ and ν are absolutely continuous, then, as shown by Brenier and McCann (see [36]), there exists an optimal transportation T which takes μ to ν. Moreover, this mapping is μ-unique and has the form $T = \nabla W$, where W is convex. Assuming smoothness of W, one can easily verify that W solves the following nonlinear PDE (the Monge–Ampère equation):

$$\det D^2 W = \frac{\rho_0}{\rho_1(\nabla W)}.$$

In fact, this equation is satisfied in a certain sense without any smoothness assumptions (see Section 3).

2) Geometric flows.

We refer to [16], [17] for an account in geometric flows. Let $\{\Gamma_t\} \subset \mathbb{R}^d$ be a family of embedded hypersurfaces. Denote by $V(x,t)$ the velocity in the direction of the inward normal $-\mathrm{n}(x)$ at a point $x \in \Gamma_t$. We say that $\{\Gamma_t\}$ satisfies a surface evolution equation (or $\{\Gamma_t\}$ is a geometric flow) if V satisfies

$$V = f(x,t,\mathrm{n},\mathrm{Dn}) \tag{2}$$

for some given function f. If $f = H$ is the mean curvature, then Γ_t is called the mean curvature flow. If $f = K$ is the Gauss curvature, then Γ_t is called the Gauss curvature flow.

The Gauss curvature flows have been introduced by Firey [15] as a model of the wearing stone on a beach. The existence and uniqueness of a Gauss curvature flow starting from a smooth initial convex surface has been obtained by Tso [34] by solving a corresponding parabolic Monge–Ampère equation. He proved, in particular, that Γ_t remains convex and shrinks to a point in finite time. The same result for the mean curvature flow has been obtained by Huisken [21]. More on Gauss curvature flows see in [3].

The main problem arising in respect with non-convex initial surfaces is the eventual singularity of the solution. It turns out that in general Γ_t becomes singular in finite time. To overcome this problem several notions of generalized solutions have been proposed. A weak notion of a solution to (2) have been introduced by Brakke [11]. He proved the existence of the mean curvature flow for any initial data in some generalized measure-theoretical sense. According to the level-set method (see [17]), the family $\{\Gamma_t\}$ is considered as level sets of some function $u(t,x)$ satisfying a nonlinear parabolic equation in viscosity sense. Finally, it is known that sometimes the solutions to curvature flows can be obtained as scaling limits of certain elliptic or parabolic equations. For instance, the mean curvature flow can be obtained as a singular limit of the solutions to Allen–Cahn or Ginzburg–Landau equations (see [22], [31]). It has been shown in [9] that Gauss curvature flows starting from convex surfaces are singular limits of some optimal transportation problems. More precisely, the following result has been proved in [9].

Theorem. *Let $A \subset \mathbb{R}^d$ be a compact convex set and let $\mu = \rho_0\, dx$ be a probability measure on A equivalent to the restriction of Lebesgue measure. Let $\nu = \rho_1\, dx$ be a probability measure on $B_R = \{x\colon\ |x| \le R\}$ equivalent to the restriction of Lebesgue measure. Then, there exist a Borel mapping $T\colon\ A \to B_R$ and a continuous function $\varphi\colon\ A \to [0,R]$ with convex sub-level sets $A_s = \{\varphi \le s\}$ such that $\nu = \mu \circ T^{-1}$ and*

$$T = \varphi \cdot \mathrm{n} \quad \mathcal{H}^d\text{-almost everywhere,}$$

where $\mathrm{n} = \mathrm{n}(x)$ is a unit outer normal vector to the level set $\{y\colon\ \varphi(y) = \varphi(x)\}$ at the point x.

If φ is smooth, the level sets of φ are moving according to the following Gauss curvature flow equation:

$$\dot{x}(s) = -s^{d-1}\ \frac{\rho_1(s\mathrm{n})}{\rho_0(x)} K(x) \cdot \mathrm{n}(x) \tag{3}$$

where $x(s) \in \partial A_{R-s}$, $0 \le s \le r$, $x(0) \in \partial A$ is any initial point satisfying $\varphi(x(0)) = R$.

Remark 2.1.
1) The theorem does not guarantee that the boundary ∂A is exactly the level set $\{\varphi = R\}$. Nevertheless, one can easily check that this is indeed the case when A is strictly convex.
2) It is not clear in general whether $\{x\colon \varphi(x) = 0\}$ contains a unique point or just has Lebesgue measure zero.
3) The case $\rho_1 = \frac{1}{r^{d-1}}$, $\rho_0 = C$ corresponds to the standard Gauss curvature flow. The asymptotic behavior of ∂A_r for small values of r is a standard problem in differential geometry. For the classical Gauss flow it is known that ∂A_r is asymptotically spherical in shape for values of r close to 0 (see [3]). This problem has not been studied so far for the flows of the type (3).
4) Potential φ is not Sobolev in general, but admits a bounded variation (BV) (see [1]). The distributional derivative of φ can have a singular component in the n-direction.

In addition, T is unique and admits an inverse T^{-1} (see [9], Section 3).

Let us briefly describe the idea of the proof and some important related facts. The potential φ is a pointwise limit of a sequence of functions $\{\varphi_t\}$ with convex sublevel sets. To construct φ_t we consider the optimal transportation ∇W_t of μ to $\nu \circ S_t^{-1}$, where $S_t(x) = x|x|^t$. Let us set

$$T_t = \frac{\nabla W_t}{|\nabla W_t|^{\frac{t}{1+t}}}$$

Clearly, T_t pushes forward μ to ν. Choose W_t in such a way that $\min_{x\in A} W_t(x) = 0$. Define a new potential function φ_t by

$$W_t = \frac{1}{t+2}\varphi_t^{t+2}.$$

Then one has

$$T_t = \varphi_t \frac{\nabla \varphi_t}{|\nabla \varphi_t|^{\frac{t}{t+1}}}.$$

It was shown in [9] that

$$\lim_{n\to\infty} \varphi_{t_n} = \varphi, \ \lim_{n\to\infty} T_{t_n} = T$$

almost everywhere (for a suitable subsequence $t_n \to \infty$).

The dual potentials

$$W_t^*(y) = \sup_{x\in\mathbb{R}^d} \bigl(\langle x, y\rangle - W_t(x)\bigr)$$

of the corresponding dual Monge–Kantorowich problem define via renormalization another natural convergent sequence

$$H_t(y) = \frac{W_t^*(y|y|^t)}{|y|^{1+t}}.$$

It was shown in [9] that

$$H_t \to H$$

pointwise, where

$$H(r,\theta)\colon B_R = [0,R] \times S^{d-1} \to \mathbb{R},$$

$$H(r,\theta) = \sup_{x\colon \varphi(x)\le r} \langle \theta, x\rangle.$$

We warn the reader that in [9] we deal with a slightly different potential $\psi_t = H_t(r,\theta)r$.

Let us describe the expression for T_t^{-1} in terms of H_t. To this end we fix $\mathrm{n} \in S^{d-1}$ and introduce local coordinates $(\theta_1, \cdots, \theta_{d-1})$ on S^{d-1} in a neighborhood of n. We assume everywhere below that

$$e_i = \frac{\partial \mathrm{n}}{\partial \theta_i}$$

constitute an orthonormal basis in the tangent space of S^{d-1} at n. Then the following relation holds

$$T_t^{-1}(y) = \Bigl(H_t + \frac{r}{t+1}(H_t)_r\Bigr)\cdot \mathrm{n} + \sum_{i=1}^{d-1} (H_t)_{\theta_i} \cdot e_i.$$

In the limit $t \to \infty$ one has

$$T^{-1}(y) = H \cdot \mathrm{n} + \sum_{i=1}^{d-1} H_{\theta_i} \cdot e_i = H \cdot \mathrm{n} + D_\theta H.$$

Remark 2.2. In what follows we often choose the following convenient local coordinate system on S^{d-1}. We take the center of S^{d-1} for the origin and introduce the standard Euqlidean coordinates in $\mathbb{R}^d$ such that n becomes the North Pole: $\mathrm{n} = (0, 0, \dots, 1)$. A neighborhood of n can be parametrized by

$$(\theta_1, \dots, \theta_{d-1}) \to \Big(\theta_1, \dots, \theta_{d-1}, (1 - \sum_{i=1}^{d-1} \theta_i^2)^{1/2}\Big).$$

In particular, one has at n:

$$\frac{\partial e_i}{\partial \theta_i} = -\mathrm{n}, \ \frac{\partial e_j}{\partial \theta_i} = 0, \ i \neq j.$$

Clearly, $(r, \theta_1, \dots, \theta_{d-1})$ is a parametrization of a cone with the vertex at the origin.

Now we describe the relation between the Gauss mass transport and the parabolic Monge–Ampère equation.

Several parabolic analogs of the elliptic Monge–Ampère equation have been introduced by Krylov (see [25]). He also proved some forms of the parabolic maximum principle (see also [35]).

Let $\mu = \rho_0 \, dx$ be a probability measure on an strictly convex set A. Consider a Gauss mass transportation

$$T = \varphi \frac{\nabla \varphi}{|\nabla \varphi|}$$

sending μ to a measure $\nu = \rho_1 \, dx$ on $B_R \colon = \{x \colon |x| \le R\}$.

Example 2.3. Assume $d = 2$ and fix a standard coordinate system (x_1, x_2). Assume that the functions below are smooth. Introduce the global polar coordinate system (r, θ). One has

$$T^{-1} = H \cdot \mathrm{n} + H_\theta \cdot \mathrm{v},$$

$$\mathrm{n} = \big(\cos\theta, \sin\theta\big), \ \mathrm{v} = (-\sin\theta, \cos\theta).$$

Let us compute the derivative of T^{-1} in polar coordinates:

$$T^{-1}_r = H_r \cdot \mathrm{n} + H_{\theta r} \cdot \mathrm{v}$$

$$T^{-1}_\theta = H \cdot \dot{\mathrm{n}}_\theta + H_\theta \cdot \mathrm{n} + H_\theta \cdot \dot{\mathrm{v}}_\theta + H_{\theta\theta} \cdot \mathrm{v} = (H + H_{\theta\theta}) \cdot \mathrm{v}.$$

Taking into account that $\det D(r, \theta) = \frac{1}{r}$ one gets $\det DT^{-1} = \frac{H_r(H + H_{\theta\theta})}{r}$. Finally, by the change of variables formula

$$\rho_1 = \rho_0(H \cdot \mathrm{n} + H_\theta \cdot \mathrm{v}) \frac{H_r(H + H_{\theta\theta})}{r}. \tag{4}$$

Let us describe a standard trick which allows to rewrite (4) in the form of the parabolic Monge–Ampère equation. Introduce another variable on $x_2 < 0$:

$$z = -\mathrm{ctg}\ \theta, \ \pi \le \theta \le 2\pi.$$

Thus $\theta = \mathrm{arcctg}(-z)$. Instead of H it is convenient to work with

$$u = \sqrt{1 + z^2}\ H.$$

Note that u is just the restriction of the corresponding 1-homogeneous support function H_{A_r} with a fixed r to the line $x_2 = -1$. In particular, u is convex in z. Taking into account that $\frac{\partial}{\partial z} = \frac{1}{1+z^2}\frac{\partial}{\partial \theta}$, one can easily compute

$$u_z = \frac{zH + H_\theta}{\sqrt{1+z^2}}, \ u_{zz} = \frac{H + H_{\theta\theta}}{(1+z^2)^{\frac{3}{2}}}.$$

Finally, we set

$$\mathcal{T} = T^{-1} \circ (r, \mathrm{arcctg}(-z)).$$

Writing this mapping in coordinates (x_1, x_2) as a function of (z, r), one gets

$$\mathcal{T}^{-1} = (u_z, zu_z - u) = (u_z, u^*(u_z)),$$

where u^* is convex conjugated to u with respect to z-variable

$$u^*(z, r) = \sup_{x \in \mathbb{R}^1} \bigl(xz - u(z, r)\bigr).$$

The change of variables formula takes the form

$$u_r \cdot u_{zz} = \frac{1}{\rho_0(u_z, zu_z - u)} \frac{r}{1+z^2} \rho_1\Bigl(\frac{rz, -r}{\sqrt{1+z^2}}\Bigr), \ (z, r) \in \mathbb{R}^+ \times \mathbb{R}. \tag{5}$$

Note that (5) can be considered as a *parabolic Monge-Ampère* equation.

