The Anomalous Diffusion in Stochastically Bent Thin Tubes

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Abstract

In this study we investigate the processes of diffusion of impurity particles in the media with stochastic characteristics.

The exact equation with fractal derivation for middle admixture concentration in fibers with telegraph – type of curve angel is obtained.

Exact solution is found. Admixtures anomalous diffusion effect is shown.

Keywords: Anomalous diffusion, random, fractal operator.

1. Introduction

In a number of researches of the last years [1 – 5, 7-9] effective approximate methods of the analysis of process of diffusion of impurity particles in the environments with stochastic characteristics are developed. In these researches - under stochastic fields, on the base of which there is a transport of impurity, we understand either outside margin potential [1, 2], or permeability in the porous medias [3-4, 7-9], or a bend angle of tubes in tubes [5]. The received approximate equations for average concentration of impurity are contained by operators of fractional differentiation that specifies effects

of the anomalous diffusion. However, these equations are received under strong assumptions: small amplitudes of fluctuations of parameters of the environment for space scales large enough for correlative length; and for time - a lot of big the diffusive time corresponding to correlative length. Thus, it is rather difficult to estimate accuracy of the received equations and their solutions. In this sense there is a necessity to have precisely solved model which will allow comparing the exact decisions corresponding to considered model with structure of decisions, received by approximate methods. Creation of exact model also is the purpose of the present research.

2. Main Research Hypotheses

As investigated model we will consider ensemble of thin tubes with the constant cross-section section, located in a plane *(x, y)*. It is supposed that tubes are filled by a liquid in which are impurity particles, kinetics of which we are interested. The tube angle θ of slope to an axis *x* is stochastic function of coordinates with the set statistical properties which will be defined more low.

It is obvious that for any realization preservation of number of particles and Fick's law for a stream of particles leads to the usual equation of diffusion:

$$\frac{\partial \rho_0(z,t)}{\partial t} - D_0 \frac{\partial^2 \rho_0(z,t)}{\partial z^2} = 0, \ (-\infty < z < \infty), \tag{1}$$

where $\rho_0(z,t)$ - density (concentration) of impurity, *z* – longitudinal coordinate (along a tube, *t* – time variable. Solution of this equation with Решение этого уравнения с initial conditions $\rho_0(z,0) = \delta(z)$, где $\delta(z)$ - Dirac delta function (point source), and zero conditions for infinity $\rho_0(\pm \infty, t) = 0$ looks like:

$$\rho_0(z,t) = \frac{1}{\sqrt{4D_0 t}} \exp\left\{-\frac{z^2}{4D_0 t}\right\}. \tag{2}$$

Chosen normalizing on unit allows to treat $\rho_0(z,t)$ and as concentration of impurity and as density of distribution of probabilities of a finding of a particle in a point *z* at the moment of time *t*.

We ought to find the concentration of impurity $\rho(x,y,t)$ for the arbitrary space point (x,y). For these we would use formulae:

$$\rho(x,y,t) = \int_{-\infty}^{+\infty} \rho_0(z,t)\rho(x,y/z)dz, \tag{3}$$

where conditional density of distribution of probabilities on coordinates (x,y) for the given *z* along the tube might be found as:

$$\rho(x,y/z) = \langle \delta(x - x(z))\delta(y - y(z)) \rangle. \tag{4}$$

Here $x(z) = \int_0^z \cos\theta(z')dz'$, $y(z) = \int_0^z \sin\theta(z')dz'$. Brackets $\langle ... \rangle$ in (4) mean the averaging on a casual field $\theta(z)$.

