$\hbar$-EXPANSION OF KP HIERARCHY: RECURSIVE CONSTRUCTION OF SOLUTIONS

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Abstract. The $\hbar$-dependent KP hierarchy is a formulation of the KP hierarchy that depends on the Planck constant $\hbar$ and reduces to the dispersionless KP hierarchy as $\hbar \to 0$. A recursive construction of its solutions on the basis of a Riemann-Hilbert problem for the pair (L, M) of Lax and Orlov-Schulman operators is presented. The Riemann-Hilbert problem is converted to a set of recursion relations for the coefficients X_n of an $\hbar$-expansion of the operator $X = X_0 + \hbar X_1 + \hbar^2 X_2 + \cdots$ for which the dressing operator W is expressed in the exponential form $W = \exp(X/\hbar)$. Given the lowest order term X_0, one can solve the recursion relations to obtain the higher order terms. The wave function Ψ associated with W turns out to have the WKB form $\Psi = \exp(S/\hbar)$, and the coefficients S_n of the $\hbar$-expansion $S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$, too, are determined by a set of recursion relations. This WKB form is used to show that the associated tau function has an $\hbar$-expansion of the form $\log\tau = \hbar^{-2}F_0 + \hbar^{-1}F_1 + F_2 + \cdots$.

0. Introduction

The KP hierarchy can be completely solved by several methods. The most classical methods are based on Grassmann manifolds [SS], [SW], fermions and vertex operators [DJKM] and factorisation of microdifferential operators [Mu]. Unfortunately, those methods are not very suited for a "quasi-classical" ($\hbar$-dependent, where $\hbar$ is the Planck constant) formulation [TT2] of the KP hierarchy.

The $\hbar$-dependent formulation of the KP hierarchy was introduced to study the dispersionless KP hierarchy [KG], [Kr], [TT1] as a classical limit (i.e., the lowest order of the $\hbar$-expansion) of the KP hierarchy. This point of view turned out to be very useful for understanding various features of the dispersionless KP hierarchy such as Lax equations, Hirota equations, infinite dimensional symmetries, etc., in the light of the KP hierarchy. In this paper, we return to the $\hbar$-dependent KP hierarchy itself, and consider all orders of the $\hbar$-expansion.

We first address the issue of solving a Riemann-Hilbert problem for the pair (L, M) of Lax and Orlov-Schulman operators [OS]. This is a kind of "quantisation" of a Riemann-Hilbert problem that solves the dispersionless KP hierarchy [TT1]. Though the Riemann-Hilbert problem for the full KP hierarchy was formulated in our previous work [TT2], we did not consider the existence of its solution in a general setting. In this paper, we settle this issue by an $\hbar$-expansion of the dressing operator W, which is assumed to have the exponential form $W = \exp(X/\hbar)$ with an operator X of negative order. Roughly speaking, the coefficients X_n, $n = 0, 1, 2, \ldots$, of the $\hbar$-expansion of X are shown to be determined recursively from the lowest order term X_0 (in other words, from a solution of the dispersionless KP hierarchy).

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We next convert this result to the language of the wave function Ψ. This, too, is to answer a problem overlooked in our previous paper [TT2]. Namely, given the dressing operator in the exponential form $W = \exp(X/\hbar)$, we show that the associated wave function has the WKB form $\Psi = \exp(S/\hbar)$ with a phase function S expanded into nonnegative powers of $\hbar$. This is genuinely a problem of calculus of microdifferential operators rather than that of the KP hierarchy. A simplest example such as $X = x(\hbar\partial)^{-1}$ demonstrates that this problem is by no means trivial. Borrowing an idea from Aoki's "exponential calculus" of microdifferential operators [A], we show that dressing operators of the form $W = \exp(X/\hbar)$ and wave functions of the form $\Psi = \exp(S/\hbar)$ are determined from each other by a set of recursion relations for the coefficients of their $\hbar$-expansion. More precisely, we need many auxiliary quantities other than X and S, for which we can derive a large set of recursion relations. Thus our construction is essentially recursive. Consequently, the wave function of the solution of the aforementioned Riemann-Hilbert problem, too, are recursively determined by the $\hbar$-expansion.

Having the $\hbar$-expansion of the wave function, we can readily derive an $\hbar$-expansion of the tau function as stated in our previous work [TT2]. This $\hbar$-expansion is a generalisation of the "genus expansion" of partition functions in string theories and random matrices [D], [Kr], [Mo], [dFGZ].

This paper is organised as follows. Section 1 is a review of the $\hbar$-dependent formulation of the KP hierarchy. Relevant Riemann-Hilbert problems are also reviewed here. Section 2 presents the recursive solution of the Riemann-Hilbert problem. A technical clue is the Campbell-Hausdorff formula, details of which are collected in Appendix A. The construction of solution is illustrated for the case of the Kontsevich model [AvM] in Appendix B. Section 3 deals with the $\hbar$-expansion of the wave function. Aoki's exponential calculus is also briefly reviewed here. Section 4 mentions the $\hbar$-expansion of the tau function. Section 5 is devoted to concluding remarks.

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1. $\hbar$-dependent KP hierarchy: review

In this section we recall several facts on the KP hierarchy depending on a formal parameter $\hbar$ in [TT2], §1.7.

The $\hbar$-dependent KP hierarchy is defined by the Lax representation

$$\hbar\frac{\partial L}{\partial t_n} = [B_n, L], \qquad B_n = (L^n)_{\geq 0}, \qquad n = 1, 2, \ldots, \tag{1.1}$$

where the *Lax operator* L is a microdifferential operator of the form

$$L = \hbar\partial + \sum_{n=1}^{\infty} u_{n+1}(\hbar, x, t)(\hbar\partial)^{-n}, \qquad \partial = \frac{\partial}{\partial x}, \tag{1.2}$$

and "$(\quad)_{\geq 0}$" stands for the projection onto a differential operator dropping negative powers of ∂. The coefficients $u_n(\hbar, x, t)$ of L are assumed to be formally regular with respect to $\hbar$. This means that they have an asymptotic expansion of the form $u_n(\hbar, x, t) = \sum_{m=0}^{\infty} \hbar^m u_n^{(m)}(x, t)$ as $\hbar \to 0$.

We need two kinds of "order" of microdifferential operators: one is the ordinary order,

$$\operatorname{ord}\left(\sum a_{n,m}(x,t)\hbar^n\partial^m\right) \stackrel{\text{def}}{=} \max\{m \mid a_{n,m}(x,t) \neq 0\}, \tag{1.3}$$

and the other is the $\hbar$-order defined by

$$\operatorname{ord}^{\hbar}\left(\sum a_{n,m}(x,t)\hbar^n\partial^m\right) \stackrel{\text{def}}{=} \max\{m - n \mid a_{n,m}(x,t) \neq 0\}. \tag{1.4}$$

In particular, $\operatorname{ord}^{\hbar} \hbar = -1$, $\operatorname{ord}^{\hbar} \partial = 1$, $\operatorname{ord}^{\hbar} \hbar\partial = 0$. For example, the condition which we imposed on the coefficients $u_n(\hbar, x, t)$ can be restated as $\operatorname{ord}^{\hbar}(L) = 0$.

The *principal symbol* (resp. the *symbol of order l*) of a microdifferential operator $A = \sum a_{n,m}(x,t)\hbar^n\partial^m$ with respect to the $\hbar$-order is

$$\sigma^{\hbar}(A) \stackrel{\text{def}}{=} \sum_{m-n=\operatorname{ord}(A)} a_{n,m}(x,t)\xi^m \tag{1.5}$$

$$(\text{resp. } \sigma_l^{\hbar}(A) \stackrel{\text{def}}{=} \sum_{m-n=l} a_{n,m}(x,t)\xi^m). \tag{1.6}$$

When it is clear from the context, we sometimes use $\sigma^{\hbar}$ instead of $\sigma_l^{\hbar}$.

Remark 1.1. This "order" coincides with the order of an microdifferential operator if we formally replace $\hbar$ with $\partial_{t_0}^{-1}$, where t_0 is an extra variable. In fact, naively extending (1.1) to $n = 0$, we can introduce the time variable t_0 on which nothing depends. See also [KR].

As in the usual KP theory, the Lax operator L is expressed by a *dressing operator* W:

$$L = \operatorname{Ad} W(\hbar\partial) = W(\hbar\partial)W^{-1} \tag{1.7}$$

The dressing operator W should have a specific form:

$$W = \exp(\hbar^{-1}X(\hbar, x, t, \hbar\partial))(\hbar\partial)^{\alpha(\hbar)/\hbar}, \tag{1.8}$$

$$X(\hbar, x, t, \hbar\partial) = \sum_{k=1}^{\infty} \chi_k(\hbar, x, t)(\hbar\partial)^{-k}, \tag{1.9}$$

$$\operatorname{ord}^{\hbar}(X(\hbar, x, t, \hbar\partial)) = \operatorname{ord}^{\hbar} \alpha(\hbar) = 0, \tag{1.10}$$

and $\alpha(\hbar)$ is a constant with respect to x and t. (In [TT2] we did not introduce α, which will be necessary in Section 2.)

The *Orlov-Schulman operator* M is defined by

$$M = \operatorname{Ad}\left(W \exp\left(\hbar^{-1}\zeta(t, \hbar\partial)\right)\right) x = W\left(\sum_{n=1}^{\infty} nt_n(\hbar\partial)^{n-1} + x\right) W^{-1} \tag{1.11}$$

where $\zeta(t, \hbar\partial) = \sum_{n=1}^{\infty} t_n(\hbar\partial)^n$. It is easy to see that M has a form

$$M = \sum_{n=1}^{\infty} nt_n L^{n-1} + x + \alpha(\hbar)L^{-1} + \sum_{n=1}^{\infty} v_n(\hbar, x, t)L^{-n-1}, \tag{1.12}$$

and satisfies

- $\operatorname{ord}^\hbar(M) = 0$;
- the canonical commutation relation: $[L, M] = \hbar$;
- the same Lax equations as L:

$$\hbar \frac{\partial M}{\partial t_n} = [B_n, M], \quad n = 1, 2, \dots. \tag{1.13}$$

Remark 1.2. If an operator M of the form (1.12) satisfies the Lax equations (1.13) and the canonical commutation relation $[L, M] = \hbar$ with the Lax operator L of the KP hierarchy, then $\alpha(\hbar)$ in the expansion (1.12) does not depend on any t_n nor on x. In fact, expanding the canonical commutation relation, we have

$$\hbar + \hbar \frac{\partial \alpha}{\partial x} (\hbar\partial)^{-1} + (\text{lower order terms}) = \hbar,$$

which implies $\frac{\partial \alpha}{\partial x} = 0$. Similarly, from (1.13) follows $\frac{\partial \alpha}{\partial t_n} = 0$ with the help of (1.1) and $[L^n, M] = n\hbar L^{n-1}$.

The following proposition (Proposition 1.7.11 of [TT2]) is a "dispersionful" counterpart of the theorem for the dispersionless KP hierarchy found earlier (Proposition 7 of [TT1]; cf. Proposition 1.4 below).

Proposition 1.3. *(i) Suppose that operators $f(\hbar, x, \hbar\partial)$, $g(\hbar, x, \hbar\partial)$, L and M satisfy the following conditions:*

- $\operatorname{ord}^\hbar f = \operatorname{ord}^\hbar g = 0$, $[f, g] = \hbar$;
- *L is of the form* (1.2) *and M is of the form* (1.12), $[L, M] = \hbar$;
- *$f(\hbar, M, L)$ and $g(\hbar, M, L)$ are differential operators:*

$$(f(\hbar, M, L))_{<0} = (g(\hbar, M, L))_{<0} = 0. \tag{1.14}$$

Then L is a solution of the KP hierarchy (1.1) *and M is the corresponding Orlov-Schulman operator.*

(ii) Conversely, for any solution (L, M) of the $\hbar$-dependent KP hierarchy there exists a pair (f, g) satisfying the conditions in (i).

The leading term of this system with respect to the $\hbar$-order gives the *dispersionless KP hierarchy.* Namely,

$$\mathcal{L} := \sigma^\hbar(L) = \xi + \sum_{n=1}^{\infty} u_{0,n+1} \xi^{-n}, \qquad (u_{0,n+1} := \sigma^\hbar(u_{n+1})) \tag{1.15}$$

satisfies the dispersionless Lax type equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \qquad \mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \qquad n = 1, 2, \dots, \tag{1.16}$$

where $(\quad)_{\geq 0}$ is the truncation of Laurent series to its polynomial part and $\{,\}$ is the Poisson bracket defined by

$$\{a(x, \xi), b(x, \xi)\} = \frac{\partial a}{\partial \xi} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial \xi}. \tag{1.17}$$

The dressing operation (1.7) for L becomes the following dressing operation for $\mathcal{L}$:

$$\begin{aligned} &\mathcal{L} = \exp\bigl(\operatorname{ad}_{\{,\}} X_0\bigr) \exp\bigl(\operatorname{ad}_{\{,\}} \alpha_0 \log \xi\bigr) \xi = \exp\bigl(\operatorname{ad}_{\{,\}} X_0\bigr) \xi, \\ &X_0 := \sigma^\hbar(X), \ \alpha_0 := \sigma^\hbar(\alpha) \end{aligned} \tag{1.18}$$

where $\operatorname{ad}_{\{,\}}(f)(g) := \{f, g\}$.