In addition, (5) can be easily interpreted from the point of view of mass transportation. Indeed, let us set

$$\tilde{\nu} = \frac{r}{1+z^2} \rho_1\Bigl(\frac{rz, -r}{\sqrt{1+z^2}}\Bigr) \ drdz.$$

Then $\tilde{\nu}$ is a measure on $\mathbb{R} \times [0, R]$ which coincides with the image of ν under the mapping

$$(x, y) \longmapsto \frac{(rz, -r)}{\sqrt{1+z^2}}.$$

Further, μ is the image of $\tilde{\nu}$ under $\mathcal{T}^{-1}$. Function u is convex in z and increasing with respect to r.

All these computations can be generalized to the multidimensional case. One has $T^{-1} = H \cdot \mathrm{n} + \sum_{i=1}^{d-1} H_{\theta_i} \cdot e_i = H \cdot \mathrm{n} + D_\theta H$ and

$$\rho_1 = \rho_0(T^{-1}) \frac{H_r \cdot \det(H \cdot \mathrm{Id} + D^2_\theta H)}{r^{d-1}}. \tag{6}$$

Here $D^2_\theta H$ denotes the Hessian of H on the unit sphere. For computing $D^2_\theta H$ it is convenient to deal with the local polar coordinate system as described above. In this case $D^2_\theta H(\mathrm{n})$ can be represented just by the matrix $(\partial^2_{\theta_i \theta_j} H)$. Note that

$$\det(H \cdot \mathrm{Id} + D^2_\theta H) = \frac{1}{K(T^{-1})}, \ H_r = \frac{1}{|\nabla \varphi(T^{-1})|}.$$

Finally, let us define coordinates (z, r) and the corresponding chart

$$V(z, r) \colon \{-R < x_d < 0\} \to B_R,$$

$$\bigl(x_1, \ldots, x_d\bigr) = \frac{r}{\sqrt{1 + z_1^2 + \cdots + z_{d-1}^2}} \bigl(z_1, \ldots, z_{d-1}, -1\bigr) = V(z, r).$$

Now we introduce a new potential u

$$u = \sqrt{1 + z_1^2 + \cdots + z_{d-1}^2} \ S$$

and verify the following proposition by direct computations.

Proposition 2.4. *Assume that T is smooth. The following representations hold on $-R < x_d < 0$:*

1)
$$\mathcal{T}^{-1} = \big(\nabla_z u, \langle z, \nabla_z u\rangle - u\big) = \big(\nabla_z u, u^*(\nabla_z u)\big),$$
where
$$u^*(z,r) = \sup_{x \in \mathbb{R}^{d-1}} \big(\langle x, z\rangle - u(z,r)\big),$$
$$\mathcal{T}^{-1} = T^{-1} \circ V(z,r).$$

2)
$$\det\big(H \cdot \mathrm{Id} + D^2_\theta H\big) = (1 + z_1^2 + \cdots + z_{d-1}^2)^{\frac{3}{2}(d-1)} \det D^2_z u$$

3) *the change of variables takes the form*
$$u_r \cdot \det D^2_z u = \frac{\tilde{\rho}_1}{\rho_0(\nabla_z u, \langle z, \nabla_z u\rangle - u)},$$
where
$$\tilde{\rho}_1 = \Big[\frac{r}{1 + z_1^2 + \cdots + z_{d-1}^2}\Big]^{d-1} \rho_1\Big(\frac{rz_1, \cdots, rz_{d-1}, -r}{\sqrt{1 + z_1^2 + \cdots + z_{d-1}^2}}\Big).$$

More on the parabolic Monge–Ampère equation see in Section 7.

3. Change of variables

Let A be any convex compact set of positive volume and let $T\colon A \to B$ be a Gauss mass transport between two given probability measures μ and ν satisfying the assumptions specified in the introduction. To prove the change of variables formula for the Gauss mass transport we need to define the Gauss curvature for sufficiently "large" amount of points $x \in \partial A$. To this end we consider the corresponding support function
$$H_A(\theta) = \sup_{x \in A} \langle \theta, x\rangle.$$
Here we assume that $\theta \in \mathbb{R}^d$. Clearly, H_A is 1-homogeneous and convex. Hence, by the Alexandrov theorem H_A is almost everywhere twice differentiable. Recall that every convex function V is a.e. twice differentiable in the Alexandrov sense, i.e. for almost all x there exists a matrix $D^2_a V(x)$ (the absolutely continuous part of the second distributional derivative) such that

$$\Big|V(y) - V(x) - \langle \nabla V(x), y - x\rangle - \frac{1}{2}\langle D^2_a V(x)\, y - x, y - x\rangle\Big| = o(|y-x|^2),\ y \to x \tag{7}$$

(see [14]).

Remark 3.1. A parabolic analog of the Alexandrov theorem for monotone-convex functions was proved by Krylov (see [25]).

Definition 3.2. In what follows we say that $f\colon M \to \mathbb{R}$, where M is a Borel set is differentiable at $x \in M$ in the sense of Alexandrov if (7) holds for $y \in M$.

In particular, homogeneity implies that for every fixed $r > 0$ the function $H_A|_{\partial B_r}$ is twice differentiable for $\mathcal{H}^{d-1}$-almost all $x \in \partial B_r$.

Recall that H is defined as follows:

$$H(r,\theta) = \sup_{\theta \in S^d, x \in A_r} \langle \theta, x \rangle.$$

Lemma 3.3. *For μ-almost all $x \in A$ and all $0 < r \le R$ the function $H|_{\partial B_r}$ is twice differentiable at $r \cdot \mathrm{n(x)}$ in the Alexandrov sense.*

Proof. It was noted above that $H|_{\partial B_r}$ is twice differentiable in the Alexandrov sense for $\mathcal{H}^{d-1}$-almost all $y \in \partial B_r$. Hence by Fubuni's theorem the set of all y such that $H|_{\partial B_r}(y)$, $r = |y|$ is not twice differentiable in the Alexandrov sense has ν-measure zero. The claim follows from the fact that T pushes forward μ to ν. $\square$

Next we want to define the Gauss curvature for an arbitrary convex surface ∂A $\mathcal{H}^{d-1}$-almost everywhere. Let us recall how the Gauss curvature can be defined in the smooth case.

Assume that ∂A is a level set of some smooth function F. Then

$$\mathrm{n}(x) \colon = \frac{\nabla F(x)}{|\nabla F(x)|}.$$

Let $\{e_1, e_2, \dots, e_{d-1}\}$ be an orthonormal basis such that $\mathrm{n} \perp e_i$ for every $1 \le i \le d-1$. Then

$$K(x) = \det D\mathrm{n}(x),$$

where D is the differential operator on the tangent space to ∂A. Computing this expression in Euclidean coordinates one gets

$$K = \det A_{i,j}, \text{ where}$$

$$A_{i,j}(x) = \frac{1}{|\nabla F(x)|} \langle D^2 F(x) e_i, e_j \rangle, \ 1 \le i,j \le d-1. \tag{8}$$

In particular, this formula is applicable when the surface is represented locally as the graph of a convex function

$$F = W(x_1, \dots, x_{d-1}) - x_d,$$

where $\{e_i\}$ can be obtained by an orthogonalization procedure from a basis tangent to F at some point.

It is convenient to compute the Gauss curvature in terms of the support function. The following lemma (well known for smooth surfaces) gives another practical way of computing.

Lemma 3.4. *Let ∂A be a convex surface which coincides with a graph of some convex function $W(x_1, \dots, x_{d-1})$ in a neighborhood Ω of x_0 and $\mathrm{n(x_0)}$ is unique at x_0. Then the following are equivalent*

1) *W is differentiable at x_0 in the Alexandrov sense and $D^2_a W(x_0)$ is nondegenerate,*
2) *$H|_{S_{d-1}}$ is differentiable at $\mathrm{n}(x_0)$ in the Alexandrov sense and*

$$\det\Big(H \cdot \mathrm{Id} + \big(D^2_\theta\big)_a H\Big) \circ \mathrm{n}(x_0) \ne 0.$$

Proof. Choosing an appropriate coordinate system, we may assume without loss of generality that $x_0 = 0$, $W(0) = 0$ and $\nabla W(0) = 0$. Let us assume for a while that W is smooth and $D^2 W > 0$ in Ω. Introduce local coordinates $(\theta_1, \cdots, \theta_{d-1})$ on S^{d-1} satisfying the equality

$$T_W = \frac{1}{\sqrt{1+|\nabla W|^2}}\Bigl(W_{x_1}, \dots, W_{x_{d-1}}, -1\Bigr) = \Bigl(\theta_1, \theta_2, \dots, \theta_{d-1}, -\sqrt{1-\sum_{i=1}^{d-1}\theta_i^2}\Bigr).$$

Clearly, the first $d-1$ basis vectors of the ambient space constitute an orthogonal basis in the tangent space to S^{d-1} at x_0. Note that

$$\nabla W(x) = \frac{\theta}{\sqrt{1-|\theta|^2}},$$

where $\theta = (\theta_1, \dots, \theta_{d-1})$, $x = (x_1, \dots, x_d)$ and

$$H = \sum_{i=1}^{d-1} \theta_i x_i - \sqrt{1-|\theta|^2} x_d.$$

Set $W^*(x) = \sup_{y\in\Omega}(\langle x, y\rangle - W(y))$. Since ∇W and ∇W^* are reciprocal, one has

$$x = \nabla W^*\Bigl(\frac{\theta}{\sqrt{1-|\theta|^2}}\Bigr).$$

Taking into account that $x_d = W(x)$, one has

$$H = \Bigl\langle \theta, \nabla W^*\Bigl(\frac{\theta}{\sqrt{1-|\theta|^2}}\Bigr)\Bigr\rangle - \sqrt{1-|\theta|^2}\cdot W\Bigl(\nabla W^*\Bigl(\frac{\theta}{\sqrt{1-|\theta|^2}}\Bigr)\Bigr).$$

Hence

$$H(\theta) = \sqrt{1-|\theta|^2}\; W^*\Bigl(\frac{\theta}{\sqrt{1-|\theta|^2}}\Bigr)$$

on $T_W(\Omega)$. This is equivalent to

$$W^*(x) = \sqrt{1+|x|^2}\; H\Bigl(\frac{(x,-1)}{\sqrt{1+|x|^2}}\Bigr)$$

on Ω. By approximation arguments these relations remain valid for every convex W in Ω. Now assume that H is twice Alexandrov differentiable at 0. Clearly, $H(0) = 0$, $\nabla H(0) = 0$. The same holds for W^*. Using Alexandrov differentiability of H, we get

$$W^*(x) = \Bigl(\frac{x^2}{2}H(0,-1) + \frac{1}{2}\langle\bigl(D^2_\theta\bigr)_a H(0,-1)x, x\rangle\Bigr) + o(x^2).$$

This means that $D^2_a W^* = H \cdot \mathrm{Id} + \bigl(D^2_\theta\bigr)_a H$. We get 1) by the duality relations for convex functions (see, for instance, [29]). The opposite implication follows by the same arguments. $\square$

Clearly, if the surface is smooth and strictly convex, in the situation of the Lemma 3.4 one has

$$K = \frac{1}{\det\bigl(H\cdot\mathrm{Id} + D^2_\theta H\bigr)\circ \mathrm{n}}.$$

Definition 3.5. Let A be an arbitrary convex surface. We call the following quantity K "Gauss curvature of ∂A at x"

$$K(x) \colon = \frac{1}{\det\big(H \cdot \mathrm{Id} + \big(D^2_\theta\big)_a H\big) \circ \mathrm{n(x)}} \tag{9}$$

if there exists a unique normal n(x), H is twice Alexandrov differentiable at n(x) and $H \cdot \mathrm{Id} + \big(D^2_\theta\big)_a H$ is nondegenerate.