For the further we would like to restrict for the case of diffusion along *y*, so we would be interested for density function $\rho(y,t) = \int_{-\infty}^{+\infty} \rho(x,y,t)dx$. Taking into account (3) and (4), we found:

$$\rho(y,t) = \int_{-\infty}^{+\infty} \rho_0(z,t)\rho(y/z)dz; \ \rho(y/z) = \langle \delta(y - y(z)) \rangle. \tag{5}$$

It is obvious that the results are determined by stochastic characteristics for the field $\theta(z)$. For the further we suppose that $\theta(z)$ is telegraph type stochastic field [3]:

$$\theta(z) = \theta(-1)^{n(0,z)}, \tag{6}$$

where $n(0,z)$ - number of reversal of sign in the interval $(0,z)$. For non-intersecting intervals the number of reversal of sign are independent and have the Poisson distribution:

$$P(n(z_1, z_2) = k) = e^{-\mu|z_2 - z_1|} \frac{(\mu|z_2 - z_1|)^k}{k!}, \tag{7}$$

with μ - average reversal of signs for the unit lengths. Parameter μ is connected with correlation lengths l for the stochastic field by $l = 1/(2\mu)$. Moreover, we would assume that θ in (6) – is the casual variable with the density $\rho(\theta) = \frac{1}{2}(\delta(\theta - \theta_0) + \delta(\theta + \theta_0))$.

3. Equation for $\rho(y,t)$ Deriving

For the function $\rho(y,t)$ we would like according with (5) determine $\rho(y/z)$. For these we would find Fourier image for $\rho(y/z)$:

$$\rho(k/z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \rho(y/z) e^{-iky} dy = \frac{1}{2\pi} \left\langle \exp\left(-ik \sin\theta \int_0^z (-1)^{n(0,z')} dz'\right) \right\rangle.$$

Here the brackets assumes the double averaging for the sign reversal with the help of Poisson distribution (7) and for θ with the help of the density, so $\langle ... \rangle = \langle\langle ... \rangle_n\rangle_\theta$.

We also would find the average for Fourier image for the fixed θ. For these we would put the function:

$$\Phi(z) = \left\langle \exp\left(-ik\sin\theta \int_0^z (-1)^{n(0,z')} dz'\right) \right\rangle_n \quad \text{и} \quad \Psi(z) = \left\langle (-1)^{n(0,z)} \exp\left(ik\sin\theta \int_0^z (-1)^{n(0,z')} dz'\right) \right\rangle_n.$$

Here $\Phi(z)$ and $\Psi(z)$ are connected by differential equation:

$$\frac{d\Phi}{dz} = -ik\sin\theta\Psi(z). \tag{9}$$

For the Ψ we will take into account that for any functional (function of function) $A\left(z, \int_0^z (-1)^{n(0,z')} dz'\right)$ we have [3]:

$$\frac{d}{dz}\left\langle (-1)^{n(0,z)} A \right\rangle_n = \pm 2\mu \left\langle (-1)^{n(0,z)} \right\rangle_n + \left\langle (-1)^{n(0,z)} \frac{dA}{dz} \right\rangle_n,$$

where the upper «+» is for $z > 0$, and «–» – for $z < 0$. So we have

$$\frac{d\Psi}{dz} = \pm 2\mu\Psi - ik\sin\theta\Phi(z). \tag{10}$$

Solutions for (9) and (10) with the obvious initial conditions $\Phi(0) = 1$ and $\Psi(0) = 1$ are the following:

$$\Phi(z) = \frac{1}{2}\left(1 - \frac{\mu \mp ik\sin\theta}{\sqrt{\mu^2 - k^2\sin^2\theta}}\right)\exp\left(-|z|\left(\mu + \sqrt{\mu^2 - k^2\sin^2\theta}\right)\right)$$
$$+ \frac{1}{2}\left(1 + \frac{\mu \mp ik\sin\theta}{\sqrt{\mu^2 - k^2\sin^2\theta}}\right)\exp\left(-|z|\left(\mu - \sqrt{\mu^2 - k^2\sin^2\theta}\right)\right).$$

By the averaging procedure for θ with (8) and taking into account $\rho(k/z) = \frac{1}{2\pi}\langle \Phi(z)\rangle_\theta$, we have the final representation for Fourier image

$$\rho(k/z) = \frac{1}{2\pi} e^{-\mu|z|}\left[\mu \frac{\sinh\sqrt{\mu^2 - a_0^2 k^2}\,|z|}{\sqrt{\mu^2 - a_0^2 k^2}} + \cosh\sqrt{\mu^2 - a_0^2 k^2}\,|z|\right], \tag{11}$$

where $a_0 = \sin\theta_0$ - amplitude of fluctuations for telegraph stochastic field.

