HIGHER DETERMINANTS AND THE MATRIX-TREE THEOREM

YURII BURMAN

Introduction

The classical matrix-tree theorem was discovered by G. Kirchhoff in 1847, see [1]. It relates the principal minor of the $n \times n$ Laplace matrix to a particular sum of monomials indexed by the set of trees with n vertices. Kirchhoff's theorem has attracted much attention since then, and produced a wide-branching tree of new proofs, generalizations, extensions, analogs and applications; a comprehensive survey of them is far beyond the scope of the current article. To name just a few, let us mention the all-minors generalization [2], generalization to nonsymmetric Laplace matrices [3, Section 5.3], to graphs with arbitrary 2-core [4], to the Laplacian of the discrete line bundle with a connection [5, 6] (essentially anteceded in [7, Section 5]), an analog for Pfaffians [8], a generalization to matrices represented as functions of rank 1 operators, including analogs to all Coxeter groups [9], and a hyperdeterminant version [10]. The original problem by G. Kirchhoff was to find resistance of a resistor network, but later the theory found applications in so distant areas as stochastic processes [11] and embedding graphs into surfaces [12].

The aim of this paper is to present a generalization of the (nonsymmetric) matrix-tree theorem containing no trees and essentially no matrices. Instead of trees we consider acyclic directed graphs with a prescribed set of sinks, and instead of determinant, a polynomial invariant of the matrix determined by directed graph such that any two vertices of the same connected component are mutually reacheable (according to arrows). The exact formulation of the theorem is given in Section 1, its proof in Section 2, Section 3 contains some corollaries and applications.

Acknowledgements

The work was inspired by long discussions with prof. Michael Polyak of Haifa Technion, Israel, to whom the author expresses his deep gratitude.

The work was supported by the HSE Scientific Fund grant 15-01-0031 "Hurwitz numbers and graph isomorphism".

1. Main theorem

Fix positive integers n and k and a field $\mathbb{F}$. Denote by $G_{n,k}$ the set of all directed graphs with n vertices numbered $1, 2, \dots, n$ and k edges numbered $1, 2, \dots, k$; graphs $G \in G_{n,k}$ are allowed to have loops, parallel edges (i.e. edges having the same starting and terminal vertex) and isolated vertices (i.e. not incident to an edge). By $\mathcal{G}_{n,k}$ denote a vector space over $\mathbb{F}$ spanned by $G_{n,k}$.

Let K be a associative commutative $\mathbb{F}$-algebra, and let A be an $n \times n$-matrix with the matrix elements $a_{pq} \in K$. Then $\langle A| : \mathcal{G}_{n,k} \to K$ is a $\mathbb{F}$-linear map defined on the generators as follows:

$$\langle A|G\rangle = \prod_e a_{e_-e_+}; \tag{1}$$

here the product is taken over all the edges e of G, and e_- and e_+ are the starting and the terminal vertex of e. By abuse of notation we will be writing sometimes $\langle A|G\rangle$ if G contains less vertices than the size of A (the extra matrix elements of A simply are not used) or even if G contains more vertices than the size of A, but the extra vertices are all isolated.

Recall that the generators of $G_{n,k}$ are graphs where all the edges and vertices are numbered. The definition of $\langle A|G\rangle$ makes use of the vertex numbering, but not of the edge numbering; so if the graphs $G_1, G_2 \in G_{n,k}$ differ by edge numbering only, then $\langle A|G_1\rangle = \langle A|G_2\rangle$ for any A.

A *loop-breaking operator* is defined as a linear operator $\mathcal{B} : \mathcal{G}_{n,k} \to \mathcal{G}_{n,k}$ acting on the generators as follows: if $G \in G_{n,k}$ is a graph with m loops attached to the vertices $i_1, \dots, i_m$ (not necessarily distinct) then one has

$$\mathcal{B}(G) = (-1)^m \sum_{\substack{j_1 \neq i_1 \\ \dots \\ j_m \neq i_m}} G_{j_1,\dots,j_m} \tag{2}$$

where $G_{j_1,\dots,j_m}$ is a graph obtained from G by replacement of the loops with the edges $(i_1j_1), \dots, (i_mj_m)$. In particular, if G has no loops then $\mathcal{B}(G) = G$. Also $\mathcal{B}(G) = G$ by definition if $n = 1$ (G contains one vertex).