The principal symbol of the Orlov-Schulman operator is

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + x + \alpha_0 \mathcal{L}^{-1} + \sum_{n=1}^{\infty} v_{0,n} \mathcal{L}^{-n-1}, \qquad v_{0,n} := \sigma^{\hbar}(v_n), \tag{1.19}$$

which is equal to

$$\mathcal{M} = \exp\bigl(\mathrm{ad}_{\{,\}} X_0\bigr) \exp\bigl(\mathrm{ad}_{\{,\}} \alpha_0 \log \xi\bigr) \exp\bigl(\mathrm{ad}_{\{,\}} \zeta(t,\xi)\bigr) x, \tag{1.20}$$

where $\zeta(t,\xi) = \sum_{n=1}^{\infty} t_n \xi^n$. The series $\mathcal{M}$ satisfies the canonical commutation relation with $\mathcal{L}$, $\{\mathcal{L}, \mathcal{M}\} = 1$ and the Lax type equations:

$$\frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}, \qquad n = 1, 2, \dots. \tag{1.21}$$

The Riemann-Hilbert type construction of the solution is essentially the same as Proposition 1.3. (We do not need to assume the canonical commutation relation $\{\mathcal{L}, \mathcal{M}\} = 1$.)

Proposition 1.4. *(i) Suppose that functions $f_0(x,\xi)$, $g_0(x,\xi)$, $\mathcal{L}$ and $\mathcal{M}$ satisfy the following conditions:*

- $\{f_0, g_0\} = 1$*;*
- *$\mathcal{L}$ is of the form* (1.15) *and $\mathcal{M}$ is of the form* (1.19)
- *$f_0(\mathcal{M}, \mathcal{L})$ and $g_0(\mathcal{M}, \mathcal{L})$ do not contain negative powers of ξ*

$$(f_0(\mathcal{M}, \mathcal{L}))_{<0} = (g_0(\mathcal{M}, \mathcal{L}))_{<0} = 0. \tag{1.22}$$

then $\mathcal{L}$ is a solution of the KP hierarchy (1.1) *and $\mathcal{M}$ is the corresponding Orlov-Schulman function.*

(ii) Conversely, for any solution $(\mathcal{L}, \mathcal{M})$ of the dispersionless KP hierarchy, there exists a pair (f_0, g_0) satisfying the conditions in (i).

If f, g, L and M are as in Proposition 1.3, then $f_0 = \sigma^{\hbar}(f)$, $g_0 = \sigma^{\hbar}(g)$, $\mathcal{L} = \sigma^{\hbar}(L)$ and $\mathcal{M} = \sigma^{\hbar}(M)$ satisfy the conditions in Proposition 1.4. In other words, (f,g) and (L,M) are quantisation of the canonical transformations (f_0, g_0) and $(\mathcal{L}, \mathcal{M})$ respectively. (See, for example, [S] for quantised canonical transformations.)

2. Recursive construction of the dressing operator

In this section we prove that the solution of the KP hierarchy corresponding to the quantised canonical transformation (f,g) is recursively constructed from its leading term, i.e., the solution of the dispersionless KP hierarchy corresponding to the Riemann-Hilbert data $(\sigma^{\hbar}(f), \sigma^{\hbar}(g))$.

Given the pair (f,g), we have to construct the dressing operator W, or X and α in (1.8), such that operators

$$\begin{aligned} f(\hbar, L, M) &= \mathrm{Ad}\left(W \exp\left(\hbar^{-1}\zeta(t, \hbar\partial)\right)\right) f(\hbar, x, \hbar\partial) \\ g(\hbar, L, M) &= \mathrm{Ad}\left(W \exp\left(\hbar^{-1}\zeta(t, \hbar\partial)\right)\right) g(\hbar, x, \hbar\partial) \end{aligned} \tag{2.1}$$

are both differential operators (cf. Proposition 1.3). Let us expand X and α with respect to the $\hbar$-order as follows:

$$X(\hbar, x, t, \hbar\partial) = \sum_{n=0}^{\infty} \hbar^n X_n(x, t, \hbar\partial), \quad X_n(x, t, \hbar\partial) = \sum_{k=1}^{\infty} \chi_{n,k}(x, t)(\hbar\partial)^{-k}, \tag{2.2}$$

$$\alpha(\hbar) = \sum_{n=0}^{\infty} \hbar^n \alpha_n, \tag{2.3}$$

where $\chi_{n,k}$ and α_n do not depend on $\hbar$ and hence χ_k in (1.9) is expanded as $\chi_k = \sum_{n=0}^{\infty} \hbar^n \chi_{n,k}$.

Assume that the solution of the dispersionless KP hierarchy corresponding to $(\sigma^\hbar(f), \sigma^\hbar(g))$ is given. In other words, assume that a symbol $X_0 = \sum_{k=1}^{\infty} \chi_{0,k}(x,t)\xi^{-k}$ and a constant α_0 are given such that

$$\sigma^\hbar(f)(\mathcal{L}, \mathcal{M}) = \exp\bigl(\mathrm{ad}_{\{,\}} X_0\bigr) \exp\bigl(\mathrm{ad}_{\{,\}} \alpha_0 \log\xi\bigr) \exp\bigl(\mathrm{ad}_{\{,\}} \zeta(t,\xi)\bigr)\sigma^\hbar(f)(x,\xi)$$
$$\sigma^\hbar(g)(\mathcal{L}, \mathcal{M}) = \exp\bigl(\mathrm{ad}_{\{,\}} X_0\bigr) \exp\bigl(\mathrm{ad}_{\{,\}} \alpha_0 \log\xi\bigr) \exp\bigl(\mathrm{ad}_{\{,\}} \zeta(t,\xi)\bigr)\sigma^\hbar(g)(x,\xi)$$

do not contain negative powers of ξ:

$$\bigl(\sigma^\hbar(f)(\mathcal{L}, \mathcal{M})\bigr)_{<0} = \bigl(\sigma^\hbar(g)(\mathcal{L}, \mathcal{M})\bigr)_{<0} = 0. \tag{2.4}$$

(See Proposition 1.4.)

We are to construct X_n and α_n recursively, starting from X_0 and α_0. For this purpose expand $f(\hbar, L, M)$ and $g(\hbar, L, M)$ in (2.1) as follows:

$$\begin{aligned} P &:= \mathrm{Ad}\left(\exp(\hbar^{-1}X)\exp(\hbar^{-1}\alpha\log\hbar\partial)\right) f_t \\ &= P_0 + \hbar P_1 + \cdots + \hbar^k P_k + \cdots, \end{aligned} \tag{2.5}$$

$$\begin{aligned} Q &:= \mathrm{Ad}\left(\exp(\hbar^{-1}X)\exp(\hbar^{-1}\alpha\log\hbar\partial)\right) g_t \\ &= Q_0 + \hbar Q_1 + \cdots + \hbar^k Q_k + \cdots, \end{aligned} \tag{2.6}$$

where

$$f_t := \mathrm{Ad}\left(e^{\hbar^{-1}\zeta(t,\hbar\partial)}\right) f, \qquad g_t := \mathrm{Ad}\left(e^{\hbar^{-1}\zeta(t,\hbar\partial)}\right) g, \tag{2.7}$$

and P_i's and Q_i's are operators of the form $P_i = P_i(x, t, \hbar\partial)$ and $Q_i = Q_i(x, t, \hbar\partial)$ and $\mathrm{ord}^\hbar P_i = \mathrm{ord}^\hbar Q_i = 0$. Suppose that we have chosen $X_0, \dots, X_{i-1}$ and $\alpha_0, \dots, \alpha_{i-1}$ so that $P_0, \dots, P_{i-1}$ and $Q_0, \dots, Q_{i-1}$ do not contain negative powers of ∂. If an operator X_i and a constant α_i are constructed from these given $X_0, \dots, X_{i-1}$ and $\alpha_0, \dots, \alpha_{i-1}$ so that resulting P_i and Q_i do not contain negative powers of ∂, this procedure gives recursive construction of X and α in question.

We can construct such X_i and α_i as follows. (Details and meaning shall be explained in the proof of Theorem 2.1.):

- (Step 0) Assume $X_0, \dots, X_{i-1}$ and $\alpha_0, \dots, \alpha_{i-1}$ are given and set

$$X^{(i-1)} := \sum_{n=0}^{i-1} \hbar^n X_n, \qquad \alpha^{(i-1)} := \sum_{n=0}^{i-1} \hbar^n \alpha_n. \tag{2.8}$$

- (Step 1) Set

$$P^{(i-1)} := \mathrm{Ad}\left(\exp\hbar^{-1}X^{(i-1)}\right)\exp\left(\hbar^{-1}\alpha^{(i-1)}\log(\hbar\partial)\right) f_t, \tag{2.9}$$

$$Q^{(i-1)} := \mathrm{Ad}\left(\exp\hbar^{-1}X^{(i-1)}\right)\exp\left(\hbar^{-1}\alpha^{(i-1)}\log(\hbar\partial)\right) g_t. \tag{2.10}$$

Expand $P^{(i-1)}$ and $Q^{(i-1)}$ with respect to the $\hbar$-order as

$$P^{(i-1)} = P_0^{(i-1)} + \hbar P_1^{(i-1)} + \cdots + \hbar^k P_k^{(i-1)} + \cdots \tag{2.11}$$

$$Q^{(i-1)} = Q_0^{(i-1)} + \hbar Q_1^{(i-1)} + \cdots + \hbar^k Q_k^{(i-1)} + \cdots . \tag{2.12}$$

($\mathrm{ord}^\hbar P_k^{(i-1)} = \mathrm{ord}^\hbar Q_k^{(i-1)} = 0$.)

- (Step 2) Put $\mathcal{P}_0 := \sigma^\hbar(P_0^{(i-1)})$, $\mathcal{Q}_0 := \sigma^\hbar(Q_0^{(i-1)})$, $\mathcal{P}_i^{(i-1)} := \sigma^\hbar(P_i^{(i-1)})$, $\mathcal{Q}_i^{(i-1)} := \sigma^\hbar(Q_i^{(i-1)})$ and define a constant α_i and a series $\tilde{\mathcal{X}}_i(x,t,\xi) = \sum_{k=1}^\infty \tilde{\chi}_{i,k}(x,t)\xi^{-k}$ by

$$\alpha_i \log\xi + \tilde{\mathcal{X}}_i := \int^\xi \left(\frac{\partial \mathcal{Q}_0}{\partial \xi} \mathcal{P}_i^{(i-1)} - \frac{\partial \mathcal{P}_0}{\partial \xi} \mathcal{Q}_i^{(i-1)} \right)_{\leq -1} d\xi. \tag{2.13}$$

The integral constant of the indefinite integral is fixed so that the right hand side agrees with the left hand side.
- (Step 3) Define a series $\mathcal{X}_i(x,t,\xi) = \sum_{k=1}^\infty \chi_{i,k}(x,t)\xi^{-k}$ by

$$\begin{aligned} \mathcal{X}_i &= \tilde{\mathcal{X}}_i' - \frac{1}{2}\{\sigma^\hbar(X_0), \tilde{\mathcal{X}}_i'\} + \sum_{p=1}^\infty K_{2p}(\mathrm{ad}_{\{,\}}(\sigma^\hbar(X_0)))^{2p}\tilde{\mathcal{X}}_i', \\ \tilde{\mathcal{X}}_i' &:= \alpha_i \log\xi + \tilde{\mathcal{X}}_i(x,\xi) - \exp(\mathrm{ad}_{\{,\}}\,\sigma^\hbar(X_0))(\alpha_i\log\xi). \end{aligned} \tag{2.14}$$

Here K_{2p} is determined by the generating function

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{p=1}^\infty K_{2p} z^{2p}, \tag{2.15}$$

or $K_{2p} = B_{2p}/(2p)!$, where B_{2p}'s are the Bernoulli numbers.
- (Step 4) The operator $X_i(x,t,\hbar\partial)$ is defined as the operator with the principal symbol $\mathcal{X}_i$:

$$X_i = \sum_{k=1}^\infty \chi_{i,k}(x,t)(\hbar\partial)^{-k}. \tag{2.16}$$

The main theorem is the following:

Theorem 2.1. *Assume that X_0 and α_0 satisfy* (2.4) *and construct X_i's and α_i's by the above procedure recursively. Then X and α defined by* (2.2) *satisfy* (1.14). *Namely $W = \exp(X/\hbar)(\hbar\partial)^{\alpha/\hbar}$ is a dressing operator of the $\hbar$-dependent KP hierarchy.*

The rest of this section is the proof of Theorem 2.1 by induction.

Let us denote the "known" part of X and α by $X^{(i-1)}$ and $\alpha^{(i-1)}$ as in (2.8) and, as intermediate objects, consider $P^{(i-1)}$ and $Q^{(i-1)}$ defined by (2.9) and (2.10), which are expanded as (2.11) and (2.12).

