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A Generalization of the Helly Theorem for Functions with Values in a Uniform Space

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Abstract—In this paper we consider sequences of functions that are defined on a subset of the real line and take on values in a uniform Hausdorff space. For such sequences we obtain a sufficient condition for the existence of pointwise convergent subsequences. We prove that this generalization of the Helly theorem includes many results of the recent research. In addition, we prove that the sufficient condition is also necessary for uniformly convergent sequences of functions. We also obtain a representation for regular functions whose values belong to the uniform space.

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1. INTRODUCTION

According to the classical Helly selection principle [1] ([2], P. 208, lemma 2), each infinite uniformly bounded family of monotone real functions defined on a segment $[a, b]$ from $\mathbb{R}$ contains a pointwise convergent on $[a, b]$ subsequence. This Helly theorem is also valid for an arbitrary nonempty subset T from $\mathbb{R}$ (see, e.g., [3]) and for a uniformly bounded sequence of functions with uniformly bounded Jordan variations. Most generalizations of the Helly selection principle are based on the uniform boundedness of a sequence of functions and their generalized variations (see [4–7] for real-valued functions and [3, 8–16] for functions that take on values in a metric or Banach space). Such selection principles have many applications [3, 8–10, 12–14], because they are efficient in proving the existence theorems; for example, they are widely used in the convergence theory for Fourier series and in the theory of stochastic processes. Generalizations of the Helly theorems are also applied in the multivalued analysis for proving the existence of regular selections for multifunctions of bounded generalized variation, and in studying nonlinear superposition operators [3].

In [17, 18] one first presents a selection principle for one-variable functions with values in a uniform space. This principle implies most known generalizations of the Helly theorem with restrictions on generalized variations [3–16]. Moreover, the restriction on the *modulus of variation* of functions from the initial sequence that is a base of the selection principle proposed in [17, 18] is not only a sufficient condition for the existence of a pointwise convergent subsequence, but also the necessary condition for the uniform convergence of the sequence of functions (as distinct from the known selection principles [3–16]).

In this paper we generalize the Helly theorem for a sequence of functions with values in a Hausdorff uniform space; the obtained result includes the selection principle proposed in [17, 18].

Section 2 contains the main definitions and statements of the obtained results. In Section 3 we adduce some auxiliary assertions and study regular functions with respect to a dense subset of the definition domain. In Section 4 we prove the basic theorems and compare the main result (Theorem 1) with the selection principle proposed in [17, 18].

Results of this paper were announced in [19, 20].

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2. MAIN DEFINITIONS AND RESULTS

In what follows we assume that $(X, \mathcal{U})$ is a Hausdorff uniform space with a set of pseudometrics $\{d_p\}_{p\in\mathcal{P}}$ of uniformity $\mathcal{U}$ ([21], Chap. 6), where $\mathcal{P}$ is some index set; in particular, each pseudometric d_p is uniformly continuous on $X \times X$ with respect to the product uniformity ([21], P. 243, theorem 11), i.e., $V_{p,r} \in \mathcal{U}$ for each $r > 0$, where $V_{p,r} = \{(x, y) \in X \times X \mid d_p(x, y) < r\}$. Recall that if $\{d_p\}_{p\in\mathcal{P}}$ is a set of pseudometrics of a uniformity $\mathcal{U}$, then the family $\{V_{p,r} \mid p \in \mathcal{P}, r > 0\}$ is a base of the uniformity $\mathcal{U}$, i.e., for any $U \in \mathcal{U}$ there exist $p \in \mathcal{P}$ and $r > 0$ such that $V_{p,r} \subset U$ ([21], P. 250, theorem 19). A uniform space $(X, \mathcal{U})$ is said to be a *Hausdorff* (*or separable*) one, if conditions $x, y \in X$ and $d_p(x, y) = 0$ satisfied for all $p \in \mathcal{P}$ imply that $x = y$. A sequence of elements $\{x_j\}_{j=1}^{\infty}$ of a uniform space $(X, \mathcal{U})$ converges to an element $x \in X$ (as $j \to \infty$), if $\lim_{j\to\infty} d_p(x_j, x) = 0$ for all $p \in \mathcal{P}$. Since X is Hausdorff, this limit element x is unique. A subset $Y \subset X$ of a uniform space $(X, \mathcal{U})$ is called *relatively sequentially compact*, if each sequence of elements of Y contains a subsequence that converges in X to some element of X.