For two vertices a, b in a directed graph G write $a \sim b$ if G contains a directed cycle passing through a and b (equivalently, there is a directed path joining a with b and a directed path joining b with a); also $a \sim a$ by definition for any a. Obviously, $\sim$ is an equivalence relation; so, the set of vertices of any directed graph G is split into equivalence classes $V_1, \dots, V_N$. If $N = 1$ (any two vertices of G belong to a directed cycle) then G is called strongly connected. Denote by $C(G)$ the graph with the vertices $V_1, \dots, V_N$; vertices V_i and V_j. $i \neq j$, are joined by an edge if G contains an edge (ab) with $a \in V_i$ and $b \in V_j$.

A directed graph G is called acyclic if it has no directed cycles (in particular, no loops); an equivalent definition is that $G = C(G)$. Apparently, for any graph G the graph $C(G)$ is acyclic (and contains no parallel edges).

Fix integers $n \geq 1$, $k \geq 1$, $0 \leq s \leq n$, and $1 \leq i_1 < \dots < i_s \leq n$, and denote by by $\mathcal{AC}_{n,k}^{(i_1,\dots,i_s)}$ the set of all acyclic graphs with n vertices and k edges such that the vertices $i_1, \dots, i_s$, and only they, are sinks (have no edges starting from them; isolated vertices are sinks, too). Also denote

$$x_{n,k}^{(i_1,\dots,i_s)} \stackrel{\text{def}}{=} \sum_{G \in \mathcal{AC}_{n,k}^{(i_1,\dots,i_s)}} G \in \mathcal{G}_{n,k}$$

Example 1. An acyclic graph has at least one sink in every connected component. So, if $G \in \mathcal{AC}_{n,k}^{(i_1,\dots,i_s)}$ then $k \geq n - \beta_0(G) \geq n - s$, where $\beta_0(G)$ is the number of connected components of G. If $k = n - s$ then G is a forest with s connected components. Every its component is a tree containing exactly one vertex i_p, and all the edges in this component are oriented towards i_p.

A directed graph G is called strongly semiconnected if every its connected component is strongly connected. (Equivalently, $C(G)$ contains no edges). Again, fix n,

k, s and $i_1, \dots, i_s$ and denote by $\mathcal{SSC}_{n,k}^{(i_1,\dots,i_s)}$ the set of all strongly semiconnected graphs G with n vertices and k edges, such that the vertices $i_1, \dots, i_s$, and only they, are isolated. Also denote

$$y_{n,k}^{(i_1,\dots,i_s)} \stackrel{\text{def}}{=} \sum_{G \in \mathcal{SSC}_{n,k}^{(i_1,\dots,i_s)}} (-1)^{\beta_0(G)} G \in \mathcal{G}_{n,k}.$$

Example 2. If $G \in \mathcal{SSC}_{n,k}^{(i_1,\dots,i_s)}$ then every connected component h of G either is an isolated vertex or contains at least as many edges as vertices. So, $k \geq n - s$; if $k = n - s$ then G is a collection of oriented cycles (of any length, possibly including 1), and the isolated vertices $i_1, \dots, i_s$. Such a graph G defines a transposition σ in the set of all non-isolated vertices: if $a \neq i_1, \dots, i_s$, then G contains exactly one edge e with $a = e_-$; then $\sigma(a) \stackrel{\text{def}}{=} e_+$. Apparently, $(-1)^{\beta_0(G)} = (-1)^s \operatorname{sgn}(\sigma)$; here $\operatorname{sgn}(\sigma) \stackrel{\text{def}}{=} \pm 1$ depending on the parity of σ. For the graph G there are $(n-s)!$ ways to number its edges; thus,