If X and α are expanded as (2.2) and (2.3), the dressing operator $W = \exp(X/\hbar)\exp(\alpha\log(\hbar\partial)/\hbar)$ is factored as follows by the Campbell-Hausdorff theorem:

$$\begin{aligned} W = \exp&\left(\hbar^{i-1}(\alpha_i \log(\hbar\partial) + \tilde{X}_i) + \hbar^i X_{>i}\right) \times \\ &\times \exp\left(\hbar^{-1}X^{(i-1)}\right)\exp\bigl(\hbar^{-1}\alpha^{(i-1)}\log(\hbar\partial)\bigr), \end{aligned} \tag{2.17}$$

where $\operatorname{ord}^\hbar(\alpha_i \log(\hbar\partial) + \tilde{X}_i(x, \hbar\partial)) = 0$, $\operatorname{ord}^\hbar(X_{>i}) \leqq 0$ and the principal symbol of $\alpha_i \log(\hbar\partial) + \tilde{X}_i(x, \hbar\partial)$ is defined by

$$
\begin{aligned}
(2.18)\quad & \sigma^\hbar(\alpha_i \log(\hbar\partial) + \tilde{X}_i)(x, \xi) \\
&= \sum_{n=1}^\infty \frac{(\operatorname{ad}_{\{,\}} \sigma^\hbar(X_0))^{n-1}}{n!} \sigma^\hbar(X_i) + \exp\bigl(\operatorname{ad}_{\{,\}} \sigma^\hbar(X_0)\bigr)(\alpha_i \log \xi).
\end{aligned}
$$

Note that the only log term in (2.18) is $\alpha_i \log \xi$ and the rest is sum of negative powers of ξ. The principal symbol of X_i is recovered from $\tilde{X}_i$ by the formula

(2.19)

$$
\begin{aligned}
\sigma^\hbar(X_i) &= \sigma^\hbar(\tilde{X}'_i) - \frac{1}{2}\{\sigma^\hbar(X_0), \sigma^\hbar(\tilde{X}'_i)\} + \sum_{p=1}^\infty K_{2p} (\operatorname{ad}_{\{,\}}(\sigma^\hbar(X_0)))^{2p} \sigma^\hbar(\tilde{X}'_i), \\
\sigma^\hbar(\tilde{X}'_i) &:= \sigma^\hbar(\tilde{X}_i)(x, \xi) - \exp(\operatorname{ad}_{\{,\}} \sigma^\hbar(X_0))(\alpha_i \log \xi) \\
&= \sum_{n=1}^\infty \frac{(\operatorname{ad}_{\{,\}} \sigma^\hbar(X_0))^{n-1}}{n!} \sigma^\hbar(X_i).
\end{aligned}
$$

Here coefficients K_{2p} are defined by (2.15). This inversion relation is the origin of (2.14). (Note that the principal symbol determines the operator X_i, since it is a homogeneous term in the expansion (2.2).) We prove formulae (2.17) and (2.19) in Appendix A.

The factorisation (2.17) implies

$$
\begin{aligned}
P &= \operatorname{Ad}\Bigl(\exp\bigl(\hbar^{i-1}(\alpha_i \log(\hbar\partial) + \tilde{X}_i) + \hbar^i X_{>i}\bigr)\Bigr) P^{(i-1)} \\
&= P^{(i-1)} + \hbar^{i-1}[(\alpha_i \log(\hbar\partial) + \tilde{X}_i) + \hbar X_{>i}, P^{(i-1)}] + (\text{terms of } \hbar\text{-order} < -i).
\end{aligned}
$$

Thus, substituting the expansion (2.11) in the step 1, we have

$$
\begin{aligned}
P =& P_0^{(i-1)} + \hbar P_1^{(i-1)} + \cdots + \hbar^i P_i^{(i-1)} + \cdots \\
&+ \hbar^{i-1}[\alpha_i \log(\hbar\partial) + \tilde{X}_i, P_0^{(i-1)}] \\
&+ (\text{terms of } \hbar\text{-order} < -i).
\end{aligned}
\tag{2.20}
$$

Comparing this with the $\hbar$-expansion of P (2.5), we can express P_i's in terms of $P_j^{(i-1)}$, $\tilde{X}_i$ and α_i as follows:

$$
P_j = P_j^{(i-1)} \qquad (j = 0, \dots, i-1), \tag{2.21}
$$

$$
\sigma_0(P_i) = \sigma_0(P_i^{(i-1)} + h^{-1}[\alpha_i \log(\hbar\partial) + \tilde{X}_i, P_0^{(i-1)}]). \tag{2.22}
$$

Similar equations for Q are obtained in the same way. The first equations (2.21) show that the terms of $\hbar$-order greater than $-i$ in (2.5) are already fixed by $X_0, \dots, X_{i-1}$ and $\alpha_0, \dots, \alpha_{i-1}$, which justifies the inductive procedure. That is to say, we are assuming that $X_0, \dots, X_{i-1}$ and $\alpha_0, \dots, \alpha_{i-1}$ have been already determined so that $P_j = P_j^{(i-1)}$ and $Q_j = Q_j^{(i-1)}$ for $j = 0, \dots, i-1$ are differential operators.

The operator X_i and constant α_i should be chosen so that the right hand side of (2.22) and the corresponding expression for Q are differential operators. Taking equations $P_0^{(i-1)} = P_0$ and $Q_0^{(i-1)} = Q_0$ into account, we define

$$\begin{aligned} \tilde{P}_i^{(i)} &:= P_i^{(i-1)} + \hbar^{-1}[\alpha_i \log(\hbar\partial) + \tilde{X}_i, P_0], \\ \tilde{Q}_i^{(i)} &:= Q_i^{(i-1)} + \hbar^{-1}[\alpha_i \log(\hbar\partial) + \tilde{X}_i, Q_0]. \end{aligned} \tag{2.23}$$

Then the condition for X_i and α_i is written in the following form of equations for symbols:

$$(\sigma_0^\hbar(\tilde{P}_i^{(i)}))_{\le -1} = 0, \qquad (\sigma_0^\hbar(\tilde{Q}_i^{(i)}))_{\le -1} = 0. \tag{2.24}$$

(The parts of $\hbar$-order less than -1 should be determined in the next step of the induction.) To simplify notations, we denote the symbols $\sigma_0^\hbar(\tilde{P}_i^{(i)})$, $\sigma_0^\hbar(P_i^{(i-1)})$ and so on by the corresponding calligraphic letters as $\mathcal{P}_i^{(i)}$, $\mathcal{P}_i^{(i-1)}$ etc. By this notation we can rewrite the equations (2.24) in the following form:

$$\begin{aligned} (\tilde{\mathcal{P}}_i^{(i)})_{\le -1} &= 0, & \tilde{\mathcal{P}}_i^{(i)} &:= \mathcal{P}_i^{(i-1)} + \{\alpha_i \log\xi + \tilde{\mathcal{X}}_i, \mathcal{P}_0\} = 0, \\ (\tilde{\mathcal{Q}}_i^{(i)})_{\le -1} &= 0, & \tilde{\mathcal{Q}}_i^{(i)} &:= \mathcal{Q}_i^{(i-1)} + \{\alpha_i \log\xi + \tilde{\mathcal{X}}_i, \mathcal{Q}_0\} = 0. \end{aligned} \tag{2.25}$$

The above definitions of $\tilde{\mathcal{P}}_i^{(i)}$ and $\tilde{\mathcal{Q}}_i^{(i)}$ are written in the matrix form:

$$\begin{pmatrix} \dfrac{\partial \mathcal{P}_0}{\partial x} & -\dfrac{\partial \mathcal{P}_0}{\partial \xi} \\ \\ \dfrac{\partial \mathcal{Q}_0}{\partial x} & -\dfrac{\partial \mathcal{Q}_0}{\partial \xi} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \xi}(\alpha_i \log\xi + \tilde{\mathcal{X}}_i) \\ \\ \dfrac{\partial}{\partial x}(\alpha_i \log\xi + \tilde{\mathcal{X}}_i) \end{pmatrix} = \begin{pmatrix} \mathcal{P}_i^{(i)} - \mathcal{P}_i^{(i-1)} \\ \\ \mathcal{Q}_i^{(i)} - \mathcal{Q}_i^{(i-1)} \end{pmatrix}. \tag{2.26}$$

Recall that operators $P^{(i-1)}$ and $Q^{(i-1)}$ are defined by acting adjoint operation to the canonically commuting pair (f, g) in (2.9), (2.10) and (2.7). Hence they also satisfy the canonical commutation relation: $[P^{(i-1)}, Q^{(i-1)}] = \hbar$. The principal symbol of this relation gives

$$\{\mathcal{P}_0^{(i-1)}, \mathcal{Q}_0^{(i-1)}\} = \{\mathcal{P}_0, \mathcal{Q}_0\} = 1,$$

which means that the determinant of the matrix in the left hand side of (2.26) is equal to 1. Hence its inverse matrix is easily computed and we have

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi}(\alpha_i \log\xi + \tilde{\mathcal{X}}_i) \\ \\ \dfrac{\partial}{\partial x}(\alpha_i \log\xi + \tilde{\mathcal{X}}_i) \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial \mathcal{Q}_0}{\partial \xi} & \dfrac{\partial \mathcal{P}_0}{\partial \xi} \\ \\ -\dfrac{\partial \mathcal{Q}_0}{\partial x} & \dfrac{\partial \mathcal{P}_0}{\partial x} \end{pmatrix} \begin{pmatrix} \mathcal{P}_i^{(i)} - \mathcal{P}_i^{(i-1)} \\ \\ \mathcal{Q}_i^{(i)} - \mathcal{Q}_i^{(i-1)} \end{pmatrix}. \tag{2.27}$$

We are assuming that $\mathcal{P}_0$ and $\mathcal{Q}_0$ do not contain negative powers of ξ and we are searching for $\alpha_i \log\xi + \tilde{\mathcal{X}}_i$ such that $\mathcal{P}_i^{(i)}$ and $\mathcal{Q}_i^{(i)}$ are series of ξ without negative powers. Since α_i is constant with respect to x, the left hand side of (2.27) contain only negative powers of ξ. Thus taking the negative power parts of the both hand sides in (2.27), we have

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi}(\alpha_i \log\xi + \tilde{\mathcal{X}}_i) \\ \\ \dfrac{\partial}{\partial x}(\alpha_i \log\xi + \tilde{\mathcal{X}}_i) \end{pmatrix} = \begin{pmatrix} \left(\dfrac{\partial \mathcal{Q}_0}{\partial \xi} \mathcal{P}_i^{(i-1)} - \dfrac{\partial \mathcal{P}_0}{\partial \xi} \mathcal{Q}_i^{(i-1)} \right)_{\le -1} \\ \\ \left(\dfrac{\partial \mathcal{Q}_0}{\partial x} \mathcal{P}_i^{(i-1)} - \dfrac{\partial \mathcal{P}_0}{\partial x} \mathcal{Q}_i^{(i-1)} \right)_{\le -1} \end{pmatrix}. \tag{2.28}$$

This is the equation which determines α_i and X_i.

The system (2.28) is solvable thanks to Lemma 2.2 below. Hence, integrating the first element of the right hand side with respect to ξ, we obtain $\alpha_i \log\xi + \tilde{\mathcal{X}}_i$. This is Step 2, (2.13). In the end, the principal symbol of X_i is determined by (2.19) or (2.14) in Step 3 and thus X_i is defined as in Step 4. This completes the construction of X_i and α_i and the proof of the theorem.

Lemma 2.2. *The system* (2.28) *is compatible.*

Proof. We check:

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{Q}_0}{\partial \xi}\mathcal{P}_i^{(i-1)} - \frac{\partial \mathcal{P}_0}{\partial \xi}\mathcal{Q}_i^{(i-1)}\right)_{\le -1}\\ =&\frac{\partial}{\partial \xi}\left(\frac{\partial \mathcal{Q}_0}{\partial x}\mathcal{P}_i^{(i-1)} - \frac{\partial \mathcal{P}_0}{\partial x}\mathcal{Q}_i^{(i-1)}\right)_{\le -1}.\end{aligned} \tag{2.29}$$

Since differentiation commutes with truncation of power series, condition (2.29) is equivalent to saying that the negative power part of the following is zero:

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{Q}_0}{\partial \xi}\mathcal{P}_i^{(i-1)} - \frac{\partial \mathcal{P}_0}{\partial \xi}\mathcal{Q}_i^{(i-1)}\right) - \frac{\partial}{\partial \xi}\left(\frac{\partial \mathcal{Q}_0}{\partial x}\mathcal{P}_i^{(i-1)} - \frac{\partial \mathcal{P}_0}{\partial x}\mathcal{Q}_i^{(i-1)}\right)\\ =&\frac{\partial \mathcal{Q}_0}{\partial \xi}\frac{\partial \mathcal{P}_i^{(i-1)}}{\partial x} - \frac{\partial \mathcal{P}_0}{\partial \xi}\frac{\partial \mathcal{Q}_i^{(i-1)}}{\partial x} - \frac{\partial \mathcal{Q}_0}{\partial x}\frac{\partial \mathcal{P}_i^{(i-1)}}{\partial \xi} + \frac{\partial \mathcal{P}_0}{\partial x}\frac{\partial \mathcal{Q}_i^{(i-1)}}{\partial \xi}\\ =&-\{\mathcal{P}_i^{(i-1)}, \mathcal{Q}_0\} - \{\mathcal{P}_0, \mathcal{Q}_i^{(i-1)}\}.\end{aligned} \tag{2.30}$$

Defined from canonically commuting pair (f, g) by adjoint action (2.9) and (2.10), the pair of operators $(P^{(i-1)}, Q^{(i-1)})$ is canonically commuting: $[P^{(i-1)}, Q^{(i-1)}] = \hbar$. The negative order part of this relation is zero. On the other hand, substituting the expansions $P^{(i-1)} = \sum_{n=0}^{\infty} \hbar^n P_n^{(i-1)}$ and $Q^{(i-1)} = \sum_{n=0}^{\infty} \hbar^n Q_n^{(i-1)}$ in this canonical commutation relation and noting that $P_j^{(i-1)}$ and $Q_j^{(i-1)}$ $(j = 0, \dots, i-1)$ do not contain negative order part by the induction hypothesis, we have

$$\begin{aligned}0 &= ([P^{(i-1)}, Q^{(i-1)}])_{\le -1}\\ &= [\hbar^i P_i^{(i-1)}, Q_0^{(i-1)}] + [P_0^{(i-1)}, \hbar^i Q_i^{(i-1)}] + (\text{terms of } \hbar\text{-order} < -i-1).\end{aligned}$$

Taking the principal symbol of this equation, we have

$$0 = \{\mathcal{P}_i^{(i-1)}, \mathcal{Q}_0\} + \{\mathcal{P}_0, \mathcal{Q}_i^{(i-1)}\},$$

which proves that (2.30) vanishes. □

3. Asymptotics of the wave function

In this section we prove that the dressing operator of the form

$$W(\hbar, x, t, \hbar\partial) = \exp(X(\hbar, x, \hbar\partial)/\hbar), \qquad \operatorname{ord}^\hbar X \leqq 0, \qquad \operatorname{ord} X \leqq -1, \tag{3.1}$$

gives a wave function of the form

$$\Psi(\hbar, x, t; z) = W e^{(xz+\zeta(t,z))/\hbar} = \exp(S(\hbar, x, t, z)/\hbar), \qquad \operatorname{ord}^\hbar S \leqq 0, \tag{3.2}$$

$$S(\hbar, x, t; z) = \sum_{n=0}^{\infty} \hbar^n S_n(x, t; z) + \zeta(t, z), \qquad \zeta(t, z) := \sum_{n=1}^{\infty} t_n z^n, \tag{3.3}$$

and vice versa.