If n(x) is unique, but H is not twice Alexandrov differentiable at n(x), we set $K(x) = 0$. The latter is equivalent to $\det D^2_a W(x_1, \ldots, x_{d-1}) = 0$ if ∂A coincides locally with a graph of $W \colon \mathbb{R}^{d-1} \to \mathbb{R}$.

Remark 3.6. Clearly, by Lemma 3.4 $K(x)$ is well-defined for $\mathcal{H}^{d-1}|_{\partial A}$-almost all x, since W is $\mathcal{H}^{d-1}$-a.e. differentiable.

Remark 3.7. In the special coordinate system considered in the proof of Lemma 3.4 one has $K = \det D^2_a W$. Following the proof of Lemma 3.4 one can easily understand that (8) holds almost everywhere in a non-smooth setting with $F = W - x_d$ if the second derivative of W is understood in the Alexandrov sense.

Recall an important result of McCann [29].

Theorem (McCann). (Change of variables formula for convex functions.) *Let $\mu = f\ dx$ and $\nu = g\ dx$ be two probability measures and V be a convex function such that $\nu = \mu \circ \nabla V^{-1}$. Then for μ-almost all x one has*

$$g(\nabla V) \det D^2_a V = f,$$

where $D^2_a V$ is the second Alexandrov derivative of V.

In the following proposition we deal with the Gauss map $\mathrm{n} \colon \partial A \to S^{d-1}$ (non multivaled!) which is $\mathcal{H}^{d-1}$-a.e. well defined.

Proposition 3.8. *For every $A_t = \{x \colon \varphi(x) \le t\}$ the measure $\Big(K \cdot \mathcal{H}^{d-1}|_{\partial A_t}\Big) \circ \mathrm{n}^{-1}$ is absolutely continuous with respect to $\mathcal{H}^{d-1}$ and the following change of variables formula holds for every bounded Borel function $f \colon S^{d-1} \to \mathbb{R}$:*

$$\int_{\partial A_t} f(\mathrm{n}) K \ d\mathcal{H}^{d-1} = \int_{\mathrm{n}(\partial A_t)} f \ d\mathcal{H}^{d-1}.$$

Proof. It is sufficient to prove this result for $\partial A_t \cap V$ instead of ∂A_t, where V is a small neighborhood of a point $x_0 \in \partial A_t$ with unique $n(x_0)$. Fix such a point and choose a coordinate system in such a way that $n(x_0) = e_d$ and the surface ∂A_t coincides (locally) with the graph of a convex function $W \colon \mathrm{U} \subset \mathbb{R}^{d-1} \to \mathbb{R}$, where U is an open ball containing 0 and W attains its minimum at 0. In addition, we may assume that $\partial W(\mathrm{U})$ is a bounded set. Let U be a local chart of $\partial A_t \cap V$ and parametrize a part of $\partial A_t \cap V$ in the following way

$$\mathrm{U} \ni (x_1, \ldots, x_{d-1}) \to (x_1, \ldots, x_{d-1}, W(x)).$$

Since W is Lipschitz on U, the surface measure $\mathcal{H}^{d-1}$ on $\partial A_t \cap V$ can be computed in this chart by $m_0 = (1 + |\nabla W|^2)^{\frac{1}{2}} \mathcal{H}^{d-1}$. The Gauss map n is given by

$$\mathrm{n} = \frac{1}{\sqrt{1 + |\nabla W|^2}} (-\partial_{x_1} W, \ldots, -\partial_{x_{d-1}} W, 1).$$

This holds for almost every $(x_1, \ldots, x_{d-1})$.

Identify the half-sphere $S^{d-1}\cap\{x_d \le 0\}$ with its projection B_1^{d-1} on $(x_1, \dots x_{d-1})$ and n with the mapping ñ: $-\frac{\nabla W}{\sqrt{1+|\nabla W|^2}}$ taking values in B_1^{d-1}. Note that the surface measure on S^{d-1} can be computed in the local chart

$$(x_1, \dots, x_{d-1}) \to (x_1, \dots, x_{d-1}, \sqrt{1 - x_1^2 - \dots - x_{d-1}^2})$$

by $m_1 = \frac{1}{\sqrt{1-|x|^2}} \mathcal{H}^{d-1}$.

Note that $\tilde{n} = F \circ \nabla W$ is the composition of ∇W with the smooth mapping

$$F(x) = -\frac{x}{\sqrt{1+|x|^2}}$$

which is nondegenerate everywhere.

Writing the local chart expressions we get that the claim is equivalent to the equality $m_1|_{\tilde{n}(M_+)} = (K \cdot m_0)|_{M_+} \circ \tilde{n}$, where $M_+ = \{x \in V\colon \det D_a^2 W > 0\}$.

Note that K is well-defined on M_+. By Remark 3.7 we have

$$K = \det D_a \tilde{n} = \det D_a W \cdot \det DF(x) \circ (\nabla W).$$

By the result of McCann the optimal transport ∇W pushes forward

$$\det D_a^2 W \cdot \mathcal{H}^{d-1}|_{M_+}$$

to $\mathcal{H}^{d-1}|_{\nabla W(M_+)}$. Hence we obtain that the image of the measure

$$K(1+|\nabla W|^2)^{\frac{1}{2}} \mathcal{H}^{d-1}|_{M_+} = K \cdot m_0|_{M_+}$$

under ∇W coincides with

$$\det F(x)(1+|x|^2)^{\frac{1}{2}} \mathcal{H}^{d-1}|_{\nabla W(M_+)}.$$

Now applying the standard change of variables formula we get that the image of

$$\det F(x)(1+|x|^2)^{\frac{1}{2}} \mathcal{H}^{d-1}|_{\partial W(M_+)}$$

under $y = F(x)$ coincides with $\frac{1}{\sqrt{1-|y|^2}} \mathcal{H}^{d-1}|_{\tilde{n}(M_+)} = m_1|_{\tilde{n}(M_+)}$. The proof is complete. $\square$

The fact below follows easily from the McCann's theorem.

Corollary 3.9. *If V is a convex function satisfying $\det D_a^2 V|_M = 0$ for some set M with $\lambda(M) > 0$, then the image of $\lambda|_M$ under ∇V is singular to λ.*

The proof of the following lemma can be found, for instance, in [7].

Lemma 3.10. *If V is a convex function satisfying $D_a^2 V > 0$ on M, then $\lambda|_M \circ \nabla V^{-1}$ is an absolutely continuous measure.*

We prove an analog of the McCann's theorem for the Gauss mass transport. We start with a change of variables formula for the mapping $\mathcal{T}^{-1}$ defined by a monotone-convex potential u. Since u and H are related by a smooth change of variables, it gives immediately a change of variables formula for H.

Remark 3.11. We recall that $u(z, r)$ (see Section 2) is convex in z and increasing in r. Hence one can define u_r and $(D_z^2)_a u$ $\mathcal{H}^d$-almost everywhere, where u_r means a partial derivative of u in the classical sense.

Theorem 3.12. **(Change of variables formula for u)** *The potential u satisfies the change of variables formula for $\mathcal{H}^d$-almost all $(z,r) \subset \mathbb{R}^{d-1} \times [0,R]$*

$$u_r \cdot \det{(D^2_z)_a u} = \frac{\tilde{\rho}_1}{\rho_0(\mathcal{T}^{-1})},$$

where

$$\tilde{\rho}_1 = \Bigl[\frac{r}{1+z_1^2+\ldots+z_{d-1}^2}\Bigr]^{d-1} \rho_1\Bigl(\frac{rz_1,\ldots,rz_{d-1},-r}{\sqrt{1+z_1^2+\ldots+z_{d-1}^2}}\Bigr).$$

Proof. Fix an orthogonal coordinate system $(x_1,\ldots,x_d)$ and denote by $\tilde{e}_i$ the corresponding basis. Recall that mapping $\mathcal{T}^{-1}$ sends $\tilde{\nu} = \tilde{\rho}_1 dx|_{\mathbb{R}^d \times [0,R]}$ to $\mu|_{\mathcal{T}^{-1}(\{x_d<0\})}$ and admits a.e. the representation

$$\mathcal{T}^{-1} = \sum_{i=1}^{d-1} u_{z_i} \cdot \tilde{e}_i + u^*(\nabla_z u) \cdot \tilde{e}_d,$$

where $u^*(z) = \sup_{x\in\mathbb{R}^{d-1}}\bigl(\langle x,z\rangle - u(x)\bigr)$. Let us represent $\mathcal{T}^{-1}$ as the composition of two mappings $\mathcal{T}^{-1} = S_2 \circ S_1$, where $S_1 \colon \mathbb{R}^{d-1}\times[0,R] \to \mathbb{R}^{d-1}\times[0,R]$ has the form

$$S_1(z,r) = (\nabla_z u, r)$$

(all expressions are written in the Euclidean (z,r)-coordinates!) and

$$S_2(z,r) = \sum_{i=1}^{d-1} z_i \cdot \tilde{e}_i + u^*(z,r)\cdot \tilde{e}_d.$$

Let us show that $\det\bigl(D^2_z\bigr)_a u > 0$ almost everywhere. Indeed, set

$$M \colon = \{(z,r)\colon \det\bigl(D^2_z\bigr)_a u = 0\}.$$

Assume that $\lambda(M) > 0$. Then by Corollary 3.9 and Fubini's theorem

$$\tilde{\nu} = \tilde{\rho}_1 \cdot \mathcal{H}^{d-1}|_M \circ S_1^{-1}$$

is a singular measure. Let us disintegrate $\tilde{\nu}$ along the r-axis:

$$\tilde{\nu}(r,z) = \nu^z(dr)\cdot\mu_0(dz).$$

Here μ_0 is the projection of $\tilde{\nu}$ onto $\mathbb{R}^{d-1}$ and $\nu^z(dr)$ are the corresponding conditional measures.

Denote by $\tilde{\nu}_0$ the projection of $\tilde{\nu}$ onto $(z_1,\ldots,z_{d-1})$. It follows from the relation $\tilde{\nu}\circ S_2^{-1} = \mu|_{\mathcal{T}^{-1}(\{x_d<0\})}$ that the image of $\tilde{\nu}_0 = \bigl(\int \nu^z(dr)\bigr)\cdot\mu_0(dz)$ under

$$(z_1,\ldots,z_{d-1}) \to \sum_{i=1}^{d-1} z_i \cdot \tilde{e}_i \tag{10}$$

coincides with the projection of $\mu|_{\mathcal{T}^{-1}(\{x_d<0\})}$ onto $(x_1,\ldots,x_{d-1})$. Since the latter admits a Lebesgue density and (10) is smooth and nondegenerate, one gets that $\mu_0(dz)$ admits a Lebesgue density $f_0(z)$. Hence for $\mathcal{H}^{d-1}$-almost all z the one-dimensional measure $\nu^z(dr)|_{S_1(M)}$ is singular. Note for $\mathcal{H}^{d-1}$-almost every fixed z the mapping $r \mapsto u^*(z,r)$ pushes forward $\nu^z(dr)|_{S_1(M)}$ to a one-dimensional absolutely continuous measure. Since $\nu^z(dr)|_{S_1(M)}$ is a singular measure, one has

$$u^*_r(r,z) = -\infty$$

(note that u^* is decreasing in r) for $\mathcal{H}^{d-1}$-almost all z and $\nu^z(dr)_{S_1(M)}$-almost all r. This follows by duality from Corollary 3.9.