$$\langle A | y_{n,n-s}^{(i_1,\dots,i_s)} \rangle = (-1)^s (n-s)! \det A_{i_1,\dots,i_s}$$

where $A_{i_1,\dots,i_s}$ is the diagonal minor of the matrix A obtained by deletion of rows and columns numbered $i_1, \dots, i_s$. In particular, $\langle A | y_{n,n}^{\varnothing} \rangle = n! \det A$. This allows us to call the expression $\langle A | y_{n,k}^{\varnothing} \rangle$ a *higher determinant (of degree k)* of the matrix A, and $\langle A | y_{n,k}^{(i_1,\dots,i_s)} \rangle$, a *higher minor.*

The main result of this article is

Theorem 1. $\mathcal{B}(y_{n,k}^{(i_1,\dots,i_s)}) = x_{n,k}^{(i_1,\dots,i_s)}$.

We are going to prove it in Section 2.

Let $w = \{w_{ij} \mid 1 \leq i \neq j \leq n\}$ be the set of independent variables; denote by K the $\mathbb{F}$-algebra of polynomials on all the w_{ij}. Let $L_n(w)$ be a $n \times n$-matrix with the matrix elements $\ell_{ij} = w_{ij}$ for $i \neq j$ and $\ell_{ii} = -\sum_{p \neq i} w_{ip}$; $L_n(w)$ is called a (nonsymmetric) Laplace matrix. For any graph $G \in G_{n,k}$ the element $\mathcal{B}(G)$ is a sum of graphs having no loops. Thus, $\langle A | \mathcal{B}(u) \rangle$ for $u \in \mathcal{G}_{n,k}$ does not depend on the diagonal elements of A. The definitions of the Laplace matrix and of the loop-breaking operator imply now that

$$\langle L_n(w) | u \rangle = \langle L_n(w) | \mathcal{B}(u) \rangle.$$

At the same time, once w_{ij} are independent variables, if $\langle L_n(w) | G_1 \rangle = \langle L_n(w) | G_2 \rangle$ for $G_1, G_2 \in G_{n,k}$ where the graphs G_1 and G_2 contain no loops then $G_1 = G_2$. So, the following statement is equivalent to Theorem 1:

Theorem 1, second formulation. *The higher minor of the Laplace matrix is* $\langle L_n(w) | y_{n,k}^{(i_1,\dots,i_s)} \rangle = \langle L_n(w) | x_{n,k}^{(i_1,\dots,i_s)} \rangle$.

Example 3. Let $k = n - s$. Then by Example 2 $\langle L_n(w) | y_{n,n-s}^{(i_1,\dots,i_s)} \rangle = (-1)^s (n - s)! \det(L_n(w))_{i_1,\dots,i_s}$. A graph $G \in \mathcal{AC}_{n,n-s}^{(i_1,\dots,i_s)}$ contains $n - s$ edges, and there are $(n - s)!$ ways to number them. So,

$$\langle L_n(w) | x_{n,n-s}^{(i_1,\dots,i_s)} \rangle = (n-s)! \sum_{(p_1,q_1),\dots,(p_{n-s},q_{n-s})} w_{p_1 q_1} \dots w_{p_{n-s} q_{n-s}}$$

where the set $(p_1, q_1), \dots, (p_{n-s}, q_{n-s})$ is included into the summation if the graph with the edges $(p_1, q_1), \dots, (p_{n-s}, q_{n-s})$ is a s-component forest, each component containing exactly one vertex i_α. So, this particular case of Theorem 1 (in the second formulation) is just the classical Matrix-tree theorem (the "diagonal-minors" version, see [2]). We will use the Matrix-tree theorem as the base of induction in the proof of Theorem 1.