Since the time variables t_n do not play any role in this section, we set them to zero. As the factor $(\hbar\partial)^{\alpha/\hbar}$ in (1.8) becomes a constant factor z^α when it is applied to $e^{xz/\hbar}$, we also omit it here.

Let $A(\hbar, x, \hbar\partial) = \sum_n a_n(\hbar, x)(\hbar\partial)^n$ be a microdifferential operator. The *total symbol* of A is a power series of ξ defined by

$$\sigma_{\mathrm{tot}}(A)(\hbar, x, \xi) := \sum_n a_n(\hbar, x)\xi^n. \tag{3.4}$$

Actually, this is the factor which appears when the operator A is applied to $e^{xz/\hbar}$:

$$Ae^{xz/\hbar} = \sigma_{\mathrm{tot}}(A)(\hbar, x, z)e^{xz/\hbar}. \tag{3.5}$$

Using this terminology, what we show in this section is that a operator of the form $e^{X/\hbar}$ has a total symbol of the form $e^{S/\hbar}$ and that an operator with total symbol $e^{S/\hbar}$ has a form $e^{X/\hbar}$. Exactly speaking, the main results in this section are the following two propositions.

Proposition 3.1. *Let $X = X(\hbar, x, \hbar\partial)$ be a microdifferential operator such that $\operatorname{ord} X = -1$ and $\operatorname{ord}^\hbar X = 0$. Then the total symbol of $e^{X/\hbar}$ has such a form as*

$$\sigma_{\mathrm{tot}}(\exp(\hbar^{-1}X(\hbar, x, \hbar\partial))) = e^{S(\hbar,x,\xi)/\hbar}, \tag{3.6}$$

where $S(\hbar, x, \xi)$ is a power series of ξ^{-1} without non-negative powers of ξ and has an $\hbar$-expansion

$$S(\hbar, x, \xi) = \sum_{n=0}^{\infty} \hbar^n S_n(x, \xi). \tag{3.7}$$

Moreover, the coefficient S_n is determined by $X_0, \dots, X_n$ in the $\hbar$-expansion (2.2) *of $X = \sum_{n=0}^{\infty} \hbar^n X_n$.*

Explicitly, S_n is determined as follows:

- (Step 0) Assume that $X_0, \dots, X_n$ are given. Let $X_i(x, \xi)$ be the total symbol $\sigma_{\mathrm{tot}}(X_i(x, \hbar\partial))$.
- (Step 1) Define $Y^{(l)}_{k,m}(x, y, \xi, \eta)$ and $S^{(l)}(x, \xi)$ by the following recursion relations:

$$Y^{(l)}_{k,-1} = 0 \tag{3.8}$$

$$S^{(0)}_m = 0, \tag{3.9}$$

$$Y^{(l)}_{0,m}(x, y, \xi, \eta) = \delta_{l,0} X_m(x, \xi) \tag{3.10}$$

for $l \geqq 0$, $m = 0, \dots, n$,

$$\begin{aligned} &Y^{(l)}_{k+1,m}(x, y, \xi, \eta) \\ &= \frac{1}{k+1}\left(\partial_\xi\partial_y Y^{(l)}_{k,m-1}(x, y, \xi, \eta) + \sum_{\substack{0\le l'\le l-1 \\ 0\le m'\le m}} \partial_\xi Y^{(l')}_{k,m'}(x, y, \xi, \eta)\partial_y S^{(l-l')}_{m-m'}(y, \eta)\right) \end{aligned} \tag{3.11}$$

for $k \geqq 0$, and

$$S^{(l+1)}_m(x, \xi) = \frac{1}{l+1}\sum_{k=0}^{l+m} Y^{(l)}_{k,m}(x, x, \xi, \xi). \tag{3.12}$$

(We shall prove that $Y^{(l)}_{k,m} = 0$ if $k > l + m$.) Schematically this procedure goes as follows:

$$
\begin{array}{ccccccccc}
 & & Y^{(l)}_{0,0} = \delta_{l,0} X_0 & & Y^{(l)}_{0,1} = \delta_{l,0} X_1 & & Y^{(l)}_{0,2} = \delta_{l,0} X_2 & & \\
 & & + & \searrow & + & \searrow & + & & \\
Y^{(l)}_{k,-1} = 0 & \rightarrow & Y^{(l)}_{k,0} & \rightarrow & Y^{(l)}_{k,1} & \rightarrow & Y^{(l)}_{k,2} & & \cdots \\
 & & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & \\
 & & S^{(l+1)}_0 & & S^{(l+1)}_1 & & S^{(l+1)}_2 & &
\end{array}
$$

- (Step 2) $S_n(x,\xi) = \sum_{l=1}^{\infty} S^{(l)}_n(x,\xi)$. (The sum makes sense as a power series of ξ.)

Proposition 3.2. *Let $S = \sum_{n=0}^{\infty} \hbar^n S_n$ be a power series of ξ^{-1} without non-negative powers of ξ. Then there exists a microdifferential operator $X(\hbar, x, \hbar\partial)$ such that* $\operatorname{ord} X \leqq -1$, $\operatorname{ord}^{\hbar} X \leqq 0$ *and*

$$\sigma_{\mathrm{tot}}(\exp(\hbar^{-1} X(\hbar, x, \hbar\partial))) = e^{S(\hbar,x,\xi)/\hbar}. \tag{3.13}$$

Moreover, the coefficient $X_n(x,\xi)$ in the $\hbar$-expansion $X = \sum_{n=0}^{\infty} \hbar^n X_n$ of the total symbol $X = X(\hbar, x, \xi)$ is determined by $S_0, \dots, S_n$ in the $\hbar$-expansion of S.

Explicit procedure is as follows:

- (Step 0) Assume that $S_0, \dots, S_n$ are given. Expand them into homogeneous terms with respect to powers of ξ: $S_n(x,\xi) = \sum_{j=1}^{\infty} S_{n,j}(x,\xi)$, where $S_{n,j}$ is a term of degree $-j$.
- (Step 1) Define $Y^{(l)}_{k,n,j}(x,y,\xi,\eta)$ as follows:

$$Y^{(l)}_{k,-1,j}(x,y,\xi,\eta) = 0, \tag{3.14}$$

$$Y^{(l)}_{k,m,1}(x,y,\xi,\eta) = \delta_{l,0}\delta_{k,0} S_{m,1}(x,\xi) \tag{3.15}$$

for $m = 0, \dots, n$, $k \geqq 0$, $l \geqq 0$ and

$$Y^{(l)}_{0,m,j} = 0 \tag{3.16}$$

for $m = 0, \dots, n$, $l > 0$, $j \geqq 1$. For other (l,k,m,j), $(l,k) \neq (0,0)$, $Y^{(l)}_{k,m,j}$ are determined by the recursion relation:

$$
\begin{aligned}
Y^{(l)}_{k+1,m,j}(x,y,\xi,\eta) = \frac{1}{k+1} \Bigg(& \partial_\xi \partial_y Y^{(l)}_{k,m-1,j-1}(x,y,\xi,\eta) + \\
+ \sum_{\substack{0 \le l' \le l-1 \\ 0 \le j' \le j-1, 0 \le m' \le m \\ 0 \le k'' \le j-j'-l+l'}} & \frac{1}{l-l'} \partial_\xi Y^{(l')}_{k,m',j'}(x,y,\xi,\eta) \partial_y Y^{(l-l'-1)}_{k'',m-m',j-j'-1}(x,x,\xi,\xi) \Bigg).
\end{aligned}
\tag{3.17}
$$

The remaining $Y^{(0)}_{0,m,j}$ is determined by:

$$Y^{(0)}_{0,m,j}(x,y,\xi,\eta) = S_{m,j}(x,\xi) - \sum_{\substack{(l,k) \neq (0,0) \\ l,k \geq 0, l+k \leq j}} \frac{1}{l+1} Y^{(l)}_{k,m,j}(x,x,\xi,\xi). \tag{3.18}$$

(We shall show that $Y^{(l)}_{k,m,j}=0$ for $l+k>j$.) Schematically this procedure goes as follows:

$$\begin{array}{ccccccc}
 & & Y^{(l)}_{k,m,1}=\delta_{l,0}\delta_{k,0}S_{m,1} & & & & \\
 & & \downarrow & & & & \\
Y^{(l')}_{k',m',1}(m'<m) & \to & Y^{(l)}_{k,m,2}\ (k,l\neq 0) & \to & Y^{(0)}_{0,m,2} & \leftarrow & S_{m,2} \\
 & & \downarrow & \swarrow & & & \\
Y^{(l')}_{k',m',1}, Y^{(l')}_{k',m',2}(m'<m) & \to & Y^{(l)}_{k,m,3}\ (k,l\neq 0) & \to & Y^{(0)}_{0,m,3} & \leftarrow & S_{m,3} \\
 & & \vdots & & & &
\end{array}$$

- (Step 2) $X_n(x,\xi)=\sum_{j=1}^{\infty} Y^{(0)}_{0,n,j}(x,x,\xi,\xi)$. (The infinite sum is the homogeneous expansion in terms of powers of ξ.)

Combining these propositions with the results in Section 2, we can, in principle, make a recursion formula for S_n ($n=0,1,2,\dots$) of the wave function of the solution of the KP hierarchy corresponding to the quantised canonical transformation (f,g) as follows: let $S_0,\dots,S_{i-1}$ be given.

(1) By Proposition 3.2 we have $X_0,\dots,X_{i-1}$.
(2) We have a recursion formula for X_i by Theorem 2.1.
(3) Proposition 3.1 gives a formula for S_i.

If we take the factor $(\hbar\partial)^{\alpha/\hbar}$ into account, this process becomes a little bit complicated, but essentially the same.

The rest of this section is devoted to the proof of Proposition 3.1 and Proposition 3.2.

These statements might seem obvious, but since the multiplication in the definition of

$$e^{X/\hbar}=\sum_{n=0}^{\infty}\frac{X(\hbar,x,\hbar\partial)^n}{\hbar^n n!} \tag{3.19}$$

is non-commutative, while the multiplication of total symbols in the series

$$e^{S/\hbar}=\sum_{n=0}^{\infty}\frac{S(\hbar,x,\xi)^n}{\hbar^n n!} \tag{3.20}$$

is commutative, it needs to be proved. In fact, computation of the simplest example of $X=x(\hbar\partial)^{-1}$ would show how complicated the formula can be:

$$\begin{aligned}
&\sigma_{\mathrm{tot}}(e^{x(\hbar\partial)^{-1}/\hbar})=\sum_{n=0}^{\infty}\frac{1}{n!\hbar^n}\sigma_{\mathrm{tot}}(X^n)\\
=&1+\frac{1}{1!\hbar}x\xi^{-1}+\frac{1}{2!\hbar^2}(x^2\xi^{-2}-\hbar x\xi^{-3})\\
&+\frac{1}{3!\hbar^3}(x^3\xi^{-3}-3\hbar x^2\xi^{-4}+3\hbar^2x\xi^{-5})\\
&+\frac{1}{4!\hbar^4}(x^4\xi^{-4}-6\hbar x^3\xi^{-5}+15\hbar^2x^2\xi^{-6}-15\hbar^3x\xi^{-7})+\cdots\\
=&\exp\frac{1}{\hbar}\left(x\xi^{-1}-\frac{x\xi^{-3}}{2}+\frac{x\xi^{-5}}{2}-\frac{5x\xi^{-7}}{8}+\cdots\right).
\end{aligned}$$

It is not obvious, at least to authors, why there is no more negative powers of $\hbar$ in the last expression, which can be obtained by calculating the logarithm of the previous expression.