The reverse Fourier transformation [6] gives:

$$\rho(y/z) = \frac{\mu}{2a_0} e^{-\mu|z|}\left[I_0\left(\mu\sqrt{z^2 - \frac{y^2}{a_0^2}}\right) + \frac{|z|}{\sqrt{z^2 - y^2/a_0^2}} I_1\left(\mu\sqrt{z^2 - \frac{y^2}{a_0^2}}\right)\right]\cdot\Theta\left(|z| - \frac{|y|}{a_0}\right)$$
$$+ \frac{1}{2a_0} e^{-\mu|z|}\delta\left(|z| - \frac{|y|}{a_0}\right) \tag{12}$$

where $I_0(u)$ и $I_1(u)$ - modified Bessel functions, $\Theta(x)$ - unit function.

Учитывая (12), (2) и (5), находим плотность распределения

$$\rho(y,t) = \frac{1}{a_0\sqrt{D_0 t}}\int\limits_{|y|/a_0}^{\infty}\left(\mu + \frac{z}{4D_0 t}\right)\cdot\exp\left(-\mu z - \frac{z^2}{4D_0 t}\right)\cdot I_0\left(\mu\sqrt{z^2 - \frac{y^2}{a_0^2}}\right)dz. \tag{13}$$

We can observe that density $\rho(y,t)$ reduces when y, t increases. We put below the figure for $\rho(y,t)$. For convenience we put the dimensionless variables t', y', z', where $t = t'/(4\mu^2 D_0) = t'\tau$, $y = y' a_0/(2\mu) = y' a_0 l$, $z = z'/(2\mu) = z'l$. So instead (13) we have:

$$\rho(y',t') = \frac{1}{2\sqrt{\pi t'}}\int\limits_{|y'|}^{\infty}\left(1 + \frac{z'}{2t'}\right)\cdot\exp\left(-\frac{z'}{2} - \frac{(z')^2}{4t'}\right)\cdot I_0\left(\frac{1}{2}\sqrt{(z')^2 - (y')^2}\right)dz'.$$

Figure 1: Dependence of reduced concentration $\rho(y',t')$ of normalized coordinate y'. Line *1* means *t'=0.1*; line *2 – t'=0.5*; line *3 – t'=2*; line *4 – t'=10*.
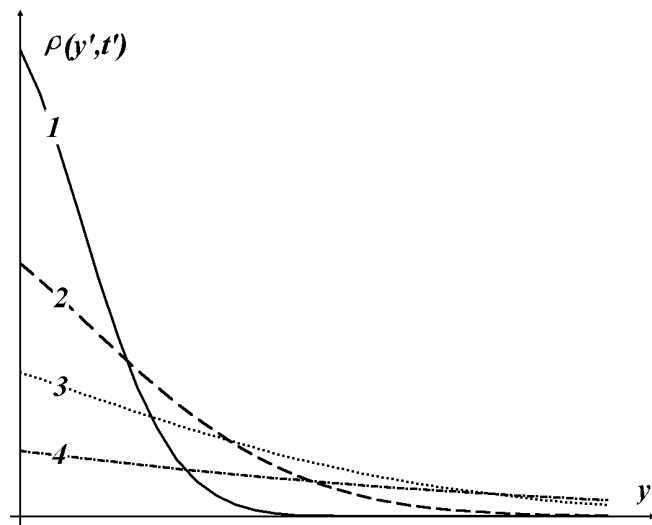
For the equation for (13) we would use Fourier – Laplace variables (k,q): it means the Fourier transformation for coordinate *y* and Laplace transformation for time. We also use:

$$\rho(k,q)=\int_{-\infty}^{+\infty}\rho_0(z,q)\rho(k/z)dz,\tag{14}$$

where $\rho(k/z)$ is determined by (11), and so for $\rho_0(z,q)$ by using (2) we have:

$$\rho_0(z,q)=\frac{1}{\sqrt{4D_0}}q^{-1/2}\exp\left(-\frac{|z|}{\sqrt{D_0}}q^{1/2}\right).\tag{15}$$

Using (15) and (11) for (14), finally we have:

$$\rho(k,q)=\frac{q^{-1/2}\left(2\mu\sqrt{D_0}+q^{1/2}\right)}{q+2\mu\sqrt{D_0}q^{1/2}+D_0a_0^2k^2}\bullet\varphi(k),\tag{16}$$

where $\varphi(k)=1/2\pi$ - Fourier image for initial distribution $\varphi(y)=\rho(y,0)=\delta(y)$. We rewrite (16) as following:

$$\rho q-\varphi=-\frac{1}{2\mu\sqrt{D_0}}q^{1/2}\left[\rho q-\varphi+D_0a_0^2k^2\rho\right].\tag{17}$$

By passing in (17) for initial variables y and t, we have:

$$\frac{\partial\rho}{\partial t}=-\frac{1}{2\mu\sqrt{D_0}}D_t^{1/2}\left(\frac{\partial\rho}{\partial t}-D_0a_0^2\frac{\partial^2\rho}{\partial y^2}\right),\tag{18}$$

where $D_t^{1/2}$ - fractional operator for time of the order $1/2$ in Riemann – Liouville sense [6]. The analogous equations are used in [3-5, 7-9]. The differences are determined by the concrete model and stochastic field behavior. We would like to mention that equation (18) is the exact one and for above cited researches the approximations were used for the length (more than correlation length) and for time (more than diffusion time which corresponds for correlation length).

4. Abnormal Diffusion

From (18) it is obvious that when we have no fluctuations $(a_0=0)$ we have $\rho(y,t)=\rho(y,0)=\delta(y)$ and there is no transport of porosity for direction y. Transport of porosity is determined by fluctuations. So here we have only abnormal diffusion. For the characteristics of the abnormal diffusion we can receive the average number displacement for the particle for the time t. For the typical medias, like (1) it looks usually like $\overline{y^2}\sim t$ for all times. In stochastic medias like we have it not so (formally we have another type of fractional differential operators

Let us determine $\overline{y^2}(t)$. From (16) by using Fourier transformations we have:

$$\rho(y,q)=\frac{\sqrt{2\mu\sqrt{D_0}+q^{1/2}}}{2\sqrt{D_0a_0^2}q^{3/4}}\exp\left(-\frac{|y|}{\sqrt{D_0a_0^2}}\sqrt{q^{1/2}\left(q^{1/2}+2\mu\sqrt{D_0}\right)}\right).$$

From this distribution we have

$$\overline{y^2}(q)=\frac{2D_0a_0^2}{q^{3/2}\left(q^{1/2}+2\mu\sqrt{D_0}\right)}.\tag{19}$$

The reverse Laplace transform for (19) gives:

$$\overline{y^2}(t)=2a_0^2l^2\left[\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{\tau}}-1+e^{t/\tau}\left(1-\Phi\left(\sqrt{\frac{t}{\tau}}\right)\right)\right],\tag{20}$$

where $\Phi(u) = \dfrac{2}{\sqrt{\pi}} \int\limits_0^u e^{-x^2} dx$ - probability integral, $l = 1/(2\mu)$ - correlation length, $\tau = l^2/D_0$ - diffusion time. From (20) for $t \ll \tau$ we have $\overline{y^2}(t) = 2a_0^2 D_0 t$, that corresponds for standard diffusion for normalized diffusion coefficient $D = D_0 a_0^2$. But for $t \gg \tau$, we have $\overline{y^2}(t) = 4a_0^2 l^2 \sqrt{\dfrac{t}{\pi\tau}} - 2a_0^2 l^2$, which means the abnormal slow diffusion.