Remark. Theorem 1 includes the case $s = 0$. Since an acyclic graph must have a sink, $x^{\varnothing}_{n,k} = 0$, so one has $\mathcal{B}y^{\varnothing}_{n,k} = 0$. This is evident for $k = n$ (a particular case of Example 3): the Laplace matrix $L_n(w)$ is degenerate (the sum of its columns is 0), so $\det L_n(w) = 0$. For other k the result does not seem to have an independent proof which is simpler than a general proof of Theorem 1.

2. Proofs

Let $G \in G_{n,k}$, and $E = \{e_1, \dots, e_m\}$ be some set of its edges. Denote by $\mu_E(G)$ the graph obtained from G by breaking every edge $e_i \in E$ and replacing it by a loop attached to the the starting vertex $(e_i)_-$ of e_i. Apparently, the operation μ_E has the following properties:

1. $\beta_0(\mu_E(G)) = \beta_0(G \setminus E)$ (the graph G with all the edges $e_i \in E$ deleted).
2. The sinks of G and $\mu_E(G)$ are the same.
3. For any pair of directed graphs $G, H \in G_{n,k}$ denote by $(G|\mathcal{B}|H)$ the coefficient at G in $\mathcal{B}(H)$. Then $(G|\mathcal{B}|\mu_E(G)) = (-1)^{\#E}$, and $(G|\mathcal{B}|H) = 0$ is $H \neq \mu_E(G)$ for any E.

Fix, as above, the vertices $i_1, \dots, i_s$. Call a set of edges E of the graph G admissible if $\mu_E(G) \in \mathcal{SSC}^{(i_1,\dots,i_s)}_{n,k}$. It follows from Property 2 above that an admissible set of edges exists if the vertices $i_1, \dots, i_s$, and only they, are the sinks of G (sinks of a strongly semiconnected graph are exactly its isolated vertices). The set of all admissible sets of edges of the graph G will be denoted $\mathrm{Adm}^{(i_1,\dots,i_s)}(G)$, or just $\mathrm{Adm}(G)$.

For any set R of sets of edges of the graph G denote

$$\mathcal{Z}(R) \stackrel{\text{def}}{=} \sum_{E \in R} (-1)^{\#E + \beta_0(G \setminus E)}.$$

Properties 1 and 3 of the operator μ_E imply that the following statement is equivalent to Theorem 1:

Theorem 1, third formulation. *$\mathcal{Z}(\mathrm{Adm}(G)) = (-1)^k$ if $G \in \mathcal{AC}^{(i_1,\dots,i_s)}_{n,k}$ and $\mathcal{Z}(\mathrm{Adm}(G)) = 0$ otherwise.*

We are going to prove Theorem 1 in its third formulation by a simultaneous induction by n (the number of vertices) and k (the number of edges).

2.1. Induction base.

2.1.1. *Vertices.* The induction base are graphs with $n = 2$ vertices a and b and arbitrary number of edges. Let the graph G contain p edges ab and q edges ba; there are $\binom{p+q}{p}$ such graphs that differ by edge numbering.

The graph belongs to $\mathcal{AC}^{(i_1,\dots,i_s)}_{n,p+q}$ if and $p = 0$ or $q = 0$. In this case the only edge set $E \in \mathrm{Adm}(G)$ consists of all the edges, so $\sum_{E \in \mathrm{Adm}(G)} (-1)^{\#E + \beta_0(G \setminus E)} = (-1)^p$ ($\#E = p$, $\beta_0 = 2$).

Let $p, q > 0$. A set E of edges belongs to Adm either if it contains $u < p$ edges ab and $v < q$ edges ba or if it consists of all the edges. In the first case there are $\binom{p}{u}\binom{q}{v}$ ways to choose edges for E. So, the sum in question is

$$\begin{aligned}
\mathcal{Z}(\mathrm{Adm}(G)) &= \binom{p+q}{p}\Big((-1)^{p+q} + \sum_{u=0}^{p-1}\binom{p}{u}\sum_{v=0}^{q-1}\binom{q}{v}(-1)^{u+v+1}\Big) \\
&= \binom{p+q}{p}\Big((-1)^{p+q} - \sum_{u=0}^{p-1}\binom{p}{u}(-1)^u\sum_{v=0}^{q-1}\binom{q}{v}(-1)^v\Big) \\
&= \binom{p+q}{p}\Big((-1)^{p+q} - ((1-1)^p - (-1)^p)((1-1)^q - (-1)^q)\Big) = 0
\end{aligned}$$

($p, q > 0$, so $(1-1)^p = (1-1)^q = 0$).