To avoid confusion, the commutative multiplication of total symbols $a(\hbar,x,\xi)$ and $b(\hbar,x,\xi)$ as power series is denoted by $a(\hbar,x,\xi)\,b(\hbar,x,\xi)$ and the non-commutative multiplication corresponding to the operator product is denoted by $a(\hbar,x,\xi)\circ b(\hbar,x,\xi)$. Recall that the latter multiplication is expressed (or defined) as follows:

$$\begin{aligned} a(\hbar,x,\xi)\circ b(\hbar,x,\xi) &= e^{\hbar\partial_\xi\partial_y} a(\hbar,x,\xi)b(\hbar,y,\eta)|_{y=x,\eta=\xi} \\ &= \sum_{n=0}^{\infty}\frac{\hbar^n}{n!}\partial_\xi^n a(\hbar,x,\xi)\partial_y^n b(\hbar,y,\eta)|_{y=x,\eta=\xi}. \end{aligned} \tag{3.21}$$

(See, for example, [S], [A] or [KR].) The order of an operator corresponding to symbol $a(\hbar,x,\xi)$ is denoted by $\operatorname{ord}_\xi a(\hbar,x,\xi)$, which is the same as the order of $a(\hbar,x,\xi)$ as a power series of ξ. The $\hbar$-order is the same as that of the operator: $\operatorname{ord}^\hbar x = \operatorname{ord}^\hbar \xi = 0$, $\operatorname{ord}^\hbar \hbar = -1$.

The main idea of proof of propositions is due to Aoki [A], where exponential calculus of pseudodifferential operators is considered. He considered analytic symbols of exponential type, while our symbols are formal ones. Therefore we have only to confirm that those symbols make sense as formal series.

First, we prove the following lemma.

Lemma 3.3. *Let $a(\hbar,x,\xi)$ and $b(\hbar,x,\xi)$ be two symbols such that $\operatorname{ord}_\xi a(\hbar,x,\xi) = M$, $\operatorname{ord}^\hbar a(\hbar,x,\xi) = 0$, $\operatorname{ord}_\xi b(\hbar,x,\xi) = N$, $\operatorname{ord}^\hbar b(\hbar,x,\xi) = 0$, $\operatorname{ord}_\xi p(\hbar,x,\xi) = \operatorname{ord}_\xi q(\hbar,x,\xi) = 0$, $\operatorname{ord}^\hbar p(\hbar,x,\xi) = \operatorname{ord}^\hbar q(\hbar,x,\xi) = 0$.*

Then there exists a symbol c ($\operatorname{ord}_\xi c = N+M$, $\operatorname{ord}^\hbar c = 0$), r ($\operatorname{ord}_\xi r = 0$, $\operatorname{ord}^\hbar r = 0$) such that

$$\left(a(\hbar,x,\xi)e^{p(\hbar,x,\xi)/\hbar}\right)\circ\left(b(\hbar,x,\xi)e^{q(\hbar,x,\xi)/\hbar}\right) = c(\hbar,x,\xi)e^{r(\hbar,x,\xi)/\hbar}. \tag{3.22}$$

In the proof of Proposition 3.1 and Proposition 3.2, we use the construction of c and r in the proof of Lemma 3.3.

Proof. Following [A], we introduce a parameter t and consider

$$\pi(t) = \pi(t;\hbar,x,y,\xi,\eta) := e^{\hbar t\partial_\xi\partial_y} a(\hbar,x,\xi)b(\hbar,y,\eta)e^{\left(p(\hbar,x,\xi)+q(\hbar,y,\eta)\right)/\hbar}. \tag{3.23}$$

If we set $t=1$, $y=x$ and $\eta=\xi$, this reduces to the operator product of (3.21). The series $\pi(t)$ satisfies a differential equation with respect to t:

$$\partial_t\pi = \hbar\partial_\xi\partial_y\pi, \qquad \pi(0) = a(\hbar,x,\xi)b(\hbar,y,\eta)e^{\left(p(\hbar,x,\xi)+q(\hbar,y,\eta)\right)/\hbar}. \tag{3.24}$$

We construct its solution in the following form:

$$\begin{aligned} \pi(t) &= \psi(t)e^{w(t)/\hbar}, \\ \psi(t) &= \psi(t;\hbar,x,y,\xi,\eta) = \sum_{n=0}^{\infty}\psi_n t^n, \\ w(t) &= w(t;\hbar,x,y,\xi,\eta) = \sum_{k=0}^{\infty} w_k t^k. \end{aligned} \tag{3.25}$$

Later we set $t = 1$ and prove that $\psi(1)$ and $w(1)$ is meaningful as a formal power series of ξ and η. The differential equation (3.24) is rewritten as

$$\begin{aligned}&\frac{\partial\psi}{\partial t} + \hbar^{-1}\psi\frac{\partial w}{\partial t}\\ =&\hbar\partial_\xi\partial_y\psi + \partial_\xi\psi\partial_y w + \partial_y\psi\partial_\xi w + \psi\left(\partial_\xi\partial_y w + \hbar^{-1}\partial_\xi w\partial_y w\right).\end{aligned} \tag{3.26}$$

Hence it is sufficient to construct $\psi(t) = \psi(t;\hbar,x,y,\xi,\eta)$ and $w(t) = w(t;\hbar,x,y,\xi,\eta)$ which satisfy

$$\frac{\partial w}{\partial t} = \hbar\partial_\xi\partial_y w + \partial_\xi w\partial_y w, \tag{3.27}$$

$$\frac{\partial\psi}{\partial t} = \hbar\partial_\xi\partial_y\psi + \partial_\xi\psi\partial_y w + \partial_y\psi\partial_\xi w. \tag{3.28}$$

(This is a *sufficient* condition but not a necessary condition for $\pi = \psi e^w$ to be a solution of (3.24). The solution of (3.24) is unique, but ψ and w satisfying (3.26) are not unique at all.)

To begin with, we solve (3.27) and determine $w(t)$. Expanding it as $w(t) = \sum_{k=0}^\infty w_k t^k$, we have a recursion relation and the initial condition

$$\begin{aligned}w_{k+1} &= \frac{1}{k+1}\left(\hbar\partial_\xi\partial_y w_k + \sum_{\nu=0}^{k}\partial_\xi w_\nu\partial_y w_{k-\nu}\right),\\ w_0 &= p(x,\xi) + q(y,\eta),\end{aligned} \tag{3.29}$$

which determine $w_k = w_k(\hbar,x,y,\xi,\eta)$ inductively. Note that $\operatorname{ord}_\xi w_0 \leqq 0$ and ∂_ξ lowers the order by one, which implies

$$\operatorname{ord}_\xi w_k \leqq -k. \tag{3.30}$$

(Here ord_ξ denotes the order with respect to ξ and η.) This shows that $w(1) = \sum_{k=0}^\infty w_k$ makes sense as a formal series of ξ and η. Moreover it is obvious that w_k and $w(1)$ are formally regular with respect to $\hbar$.

As a next step, we expand $\psi(t)$ as $\psi(t) = \sum_{k=0}^\infty \psi_k t^k$ and rewrite (3.28) into a recursion relation and the initial condition:

$$\begin{aligned}\psi_{k+1} &= \frac{1}{k+1}\left(\hbar\partial_\xi\partial_y\psi_k + \sum_{\nu=0}^{k}(\partial_\xi\psi_\nu\partial_y w_{k-\nu} + \partial_y\psi_\nu\partial_\xi w_{k-\nu})\right),\\ \psi_0 &= a(x,\xi)b(y,\eta).\end{aligned} \tag{3.31}$$

In this case we have estimate of the order of the terms:

$$\operatorname{ord}_\xi \psi_k \leqq N + M - k, \tag{3.32}$$

which shows that the inifinite sum $\psi(1) = \sum_{k=0}^\infty \psi_k$ makes sense. The regularity of ψ_k and $\psi(1)$ is also obvious.

Thus, we have constructed $\pi(t) = \pi(t;\hbar,x,y,\xi,\eta) = \psi(t;\hbar,x,y,\xi,\eta)e^{w(t;\hbar,x,y,\xi,\eta)}$, which is meaningful also at $t = 1$. Hence the product $a(\hbar,x,\xi)\circ b(\hbar,x,\xi) = \pi(1;\hbar,x,x,\xi,\xi)$ is expressed in the form $c(\hbar,x,\xi)e^{r(\hbar,x,\xi)/\hbar}$, where $c(\hbar,x,\xi) = \psi(1;\hbar,x,x,\xi,\xi)$, $r(\hbar,x,\xi) = w(1;\hbar,x,x,\xi,\xi)/\hbar$. □

Proof of Proposition 3.1. We make use of differential equations satisfied by the operator

$$E(s) = E(s;\hbar,x,\hbar\partial) := \exp\left(\frac{s}{\hbar}X(\hbar,x,\hbar\partial)\right), \tag{3.33}$$

depending on a parameter s. The total symbol of $E(s)$ is defined as

$$E(s;\hbar,x,\xi)=\sum_{k=0}^{\infty}\frac{s^k}{\hbar^k k!}X^{(k)}(\hbar,x,\xi),\qquad X^{(0)}=1,\qquad X^{(k+1)}=X\circ X^{(k)}. \tag{3.34}$$

Taking the logarithm (as a function, not as an operator) of this, we can define $S(s)=S(s;\hbar,x,\xi)$ by

$$E(s;\hbar,x,\xi)=e^{\hbar^{-1}S(s;\hbar,x,\xi)}. \tag{3.35}$$

What we are to prove is that $S(s)$, constructed as a series, makes sense at $s=1$ and formally regular with respect to $\hbar$.

Differentiating (3.35), we have

$$X(\hbar,x,\xi)\circ E(s;\hbar,x,\xi)=\frac{\partial S}{\partial s}e^{S(s;\hbar,x,\xi)/\hbar}. \tag{3.36}$$

By Lemma 3.3 ($a\mapsto X$, $b\mapsto 1$, $p\mapsto 0$, $q\mapsto S$) and the technique in its proof, we can rewrite the left hand side as follows:

$$X(\hbar,x,\xi)\circ E(s;\hbar,x,\xi)=Y(s;\hbar,x,x,\xi,\xi)e^{S(s;\hbar,x,\xi)/\hbar}, \tag{3.37}$$

where $Y(s;\hbar,x,y,\xi,\eta)=\sum_{k=0}^{\infty}Y_k$ and $Y_k(s;\hbar,x,y,\xi,\eta)$ are defined by

$$\begin{aligned}&Y_{k+1}(s;\hbar,x,y,\xi,\eta)\\&\quad=\frac{1}{k+1}(\hbar\partial_\xi\partial_y Y_k(s;\hbar,x,y,\xi,\eta)+\partial_\xi Y_k(s;\hbar,x,y,\xi,\eta)\partial_y S(s;\hbar,y,\eta)),\\&Y_0(s;\hbar,x,y,\xi,\eta)=X(\hbar,x,\xi).\end{aligned} \tag{3.38}$$

$Y_k(s)$ corresponds to ψ_k in the proof of Lemma 3.3, while w_k there corresponds to $\delta_{k,0}S(s)$. On the other hand, substituting (3.37) into the left hand side of (3.36), we have

$$\frac{\partial S}{\partial s}(s;\hbar,x,\xi)=Y(s;\hbar,x,x,\xi,\xi). \tag{3.39}$$

We rewrite the system (3.38) and (3.39) in terms of expansion of $S(s;\hbar,x,\xi)$ and $Y_k(s;\hbar,x,y,\xi,\eta)$ in powers of s and $\hbar$:

$$\begin{aligned}S(s;\hbar,x,\xi)&=\sum_{l=0}^{\infty}S^{(l)}(\hbar,x,\xi)s^l=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}S^{(l)}_n(x,\xi)\hbar^n s^l,\\Y_k(s;\hbar,x,y,\xi,\eta)&=\sum_{l=0}^{\infty}Y^{(l)}_k(\hbar,x,y,\xi,\eta)s^l=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}Y^{(l)}_{k,n}(x,y,\xi,\eta)\hbar^n s^l,\end{aligned} \tag{3.40}$$

The coefficient of $\hbar^n s^l$ in (3.38) is

$$\begin{aligned}&Y^{(l)}_{k+1,n}(x,y,\xi,\eta)\\&\quad=\frac{1}{k+1}\left(\partial_\xi\partial_y Y^{(l)}_{k,n-1}(x,y,\xi,\eta)+\sum_{\substack{l'+l''=l\\n'+n''=n}}\partial_\xi Y^{(l')}_{k,n'}(x,y,\xi,\eta)\partial_y S^{(l'')}_{n''}(y,\eta)\right),\end{aligned} \tag{3.41}$$

($Y^{(l)}_{k,-1}=0$) and

$$Y^{(l)}_{0,n}(x,y,\xi,\eta)=\delta_{l,0}X_n(x,\xi), \tag{3.42}$$

while (3.39) gives

$$S_n^{(l+1)}(x,\xi) = \frac{1}{l+1}\sum_{k=0}^{\infty} Y_{k,n}^{(l)}(x,x,\xi,\xi). \tag{3.43}$$

We first show that these recursion relations consistently determine $Y_{k,n}^{(l)}$ and $S_n^{(l)}$. Then we prove that the infinite sum in (3.43) is finite.

Fix $n \geqq 0$ and assume that $Y_{k,0}^{(l)}, \dots, Y_{k,n-1}^{(l)}$ and $S_0^{(l)}, \dots, S_{n-1}^{(l)}$ have been determined for all (l,k). (When $n=0$, $Y_{k,-1}^{(l)} = 0$ as mentioned above and $S_{-1}^{(l)}$ can be ignored as it does not appear in the induction.)