5. Details of Deriving

For the details of displacement $\overline{y^2}(t)$ deriving we would use (18)-type relation:

$$J_t^{1/2} \frac{\partial \rho}{\partial t} = \frac{1}{2\mu\sqrt{D_0}}\left(D_0 a_0^2 \frac{\partial^2 \rho}{\partial y^2} - \frac{\partial \rho}{\partial t} \right), \tag{21}$$

where $J_t^{1/2}$ - fractional operator for time of the order $1/2$ in Riemann – Liouville sense. By using on both parts of (21) by $D_t^{1/2}$ operator and taking into account $D_t^{1/2} J_t^{1/2}$ - unit operator, we would receive (18). If we multiply (21) by y^2 and integrate the result on y for $(-\infty;\infty)$, we receive:

$$J_t^{1/2} V = \frac{\sqrt{D_0}\,a_0^2}{\mu} - \frac{1}{2\mu\sqrt{D_0}} V, \tag{22}$$

where $V = \dfrac{d\overline{y^2}}{dt}$ and $\int\limits_{-\infty}^{\infty} y^2 \dfrac{\partial^2 \rho}{\partial y^2} dy = 2$.

Looking at (22) we have the initial condition $V(0) = 2D_0 a_0^2$, because we have zero integral of $J_t^{1/2} V$ for $t = 0$. By acting with operator $D_t^{1/2}$ on both parts of (22), we have:

$$V = \frac{\sqrt{D_0}\,a_0^2}{\mu\sqrt{\pi}} \frac{1}{\sqrt{t}} - \frac{1}{2\mu\sqrt{D_0}} D_t^{1/2} V, \tag{23}$$

where according to [6], we have $D_t^{1/2} 1 = \dfrac{1}{\sqrt{\pi t}}$. By using $D_t^{1/2}$ for (23), and taking into account $D_t^{1/2} t^{-1/2} = 0$ and $D_t^{1/2} D_t^{1/2} = \dfrac{\partial}{\partial t}$, we have

$$D_t^{1/2} V = -\frac{1}{2\mu\sqrt{D_0}} \frac{\partial V}{\partial t}. \tag{24}$$

By substituting the above in to (23), we have finally:

$$\frac{\partial V}{\partial t} - 4\mu^2 D_0 V = -\frac{4 D_t^{3/2} a_0^2 \mu}{\sqrt{\pi}} \frac{1}{\sqrt{t}}. \tag{25}$$

This is non-uniform linear equation of the first order. It's solution for the $V(0) = 2D_0 a_0^2$ looks like:

$$V = e^{4\mu^2 D_0 t}\left[2D_0 a_0^2 - \frac{4 D_0^{3/2} a_0^2 \mu}{\sqrt{\pi}} \int\limits_0^t e^{-4\mu^2 D_0 t'} \frac{dt'}{\sqrt{t'}} \right]. \tag{26}$$

By integrating (26) for time t interval $(0;t)$, we have (20) for $\overline{y^2}(t)$.