Next we note that

$$u_r^*(\nabla_z u, r) = -u_r(z, r)$$

λ-a.e. Indeed, in the case of smooth functions with non-degenerated second derivative it follows by differentiating the duality relation

$$u^*(\nabla_z u, r) + u = \langle \nabla_z u, z \rangle$$

in r. The general case easily follows by approximations.

Thus $u_r = \infty$ $\tilde{\nu}$-almost surely on M. But this contradicts the assumption $\lambda(M) > 0$.

Since $\det\big(D_z^2\big)_a u > 0$, by Lemma 3.10 the image $\tilde{\nu}$ under S_1 is absolutely continuous. Then one can apply the McCann's change of variables formula for $\mathcal{H}^1$-almost every fixed value of r. Applying the same theorem once again to S_2 (for $\mathcal{H}^{d-1}$-almost every fixed z) one gets the result. $\square$

Corollary 3.13. **(Change of variables formula for H)** *Since $\mathcal{T}^{-1}$ and T^{-1} are related by a smooth change of variables, one immediately gets*

$$\rho_1 = \rho_0(T^{-1}) \frac{H_r \cdot \det(H \cdot \mathrm{Id} + \big(D_\theta^2\big)_a H)}{r^{d-1}}$$

$\mathcal{H}^d$-almost everywhere on B_R.

Corollary 3.14. *The Gauss curvature $K(x) = \det(H \cdot \mathrm{Id} + \big(D_\theta^2\big)_a H)$ is well-defined and positive for μ-almost all x.*

Recall that $D\varphi$ denotes the generalized derivative of φ in the distributional sense. Since φ has convex sublevel sets, it is a BV function (see [1]). Hence $D\varphi$ can be understood as a vector-valued measure satisfying

$$\int \langle D\varphi, \xi \rangle \ dx = -\int \varphi \ \mathrm{div}\xi \ dx$$

for every smooth compactly supported vector field ξ. We denote by $\|D\varphi\|$ the corresponding total variational measure and by $|D_a\varphi|$ its absolutely continuous component and by $|D_s\varphi|$ its singular component.

Theorem 3.15. **(Change of variables formula for φ)** *The following change of variables formula holds for μ-almost all $x \in A$:*

$$\big(K|D_a\varphi|\varphi^{d-1}\big)(x) \ \rho_1(T(x)) = \rho_0(x).$$

Proof. Let $\tilde{A} \subset A$ be a set, where K is well-defined and positive. By the previous corollary $\mu(\tilde{A}) = 1$. Let us show that $|D_s\varphi|(\tilde{A}) = 0$. Indeed, otherwise we can find a set $M_s \subset \tilde{A}$ with $\lambda(M_s) = 0$ and $|D_s\varphi|(M_s) > 0$. By the coarea formula for BV functions (see [1], p. 159)

$$0 < \int_{M_s} K \ d|D_s\varphi| = \int_{M_s} K \ d\|D\varphi\| = \int_0^r \int_{\partial A_t \cap M_s} K \ d\mathcal{H}^{d-1} dt.$$

By Proposition 3.8 and Fubini's theorem the latter equals

$$\int_0^r \int_{\partial B_t \cap T(M_s)} t^{-(d-1)} \ d\mathcal{H}^{d-1} dt = \int_{T(M_s)} |y|^{-(d-1)} d\mathcal{H}^d.$$

Since T pushes forward μ to ν, one has $\lambda(T(M_s)) = 0$. We get a contradiction.

Applying again the coarea formula for BV functions we get

$$\int_{\tilde{A}} \xi(T)K\ |D_a\varphi| dx = \int_0^r \int_{\partial A_t \cap \tilde{A}} \xi(T)K\ d\mathcal{H}^{d-1} dt,$$

for any Borel bounded function ξ. By Proposition 3.8

$$\int_{\partial A_t \cap \tilde{A}} \xi(T)K\ d\mathcal{H}^{d-1} = \int_{B_t} \xi(y)\ d\mathcal{H}^{d-1}(y)$$

for almost all $t \in [0, R]$. Since T takes $\rho_0 dx$ to $\rho_1 dx$, one gets

$$\int_{B_r} \xi(y)\ \frac{K|D_a\varphi|}{\rho_0} \circ T^{-1}(y)\ \rho_1(y) dy = \int_0^r \Big(\int_{B_t} \xi(y)\ d\mathcal{H}^{d-1}(y) \Big) |t|^{-(d-1)} dt,$$

Hence, for μ-almost all $y \in B_r$, one has $\frac{K|D_a\varphi|}{\rho_0} \circ T^{-1}(y)\ \rho_1(y) = |y|^{-(d-1)}$. The proof is complete. □

Corollary 3.16. *Comparing different change of variables formulae, one gets*

$$|D_a\varphi| = \frac{1}{H_r(T)}$$

μ-almost everywhere.

4. Sobolev estimates for φ

The main goal of this section is to establish some natural Sobolev estimates for φ (Theorem 4.5). The proof is based on the integration-by-parts and change of variables formulae.

Before proving Theorem 4.5 we establish some $|\nabla\varphi|_{L^\infty}$-bounds with the help of the classical maximum principle. These estimates have an interest in their own, they will also serve as an intermediate step in Theorem 7.1.

It will be assumed below that $H_r, H_\theta, H_{\theta\theta}$ are continuous and continuously differentiable in r (except, maybe, the origin) up to the boundary. We also assume without loss of generality that $H \geq 0$ and $H(0) = 0$ (this can be achieved just by shifting A and assuming that $\varphi(0) = 0$). The estimates obtained below do not depend, however, on higher derivatives (see in this respect Remark 4.4).

Let us set

$$P = \rho_1 r^{d-1}.$$

Since H is smooth, it satisfies

$$P = \rho_0(T^{-1})\ H_r \cdot \det(H \cdot \mathrm{Id} + D^2_\theta H)$$

up to ∂B_R. We recall that $H_r = 1/|\nabla\varphi(T^{-1})|$.

Proposition 4.1. *a) Assume that for some $C > 0$*

$$|\nabla\rho_0| \leq C\rho_0^{1+\frac{1}{d}}, \quad P \leq C,$$

and there exists $u\colon (0, R] \to \mathbb{R}$ with $u \in L^1([a, R])$ for every $R > a > 0$ such that

$$\frac{P_r}{P} \leq u(r).$$

In addition, assume that ∂A is smooth, $\lambda_0 = \inf_{x \in \partial A} K(x) > 0$ and $\rho_0|_{\partial A} \leq C$, $P|_{\partial B_R} \geq \frac{1}{C}$. Then

$$H_r \geq D_1 \exp\big(-\int_r^R u(s) ds\big).$$

In particular

$$|\nabla\varphi| \le D_2 \exp\Big(\int_{\varphi}^{R} u(s)ds\Big)$$

with D_1, D_2 depending on d, C, λ_0, R.

b) Assume that for some $C > 0$

$$\frac{|\nabla\rho_0|}{\rho_0} \le C,\ \ \rho_0 \ge \frac{1}{C},\ \ P \le C$$

and

$$\frac{P_r}{P} \ge -C.$$

In addition, assume that ∂A is smooth, $\Lambda_0 = \sup_{x\in\partial A} K(x) < \infty$, $\rho_0|_{\partial A} \ge C$, $P|_{\partial B_R} \le \frac{1}{C}$ and

$$H \ge \varepsilon r$$

for some $\varepsilon > 0$. Then

$$H_r \le \frac{D_1}{r^d},\ \ |\nabla\varphi| \ge D_2\varphi^d$$

with D_1, D_2 depending on $d, C, R, \varepsilon, \Lambda_0$.

Proof. a) We are looking for the minimum of $H_r e^f$ on $B_R \setminus B_{r_0}$ $r_0 > 0$, where $f = f(r)$ is a radially symmetric function to be chosen later. Assume that the minimum is attained at some point $x_0 \notin \partial B_R$. We deal with the local coordinate system (r,θ) as described at Section 2. Let us differentiate $\log H_r + f(r)$ along r and every θ_i at this point. One has

$$H_{r\theta_i} = 0,\ \ \frac{H_{rr}}{H_r} \ge -f_r.$$

The second differentiation yields $H_{r\theta_i\theta_i} \ge 0$. Rotating the coordinate system when necessary we may assume that $D^2_{\theta\theta}H$ is diagonal at x_0. Differentiating the change of variables formula in r yields

$$\frac{P_r}{P} = \frac{\langle \nabla\rho_0(T^{-1}), H_r\cdot \mathrm{n} + \sum_{i=1}^{d-1} H_{r\theta_i}\cdot e_i\rangle}{\rho_0(T^{-1})} + \frac{H_{rr}}{H_r} + \sum_{i=1}^{d-1}\frac{H_r + H_{r\theta_i\theta_i}}{H + H_{\theta_i\theta_i}}.$$

Hence

$$\begin{aligned}\frac{P_r}{P} &\ge H_r\Big[\frac{\langle\nabla\rho_0(T^{-1}), n\rangle}{\rho_0(T^{-1})}\Big] - f_r + H_r\sum_{i=1}^{d-1}\frac{1}{H+H_{\theta_i\theta_i}}\\ &\ge H_r\Big[\frac{\langle\nabla\rho_0(T^{-1}), n\rangle}{\rho_0(T^{-1})}\Big] - f_r + (d-1)H_r\Big[\frac{1}{\det(H + D^2_{\theta\theta}H)}\Big]^{\frac{1}{d-1}}\\ &\ge H_r\Big[\frac{\langle\nabla\rho_0(T^{-1}), n\rangle}{\rho_0(T^{-1})}\Big] - f_r + (d-1)H_r^{\frac{d}{d-1}}\Big[\frac{\rho_0(T^{-1})}{P}\Big]^{\frac{1}{d-1}}.\end{aligned}$$

This implies

$$\begin{aligned}H_r^{\frac{d}{d-1}} \le{}& \frac{1}{d-1}\Big[\frac{P}{\rho_0(T^{-1})}\Big]^{\frac{1}{d-1}}\Big[\frac{P_r}{P} + f_r\Big]\\ &- H_r\frac{P^{\frac{1}{d-1}}}{(d-1)}\Big[\langle\nabla\rho_0(T^{-1}), n\rangle\rho_0^{-\frac{d}{d-1}}(T^{-1})\Big].\end{aligned}$$

Applying Hölder's inequality one gets

$$H_r^{\frac{d}{d-1}} \le \frac{1}{d-1}\Big[\frac{P}{\rho_0(T^{-1})}\Big]^{\frac{1}{d-1}}\Big[\frac{P_r}{P}+f_r\Big]+\frac{1}{2}H_r^{\frac{d}{d-1}}$$
$$+\,C_1\Big[P^{\frac{1}{d-1}}\big|\nabla\rho_0(T^{-1})\big|\rho_0^{-\frac{d}{d-1}}(T^{-1})\Big]^d.$$

where C_1 depends only on d. Let f be of the type

$$f=-C_2r-\int_0^r u(s)ds.$$

One gets

$$\frac{1}{2}H_r^{\frac{d}{d-1}} \le -\frac{C_2}{d-1}\Big[\frac{P}{\rho_0(T^{-1})}\Big]^{\frac{1}{d-1}}+C_1\Big[P^{\frac{1}{d-1}}\big|\nabla\rho_0(T^{-1})\big|\rho_0^{-\frac{d}{d-1}}(T^{-1})\Big]^d.$$

Then it follows from the assumption of the proposition that the right-hand is negative for a sufficiently large $C_2>0$. This contradicts the estimate $H_r\ge 0$.