(1) Since $E(s=0)=1$ by the definition (3.33), we have $S^{(0)}=0$. Hence

$$S_n^{(0)} = 0. \tag{3.44}$$

(2) Note that

$$\operatorname{ord}_\xi Y_{0,n}^{(0)} \leqq -1 \tag{3.45}$$

because of (3.42) and the assumption $\operatorname{ord} X \leqq -1$.

(3) When $l=0$, the second sum in the right hand side of the recursion relation (3.41) is absent because of (3.44). Hence if $n \geqq k+1$, we have

$$Y_{k+1,n}^{(0)} = \frac{1}{k+1}\partial_\xi \partial_y Y_{k,n-1}^{(0)} = \cdots = \frac{1}{(k+1)!}(\partial_\xi\partial_y)^{k+1} Y_{0,n-k-1}^{(0)} = 0,$$

since $Y_{0,n-k-1}^{(0)}$ does not depend on y thanks to (3.42). If $n < k+1$, the above expression becomes zero by $Y_{k-n+1,-1}^{(0)} = 0$. Hence together with (3.42), we obtain

$$Y_{k,n}^{(0)} = \delta_{k,0} X_n. \tag{3.46}$$

(4) By (3.43) we can determine $S_n^{(1)}$:

$$S_n^{(1)} = \sum_{k=0}^{\infty} Y_{k,n}^{(0)} = Y_{0,n}^{(0)} = X_n. \tag{3.47}$$

(5) Fix $l_0 \geqq 1$ and assume that for all $l = 0, \dots, l_0 - 1$ and for all $k = 0,1,2,\dots$, we have determined $Y_{k,n}^{(l)}$ and that for all $l = 0,\dots,l_0$ we have determined $S_n^{(l)}$. (The steps (3) and (4) are for $l_0 = 1$.)

Since $S_{n''}^{(0)} = 0$ by (3.44), the index l' in the right hand side of the recursion relation (3.41) (with $l = l_0$) runs essentially from 0 to $l_0 - 1$. Hence this relation determines $Y_{k+1,n}^{(l_0)}$ from known quantities for all $k \geqq 0$.

Because of the initial condition $Y_0(s;x,y,\xi,\eta) = X(x,\xi)$ (cf. (3.38)) Y_0 does not depend on s, which means that its Taylor coefficients $Y_{0,n}^{(l_0)}$ vanish for all $l_0 \geqq 1$:

$$Y_{0,n}^{(l_0)} = 0. \tag{3.48}$$

Thus we have determined all $Y_{k,n}^{(l_0)}$ $(k = 0,1,2,\dots)$.

(6) We shall prove below that $Y_{k,n}^{(l_0+1)} = 0$ if $k > l_0 + n + 1$. Hence the sum in (3.43) is finite and $S_n^{(l_0+1)}$ can be determined. The induction proceeds by incrementing l_0 by one.

Let us prove that $Y^{(l)}_{k,n}$'s determined above satisfy

$$Y^{(l)}_{k,n} = 0, \qquad \text{if } k > l + n, \tag{3.49}$$

$$\operatorname{ord}_\xi Y^{(l)}_{k,n} \leqq -k - l - 1. \tag{3.50}$$

(We define that $\operatorname{ord}_\xi 0 = -\infty$.) In particular, the sum in (3.43) is well-defined and

$$\operatorname{ord}_\xi S^{(l+1)}_n \leqq -l - 1. \tag{3.51}$$

If $n = -1$, both (3.49) and (3.50) are obvious. Fix $n_0 \geqq 0$ and assume that we have proved (3.49) and (3.50) for $n < n_0$ and all (l, k).

When $l = 0$, (3.49) and (3.50) are true for all k because of (3.46) and (3.45), respectively. Fix $l_0 \geqq 0$ and assume that we have prove (3.49) and (3.50) for $l \leqq l_0$ and all k. As a result (3.51) is true for $l \leqq l_0$.

The recursion relation (3.41) implies (3.49) and (3.50) for $l = l_0 + 1$ and all $k > 0$. In fact, if $k + 1 > (l_0 + 1) + n$, then $k > (l_0 + 1) + (n - 1)$ and $k > l' + n'$ which guarantees that $Y^{(l_0+1)}_{k,n-1} = 0$ and $Y^{(l')}_{k,n'} = 0$ in the recursion relation (3.41) for $l = l_0 + 1$ by the induction hypothesis. The estimate (3.50) is easy to check for $l_0 + 1$. (Recall once again that ∂_ξ lowers the order by one.)

For $k = 0$, (3.49) is void and (3.50) is true because of (3.42) and $\operatorname{ord} X_n \leqq -1$.

The step $l = l_0 + 1$ being proved, the induction proceeds with respect to l and consequently with respect to n.

In summary we have constructed $Y(s; \hbar, x, y, \xi, \eta)$ and $S(s; \hbar, x, \xi)$ satisfying (3.37) and (3.39). Thanks to (3.51), $S_n(x, \xi) = \sum_{l=0}^\infty S^{(l)}_n(x, \xi)$ is meaningful as a power series of ξ. Thus Proposition 3.1 is proved. □

Proof of Proposition 3.2. We reverse the order of the previous proof. Namely, given $S(\hbar, x, \xi)$, we shall construct $X(\hbar, x, \xi)$ such that the corresponding $S(1; \hbar, x, \xi)$ in the above proof coincides with it.

Suppose we have such $X(\hbar, x, \xi)$. Then the above procedure determine $Y^{(l)}_{k,n}$ and $S^{(l)}_n$. We expand them as follows:

$$\begin{aligned}
S(\hbar, x, \xi) &= \sum_{n=0}^\infty S_n(x, \xi)\hbar^n, & S_n(x, \xi) &= \sum_{j=1}^\infty S_{n,j}(x, \xi), \\
X(\hbar, x, \xi) &= \sum_{n=0}^\infty X_n(x, \xi)\hbar^n, & X_n(x, \xi) &= \sum_{j=1}^\infty X_{n,j}(x, \xi), \\
S(s; \hbar, x, \xi) &= \sum_{l=0}^\infty \sum_{n=0}^\infty S^{(l)}_n(x, \xi)\hbar^n s^l, & S^{(l)}_n(x, \xi) &= \sum_{j=1}^\infty S^{(l)}_{n,j}(x, \xi), \\
&Y_k(s; \hbar, x, y, \xi, \eta) & &Y^{(l)}_{k,n}(x, y, \xi, \eta) \\
&= \sum_{l=0}^\infty \sum_{n=0}^\infty Y^{(l)}_{k,n}(x, y, \xi, \eta)\hbar^n s^l, & &= \sum_{j=1}^\infty Y^{(l)}_{k,n,j}(x, y, \xi, \eta).
\end{aligned}$$

Here terms with index j are homogeneous terms of degree $-j$ with respect to ξ and η.

At the end of this proof we shall determine X_n by (3.42),

$$X_n(x, \xi) = Y^{(0)}_{0,n}(x, y, \xi, \eta). \tag{3.52}$$

(In particular, $Y^{(0)}_{0,n}(x,y,\xi,\eta)$ should not depend on y and η.) For this purpose, $Y^{(0)}_{0,n}$ should be determined by

$$Y^{(0)}_{0,n}(x,y,\xi,\eta) = S_n(x,\xi) - \sum_{\substack{(l,k)\neq(0,0)\\ l,k\geq 0}} \frac{1}{l+1} Y^{(l)}_{k,n}(x,x,\xi,\xi) \tag{3.53}$$

because of (3.43) and $S_n(x,\xi) = \sum_{l=1}^{\infty} S_n^{(l)}(x,\xi)$.

Since $\operatorname{ord}_\xi Y^{(l)}_{k,n}$ should be less than $-l-k$ (cf. (3.50)), we expect

$$Y^{(l)}_{k,n,1} = 0 \tag{3.54}$$

for $(l,k)\neq(0,0)$. Hence picking up homogeneous terms of degree -1 with respect to ξ, the following equation should hold:

$$Y^{(0)}_{0,n,1} = S_{n,1} \tag{3.55}$$

All $Y^{(l)}_{k,n,1}$ are determined by the above two conditions, (3.54) and (3.55). Note also that

$$Y^{(l)}_{0,n,j} = 0 \text{ for } l \neq 0 \tag{3.56}$$

because Y_0 should not depend on s because of (3.42).

Having determined initial conditions in this way, we shall determine $Y^{(l)}_{k,n,j}$ inductively. To this end we rewrite the recursion relation (3.41) by (3.43) and pick up homogeneous terms of degree j:

$$\begin{aligned} Y^{(l)}_{k+1,n,j}(x,y,\xi,\eta) = \frac{1}{k+1}\Bigg(& \partial_\xi \partial_y Y^{(l)}_{k,n-1,j-1}(x,y,\xi,\eta) + \\ & + \sum_{\substack{l'+l''=l,\, l''\geqq 1\\ j'+j''=j-1,\, n'+n''=n\\ 0\leqq k''}} \frac{1}{l''} \partial_\xi Y^{(l')}_{k,n',j'}(x,y,\xi,\eta) \partial_y Y^{(l''-1)}_{k'',n'',j''}(x,x,\xi,\xi) \Bigg) \end{aligned} \tag{3.57}$$

(As before, terms like $Y^{(l)}_{k,-1,j-1}$ appearing the above equation for $n=0$ can be ignored.)

Fix $n_0 \geqq 0$ and assume that $Y^{(l)}_{k,0,j},\dots,Y^{(l)}_{k,n_0-1,j}$ are determined for all (l,k,j).

(1) First we determine $Y^{(l)}_{k,n_0,1}$ for all (l,k) by (3.55) and (3.54).

(2) Fix $j_0 \geqq 2$ and assume that $Y^{(l)}_{k,n_0,j}$ are determined for $j=1,\dots,j_0-1$ and all (l,k). (The above step is for $j_0=2$.)

    Since all the quantities in the right hand side of the recursion relation (3.57) with $j=j_0$ are known by the induction hypothesis, we can determine $Y^{(l)}_{k,n_0,j_0}$ for $l=0,1,2,\dots$ and $k=1,2,\dots$.

(3) Together with (3.56), $Y^{(l)}_{0,n_0,j_0}=0$ for $l=1,2,\dots$, we have determined all $Y^{(l)}_{k,n_0,j_0}$ except for the case $(l,k)=(0,0)$.

(4) It follows from (3.57), (3.55) and (3.56) by induction that for all $Y^{(l)}_{k,n_0,j}$ determined in (1), (2) and (3),

$$Y^{(l)}_{k,n,j} = 0 \text{ for } l+k+1 > j. \tag{3.58}$$

This corresponds to $\operatorname{ord}_\xi Y^{(l)}_{k,n_0} \leqq -l-k-1$ (3.50) in the proof of Proposition 3.1.

(5) We determine $Y^{(0)}_{0,n_0,j_0}$ by

$$Y^{(0)}_{0,n_0,j_0} = S_{n_0,j_0} - \sum_{\substack{(l,k)\neq(0,0)\\ l,k\geq 0}} \frac{1}{l+1} Y^{(l)}_{k,n_0,j_0}(x,x,\xi,\xi), \tag{3.59}$$

which is the homogeneous part of degree $-j_0$ in (3.53). The sum in the right hand side is finite thanks to (3.58).

(6) The induction with respect to j proceeds by incrementing j_0.

Thus all $Y^{(l)}_{k,n_0,j}$ are determined and X_{n_0} is determined by $X_{n_0}(x,\xi) = \sum_{j=1}^\infty Y^{(0)}_{0,n_0,j}$ (cf. (3.52)), which completes the proof of Proposition 3.2. □

4. Asymptotics of the tau function

In this section we derive an $\hbar$ expansion

$$\log\tau(\hbar,t) = \sum_{n=0}^\infty \hbar^{n-2} F_n(t) \tag{4.1}$$

of the tau function from the $\hbar$ expansion (3.7) of the S-function. Note that we have suppressed the variable x, which is understood to be absorbed in t_1.

Let us recall the fundamental relation [DJKM]

$$\Psi(t;z) = \frac{\tau(t-\hbar[z^{-1}])}{\tau(t)} e^{\hbar^{-1}\zeta(t,z)}, \tag{4.2}$$

where $[z^{-1}] = (1/z, 1/2z^2, 1/3z^3, \dots)$ and $\zeta(t,z) = \sum_{n=1}^\infty t_n z^n$. This implies that

$$\hbar^{-1}\hat S(t;z) = \left(e^{-\hbar D(z)} - 1\right)\log\tau(t) \tag{4.3}$$

where $\hat S(t;z) = S(t;z) - \zeta(t,z)$ and $D(z) = \sum_{j=1}^\infty \frac{z^{-j}}{j}\frac{\partial}{\partial t_j}$. Differentiating this with respect to z, we have

$$\begin{aligned} \hbar^{-1}\frac{\partial}{\partial z}\hat S(t;z) &= -\hbar D'(z) e^{-\hbar D(z)}\log\tau(t) \\ &= -\hbar D'(z)(\log\hat w(t;z) + \log\tau(t)), \end{aligned} \tag{4.4}$$

where $D'(z) := \frac{\partial}{\partial z}D(z) = -\sum_{j=1}^\infty z^{-j-1}\frac{\partial}{\partial t_j}$. Hence

$$-\hbar D'(z)\log\tau(t) = \hbar^{-1}\left(\frac{\partial}{\partial z} + \hbar D'(z)\right)\hat S(t;z) \tag{4.5}$$

Multiplying z^n to this equation and taking the residue, we obtain a system of differential equations

$$\hbar\frac{\partial}{\partial t_n}\log\tau(t) = \hbar^{-1}\operatorname{Res} z^n\left(\frac{\partial}{\partial z} + \hbar D'(z)\right)\hat S(t;z)\,dz, \qquad n=1,2,\dots \tag{4.6}$$

which is known to be integrable [DJKM]. We can thus define the tau function $\tau(t)$, up to multiplication $\tau(t)\to c\tau(t)$ by a nonzero constant c, as a solution of (4.5).