2.1.2. *Edges.* It follows from Example 2 that the minimum number of edges in a strongly semiconnected graph is $k = n - s$. It was shown in Example 3 that for $k = n - s$ Theorem 1 is equivalent to the Matrix-tree theorem of [2].

2.2. **Induction step.** Let the statement be proved for all graphs G containing less than n vertices or n vertices and less than k edges.

Consider several particular cases.

2.2.1. *$G \in G_{n,k}$ is strongly connected (and therefore $s = 0$).* Let e be an edge of G, and not a loop. Denote by $\mathrm{Adm}_+(G) \subset \mathrm{Adm}(G)$ the set of all $E \in \mathrm{Adm}(G)$ such that $e \in E$, and by $\mathrm{Adm}_-(G) \subset \mathrm{Adm}(G)$ the set of all $E \in \mathrm{Adm}(G)$ such that $e \notin E$.

The assertion $E \in \mathrm{Adm}_+(G)$ is equivalent to $(E \setminus \{e\}) \in \mathrm{Adm}(G \setminus e)$, where $G \setminus e$ is the graph G with e deleted. So,

$$\mathcal{Z}(\mathrm{Adm}_+(G)) = -\mathcal{Z}(\mathrm{Adm}(G \setminus \{e\})) = 0,$$

by the induction hypothesis.

If $E \in \mathrm{Adm}_-(G)$ then $E \in \mathrm{Adm}(G/e)$ where G/e is the graph G with the edge e contracted. If, vice versa, $E \in \mathrm{Adm}(G/e)$ then either $E \in \mathrm{Adm}_-(G)$ or $E \in \mathrm{Adm}^{a\to b}(G)$ where by $\mathrm{Adm}^{a\to b}(G)$ we denote the set of sets E of edges such that $C(\mu_E(G))$ contains a single edge joining a strongly connected component V_a containing a with the component V_b containing b. The graphs $\mu_E(G/e)$ and $\mu_E(G)$ are homotopy equivalent, and therefore contain the same number of connected components. Thus,

$$\mathcal{Z}(\mathrm{Adm}_-(G)) = \mathcal{Z}(\mathrm{Adm}(G/e)) - \mathcal{Z}(\mathrm{Adm}^{a\to b}(G)) = -\mathcal{Z}(\mathrm{Adm}^{a\to b}(G))$$

by the induction hypothesis.

Denote by V a partition $V_a \sqcup V_b \sqcup V_3 \sqcup \dots \sqcup V_N$ of the set of vertices of the graph G such that $a \in V_a$ and $b \in V_b$. Also denote by $\mathrm{Adm}^V(G)$ the set of edge sets E such that $V_a, \dots, V_N$ are strongly connected components of the graph $\mu_E(G)$ (i.e. vertices of the graph $C(\mu_E(G))$); then

$$\mathcal{Z}(\mathrm{Adm}^{a\to b}(G)) = \sum_V \mathcal{Z}(\mathrm{Adm}^V(G)).$$

On the other hand,

$$\mathcal{Z}(\mathrm{Adm}^V(G)) = (-1)^{\#E_V}\mathcal{Z}(\mathrm{Adm}(G \setminus E_V)) \tag{3}$$

where E_V is the set of all edges $f \neq e$ of G starting at the vertex $f_- \in V_i$ and terminating at the vertex $f_+ \in V_j$ such that $i \neq j$. Since G is strongly connected, $E_V \neq \varnothing$ for any V, so the right-hand side of (3) is 0 by the induction hypothesis.