Let X^T stand for the set of all functions $f : T \to X$ that act from a nonempty subset $T \subset \mathbb{R}$ in X. Recall also that a sequence of functions $\{f_j\} \equiv \{f_j\}_{j=1}^{\infty} \subset X^T$ *converges pointwise on* T to some function $f \in X^T$ (we write $f_j \to f$ on T as $j \to \infty$) if $\lim_{j\to\infty} d_p(f_j(t), f(t)) = 0$ for all $p \in \mathcal{P}$ and $t \in T$; but if $\lim_{j\to\infty} \sup_{t\in T} d_p(f_j(t), f(t)) = 0$ for all $p \in \mathcal{P}$, then $\{f_j\}$ *converges uniformly on* T to the function f. A sequence $\{f_j\} \subset X^T$ is said to be *pointwise relatively sequentially compact*, if the sequence $\{f_j(t)\}$ is relatively sequentially compact with any $t \in T$.

In order to formulate a generalization of the Helly theorem and other results of this paper, we introduce the value $\{N_p(\varepsilon, f, T)\}_{p\in\mathcal{P}}$.

For a natural number $n \in \mathbb{N}$ we denote by $\{I_i\}_1^n \prec T$ an ordered set of n non-overlapping segments $I_i = [s_i, t_i] \subset \mathbb{R}$, $i = 1, \dots, n$, whose endpoints s_i and t_i belong to T so that $s_1 < t_1 \le s_2 < t_2 \le \dots \le s_{n-1} < t_{n-1} \le s_n < t_n$ (with $n = 1$ for brevity we write $I = \{I_i\}_1^1$ and $I \prec T$). If $I_i \in \{I_i\}_1^n$, $f \in X^T$, and $p \in \mathcal{P}$, then we put $|f(I_i)|_p \equiv d_p(f(s_i), f(t_i))$.

For $p \in \mathcal{P}$, $\varepsilon > 0$, and $f \in X^T$ we define the value $N_p(\varepsilon, f, T) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ by the following rule:

$$N_p(\varepsilon, f, T) = \sup\big\{n \in \mathbb{N} \mid \exists\, \{I_i\}_1^n \prec T \text{ such that } |f(I_i)|_p > \varepsilon \ \ \forall\, i = 1, \dots, n\big\}, \tag{1}$$

where $\sup \emptyset = 0$; here if $\emptyset \ne E \subset T$ and $f \in X^T$, then we put $N_p(\varepsilon, f, E) = N_p(\varepsilon, f|_E, E)$, where $f|_E : E \to X$ is the restriction of the function f on the set E. In the case when $T = [a, b]$ and $X = \mathbb{R}$, value (1) is considered in [22], Part III.

One of the most important properties of the value $\{N_p(\varepsilon, f, T)\}_{p\in\mathcal{P}}$ is the following one: With its help one can describe regular functions, i.e., those that have one-sided left and right limits (we clarify their meaning below). Let S be an everywhere dense subset of $[a, b]$. Denote by $U_S([a, b]; X)$ the set of all functions $f : [a, b] \to X$ that satisfy the Cauchy conditions with respect to S

$$\lim_{S\ni s,t\to\tau-0} d_p(f(s), f(t)) = 0 \;\forall\, p \in \mathcal{P} \text{ at each point } \tau \in (a, b] \tag{2}$$

and

$$\lim_{S\ni s,t\to\tau+0} d_p(f(s), f(t)) = 0 \;\forall\, p \in \mathcal{P} \text{ at each point } \tau \in [a, b). \tag{3}$$

Then the following equality is valid (we adduce its proof below in Section 3):

$$U_S([a, b]; X) = \big\{f : [a, b] \to X \mid \ N_p(\varepsilon, f, S) < \infty \text{ for all } p \in \mathcal{P} \text{ and } \varepsilon > 0\big\}. \tag{4}$$

The following three theorems represent the main results of this paper. The first one is the selection principle for one-variable functions with values in a uniform space stated in terms of the value $\{N_p(\varepsilon, f, T)\}_{p\in\mathcal{P}}$.