By substituting the $\hbar$-expansions

$$\log\tau(t) = \sum_{n=0}^\infty \hbar^{n-2}F_n(t), \qquad \hat S(t;z) = \sum_{n=0}^\infty \hbar^n S_n(t;z), \tag{4.7}$$

into (4.5), we have

$$\sum_{j=1}^{\infty}\sum_{n=0}^{\infty} z^{-j-1}\hbar^{n-1}\frac{\partial F_n}{\partial t_j} = \sum_{n=0}^{\infty}\hbar^{n-1}\frac{\partial S_n}{\partial z} - \sum_{j=1}^{\infty} z^{-j-1}\hbar^{n}\frac{\partial S_n}{\partial t_j}. \tag{4.8}$$

Let us expand $S_n(t;z)$ into a power series of z^{-1}:

$$S_n(t;z) = -\sum_{k=1}^{\infty}\frac{z^{-k}}{k}v_{n,k}. \tag{4.9}$$

(The notation is chosen to be consistent with our previous work, e.g., [TT2].) Comparing the coefficients of $z^{-j-1}\hbar^{n-1}$ in (4.8), we have the equations

$$\frac{\partial F_n}{\partial t_j} = v_{n,j} + \sum_{\substack{k+l=j\\ k\geq 1, l\geq 1}}\frac{1}{l}\frac{\partial v_{n-1,l}}{\partial t_k} \qquad (v_{-1,j}=0), \tag{4.10}$$

which may be understood as defining equations of $F_n(t)$. According to what we have seen above, this system of differential equations is integrable and determines F_n up to integration constants.

Remark 4.1. Tau functions in string theory and random matrices are known to have a *genus expansion* of the form

$$\log\tau = \sum_{g=0}\hbar^{2g-2}\mathcal{F}_g, \tag{4.11}$$

where $\mathcal{F}_g$ is the contribution from Riemann surfaces of genus g. In contrast, general tau functions of the $\hbar$-dependent KP hierarchy is not of this form, namely, odd powers of $\hbar$ can appear in the $\hbar$-expansion of $\log\tau$. To exclude odd powers therein, we need to impose conditions

$$0 = v_{2m+1,j} + \sum_{\substack{k+l=j\\ k\geq 1, l\geq 1}}\frac{1}{l}\frac{\partial v_{2m,l}}{\partial t_k}$$

on $v_{n,j}$ or

$$0 = \frac{\partial S_{2m+1}}{\partial z} - \sum_{j=1}^{\infty} z^{-j-1}\frac{\partial S_{2m}}{\partial t_j}$$

on S_n.

5. Concluding remarks

We have presented a recursive construction of solutions of the $\hbar$-dependent KP hierarchy. The input of this construction is the pair (f,g) of quantised canonical transformation. The main outputs are the dressing operator W in the exponential form (1.8), the wave function Ψ in the WKB form (3.2) and the tau function with the quasi-classical expansion (4.1). Thus the $\hbar$-dependent KP hierarchy introduced in our previous work [TT2] is no longer a heuristic framework for deriving the dispersionless KP hierarchy, but has its own raison d'être.

A serious problem of our construction is that the recursion relations are extremely complicated. In Appendix B, calculations are illustrated for the Kontsevich model [Ko], [D], [AvM]. As this example shows, this is by no means a practical way to construct a solution. We believe that one cannot avoid this difficulty as far as general solutions are considered.

Special solutions stemming from string theory and random matrices [Mo], [dFGZ] (e.g. the Kontsevich model) can admit a more efficient approach such as the method of Eynard and Orantin [EO]. Those methods are based on a quite different principle. In the method of Eynard and Orantin, it is the so called "loop equation" for correlation functions of random matrices. The loop equations amount to "constraints" on the tau function. Eynard and Orantin's "topological recursion relations" determine a solution of those constraints rather than of an underlying integrable hierarchy; it is somewhat surprising that a solution of those constraints gives a tau function.

Lastly, let us mention that the results of this paper can be extended to the Toda hierarchy. That case will be treated in a forthcoming paper.

Appendix A. Proof of formulae (2.17) and (2.19)

In this appendix we prove the factorisation of W (2.17) and an auxiliary formula (2.19).

The main tool in this appendix is the Campbell-Hausdorff theorem:

$$\exp(X)\exp(Y) = \exp\left(\sum_{n=0}^{\infty} c_n(X,Y)\right), \tag{A.1}$$

where $c_n(X,Y)$ is determined recursively:

$$\begin{aligned} &c_1(X,Y) = X+Y,\\ &c_{n+1}(X,Y) = \frac{1}{n+1}\Biggl(\frac{1}{2}[X-Y,c_n]+\\ &+\sum_{p\geq 1, 2p\leq n} K_{2p} \sum_{\substack{(k_1,\dots,k_{2p})\\ k_1+\cdots+k_{2p}=n}} [c_{k_1},[\cdots,[c_{k_{2p}},X+Y]\cdots]]\Biggr). \end{aligned} \tag{A.2}$$

The coefficients K_{2p} are defined by (2.15). See, for example, [B].

First we prove

$$\begin{aligned} &\exp(\hbar^{-1}X(x,t,\hbar\partial))\\ &\quad= \exp\left(\hbar^{i-1}\tilde{X}'_i + (\text{terms of } \hbar\text{-order} < -i+1)\right)\exp\left(\hbar^{-1}X^{(i-1)}\right), \end{aligned} \tag{A.3}$$

where the principal symbol of $\tilde{X}'_i$ is

$$\sigma^\hbar(\tilde{X}'_i) := \sum_{n=1}^{\infty} \frac{(\mathrm{ad}_{\{,\}}\,\sigma^\hbar(X_0))^{n-1}}{n!}\sigma^\hbar(X_i), \tag{A.4}$$

as is defined in (2.19). For simplicity, let us denote

$$A := \frac{1}{\hbar}X^{(i-1)} = \frac{1}{\hbar}\sum_{j=0}^{i-1}\hbar^j X_j, \qquad B := \frac{1}{\hbar}\sum_{j=i}^{\infty}\hbar^j X_j. \tag{A.5}$$

Note that $A+B = X/\hbar$ and $\operatorname{ord}^\hbar A \leqq 1$, $\operatorname{ord}^\hbar B \leqq -i+1$. We prove the following by induction:

$$C_n := c_n(A+B,-A) = \frac{(\mathrm{ad}\,A)^{n-1}}{n!}(B) + (\text{terms of } \hbar\text{-order} < -i+1). \tag{A.6}$$

This is obvious for $n=1$ since $C_1 = (A+B)+(-A) = B$. Assume that (A.6) is true for $n = 1,\dots,N$. This means, in particular, $\operatorname{ord}^\hbar C_n \leqq \operatorname{ord}^\hbar B \leqq 0$ $(1 \leqq n \leqq N)$, which implies that for any operator Z $\operatorname{ord}^\hbar[C_n, Z]$ is less than $\operatorname{ord}^\hbar Z$ by more than one. Hence the term of the highest $\hbar$-order in the recursive definition (A.2) is the first term. More precisely, it is decomposed as

$$\frac{1}{N+1}\cdot\frac{1}{2}[(A+B)-(-A), C_N] = \frac{1}{N+1}[A, C_N] + \frac{1}{2(N+1)}[B, C_N],$$

and the first term in the right hand side has the highest $\hbar$-order. By the induction hypothesis and $\operatorname{ord}^\hbar A \leqq 1$, we have

$$\begin{aligned}\frac{1}{N+1}[A, C_N] &= \frac{1}{N+1}\left[A, \frac{(\operatorname{ad} A)^{N-1}}{N!}(B) + (\text{terms of } \hbar \text{ order} < -i+1)\right] \\ &= \frac{(\operatorname{ad} A)^N}{(N+1)!}(B) + (\text{terms of } \hbar\text{-order} < -i+1).\end{aligned} \tag{A.7}$$

This proves (A.6) for all n. Taking its symbol of order $-i+1$, we have

$$\sigma^\hbar(c_n(A+B, -A)) = \frac{(\operatorname{ad}_{\{,\}} \sigma^\hbar(A))^{n-1}}{n!}\sigma^\hbar(B), \tag{A.8}$$

which gives the terms of (A.4). Substituting this into the Campbell-Hausdorff formula (A.1), we have (A.3).

By factorisation (A.3), we can factorise $W = \exp(X/\hbar)(\hbar\partial)^\alpha$ as follows ($\alpha^{(i-1)} := \sum_{j=0}^{i-1} \hbar^j \alpha_j$):

$$\begin{aligned}&\exp(\hbar^{-1}X(x,t,\hbar\partial))(\hbar\partial)^\alpha \\ =&\exp\left(\hbar^{i-1}\tilde{X}_i' + (\text{terms of } \hbar\text{-order} < -i+1)\right)\exp\left(\hbar^{-1}X^{(i-1)}\right)\times \\ &\times\exp\left(\hbar^{i-1}\alpha_i\log(\hbar\partial) + (\text{terms of } \hbar\text{-order} < -i+1)\right)\times \\ &\times\exp\left(\hbar^{-1}\alpha^{(i-1)}\log(\hbar\partial)\right) \\ =&\exp\left(\hbar^{i-1}\tilde{X}_i' + (\text{terms of } \hbar\text{-order} < -i+1)\right)\times \\ &\times\exp\left(e^{\operatorname{ad}(\hbar^{-1}X^{(i-1)})}\left(\hbar^{i-1}\alpha_i\log(\hbar\partial) + (\text{terms of } \hbar\text{-order} < -i+1)\right)\right)\times \\ &\times\exp\left(\hbar^{-1}X^{(i-1)}\right)\exp\left(\hbar^{-1}\alpha^{(i-1)}\log(\hbar\partial)\right).\end{aligned} \tag{A.9}$$

Since the symbol of order $-i+1$ of $e^{\operatorname{ad}(\hbar^{-1}X^{(i-1)})}\left(\hbar^{i-1}\alpha_i\log(\hbar\partial)\right)$ is $e^{\operatorname{ad}_{\{,\}} X_0}(\alpha_i \log\xi)$, (A.9) is rewritten as (2.17) by using the Campbell-Hausdorff formula (A.1) once again.

In order to recover X_i from $\tilde{X}_i$ (or $\tilde{X}_i'$), we have only to invert the definition (A.4). In the definition (A.4) we substitute $\operatorname{ad}_{\{,\}}(\sigma^\hbar(X_0))$ in the equation

$$\frac{e^t-1}{t} = \sum_{n=1}^{\infty}\frac{t^{n-1}}{n!}.$$

Hence substitution $t = \operatorname{ad}_{\{,\}}(\sigma^\hbar(X_0))$ in its inverse

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} + \sum_{p=1}^{\infty} K_{2p}t^{2p}$$

gives the inverse map $X_i \mapsto \tilde{X}_i$. Here the coefficients K_{2p} are defined in (2.15). Hence equation (2.19):

$$\sigma^\hbar(X_i) = \sigma^\hbar(\tilde{X}'_i) - \frac{1}{2}\{\sigma^\hbar(X_0), \sigma^\hbar(\tilde{X}'_i)\} + \sum_{p=1}^{\infty} K_{2p}(\mathrm{ad}_{\{,\}}(\sigma^\hbar(X_0)))^{2p}\sigma^\hbar(\tilde{X}'_i)$$

gives the symbol of X_i.

Appendix B. Example (Kontsevich model)

According to Adler and van Moerbeke [AvM], the solution of the KP hierarchy arising in the Witten-Kontsevich theorem [Ko] satisfies

$$(L^2)_{\le -1} = 0, \qquad \left(\frac{1}{2}ML^{-1} - \frac{1}{4}\hbar L^{-2} - L\right)_{\le -1} = 0. \tag{B.1}$$

This corresponds to the case where

$$f(x,\hbar\partial) = (\hbar\partial)^2, \qquad g(x,\hbar\partial) = \frac{1}{2}x(\hbar\partial)^{-1} - \frac{\hbar}{4}(\hbar\partial)^{-2} + \hbar\partial. \tag{B.2}$$

Let us apply our procedure to this case. We fix all the time variables t_n to 0, which means that we restrict ourselves to the so-called "small phase space" in topological string theory [D]. (The first time variable t_1 can be re-introduced by shifting x to $t_1 + x$.)