2.2.2. *$G \in G_{n,k}$ is strongly semiconnected.* Let $G_1, \dots, G_N$ be connected components of G. Then $E \in \mathrm{Adm}(G)$ if and only if $E = E_1 \cup \dots \cup E_N$ where $E_i \in \mathrm{Adm}(G_i)$. So,

$$\mathcal{Z}(\mathrm{Adm}(G)) = \mathcal{Z}(\mathrm{Adm}(G_1)) \dots \mathcal{Z}(\mathrm{Adm}(G_N)).$$

It follows from Section 2.2.1 that if at least one G_i contains more than one vertex then $\mathcal{Z}(\mathrm{Adm}(G_i)) = 0$, hence $\mathcal{Z}(\mathrm{Adm}(G)) = 0$. If every component contains one vertex then $\mathcal{Z}(\mathrm{Adm}(G)) = 1$.

2.2.3. *General case.* Let $V_1, \dots, V_N$ be sets of vertices of strongly connected components of the graph G; $N \geq 2$. Then if $E \in \mathrm{Adm}(G)$ then $E_V \subset E$ in the notation of Section 2.2.1, and $E \setminus E_V \in \mathrm{Adm}(G \setminus E_V)$, so that

$$\mathcal{Z}(\mathrm{Adm}(G)) = (-1)^{\# E_V} \mathcal{Z}(\mathrm{Adm}(G \setminus E_V)).$$

If G is not strongly semiconnected then $E_V \neq \varnothing$ and by the induction hypothesis $\mathcal{Z}(\mathrm{Adm}(G \setminus E_V)) = 0$ if $\# V_i > 1$ for at least one i. If $\# V_i = 1$ for all i, which is equivalent to $G \in \mathcal{AC}_{n,k}^{(i_1,\dots,i_s)}$, then $\mathcal{Z}(\mathrm{Adm}(G \setminus E_V)) = 1$, and the theorem is proved.

3. Corollaries

3.1. **Undirected graphs.** Denote by $U_{n,k}$ the set of *undirected* graphs with n vertices numbered $1, \dots, n$ and k edges numbered $1, \dots, k$, and by $\mathcal{U}_{n,k}$, the vector space spanned by $U_{n,k}$. If A is a symmetric $n \times n$-matrix then for any $u \in \mathcal{U}_{n,k}$ one can define $\langle A | u \rangle$ by (1) as before. Denote by $\mathrm{LS}_n(w)$ the symmetric Laplace matrix, i.e. the symmetric $n \times n$-matrix obtained from $L_n(w)$ by substitution $w_{ji} = w_{ij}$ for all $1 \leq i < j \leq n$.

The loop-breaking operator $\mathcal{B} : \mathcal{U}_{n,k} \to \mathcal{U}_{n,k}$ is defined by the same formula (2) as for directed graphs. Apparently, $\langle \mathrm{LS}_n(w) | u \rangle = \langle \mathrm{LS}_n(w) | \mathcal{B}(u) \rangle$ for any $u \in \mathcal{U}_{n,k}$.

For a graph $G \in U_{n,k}$ and a set of vertices $1 \leq i_1, \dots, i_s \leq n$ denote by $A_{i_1,\dots,i_s}(G)$ the number of ways to orient the edges of G (not including loops) so that the resulting directed graph would belong to $\mathcal{AC}_{n,k}^{(i_1,\dots,i_s)}$ (i.e. becomes acyclic and has $i_1, \dots, i_s$, and only them, as sinks), and $C_{i_1,\dots,i_s}(G)$, the number of ways to orient the edges of G so that the result would belong to $\mathcal{SSC}_{n,k}^{(i_1,\dots,i_s)}$.