This means that

$$H_r\exp\big(-C_2r-\int_0^r u(s)ds\big)$$

can attain its minimum only at ∂B_R. Taking into account that

$$H_r|_{\partial B_R}\ge C_3(C,R)\inf_{x\in\partial A}K(x)$$

one gets the desired estimate.

b) In the proof we use an idea from [34]. We are looking for the maximum of

$$\frac{H_r}{H-g(r)}$$

on $B_R\setminus B_{r_0}$, where $g=\frac{\varepsilon}{2}r$. Note that $H-g\ge\frac{\varepsilon}{2}r$. Assume that $\log H_r-\log(H-g(r))$ attains its maximum at x_0 with $|x_0|<R$ (otherwise the estimate is trivial). Then at this point

$$\frac{H_{rr}}{H_r}-\frac{H_r-g'}{H-g}\le 0,\ \ \frac{H_{r\theta_i}}{H_r}-\frac{H_{\theta_i}}{H-g}=0.$$

The second differentiation gives

$$\frac{H_{r\theta_i\theta_i}}{H_r}\le\frac{H_{\theta_i\theta_i}}{H-g}.$$

Differentiating the change of variables formula one obtains

$$\frac{P_r}{P}=\frac{H_{rr}}{H_r}+\sum_{i=1}^{d-1}\frac{H_r+H_{r\theta_i\theta_i}}{H+H_{\theta_i\theta_i}}+\frac{1}{\rho_0(T^{-1})}\langle\nabla\rho_0(T^{-1}),T_r^{-1}\rangle.$$

Hence

$$\sum_{i=1}^{d-1}\frac{H_r+H_{r\theta_i\theta_i}}{H+H_{\theta_i\theta_i}}\le\frac{H_r}{H-g}\sum_{i=1}^{d-1}\frac{H-g+H_{\theta_i\theta_i}}{H+H_{\theta_i\theta_i}}=\frac{H_r}{H-g}\Big(d-1-g\sum_{i=1}^{d-1}\frac{1}{H+H_{\theta_i\theta_i}}\Big)$$
$$\le(d-1)\frac{H_r}{H-g}\Big(1-g\sqrt[d-1]{\prod_{i=1}^{d-1}\frac{1}{H+H_{\theta_i\theta_i}}}\Big)$$
$$=(d-1)\frac{H_r}{H-g}\Big(1-g\sqrt[d-1]{H_r}\sqrt[d-1]{\frac{\rho_0(T^{-1})}{P}}\Big).$$

Next using

$$T_r^{-1} = H_r \cdot \mathrm{n} + \sum_{i=1}^{d-1} H_{r\theta_i} \cdot \mathrm{e_i}$$

we get

$$\frac{1}{\rho_0(T^{-1})}\langle \nabla\rho_0(T^{-1}), T_r^{-1}\rangle = \frac{\langle \nabla\rho_0(T^{-1}), \mathrm{n}\rangle}{\rho_0(T^{-1})} H_r + \sum_{i=1}^{d-1} H_{r\theta_i} \frac{\langle \nabla\rho_0(T^{-1}), \mathrm{e_i}\rangle}{\rho_0(T^{-1})}$$
$$= \frac{H_r}{\rho_0(T^{-1})}\Big(\langle \nabla\rho_0(T^{-1}), \mathrm{n}\rangle + \frac{1}{H-g}\sum_{i=1}^{d-1} H_{\theta_i}\langle \nabla\rho_0(T^{-1}), \mathrm{e_i}\rangle\Big).$$

Taking into account the assumptions, boundedness of H_{θ_i} and H, we get

$$\frac{1}{\rho_0(T^{-1})}\langle \nabla\rho_0(T^{-1}), T_r^{-1}\rangle \le C_1 \frac{H_r}{H-g}.$$

Thus we obtain

$$\frac{P_r}{P} \le \frac{H_r - g'}{H-g} + (d-1)\frac{H_r}{H-g}\Big(1 - g \sqrt[d-1]{H_r} \sqrt[d-1]{\frac{\rho_0(T^{-1})}{P}}\Big) + C_1 \frac{H_r}{H-g}.$$

Multiplying this inequality by $H-g$, using the assumptions of the theorem and boundedness of H we get

$$-C(H-g) \le C_2 H_r - g' - (d-1) g H_r^{\frac{d}{d-1}} \sqrt[d-1]{\frac{\rho_0(T^{-1})}{P}}.$$

Thus implies

$$H_r^{\frac{d}{d-1}} \le C_4 \frac{(H-g)}{g} + \frac{C_4}{g}\Big(-g' + C_2 H_r\Big) \le \frac{C_5 + C_6 H_r}{r} \le \frac{1}{2} H_r^{d/(d-1)} + C_7\Big(\frac{1}{r}\Big)^d.$$

Hence

$$\Big(\frac{H_r}{H-g}\Big)^{d/(d-1)} \le \frac{C_8}{r^{d+\frac{d}{d-1}}}.$$

This gives the desired result. $\square$

Remark 4.2. The proof of a) can be generalized to the case of pre-limiting potentials H_t (see Section 2). Since the computations are quite involved, we give only some intermediate results. For simplicity let us skip the index t and write H instead of H_t. Choose a function f in such a way that $f(x) \sim -(d-1)\ln(r-r_0)^+$ for x close to ∂B_{r_0} and assume that the minimum point x_0 does not belong to ∂B_R. One has

$$T^{-1} = \Big(H + \frac{r}{t+1}H_r\Big)\cdot \mathrm{n} + \sum_{i=1}^{d-1} H_{\theta_i} \cdot \mathrm{e_i}.$$

The derivatives of T^{-1} at x_0 satisfy

$$T_r^{-1} = \Big(\frac{t+2}{t+1}H_r + \frac{r}{t+1}H_{rr}\Big)\cdot \mathrm{n} + \sum_{i=1}^{d-1} H_{r\theta_i} \cdot \mathrm{e_i}.$$

$$T_{\theta_i}^{-1} = \Big(\frac{r}{t+1}H_{r\theta_i}\Big)\cdot \mathrm{n} + \sum_{i \ne j} H_{\theta_i\theta_j} \cdot \mathrm{e_j}. + \Big(\frac{t}{t+1}H + \frac{r}{t+1}H_r + H_{\theta_i\theta_i}\Big)\mathrm{e_i}.$$

Choosing an appropriate basis, we may assume without loss of generality that

$$H_{\theta_i\theta_j} = 0$$

for $i \neq j$. Then

$$\det DT^{-1} = \Big(\frac{t+2}{t+1}H_r + \frac{rH_{rr}}{1+t}\Big)\prod_{i=1}^{d-1}\Big[\Big(\frac{t}{1+t}H + \frac{rH_r}{1+t}\Big) + H_{\theta_i\theta_i}\Big] +$$
$$- \sum_{i=1}^{d} \frac{rH^2_{r\theta_i}}{1+t}\prod_{i\neq j}\Big(\Big[\frac{t}{1+t}H + \frac{rH_r}{1+t}\Big] + H_{\theta_j\theta_j}\Big).$$

At the minimum point one has

$$\frac{H_{rr}}{H_r} = -f', \ H_{r\theta_i} = 0, \tag{11}$$

$$\frac{H_{rrr}}{H_r} + (f'' - (f')^2) \geq 0, \ H_{r\theta_i\theta_i} \geq 0. \tag{12}$$

The reasoning from the above proposition leads to the following estimate:

$$\frac{(\rho_1)_r}{\rho_1} + \frac{(d-1)}{r} \geq H_r\Big(\frac{t+2}{t+1} - \frac{r}{t+1}f'\Big)\frac{\langle \mathrm{n}, \nabla\rho_0(\mathrm{T}^{-1})\rangle}{\rho_0(T^{-1})}$$
$$+ \frac{-(t+3)f' + r((f')^2 - f^{''})}{(t+2) - rf'}$$
$$+ (d-1)\frac{H_r^{\frac{d}{d-1}}}{r}\Big(1 - \frac{rf'}{t+1}\Big)\Big[\Big(\frac{t+2}{t+1} - \frac{rf'}{1+t}\Big)\Big]^{\frac{1}{d-1}}\Big[\frac{\rho_0(T^{-1})}{\rho_1}\Big]^{\frac{1}{d-1}}.$$

Choosing an appropriate f one gets the desired bound.

Corollary 4.3. *Assume that*

$$P < C, \frac{1}{C} \leq \rho_0,$$

∂A is smooth and uniformly convex,

$$\Big|\frac{\nabla\rho_0}{\rho_0}\Big|, \Big|\frac{P_r}{P}\Big| < C$$

and $\rho_0|_{\partial A} \leq C$, $\frac{1}{C} \leq P|_{\partial B_R}$. Then $D_1\varphi^d < |\nabla\varphi| < D_2$ for some $D_1, D_2 > 0$ depending only on d, C and ∂A.

Remark 4.4. We have proved the above estimates assuming smoothness of H. But the final results do not depend on the bounds of the derivatives of H. We give some sufficient conditions for H to be smooth in Section 7. Applying smooth approximations it is possible to show that the estimates remain true without extra smoothness assumption of the solution. In particular, the upper bound on $|\nabla\varphi|$ implies the absence of a singular part for $D\varphi$.

Theorem 4.5. *Assume that $\rho_1 = \frac{C}{r^{d-1}}$. Then for every $p > 0$ there exist $C_{p,R} > 0$ such that*

$$C_{p,R}\int_A |\nabla\varphi|^{p+1}\ d\mu \leq \int_A \Big|\frac{\nabla\rho_0}{\rho_0}\Big|^{p+1}\ d\mu + \int_{\partial A}|\nabla\varphi|^p\rho_0\ d\mathcal{H}^{d-1}, \tag{13}$$

$$C_{p,R}\int_A |\nabla\varphi|^{p+1}\ d\mu \leq \int_A \Big|\frac{\nabla\rho_0}{\rho_0}\Big|^{p+1}\ d\mu + \int_{\partial A} K^{-p}\rho_0^{p+1}\ d\mathcal{H}^{d-1}. \tag{14}$$

Proof. Under assumptions of the theorem the change of variables formula reads as

$$CK|\nabla\varphi| = \rho_0.$$

Computing DT is the standard frame $\{\mathrm{n}, \mathrm{v}_1, \cdots, \mathrm{v}_{\mathrm{d}-1}\}$, we get

$$DT = \begin{pmatrix} |\nabla\varphi| & 0 \\ b^t & A \end{pmatrix},$$

where

$$b = \Bigl(\frac{\varphi\varphi_{\mathrm{nv_i}}}{|\nabla\varphi|}\Bigr), \quad A = \Bigl(\frac{\varphi\varphi_{\mathrm{v_iv_j}}}{|\nabla\varphi|}\Bigr).$$

The Jacobian matrix of $S = T^{-1}$ computed (r,θ) coordinates has the form (recall that $\partial_n = \partial_r$ and $\partial_{\theta_{\mathrm{i}}} = r\partial_{\mathrm{v_i}}$)

$$DS = \begin{pmatrix} H_r & 0 \\ c^t & B \end{pmatrix},$$

with

$$c = \bigl(H_{r\theta_{\mathrm{i}}}\bigr), \quad B = \Bigl(\frac{H + H_{\theta_{\mathrm{i}}\theta_{\mathrm{j}}}}{r}\Bigr).$$

Recall that $H_r(T) = \frac{1}{|\nabla\varphi|}$. Since $DS(T) = DT^{-1}$, one can also assume that A and B are diagonal (at a fixed point). Denote by λ_i the eigenvalues of A. Then using $DT \circ DS(T) = \mathrm{Id}$ one easily obtains

$$\frac{\varphi\varphi_{\mathrm{nv_i}}}{|\nabla\varphi|^2} + H_{r\theta_{\mathrm{i}}}(T)\lambda_i = 0.$$

Next we find

$$\begin{aligned}
\varphi_{nn} = \partial_n|\nabla\varphi| = \partial_n(1/H_r(T)) &= -\frac{1}{H_r^2(T)}\Bigl(H_{rr}(T)|\nabla\varphi| + \sum_{i=1}^{d-1} H_{r\theta_i}(T)\langle\partial_{\mathrm{n}}\mathrm{n}, \mathrm{v_i}\rangle\Bigr)\\
&= -\frac{1}{H_r^2(T)}\Bigl(H_{rr}(T)|\nabla\varphi| + \sum_{i=1}^{d-1} H_{r\theta_{\mathrm{i}}}(T)\frac{\varphi_{\mathrm{nv_i}}}{|\nabla\varphi|}\Bigr)\\
&= -\frac{1}{H_r^2(T)}\Bigl(H_{rr}(T)|\nabla\varphi| - \sum_{i=1}^{d-1} \varphi\frac{\varphi^2_{nv_i}}{\lambda_i|\nabla\varphi|^3}\Bigr) \ge -\frac{H_{rr}(T)}{H_r^3(T)}.
\end{aligned}$$