To begin with, let us determine the leading terms of X and α, which are the initial data for our procedure. The dispersionless limit of (L, M) satisfies

$$(\mathcal{L}^2)_{\le -1} = 0, \qquad \left(\frac{1}{2}\mathcal{M}\mathcal{L}^{-1} - \mathcal{L}\right)_{\le -1} = 0. \tag{B.3}$$

Since $\mathcal{L}$ has the form (1.15), $\mathcal{L}^2$ should be a second order polynomial of ξ: $\mathcal{L}^2 = \xi^2 + u(x)$. When $t_n = 0$, $\mathcal{M}$ has the form

$$\mathcal{M} = x + \alpha_0\mathcal{L}^{-1} + \sum_{n=1}^{\infty} v_{0,n}\mathcal{L}^{-n-1}.$$

(See (1.19).) Using this and $\mathcal{L} = \xi(1 + u(x)\xi^{-2})^{1/2}$, we have

$$\frac{1}{2}\mathcal{M}\mathcal{L}^{-1} - \mathcal{L} = -\xi + \left(\frac{x}{2} - \frac{u}{2}\right)\xi^{-1} + \frac{\alpha_0}{2}\xi^{-2} + O(\xi^{-3}).$$

Hence, due to the second equation of (B.3), we have $u(x) = x$, $\alpha_0 = 0$ and, consequently,

$$\mathcal{P}_0 := \mathcal{L}^2 = \xi^2 + x, \qquad \mathcal{Q}_0 := \frac{1}{2}\mathcal{M}\mathcal{L}^{-1} - \mathcal{L} = -\xi. \tag{B.4}$$

Combining them, we have the following expression for $\mathcal{M}$:

$$\begin{aligned}\mathcal{M} =& 2\mathcal{L}(\mathcal{L} - \xi) = 2(\xi^2 + x) - 2\xi^2\left(1 + x\xi^{-2}\right)^{1/2}\\ =& x - \sum_{n=2}^{\infty} 2\binom{1/2}{n} x^n \xi^{-2n+2}\\ =& x + \frac{x^2}{4\xi^2} - \frac{x^3}{8\xi^4} + \frac{5x^4}{64\xi^6} - \frac{7x^5}{128\xi^8} + \frac{21x^6}{512\xi^{10}} + O(\xi^{-12})\end{aligned} \tag{B.5}$$

On the other hand, from the expression $\mathcal{M} = \exp(\mathrm{ad}_{\{,\}} X_0)x$ follows

$$
\begin{aligned}
\mathcal{M} =& \sum_{n=0}^{\infty} \frac{\mathrm{ad}_{\{,\}} X_0}{n!} x \\
=& x - \chi_{0,1}\xi^{-2} - 2\chi_{0,2}\xi^{-3} \\
&+ \left(-3\chi_{0,3} - \frac{\chi_{0,1}\chi_{0,1}'}{2}\right)\xi^{-4} + (-4\chi_{0,4} - 2\chi_{0,2}\chi_{0,1}')\xi^{-5} + \cdots ,
\end{aligned}
\tag{B.6}
$$

where $\chi_{0,k}$ are the coefficients in the expansion of X_0 (2.16). Comparing (B.5) and (B.6), we can determine $\chi_{0,k}$ inductively and hence X_0 is determined:

$$
\begin{aligned}
X_0 =& -\frac{x^2}{4}(\hbar\partial)^{-1} + \frac{x^3}{48}(\hbar\partial)^{-3} - \frac{x^4}{384}(\hbar\partial)^{-5} \\
&+ \frac{x^6}{6144}(\hbar\partial)^{-9} - \frac{x^7}{61440}(\hbar\partial)^{-11} + \cdots .
\end{aligned}
$$

Having determined X_0 and α_0, we can start the algorithm discussed in Section 2. In the step 1 for $i = 1$ we define $P^{(0)}$ and $Q^{(0)}$ by (2.9) and (2.10):

$$
\begin{aligned}
P^{(0)} =& (\hbar\partial)^2 + x + \frac{\hbar}{2}(\hbar\partial)^{-1} - \frac{\hbar x}{4}(\hbar\partial)^{-3} + \frac{\hbar^2}{8}(\hbar\partial)^{-4} + \frac{5\hbar x^2}{32}(\hbar\partial)^{-5} \\
&- \frac{41\hbar^2 x}{96}(\hbar\partial)^{-6} + \left(\frac{23\hbar^3}{96} - \frac{3\hbar x^3}{32}\right)(\hbar\partial)^{-7} + \frac{301\hbar^2 x^2}{384}(\hbar\partial)^{-8} \\
&+ \left(-\frac{83\hbar^3 x}{48} + \frac{53\hbar x^4}{1024}\right)(\hbar\partial)^{-9} + \left(\frac{191\hbar^4}{192} - \frac{543\hbar^2 x^3}{512}\right)(\hbar\partial)^{-10} \\
&+ \left(\frac{8783\hbar^3 x^2}{1536} - \frac{119\hbar x^5}{4096}\right)(\hbar\partial)^{-11} + \cdots , \\
Q^{(0)} =& -(\hbar\partial) - \frac{\hbar}{4}(\hbar\partial)^{-2} + \frac{3\hbar x}{8}(\hbar\partial)^{-4} - \frac{3\hbar^2}{8}(\hbar\partial)^{-5} - \frac{29\hbar x^2}{64}(\hbar\partial)^{-6} \\
&+ \frac{157\hbar^2 x}{96}(\hbar\partial)^{-7} + \left(-\frac{49\hbar^3}{32} + \frac{\hbar x^3}{2}\right)(\hbar\partial)^{-8} - \frac{519\hbar^2 x^2}{128}(\hbar\partial)^{-9} \\
&+ \left(\frac{4345\hbar^3 x}{384} - \frac{1077\hbar x^4}{2048}\right)(\hbar\partial)^{-10} + \left(-\frac{1339\hbar^4}{128} + \frac{3961\hbar^2 x^3}{512}\right)(\hbar\partial)^{-11} \\
&+ \cdots .
\end{aligned}
$$

We extract terms (symbols) of $\hbar$-order 0 from the $\hbar$-expansion of them:

$$
\mathcal{P}_0(x,\xi) = \xi^2 + x, \qquad \mathcal{Q}_0(x,\xi) = -\xi,
$$

and those of $\hbar$-order -1:

$$
\begin{aligned}
\mathcal{P}_1^{(0)}(x,\xi) =& \frac{1}{2}\xi^{-1} - \frac{x}{4}\xi^{-3} + \frac{5x^2}{32}\xi^{-5} - \frac{3x^3}{32}\xi^{-7} + \frac{53x^4}{1024}\xi^{-9} \\
&- \frac{119x^5}{4096}\xi^{-11} + \cdots \\
\mathcal{Q}_1^{(0)}(x,\xi) =& -\frac{1}{4}\xi^{-2} + \frac{3x}{8}\xi^{-4} - \frac{29x^2}{64}\xi^{-6} + \frac{x^3}{2}\xi^{-8} - \frac{1077x^4}{2048}\xi^{-10} + \cdots .
\end{aligned}
$$

Then, following (2.13), we determine α_0 and $\mathcal{X}_0$ by

$$\begin{aligned}\alpha_1 \log\xi + \tilde{\mathcal{X}}_1 :=& \int \left(\frac{\partial \mathcal{Q}_0}{\partial \xi} \mathcal{P}_1^{(0)} - \frac{\partial \mathcal{P}_0}{\partial \xi} \mathcal{Q}_1^{(0)} \right)_{\le -1} d\xi \\ =& \frac{x}{4}\xi^{-2} - \frac{3x^2}{16}\xi^{-4} + \frac{29x^3}{192}\xi^{-6} - \frac{x^4}{8}\xi^{-8} + \frac{1077x^5}{10240}\xi^{-10} + \cdots .\end{aligned}$$

Since log term is absent, $\alpha_1 = 0$ and the above expression is $\tilde{\mathcal{X}}_1$ itself, which is also equal to $\tilde{\mathcal{X}}_1'$ defined by (2.14). Then we can compute X_1 by the formulae (2.14) and (2.16):

$$X_1 = \frac{x}{4}(\hbar\partial)^{-2} - \frac{3x^2}{32}(\hbar\partial)^{-4} + \frac{5x^3}{192}(\hbar\partial)^{-6} - \frac{9x^5}{2048}(\hbar\partial)^{-10} + \cdots$$

We can repeat the procedure Step 1, 2, 3 for $i = 2$ again. The results are

$$\begin{aligned}P^{(1)} =& (\hbar\partial)^2 + x + \frac{3\hbar^2}{16}(\hbar\partial)^{-4} - \frac{23\hbar^2 x}{96}(\hbar\partial)^{-6} + \frac{9\hbar^3}{16}(\hbar\partial)^{-7} + \frac{19\hbar^2x^2}{384}(\hbar\partial)^{-8} \\ &- \frac{707\hbar^3 x}{384}(\hbar\partial)^{-9} + \left(\frac{861\hbar^4}{256} + \frac{155\hbar^2x^3}{512} \right)(\hbar\partial)^{-10} + \frac{2145\hbar^3x^2}{1024}(\hbar\partial)^{-11} \\ &+ \cdots , \\ Q^{(1)} =& -\hbar\partial - \frac{\hbar^2}{4}(\hbar\partial)^{-5} + \frac{143\hbar^2 x}{192}(\hbar\partial)^{-7} - \frac{611\hbar^3}{384}(\hbar\partial)^{-8} - \frac{85\hbar^2x^2}{64}(\hbar\partial)^{-9} \\ &+ \frac{2205\hbar^3 x}{256}(\hbar\partial)^{-10} + \left(-\frac{1795\hbar^4}{128} + \frac{1885\hbar^2x^3}{1024} \right)(\hbar\partial)^{-11} + \cdots .\end{aligned}$$

Collecting the terms of $\hbar$-order -2, we have

$$\begin{aligned}\mathcal{P}_2^{(1)} =& \frac{3}{16}\xi^{-4} - \frac{23x}{96}\xi^{-6} + \frac{19x^2}{384}\xi^{-8} + \frac{155x^3}{512}\xi^{-10} + \cdots , \\ \mathcal{Q}_2^{(1)} =& -\frac{1}{4}\xi^{-5} + \frac{143x}{192}\xi^{-7} - \frac{85x^2}{64}\xi^{-9} + \frac{1885x^3}{1024}\xi^{-11} + \cdots .\end{aligned}$$

From this result follows

$$\begin{aligned}\alpha_2 \log\xi + \tilde{\mathcal{X}}_2 :=& \int \left(\frac{\partial \mathcal{Q}_0}{\partial \xi} \mathcal{P}_2^{(1)} - \frac{\partial \mathcal{P}_0}{\partial \xi} \mathcal{Q}_2^{(1)} \right)_{\le -1} d\xi \\ =& -\frac{5}{48}\xi^{-3} + \frac{x}{4}\xi^{-5} - \frac{143x^2}{384}\xi^{-7} + \frac{85x^3}{192}\xi^{-9} - \frac{1885x^4}{4096}\xi^{-11} + \cdots .\end{aligned}$$

This means $\alpha_2 = 0$ and $\tilde{\mathcal{X}}_2'$ is equal to $\tilde{\mathcal{X}}_2$, which is equal to the above expression. Substituting these results in (2.14), we have

$$X_2 = -\frac{5}{48}(\hbar\partial)^{-3} + \frac{11x}{64}(\hbar\partial)^{-5} - \frac{85x^2}{768}(\hbar\partial)^{-7} + \frac{435x^4}{8192}(\hbar\partial)^{-11} + \cdots .$$

Substituting this into (2.9) and (2.10) for $i = 3$, we have

$$\begin{aligned}P^{(2)} =& (\hbar\partial)^2 + x + \frac{5\hbar^3}{48}(\hbar\partial)^{-7} + \frac{425\hbar^3 x}{768}(\hbar\partial)^{-9} + \frac{3205\hbar^4}{4096}(\hbar\partial)^{-10} \\ &- \frac{5865\hbar^3x^2}{2048}(\hbar\partial)^{-11} + \cdots , \\ Q^{(2)} =& -(\hbar\partial) - \frac{155\hbar^3}{384}(\hbar\partial)^{-8} + \frac{685\hbar^3 x}{512}(\hbar\partial)^{-10} - \frac{41395\hbar^4}{8192}(\hbar\partial)^{-11} + \cdots .\end{aligned}$$

By extracting the terms of $\hbar$-order -3, we have

$$\text{(B.7)}\qquad \begin{aligned} \mathcal{P}_3^{(2)} &= \frac{5}{48}\xi^{-7} + \frac{425x}{768}\xi^{-9} - \frac{5865x^2}{2048}\xi^{-11} + \cdots, \\ \mathcal{Q}_3^{(2)} &= -\frac{155}{384}\xi^{-8} + \frac{685x}{512}\xi^{-10} + \cdots. \end{aligned}$$

Hence,

$$\begin{aligned} \alpha_3 \log \xi + \tilde{\mathcal{X}}_3 &:= \int \left(\frac{\partial \mathcal{Q}_0}{\partial \xi}\mathcal{P}_3^{(2)} - \frac{\partial \mathcal{P}_0}{\partial \xi}\mathcal{Q}_3^{(2)} \right)_{\leq -1} d\xi \\ &= -\frac{15}{128}\xi^{-6} + \frac{155x}{384}\xi^{-8} - \frac{685x^2}{1024}\xi^{-10} + \cdots, \end{aligned}$$

which implies $\alpha_3 = 0$ and $\tilde{X}_3 = \tilde{X}_3'$ is equal to the above expression. Thus, again by (2.14), we have

$$X_3 = -\frac{15}{128}(\hbar\partial)^{-6} + \frac{175x}{768}(\hbar\partial)^{-8} + \cdots$$

Consequently, application of $\exp(\mathrm{Ad}((X_0 + \hbar X_1 + \hbar^2 X_2 + \hbar^3 X_3)/\hbar))$ to f and g gives

$$\begin{aligned} P^{(3)} &= (\hbar\partial)^2 + x - \frac{3395\hbar^4}{4096}(\hbar\partial)^{-10} + \cdots \\ Q^{(3)} &= -(\hbar\partial) - \frac{3395\hbar^4}{8192}(\hbar\partial)^{-11} + \cdots, \end{aligned}$$

which are differential operators up to $\hbar$-order -3.

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