Theorem 1. *Let$\emptyset \neq T \subset \mathbb{R}$ and let $(X,\mathcal{U})$ be a Hausdorff uniform space with at most countable set of pseudometrics $\{d_p\}_{p\in\mathcal{P}}$ of uniformity $\mathcal{U}$. Let $\{f_j\} \subset X^T$ be a pointwise relatively sequentially compact sequence of functions such that*

$$N_p(\varepsilon) \equiv \limsup_{j\to\infty} N_p(\varepsilon, f_j, T) < \infty \text{ for all } p \in \mathcal{P} \text{ and } \varepsilon > 0. \tag{5}$$

Then $\{f_j\}$ contains a subsequence that converges pointwise on T to some function $f \in X^T$ that satisfies the condition $N_p(\varepsilon, f, T) \leq N_p(\varepsilon)$ for all $p \in \mathcal{P}$ and $\varepsilon > 0$.

In the following theorem we prove that condition (5) is necessary for the uniform convergence of the sequence $\{f_j\}$.

Theorem 2. *Let $\emptyset \neq T \subset \mathbb{R}$ and let $(X,\mathcal{U})$ be a Hausdorff uniform space with a (not necessarily countable) set of pseudometrics $\{d_p\}_{p\in\mathcal{P}}$ of uniformity $\mathcal{U}$. If a sequence $\{f_j\} \subset X^T$ converges uniformly on T to a function $f \in X^T$ such that $N_p(\varepsilon, f, T) < \infty$ for all $p \in \mathcal{P}$ and $\varepsilon > 0$, then condition* (5) *is fulfilled; more precisely,*

$$\limsup_{j\to\infty} N_p(\varepsilon, f_j, T) \leq \lim_{\delta\to\varepsilon-0} N_p(\delta, f, T) \ \text{ for all } p \in \mathcal{P} \text{ and } \varepsilon > 0.$$

Recall that a sequence $\{f_j\} \subset X^T$ *converges almost everywhere* (*a. e.*) *on* T to a function $f \in X^T$ (as $j \to \infty$) if there exits a set $E \subset T$ of zero Lebesgue measure such that $\lim\limits_{j\to\infty} d_p(f_j(t), f(t)) = 0$ for all $p \in \mathcal{P}$ and $t \in T \setminus E$. Theorem 1 immediately implies that if a sequence $\{f_j\} \subset X^T$ satisfies condition (5) with $T \setminus E$ in place of T, where $E \subset T$ is some set of zero Lebesgue measure, then some subsequence in $\{f_j\}$ converges a. e. on T to a function $f \in X^T$ such that $N_p(\varepsilon, f, T \setminus E) < \infty$ for all $p \in \mathcal{P}$ and $\varepsilon > 0$. The following theorem is a selection principle for the convergence almost everywhere in terms of the value $\{N_p(\varepsilon, f, T)\}_{p\in\mathcal{P}}$ for a one-variable function with values in a uniform space.

Theorem 3. *Let T and $(X,\mathcal{U})$ satisfy conditions of Theorem* 1. *Assume that a sequence of functions $\{f_j\} \subset X^T$ is such that for a. a. $t \in T$ the set $\{f_j(t)\}$ is relatively sequentially compact and for any $\delta > 0$ there exists a measurable set $E_\delta \subset T$ of the Lebesgue measure $\mathcal{L}(E_\delta) \leq \delta$ such that*

$$\limsup_{j\to\infty} N_p(\varepsilon, f_j, T \setminus E_\delta) < \infty \ \text{ for all } p \in \mathcal{P} \text{ and } \varepsilon > 0.$$

Then in $\{f_j\}$ there exists a subsequence that converges a. e. on T to some function $f \in X^T$ with the following property: For any $\delta > 0$ there exists a measurable set $E'_\delta \subset T$ of the Lebesgue measure $\mathcal{L}(E'_\delta) \leq \delta$ such that $N_p(\varepsilon, f, T \setminus E'_\delta) < \infty$ for all $p \in \mathcal{P}$ and $\varepsilon > 0$.