Let us give the graphic for $\overline{y^2}(t)$. For convenience we put the dimensionless variables t', y', where $t = t'/(4\mu^2 D_0) = t'\tau$, $y = y'a_0/(2\mu) = y'a_0 l$. For the case instead (20) we have:

$$\overline{(y')^2}(t') = 2\left[\frac{2}{\sqrt{\pi}} \sqrt{t'} - 1 + e^{t'}\left(1 - \Phi\left(\sqrt{t'}\right)\right) \right].$$

Figure 2: Dependence for $\overline{(y')^2}(t')$ of t'. For small time $\overline{(y')^2} \sim t'$, for large time $\overline{(y')^2} \sim \sqrt{t'}$.
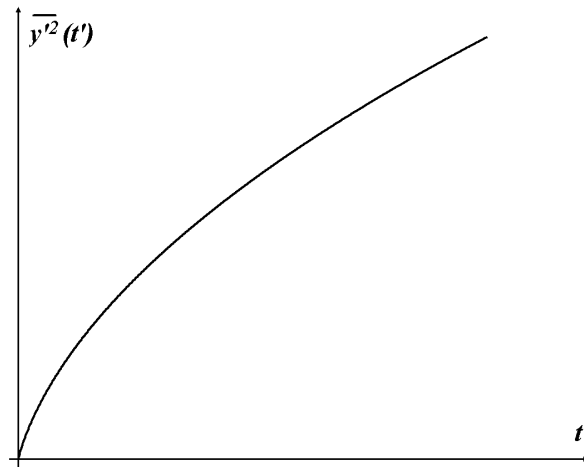
Let us also mention that the transport of porosity particles along x for our model is determined by standard diffusion with normalized diffusion coefficient $D = D_0 \cos^2 \theta_0 = D_0 (1 - a_0^2)$. From (5) we have $\rho(x,t) = \int\limits_{-\infty}^{+\infty} \rho_0(z,t)\rho(x/z)dz$, where $\rho(x/z) = \langle \delta(x - x(z)) \rangle = \delta(x - z\cos\theta_0)$, which gives solution $\rho(x,t) = \dfrac{1}{\sqrt{4\pi Dt}}\exp\left(-\dfrac{x^2}{4Dt}\right)$, that corresponds standard diffusion $\dfrac{\partial \rho(x,t)}{\partial t} - D\dfrac{\partial^2 \rho(x,t)}{\partial x^2} = 0$ for initial conditions $\rho(x,0) = \delta(x)$.

References

[1] Dean D.S., Drummong L.L. and Horgan R.R., Perturbations schemes for flow in random media. *J. Phys. A: Math. Gen.* 1994. V.27. P.5135 – 5144.

[2] Stepanyants Y.A., Teodorovich E.V., Effective hydraulic conductivity of randomly heterogeneous porous medium. *Water Research.* 2003. V.39 (3). N.12. P. 1 – 11.

[3] Erochenkova G., Lima R., A fractional diffusion equation a marker in porous media. *Chaos.* 2001. V.11. N.3. P. 495 – 499.

[4] Logvinova K., Neel M.-Cr., Fractional diffusion in heterogeneous media: iterative metod. *Izvestia Ingeneronoi akademii nauk: Applied mathematics and Mechanics.* 2004. T.7. C.33 – 44. (in Russian)

[5] Logvinova K.V., Neel M.-Cr., A fractional equation for anomalous diffusion in a randomly heterogeneous porous media. *Chaos.* 2004. V.14. N.4. P. 982 – 987.

[6] Samko S., Kilbas A. and Marichev O. Fractional integrals and derivatives: Theory and applications. *Gordon and Breach Science Publishers.* 1993. 996 P.

[7] Kozyrev O. and Logvinova K. Models for the Average of the Solutions to a P.D.E. with Stochastic Coefficients. *European Journal of Scientific Research*, 2010. T. 45. № 3. P. 383—390.

[8] Kozyrev O. and Logvinova K. Small Scale Models for the Spreading of Matter in Disordered Porous Media. *European Journal of Scientific Research*, 2010. T. 45. № 1. P. 64—78.

[9] Kozyrev O., Logvinova K. and Morozov V. Disordered Structures Models for Heterogeneous Media. *European Journal of Scientific Research*, 2011. T. 49. № 3. P. 464—467