Taking into account that φ has convex level sets (hence $\mathrm{div}\frac{\nabla\varphi}{|\nabla\varphi|} \ge 0$), we get

$$\mathrm{div}\Bigl(\varphi\frac{\nabla\varphi}{|\nabla\varphi|}|\nabla\varphi|^p\Bigr) \ge |\nabla\varphi|^{p+1} + p\varphi|\nabla\varphi|^{p-1}\varphi_{nn} \ge |\nabla\varphi|^{p+1} - p\frac{rH_{rr}}{H_r^{p+2}} \circ T.$$

Thus

$$|\nabla\varphi|^{p+1} \le \mathrm{div}\Bigl(\varphi\frac{\nabla\varphi}{|\nabla\varphi|}|\nabla\varphi|^p\Bigr) + p\frac{rH_{rr}}{H_r^{p+2}} \circ T. \tag{15}$$

Integrate (15) over A with respect to μ. One obtains

$$\begin{aligned}
\int_A \mathrm{div}\Bigl(\varphi\frac{\nabla\varphi}{|\nabla\varphi|}|\nabla\varphi|^p\Bigr)\rho_0\, dx &= R\int_{\partial A}|\nabla\varphi|^p\rho_0\, d\mathcal{H}^{d-1} - \int_A \varphi\frac{\langle\nabla\varphi, \nabla\rho_0\rangle}{|\nabla\varphi|}|\nabla\varphi|^p\, dx\\
&\le R\int_{\partial A}|\nabla\varphi|^p\rho_0\, d\mathcal{H}^{d-1} + \varepsilon\int_A|\nabla\varphi|^{p+1}d\mu + N(\varepsilon,p)\int_A \varphi^{p+1}\Bigl|\frac{\nabla\rho_0}{\rho_0}\Bigr|^{p+1}\, d\mu.
\end{aligned}$$

Applying the change of variables and integrating by parts we get

$$p\int_A \frac{rH_{rr}}{H_r^{p+2}}\circ T\ d\mu = p\int_{B_R}\frac{rH_{rr}}{H_r^{p+2}}\ d\nu = -\frac{p}{p+1}\int_{B_R}\langle \nabla H_r^{-p-1}, x\rangle \rho_1\ dx$$
$$= -\frac{pR}{p+1}\int_{\partial B_R} H_r^{-p-1}\rho_1 d\mathcal{H}^{d-1} + \frac{p}{(p+1)}\int_{B_R} H_r^{-p-1}\rho_1\ \ dx$$
$$= \frac{p}{(p+1)}\int_A |\nabla\varphi|^{p+1}\ d\mu - \frac{pR}{p+1}\int_{\partial A}|\nabla\varphi|^p\rho_0\ d\mathcal{H}^{d-1}.$$

The obtained estimates imply immediately (13). Estimate (14) follows from (13) and the change of variables formula. □

Remark 4.6. Estimates of these type are also available for the pre-limiting potentials. For instance, for $T = \varphi\frac{\nabla\varphi}{|\nabla\varphi|}|\nabla\varphi|^{\frac{1}{1+t}}$ one has

$$\varphi_{\mathrm{nn}} \geq \Bigl(\frac{1}{p}|\nabla\varphi(S)|^p\Bigr)_r \circ T$$

for $p = 2 + \frac{1}{1+t}$. Then one can show that for $q > 0$

$$\int_{\partial A}\varphi|\nabla\varphi|^q\rho_0\ d\mathcal{H}^1 + \int_A\Bigl|\frac{\nabla\rho_0}{\rho_0}\Bigr|^{q+1}\varphi^{q+1}\ d\mu \geq C_q\int_A|\nabla\varphi|^{q+1}d\mu.$$

Remark 4.7. The result can be easily generalized to the general case of a continuous rotational invariant density $\nu = \rho_\nu\ dx = \rho_\nu(r)\ dx$.

Indeed, take a mapping T sending ν to $\frac{dx}{r}$ and having the form $T(x) = f(r)\frac{x}{r}$. The function f satisfies

$$r\rho_\nu(r) = f'(r).$$

Note that $\psi\frac{\nabla\psi}{|\nabla\psi|}$, where $\psi = f(\varphi)$ sends μ to $\frac{dx}{r}$. Applying (13) to ψ we get

$$C\int_A \varphi^{p+1}\rho_\nu^{p+1}(\varphi)|\nabla\varphi|^{p+1}\ d\mu \leq \int_A\Bigl|\frac{\nabla\rho_0}{\rho_0}\Bigr|^{p+1}\ d\mu + \int_{\partial A}|\nabla\varphi|^{p+1}\rho_0\ d\mathcal{H}^{d-1}.$$

Remark 4.8. It looks possible to prove L^∞-bounds on $|\nabla\varphi|$ using the parabolic maximum principle (see the next Section) and assuming high integrability of $|\nabla\rho_0|$. Estimates of this type for the potential u have been obtained in [20]. Results from [20] are not directly applicable to our situation, since we need to consider u in unbounded domains.

5. Variants of the parabolic maximum principle

For every convex V we denote by $|\partial V|(B)$ the associated Monge–Ampère measure of the set B, which is defined as follows:

$$|\partial V|(B) = \lambda\bigl(\{\bigcup_{x\in B}\partial V(x)\}\bigr),$$

where ∂V is the subdifferential of V at x.

For smooth V one has

$$\partial V = \det D^2 V\ dx.$$

This means that ∇V sends ∂V to Lebesgue measure if $\det D^2 V \neq 0$.

Recall that for every continuous function f on a convex set A one can define its convex envelope f_* which is the supremum of all affine functions less than f. The set $\mathcal{C}_f = \{x\colon f(x) = f_*(x)\}$ is called the set of contact points of f.

According to the elliptic maximum principle (also called Alexandrov maximum principle or Alexandrov-Bakelman-Pucci principle) every continuous function f on a convex set $A \subset \mathbb{R}^d$ satisfies

$$\sup_A f \le \sup_{\partial A} f + C \cdot \mathrm{diam}(A) \Big[\partial f_*(\mathcal{C}_f) \Big]^{\frac{1}{d}},$$

where C depends only on d. If f is twice continuously differentiable, this implies

$$\sup_A f \le \sup_{\partial A} f + C \cdot \mathrm{diam}(A) \Big[\int_{D^2 f(x) \le 0} |\det D^2 f(x)| dx \Big]^{\frac{1}{d}},$$

where C depends only on d. Equivalently, passing to $g = \sup_A f - f$, one gets that for every non-negative g

$$\inf_{\partial A} g \le C \cdot \mathrm{diam}(A) \Big[\int_{D^2 g(x) \ge 0} \det D^2 g(x) dx \Big]^{\frac{1}{d}}.$$

A parabolic version of the maximum principle was obtained by Krylov (see [25]). Later Tso [35] simplified the proof in some special cases and gave extensions in some particular cases.

In this section we prove some other variants of the parabolic maximum principle.

Definition 5.1. For a continuous function f defined on a convex set A consider its sublevel set $A_t = \{f \le t\}$ and the convex envelope $\mathrm{conv}(A_t)$ of A_t. Every point $x \in \mathrm{Int} A$ satisfying $x \in A_t \cap \partial \mathrm{conv}(A_t)$ for some t we call a contact point of A_t. The set of all such points will be denoted by $\mathcal{C}_{f,l}$.

We denote by S^{d-1}_+ the upper half of the unit sphere in $\mathbb{R}^d$. For every set $\Omega = \Big\{ (r,\theta)\colon\ R_1 \le r \le R_2,\ \theta \in Q \Big\}$, where $Q \subset S^{d-1}_+$ is a spherically convex set, we denote by

$$\partial_p \Omega = Q \times R_2 \cup \partial Q \times [R_1, R_2]$$

its parabolic boundary.

Theorem 5.2. 1) *Let v be a twice continuously differentiable function on a convex set $A \subset \mathbb{R}^d$. Then there exists a constant $C = C(d)$ depending only on d such that*

$$\sup_{x \in A} v(x) \le \sup_{x \in \partial A} v(x) + C(d) \int_{\mathcal{C}_{-v,l}} |\nabla v| K dx. \tag{16}$$

where $K(x)$ is the Gauss curvature of the set $\partial\ \mathrm{conv}\{y\colon v(x) \le v(y)\}$ at x

2) *Let Ω be a set of the type*

$$\Omega = \Big\{ (r,\theta)\colon\ R_1 \le r \le R_2,\ \theta \in Q \Big\}$$

with a spherically convex $Q \subset S^{d-1}_+$ satisfying $\mathrm{dist}(Q, \partial S^{d-1}_+) > 0$. Then for every twice continuously differentiable function $f\colon \Omega \to \mathbb{R}$ satisfying $\sup_{x \in \partial_p \Omega} f \ge 0$, one has

$$\sup_\Omega f \le C_1 \sup_{\partial_p \Omega} f + C_2 \Big[\int_{\Gamma_f} \frac{|f_r \det(f \cdot \mathrm{Id} + D^2_\theta f)|}{r^{d-1}} dx \Big]^{\frac{1}{d}}, \tag{17}$$

where $\Gamma_f = \{x \in \Omega\colon f_r \le 0,\ f \cdot \mathrm{Id} + D^2_\theta f \le 0\}$, and constants $C_1, C_2 > 0$ depend only on d and Q.

Proof. 1) Set $f = (M - v)^{1/d}$, where $M = \sup_A v$. The estimate (16) is equivalent to

$$\inf_{\partial A} f^d \le C \int_{\mathcal{C}_{f,l}} f^{d-1} |\nabla f| K dx. \tag{18}$$

For every $0 < t < \inf_{\partial A} f$ let us consider the set $A_t = \{x\colon f(x) \le t\} \subset A$ and its convex envelope $\mathrm{conv}(A_t)$. Since A is convex, $\mathrm{conv}(A_t)$ lies inside of A and, in addition, $\mathrm{dist}(\mathrm{conv}(A_t), \partial A) > 0$. Set: $C_t = \partial \mathrm{conv}(A_t) \cap A_t$. Since f is smooth, the image of C_t under the Gauss map n of $\partial \mathrm{conv}(A_t)$ covers the unit sphere. Hence the image of

$$\bigcup_{0<t<\inf_{\partial A} f} C_t = \mathcal{C}_{f,l}$$

under $T = f \cdot \mathrm{n}$ coincides with $\{x\colon \|x\| \le \inf_{\partial A} f\}$. One has $\det DT = f^{d-1}|\nabla f| K$. The result follows from the change of variables formula.