3. THE MAIN PROPERTIES OF THE VALUE $N_p(\varepsilon, f, T)$

For the proof of the stated theorems and the following comparison of Theorem 1 with the selection principle in [17, 18] we need some properties of the value $\{N_p(\varepsilon, f, T)\}_{p\in\mathcal{P}}$. Note that the fact that the value $N_p(\varepsilon, f, T)$ equals 0 or ∞ is expressed, respectively, by the conditions

$$N_p(\varepsilon, f, T) = 0 \iff |f(I)|_p \leq \varepsilon \text{ for all } I \prec T \tag{6}$$

and

$$N_p(\varepsilon, f, T) = \infty \iff \forall\, n \in \mathbb{N} \ \ \exists \{I_i\}_1^n \prec T \text{ such that } |f(I_i)|_p > \varepsilon \text{ for all } i = 1, \ldots, n. \tag{7}$$

But if $N_p(\varepsilon, f, T) \in \mathbb{N}$, then for any $n \in \mathbb{N}$ in view of definition (1) we get the correlations

$$n \leq N_p(\varepsilon, f, T) \iff \exists \{I_i\}_1^n \prec T \text{ such that } |f(I_i)|_p > \varepsilon \text{ for all } i = 1, \ldots, n; \tag{8}$$

$$n > N_p(\varepsilon, f, T) \iff \forall \{I_i\}_1^n \prec T \ \ \exists \{I_{i_k}\}_{k=1}^{n-N_p(\varepsilon,f,T)} \prec T \text{ and } \{I_{i_k}\}_{k=1}^{n-N_p(\varepsilon,f,T)} \subset \{I_i\}_1^n$$

$$\text{such that} \max_{1\le k\le n-N_p(\varepsilon,f,T)} |f(I_{i_k})|_p \le \varepsilon. \quad (9)$$

The main properties of value (1) for an arbitrary function $f \in X^T$ and $p \in \mathcal{P}$ are described in the next lemma.

Lemma. (a) *If* $0 < \varepsilon_1 < \varepsilon_2$, *then* $N_p(\varepsilon_2, f, T) \le N_p(\varepsilon_1, f, T)$.

(b) *If* $\emptyset \ne E_1 \subset E_2 \subset T$, *then* $N_p(\varepsilon, f, E_1) \le N_p(\varepsilon, f, E_2)$ *for any* $\varepsilon > 0$.

(c) *If* $\{f_j\} \subset X^T$ *and* $f_j \to f$ *on* T *as* $j \to \infty$, *then* $N_p(\varepsilon, f, T) \le \liminf\limits_{j\to\infty} N_p(\varepsilon, f_j, T)$ *for all* $\varepsilon > 0$.

(d) *If* $s, t \in T$ *and* $s < t$, *then* $n_t = N_p(\varepsilon, f, (-\infty, t] \cap T) < \infty$ *if and only if* $n_s = N_p(\varepsilon, f, (-\infty, s] \cap T) < \infty$ *and* $n_{s,t} = N_p(\varepsilon, f, [s, t] \cap T) < \infty$, *and in this case there exists* $n_* \in \{0, 1\}$ *such that* $n_t = n_s + n_{s,t} + n_*$.

Proof. Properties (a) and (b) immediately follow from definition (1).

(c) Without loss of generality we assume that $N_p(\varepsilon, f, T) > 0$.