Corollary 1. *$\mathcal{B} Y_{n,k}^{(i_1,\dots,i_s)} = X_{n,k}^{(i_1,\dots,i_s)}$ where $Y_{n,k}^{(i_1,\dots,i_s)} \stackrel{def}{=} \sum_{G \in \mathcal{U}_{n,k}} (-1)^{\beta_0(G)} C_{(i_1,\dots,i_s)}(G) G$ and $X_{n,k}^{(i_1,\dots,i_s)} \stackrel{def}{=} \sum_{G \in \mathcal{U}_{n,k}} A_{i_1,\dots,i_s}(G) G$. In particular, $\langle \mathrm{LS}_n(w) | Y_{n,k}^{(i_1,\dots,i_s)} \rangle = \langle \mathrm{LS}_n(w) | X_{n,k}^{(i_1,\dots,i_s)} \rangle$.*

Example 4. Let $k = n+1$, and consider first a strongly connected graph $H \in G_{n,n+1}$. There are two types of such graphs:

1. The 8-graph: a union of two directed cycles with one common vertex a and without common edges.
2. The Θ-graph: a union of three directed paths. Two of them start at a vertex a and end at a vertex b, and the third starts at b and ends at a. The paths have no common edges and no common vertices except a and b.

A directed graph $H \in \mathcal{SSC}_{n,n+1}^{\varnothing}$ consists of an arbitrary number of oriented cycles and one strongly connected component with n_1 vertices and n_1+1 edges. Thus, for an unidrected graph $G \in U_{n,k}$ one has $C_{\varnothing}(G) \neq 0$ if and only if G contains several connected components homeomorphic to circles and one connected component homeomorphic to either a wedge of two circes or to the union of three segments with common endpoints. We will call G a 8-graph or a Θ-graph depending on the character of this component. Denote by $m(G)$ the number of circle components that are not loops. Any such component has 2 strongly connected orientations (clockwise and counterclockwise), the 8-graph has them 4 (each cycle should be oriented clockwise or counterclockwise), and the Θ-graph has 6 (two lines directed one way, and the third line, the opposite way). So,

$$Y_{n,n+1}^{\varnothing} = 4 \sum_{G \text{ is a 8-graph}} 2^{m(G)}(-1)^{\beta_0(G)} G + 6 \sum_{G \text{ is a } \Theta\text{-graph}} 2^{m(G)}(-1)^{\beta_0(G)} G.$$

3.2. **Schrödinger matrices.** Let $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ denote the $n \times n$-matrix with $\lambda_1, \dots, \lambda_n$ on the main diagonal, all the other elements being 0. The matrix $L_n(w, \lambda) \stackrel{\text{def}}{=} L_n(w) + \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is called a Schrödinger matrix by analogy with the Schrödinger operator: a sum of the Laplace operator and the diagonal (multiplication) operator. By dimensional reasons it is clear that actually any $n \times n$-matrix is a Schrödinger matrix for some w_{ij}, $i \neq j$, and λ_i; diagonal matrix elements are related to these coordinates by a linear transformation: $w_{ii} = \lambda_i - \sum_{j \neq i} w_{ij}$.

Denote

$$\widetilde{\mathcal{AC}}_{n,k}^{(i_1,\dots,i_s)} = \bigcup_{\{j_1,\dots,j_t\} \subseteq \{i_1,\dots,i_s\}} \mathcal{AC}_{n,k}^{(j_1,\dots,j_t)}$$

the set of all graphs $G \in G_{n,k}$ containing no directed cycles and no sinks except possibly $i_1, \dots, i_s$; Also denote

$$\tilde{x}_{n,k}^{(i_1,\dots,i_s)} \stackrel{\text{def}}{=} \sum_{G \in \widetilde{\mathcal{AC}}_{n,k}^{(i_1,\dots,i_s)}} G = \sum_{\{j_1,\dots,j_t\} \subseteq \{i_1,\dots,i_s\}} x_{n,k}^{(j_1,\dots,j_t)}.$$

Corollary 2. *The higher determinant of the Schrödinger matrix is* $\langle L_n(w,\lambda) | y_{n,k}^{\varnothing} \rangle = \sum_s \sum_{i_1,\dots,i_s} \lambda_{i_1} \dots \lambda_{i_s} \langle L_n(w,\lambda) | \tilde{x}_{n,k}^{(i_1,\dots,i_s)} \rangle$.