2) Let us consider the set of vectors V satisfying

$$a)\ \big\langle v, \mathrm{n} \big\rangle < M, \ \text{for all } x \in \Omega$$

and

$$b)\ \big\langle v,\ \mathrm{n} \big\rangle > m, \ \text{for all } x \in \partial_p \Omega,$$

with $\mathrm{n} = \frac{x}{|x|}$, $M = \sup_{x \in A} f$, $m = \sup_{x \in \partial A} f$. Since $\mathrm{dist}(Q, \partial S^{d-1}_+) > 0$, the set of vectors v satisfying $b)$ is non-empty and has the form

$$(r, \theta)\colon \quad r > C(Q) m, \ \theta \in \tilde{Q}$$

for some set $\tilde{Q} \subset S^{d-1}_+$ and a constant $C(Q)$ depending on Q. If $M < C(Q)m$, the claim is proved. If not, then V is nonempty. Consider the set

$$B = \{(r, \theta)\colon \quad C(Q)m < r < M, \ \theta \in \tilde{Q}\} \subset V.$$

Clearly,

$$C_0(Q)(M - C(Q)m)^d \le \lambda(B).$$

It remains to estimate $\lambda(B)$. For every $a \in B$ define $M_a = \{x\colon f(x) = \langle a, n\rangle\}$. Conditions a) and b) imply that M_a is non-empty and contained inside of Ω. Hence, there exists a point $x_0 \in M_a$ in the interiour of Ω, where $|x|$ attains its maximum. One has at this point

$$f(x_0) = \langle a, \mathrm{n} \rangle,$$
$$f_v(x_0) = \langle a, \mathrm{n}_v \rangle = \frac{1}{|x_0|} \langle a, v \rangle.$$

for every unit $v \perp \mathrm{n}$. This implies that

$$D_\theta f = a - \langle a, \mathrm{n} \rangle \mathrm{n}.$$

In addition,

$$f_r(x_0) \le 0, \quad D^2_\theta f(x_0) \le D^2_\theta \langle a, \mathrm{n} \rangle = -\langle a, \mathrm{n} \rangle \cdot \mathrm{Id} = -f(x_0) \cdot \mathrm{Id}.$$

Hence $B \subset \Gamma_f$. Set:

$$S = f(x) n + |x| \sum_{i=1}^{d-1} f_{\mathrm{v_i}}(x) \mathrm{v_i} = f(x) n + D_\theta f(x).$$

Note that $S(x_0) = a$. This means that

$$S(\Gamma_f) = B.$$

By the change of variables formula

$$\lambda(B) \le \int_{\Gamma_f} \det DS \ dx = \int_{\Gamma_f} \frac{|f_r \det(f \cdot \mathrm{Id} + D^2_\theta f)|}{r^{d-1}} dx.$$

The proof is complete. $\square$

Remark 5.3. Inequality (17) implies a form of the parabolic maximum principle (see [35]). Assume that $\sup_{\partial_p \Omega'} f = 0$. Then

$$\sup_{\Omega'} f \le C \Bigl[\int_{\Gamma_f} \frac{|f_r \det(f \cdot \mathrm{Id} + D^2_\theta f)|}{r^{d-1}} dx \Bigr]^{\frac{1}{d}}. \tag{19}$$

Set $u = \sqrt{1+z^2} f$, where z and x are related by the change of variables described is Section 2. Using

$$\det\Bigl(f \cdot \mathrm{Id} + D^2_\theta f \Bigr) = \det\Bigl((1+z^2)^{3/2} D^2_z u \Bigr)$$

and trivial uniform estimates one gets

$$\sup_{\Omega} u \le C(d,Q) \Bigl[\int_{\Gamma_u \cap \Omega} |u_t \cdot \det D^2_x u| \ dt dx \Bigr]^{\frac{1}{d}}, \tag{20}$$

$\Gamma_u = \{u_t \le 0; D^2 u \le 0\}$, for any u with $\sup_{\partial_p \Omega} u = 0$ and a cylinder $\Omega = [0,R] \times Q$ with convex Q. To remove the restriction $\sup_{\partial_p \Omega} u = 0$ one applies the estimate to $u = v - \sup_{\partial_p \Omega} v$.

Remark 5.4. The above variants of the parabolic maximum principle are naturally related with transport mappings of the type

$$T = \varphi \frac{\nabla \varphi}{|\nabla \varphi|}, \quad S = H \cdot \mathrm{n} + D_\theta H.$$

Both variants of mappings can be obtained from the "elliptic" transportation ∇V by scaling procedures (see Section 2). The transportation by gradients are naturallly associated with the elliptic maximum principle. Is it possible to derive both parabolic maximum principles from the elliptic one?

1) **Elliptic maximum principle implies (16).**

We prove that for every continuous $f \ge 0$ on a convex set $A \subset \mathbb{R}^d$ satisfying $\inf_{x \in A} f(x) = 0$ and every $0 < p \le 1$ there exists a constant $C = C(d)$ depending only on d such that

$$\inf_{\partial A} f^{d(1+p)} \le C \ \mathrm{diam}^{dp}(A) \ |\partial W_*| \circ S_p^{-1}\Bigl(\mathcal{C}_W\Bigr),$$

where $\mathcal{C}_W$ is the set of contact points of $W = \frac{p}{p+1} f^{1+\frac{1}{p}}$ and $S_p(x) = \frac{x}{|x|^{1-p}}$. In particular, if f is twice continuously differentiable, one has

$$\inf_{\partial A} f^{d(1+p)} \le C \ \mathrm{diam}^{dp}(A) \int_{\{x : \, D^2 f^{1+\frac{1}{p}}(x) \ge 0\}} \det D\Bigl(f \frac{\nabla f}{|\nabla f|^{1-p}} \Bigr) dx. \tag{21}$$

Clearly, letting $p \to 0$ we deduce an equivalent form of (16) from (21).

Proof: Let x_0 be a point satisfying $f(x_0) = 0$. If $x_0 \in \partial A$ there is nothing to prove. Thus we assume that $x_0 \notin \partial A$. Let V be the convex function whose graph is the upside-down cone with vertex $(x_0, 0)$ and base A with $V = m$ on ∂M, where $m = \inf_{x \in \Omega} \frac{p}{p+1} f^{1+\frac{1}{p}}(x)$. It is easy to check that

$$B_{m/\mathrm{diam}(\Omega)} \subset \partial V(x_0) \subset \partial W_*(\mathcal{C}_W).$$

Note that the measure with density

$$\rho\colon = \det D\Big(\frac{x}{|x|^{1-p}}\Big)$$

is the image of Lebesgue measure under S_p. Hence

$$\int_{B_{m/\text{diam}(\Omega)}} \rho dx \le |\partial W_*| \circ S_p^{-1}\Big(\mathcal{C}_W\Big).$$

The direct computation yields $\rho = p\ r^{d(p-1)}$. This immediately gives

$$\tilde{C}\Big(\frac{m}{\text{diam}(\Omega)}\Big)^{dp} = \int_{B_{m/\text{diam}(\Omega)}} \rho\ dx \le |\partial W_*| \circ S_p^{-1}\Big(\mathcal{C}_W\Big)$$

with $\tilde{C}$ depending only on d. This proves the first part.

Finally, (21) can be obtained by direct computations. We just notice that $\{x\colon D^2 f^{1+\frac{1}{p}}(x) \ge 0\} \subset \mathcal{C}_W$.

2) **Elliptic maximum principle implies (17) with $\Omega = B_R$ and symmetric f.**

Let $f\colon B_R \to \mathbb{R}$ be a symmetric ($f(-x) = f(x)$) bounded function. Assume that f is twice continuously differentiable at every $x \ne 0$ and $\inf_{x\in B_r} f(x) \le 0$. Then there exists a constant $C = C(d)$ depending only on d such that

$$\inf_{\partial B_R} f \le C\Big[\int_{\Gamma_{-f}} \frac{f_r \det(f\cdot \text{Id} + D_\theta^2 f)}{r^{d-1}} dx\Big]^{\frac{1}{d}}. \tag{22}$$

Proof: For every $t > 0$ consider

$$w_t(x) = |x|\ f(x|x|^{-\frac{t}{1+t}})$$

defined on $B_{R^{1+t}}$. One has

$$R^{1+t} \inf_{\partial B_R} f = \inf_{z\in \partial B_R} w_t(|z|^{1+t}) = \inf_{\partial B_{R^{1+t}}} w_t.$$

Since $w_t(0) = 0$, by the elliptic maximum principle

$$\inf_{\partial B_R} f = \Big(\frac{\inf_{\partial B_{R^{1+t}}} w_t}{R^{1+t}}\Big) \le C(d)\Big(\int_{\mathcal{C}_{w_t}} \det D^2 w_t\ dx\Big)^{\frac{1}{d}}.$$

Indeed, w_t is twice continuously differentiable everywhere in B_r except, maybe, the point $x = 0$. Without loss of generality one can assume that $\inf_{B_R} w_t < 0$. Since w_t is continious, $\inf w_t$ is attained at some point $\tilde{x}$. Since w_t is symmetric, the points $(\tilde{x}, w_t(\tilde{x}))$ and $(-\tilde{x}, w_t(\tilde{x}))$ belong to a horisontal supporting hyperplane to the graph of w_t. Since $w_t(0) = 0$, clearly $0 \notin \mathcal{C}_{w_t}$. This justifies the above estimate.

Set: $S_t(y) = y|y|^t$. Then

$$\begin{aligned}\int_{\mathcal{C}_{w_t}} \det D^2 w_t\ dx &= \int_{\mathcal{C}_{w_t}(S_t)\subset B_R} \det D^2 w_t(S_t) \det DS_t\ dy \\ &= \int_{\mathcal{C}_{w_t}(S_t)\subset B_R} \det D\big(\nabla w_t(S_t)\big) dy.\end{aligned}$$

Direct computations yield

$$\nabla w_t(S_t) = \Big(f + \frac{r f_r}{1+t}\Big) \cdot \text{n} + D_\theta f.$$

Hence

$$\lim_{t\to 0} \det D\big(\nabla w_t(S_t)\big) = \frac{f_r \det(f + D^2_\theta f)}{r^{d-1}}.$$

The proof is complete.

3) **Does elliptic maximum principle imply (20)?** There are good reasons to believe that the elliptic maximum principle implies (20). This problem seems to be rather involved technically and we do not consider it here. We just give a proof in a particular simple case. Let f satisfy all the assumptions from item 2). In addition, assume that $f = C$ outside of $\Omega' \cup (-\Omega')$, where

$$\Omega' = \Big\{(r,\theta)\colon\ 0 \le r \le R,\ \theta \in Q'\Big\}$$

and $Q' \subset S^{d-1}_+$ satisfies $\mathrm{dist}(Q', \partial S^{d-1}_+) > 0$. By the previous result

$$\inf_{\partial\Omega'} f \le C\Big[\int_{\Gamma_{-f}\cap\Omega'} \frac{f_r \det(f\cdot \mathrm{Id} + D^2_\theta f)}{r^{d-1}} dx\Big]^{\frac{1}{d}}.$$

Arguing as in Remark 5.3 we get that

$$\inf_{\partial_p\Omega} u \le C(d,Q)\Big[\int_{\Gamma_{-u}} |u_t \cdot \det D^2_x u|\ dtdx\Big]^{\frac{1}{d}},$$

holds for any bounded $u\colon (0,R]\times\mathbb{R}^{d-1}$, satisfying

1) u is smooth on $(\varepsilon, R]\times\mathbb{R}^{d-1}$
2) u is constant outside of $\Omega = (0,R]\times Q$ with convex $Q\subset\mathbb{R}^{d-1}$
3) $\inf_\Omega u = 0$.

Passing to $u = \sup_\Omega \varphi - \varphi$ we obtain

$$\sup_\Omega \varphi \le C(d,Q)\Big[\int_{\Gamma_\varphi} |\varphi_t \cdot \det D^2_x \varphi|\ dtdx\Big]^{\frac{1}{d}}$$

for any smooth compactly supported φ with $\mathrm{supp}(\varphi)\subseteq\Omega$.

6. Isoperimetric inequality

We discuss two apparently different proofs of the isoperimetric Euclidean inequality for convex sets (it is well-known that the general case can be easily reduced to the convex one). First of them due Gromov. It is worth mentioning (this was pointed out to the author by S. Bobkov) that arguments of such type go back to Knothe [24]. More precisely, it has been shown in [24] that the Brunn–Minkowsky inequality can be proved by transportation arguments with the help of triangular mappings. The second proof comes from the differential geometry. Our aim is to reveal a remarkable similarity between probabilistic and geometrical points of view.

1) (**Mass transportation. Probabilistic approach.**) We follow the mass transportation arguments but use the Gauss mass transport instead of optimal (or triangular) one. Let $A\subset\mathbb{R}^d$ be a convex set and $T = \varphi\frac{\nabla\varphi}{|\nabla\varphi|}$ send Lebesgue measure on A into Lebesgue measure on B_R, where B_R is a ball of the same volume. By the change of variables (see the previous section)

$$\varphi^{d-1}|D_a\varphi|K = 1.$$

Hence by the arithmetic–geometric inequality

$$1 = \det D_a T \le \frac{1}{d-1}\ \mathrm{Tr} D_a\Big(\varphi\frac{\nabla\varphi}{|\nabla\varphi|}\Big),$$

where $D_a T$ is the absolutely continuous part of the distributional derivative DT. Clearly,

$$\lambda(A) \le \frac{1}{d-1}\int_A \operatorname{div} T\, dx = \frac{1}{d-1}\int_{\partial A} \langle T, \mathrm{n}\rangle\, d\mathcal{H}^{d-1} \le \frac{R}{d-1}\mathcal{H}^{d-1}(\partial A).$$

Taking into account that $\lambda(A) = \lambda(B_R) = c_d R^d$, one easily recovers the classical isoperimetric inequality.

2) (**Curvature flows. Geometric approach.**) The same proof can be rewritten in the language of curvature flows. The curvature flow proofs are well-known in differential geometry (see partial results on the Cartan–Hadamard conjecture in [33], [30]). Let $A_t = \{x \colon \varphi(x) \le t\}$. For convenience we assume that φ is smooth on $\{x \colon \varphi(x) > 0\}$ (which is indeed the case for smooth strictly convex ∂A). Note that A_t are expanding with the speed $\frac{1}{|\nabla \varphi|}$. The enclosed volume $\lambda(A_t)$ evolves with the speed which can be exactly computed by the Gauss–Bonnet theorem

$$\frac{d}{dt}\lambda(A_t) = t^{d-1}\int_{\partial A_t} K\, d\mathcal{H}^{d-1} = t^{d-1}\mathcal{H}^{d-1}(S^{d-1}).$$

Hence

$$\lambda(A_t) = \lambda(B_t).$$

In the other hand, it is known that

$$\frac{d}{dt}\mathcal{H}^{d-1}(\partial A_t) = t^{d-1}\int_{\partial A_t} KH\, d\mathcal{H}^{d-1},$$

where H is the mean curvature. By the arithmetic-geometric inequality $K^{1/(d-1)} \le \frac{H}{d-1}$. Hence

$$\frac{d}{dt}\mathcal{H}^{d-1}(\partial A_t) \ge (d-1)t^{d-1}\int_{\partial A_t} K^{\frac{d}{d-1}}\, d\mathcal{H}^{d-1}.$$

By Hölder's inequality

$$\begin{aligned}
&\frac{d}{dt}\mathcal{H}^{d-1}(\partial A_t)\\
&\ge (d-1)t^{d-1}\Big(\int_{\partial A_t} K\, d\mathcal{H}^{d-1}\Big)^{\frac{d}{d-1}}\Big(\mathcal{H}^{d-1}(A_t)\Big)^{-1/(d-1)}\\
&= (d-1)t^{d-1}\Big(\mathcal{H}^{d-1}(S^{d-1})\Big)^{\frac{d}{d-1}}\Big(\mathcal{H}^{d-1}(A_t)\Big)^{-1/(d-1)}.
\end{aligned}$$

Integrating in t one obtains

$$\mathcal{H}^{d-1}(A_t) \ge t^d \mathcal{H}^{d-1}(S^{d-1}) = \mathcal{H}^{d-1}(B_t).$$

The proof is complete.

7. On Hölder's regularity of the Gauss mass transport

The elliptic and parabolic Monge–Ampère equations belong to the family of the so-called fully nonlinear PDE's. See [12] (and [19] for the special case of the Monge–Ampère equation). A short survey [26] presents the developments of the main ideas of the nonlinear PDE's theory.

The connection between the variational Monge–Kantorovich problem and the elliptic Monge–Ampère equation was revealed by Brenier (see [36]). In [34] the existence of the Gauss curvature flow for smooth data was established by solving the corresponding equation of the parabolic Monge–Ampère type.

Contributions to the regularity theory of the elliptic Monge–Ampère equation were made by many authors, including Alexandrov, Calabi, Yau, Pogorelov, Krylov, Spruck, Caffarelli, Nirenberg, and Urbas. There are several approaches to the regularity theory of nonlinear equations. A classical one is based on differentiating of the underlying equation. Taking the second derivative one obtains another equation which is linear with respect to higher derivatives. Then one applies a priori estimates from the linear theory. This was a common way for studying the nonlinear PDE's before the results of Krylov, Safonov, and Evans on a priori estimates for nonlinear uniformly elliptic operators. See [26] for details.

Unfortunately, the elliptic Monge–Ampère operator

$$u \to \det D^2 u$$

is **not** uniformly elliptic even in the class of convex functions. This is the reason why the Krylov–Safonov–Evans theory is not applicable directly. The regularity problem for the elliptic Monge–Ampère equation was solved in sufficient generality by Caffarelli. Combining the nonlinear regularity theory and deep geometric considerations he proved, in particular, that the solution V of the optimal transportation problem

$$g(\nabla V) \det D^2 V = f$$

for probability measures $f\ dx$ and $g\ dy$ with compact supports X and Y is $(2+\alpha)$-Hölder continuous inside of X provided f, g are Hölder continuous, bounded away from zero and Y is convex.

Many regularity results for the Gauss curvature flows (see [34], [3]) were obtained by using the classical way of differentiating the evolution equation. Similar to the the elliptic case, the parabolic maximum principle plays a crucial role in the study of this problem.

A parabolic analog of regularity theory for uniformly operators has been developed in [40].

The regularity of the parabolic Monge–Ampère equation was studied by Krylov [25], Ivochkina, Ladyzhenskaya [23], Gutiérrez, Huang [20], R.H. Wang and G.L. Wang [38], [39] (see [26] for references). Some interesting results were proved by probabilistic methods (optimal control and stochastic differential equations), see [26], [32]. A parabolic analog of the Caffarelli theory for the elliptic Monge–Ampère was developed by R.H. Wang and G.L. Wang in [38], [39]. They studied the parabolic Monge–Ampère equation

$$u_t \det D_z^2 u = f(t, z) \tag{23}$$

on the domain $Q = \Omega \times [0, T]$ with given values $u = \varphi(t, z)$ on the parabolic boundary $\partial_p Q$. It was shown in [38] that under the assumptions that

1) Ω is compact, strictly convex with C^2-boundary
2) f is positive Lipschitz continuous on $\overline{Q}$
3) $\varphi \in C^{2,1}(\overline{Q})$ with $\varphi_t > 0$, $D_z^2 \varphi > 0$ on $\overline{Q}$

there exists a solution $u \in C_{loc}^{1+\alpha/2, 2+\alpha}$ for some $\alpha > 0$. A measure-theoretic interpretation (the parabolic Monge–Ampère measure) was given in [39].

For further generalizations and refinements, see [28]. Sobolev estimates for (23) are obtained in [20].

We analyze below the regularity result of Tso. The reasoning from [34] can be easily generalized to our situation with the help of our results from Section 4. We will not repeat the lengthy reasoning from [34] and give just a brief sketch of the proof.

Let $\tilde{C}^{k,\alpha}(B_R)$ be the parabolic Hölder norm (see [34]) on functions

$$f(r,\theta)\colon [0,R]\times S^{d-1}\mapsto \mathbb{R}.$$

It was established in [34] that for $\rho_0 = 1$, $\rho_1 = \rho_1(\theta) \in C^{2+\alpha}(S^{d-1})$ with some $\alpha > 0$ and every $R > r_0 > 0$ there exists C such that

$$|H_r|_{\tilde{C}^{\beta/2,\beta}(B_R\setminus B_{r_0})} + |D^2_\theta H|_{\tilde{C}^{\beta/2,\beta}(B_R\setminus B_{r_0})} < C$$

for some $\beta > 0$.

Using the estimate $0 < c_{r_0} < H_r < C_{r_0}$ for smooth H from Section 4 and repeating the arguments from [34] it is not hard to verify Theorem 7.1 below, which is a generalization of Theorem 4.2 from [34]. Clearly, a solution H obtained in this theorem coincides with the potential H of the corresponding Gauss mass transport by the uniqueness theorem from [9].

Theorem 7.1. *Assume that $\rho_1 \in C^{2,\alpha}(B_R), \rho_0 \in C^{2,\alpha}(A)$, A is uniformly convex and $H_A \in C^{2,\alpha}(S^{d-1})$. Then a solution H to (6) with $H|_{\partial A} = H_A$ exists. In addition,*

$$H \in \tilde{C}^{4,\varepsilon}(B_R \setminus B_{r_0})$$

for every r_0 and

$$|H_r|_{\tilde{C}^{\beta/2,\beta}(B_R\setminus B_{r_0})} + |D^2_\theta H|_{\tilde{C}^{\beta/2,\beta}(B_R\setminus B_{r_0})} < C \tag{24}$$

holds for some positive β, ε, C depending on r_0, R, the curvature of ∂A, the Hölder and uniform bounds on ρ_0, ρ_1.

Sketch of the proof: One proves the existence of a solution to (6).

1) The classical short-time existence result implies that a smooth ($\tilde{C}^{4,\varepsilon}$) solution to (6) with a given initial value $H(R,\theta) = H_A(\theta)$ exists for $t \in [R-\varepsilon, R]$ (see, for instance [16], Theorems 2.5.7, 2.5.9). Let $[R^*, R]$ be the maximal existence interval. Assume that $R^* > 0$. Applying the change of variables formula, let us estimate the volume enclosed by the hypersurface determined by $H(R^*,\theta)$. One concludes that there exists a sphere contained in all hypersurfaces determined by $H(r,\theta)$, $r > R^*$. Taking the center of this sphere as the new origin one can assume without loss of generality that H is strictly positive on $[R^*, R]$.

2) The results of Section 4 give $\frac{1}{C} < |H_r| < C$ for some $C > 0$ and every $r \in [R^*, R]$. Following [34] one obtains that $\Lambda > H + D^2_\theta H > \lambda$ for some constants $0 < \lambda < \Lambda$, $t \in [R^*, R]$. This can be shown by differentiating twice the equation in θ and applying the classical maximum principle to a suitable function (see also Pogorelov-type arguments in [18], Theorem 17.19). Alternatively, one can use Caffarelli's result [13] on bounds for principal curvatures of a smooth convex set under the assumption that the corresponding Gauss curvature is positive and bounded.

3) Differentiate (6) in r. The Krylov–Safonov estimates (see [25]) imply that the parabolic Hölder norm of H_r on $[R^*, R]$ is under control. The same holds for H_θ.

4) It remains to prove Hölder's continuity of $H + D^2_\theta H$. The arguments follow [34]. Let us indicate the main difference. To estimate oscillation of $H_{\theta\theta}$ (or u_{zz}) we need an estimate for the additional term

$$\big|\log\rho_0(T^{-1})(x) - \log\rho_0(T^{-1})(y)\big| \le \sup\frac{|\nabla\rho_0|}{\rho_0}|T^{-1}(x) - T^{-1}(y)|.$$

(See [34], Theorem 4.1, (4.3)-(4.4)). Then we estimate $T^{-1}(x) - T^{-1}(y)$ by a parabolic Hölder norm of H_θ (see item 3)). Thus we get

$$H_{\tilde{C}^{1+\delta/2,2+\delta}} \leq C\Big[(H_\theta)_{\tilde{C}^{\varepsilon/2,\varepsilon}} + 1\Big]$$

for some $\delta, \varepsilon > 0$. Then the parabolic interpolation inequalities (see [27], Theorem 8.8.1.) complete the proof.

5) Since we have managed to keep control on the norms of derivatives of H on $[R^*, R]$, the solution exists for $r < R^*$ by the short-time existence theorem. Hence $R^* = 0$. The proof is complete.

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