Proof. Let $G \in G_{n,k}$ have loops attached to the vertices $p_1, \dots, p_N$ (where some p_i may coincide). Then

$$\begin{aligned} \langle L_n(w,\lambda) | G \rangle &= (\lambda_{p_1} - \sum_q w_{p_1 q}) \dots (\lambda_{p_N} - \sum_q w_{p_N q}) \prod_{e \text{ is not a loop}} w_{e_- e_+} \\ &= \sum_{s=0}^{N} \sum_{\{\{i_1,\dots,i_s\}\} \subseteq \{\{p_1,\dots,p_N\}\}} \lambda_{i_1} \dots \lambda_{i_s} \langle L_n(w) | G_{i_1,\dots,i_s} \rangle, \end{aligned}$$

where $G_{i_1,\dots,i_s}$ is the graph G with the loops at the vertices $i_1, \dots, i_s$ deleted, and double curly braces mean "sets with repetitions" where elements are allowed to coincide (so that $\{\{1,1,2\}\} \subset \{\{1,1,2,3\}\}$ but $\{\{1,2,2\}\} \not\subset \{\{1,1,2,3\}\}$). One has $G \in \mathcal{SSC}_{n,k}^{\varnothing}$ if and only if $G_{i_1,\dots,i_s} \, \mathcal{SSC}_{n,k}^{(j_1,\dots,j_t}$ with $\{\{j_1,\dots,j_t\}\} \subseteq \{\{i_1,\dots,i_s\}\}$; now the result follows from Theorem 1 in the second formulation. $\square$

References

[1] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung det linearen Verteilung galvanischer Ströme gefurht wird, *Ann. Phys. Chem.*, **72**(1847), S. 497–508.

[2] S. Chaiken, A combinatorial proof of the all minors matrix tree theorem, *SIAM J. Alg. Disc. Meth.*, **3**(1982), no. 3, pp. 319–329.

[3] W.T. Tutte, The dissection of equilateral triangles into equilateral triangles, *Math. Proc. Cambridge Phil. Soc.*, **44** (1948) no. 4, pp. 463–482.

[4] Yu. Burman, B. Shapiro. Around matrix-tree theorem, *Math. Res. Letters*, V. 13, no. 5 (2006), p. 761.

[5] R. Forman, Determinants of Laplacians on graphs, *Topology* 32, no. 1 (1993), pp. 35–46.

[6] R. Kenyon, The Laplacian on planar graphs and graphs on surfaces, *Annals of Probability* vol. 39, no. 5 (2011), pp.1983–2017.

[7] T. Zaslavsky, Biased graphs I. Bias, balance and gains, *Journal of Combinatorial Theory, Series B*, 47 (1989), no. 1, pp. 32–52.

[8] G. Masbaum, A. Vaintrob, A new matrix-tree theorem, *Internat. Math. Res. Notices*, 27 (2002), pp. 1397–1426.

[9] Yu. Burman, A. Ploskonosov, A. Trofimova. Matrix-tree theorems and discrete path integration, *Linear Alg. Appl.*, no. 466 (2015), pp. 64–82.

[10] A.Abdessalam, The Grassmann-Berezin calculus and theorems of the matrix-tree type, *Advances in Applied Mathematics*, 33, no. 1 (2004), pp. 51–70.

[11] R. Burton, R. Pemantle, Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances, *Ann. Probab.*, 21, no. 3 (1993), pp. 1329–1371.

[12] Yu.Burman, D.Zvonkine, Cycle factorizations and 1-faced graph embeddings, *European Journal of Combinatorics*, 31, no. 1 (2010), pp. 129–144.

National Research University Higher School of Economics (Department of mathematics and Laboratory for representation theory and mathematical physics; 101000, 20 Myasnitskaya str., Moscow, Russia), and the Independent University of Moscow (119002, B.Vlassievsky per., Moscow, Russia)

E-mail address: yburman@gmail.com