If $N_p(\varepsilon, f, T) < \infty$ and $n = N_p(\varepsilon, f, T)$, then in view of property (8) there exists a set $\{I_i\}_1^n \prec T$ such that $|f(I_i)|_p > \varepsilon$ for all $i = 1, \dots, n$. Let $\varepsilon' = \varepsilon'(n, p) > 0$ be such that $\min\limits_{1\le i\le n} |f(I_i)|_p > \varepsilon' > \varepsilon$. Owing to the pointwise convergence of f_j to f on T, we can find a number $J \in \mathbb{N}$ (that depends on the set $\{I_i\}_1^n$ and $p \in \mathcal{P}$) such that

$$d_p(f(s_i), f_j(s_i)) \le \frac{\varepsilon' - \varepsilon}{2} \text{ and } d_p(f_j(t_i), f(t_i)) \le \frac{\varepsilon' - \varepsilon}{2} \text{ for all } j \ge J \text{ and } i = 1, \dots, n.$$

Due to the triangle inequality for such j and i we get

$$\begin{aligned}\varepsilon' < |f(I_i)|_p &\le d_p(f(s_i), f_j(s_i)) + d_p(f_j(s_i), f_j(t_i)) + d_p(f_j(t_i), f(t_i)) \\ &\le \frac{\varepsilon' - \varepsilon}{2} + d_p(f_j(s_i), f_j(t_i)) + \frac{\varepsilon' - \varepsilon}{2} = |f_j(I_i)|_p + \varepsilon' - \varepsilon. \quad (10)\end{aligned}$$

Therefore, $|f_j(I_i)|_p > \varepsilon$ for all $j \ge J$ and $i = 1, \dots, n$. In accordance with property (8) this means that $n \le N_p(\varepsilon, f_j, T)$ for all $j \ge J$, therefore $n \le \inf\limits_{i\ge J} N_p(\varepsilon, f_j, T) \le \liminf\limits_{j\to\infty} N_p(\varepsilon, f_j, T)$.

But if $N_p(\varepsilon, f, T) = \infty$, then we choose $n \in \mathbb{N}$ arbitrarily and make use of property (7). Reasoning as above, we conclude that $n \le \liminf\limits_{j\to\infty} N_p(\varepsilon, f_j, T)$, and it remains to take into account the arbitrariness of n.

(d) Without loss of generality we assume that $n_t > 0$.

1. Let us first prove that if $n_t < \infty$, then $n_s + n_{s,t} \le n_t$.

Proposition (b) implies that $n_s \le n_t$ and $n_{s,t} \le n_t$, therefore if $n_s = 0$ or $n_{s,t} = 0$, then the inequality is evident. But if $n_s > 0$ and $n_{s,t} > 0$, then due to property (8) there exist sets $\{I_i\}_1^{n_s} \prec (-\infty, s] \cap T$ and $\{J_k\}_1^{n_{s,t}} \prec [s, t] \cap T$ such that $|f(I_i)|_p > \varepsilon$ for all $i = 1, \dots, n_s$ and $|f(J_k)|_p > \varepsilon$ for all $k = 1, \dots, n_{s,t}$. Note that $\{I_i\}_1^{n_s} \cup \{J_k\}_1^{n_{s,t}} \prec (-\infty, t] \cap T$ and that d_p-distances mentioned above (their quantity is $n_s + n_{s,t}$) exceed ε. Then on the base of property (8) we get the desired inequality $n_s + n_{s,t} \le n_t$.

2. Assume that $n_s < \infty$ and $n_{s,t} < \infty$. Let us demonstrate that if $n \in \mathbb{N}$ and the set $\{I_i\}_1^n \prec (-\infty, t] \cap T$ is such that $|f(I_i)|_p > \varepsilon$ for all $i = 1, \dots, n$ (such I_i always exist, because $n_t > 0$), then $n \le n_s + n_{s,t} + 1$, whence due to the arbitrariness of n from definition (1) we infer $n_t \le n_s + n_{s,t} + 1$, as well as the desired equality.

With $n = 1$ the inequality is evident, therefore in what follows we assume that $n \ge 2$. If a point $s \in T$ is located so that the set $\{I_i\}_1^n$ entirely belongs to $(-\infty, s] \cap T$ or $[s, t] \cap T$, then, respectively, $n \le n_s$ or $n \le n_{s,t}$. If a point s is the endpoint of one of the segments from the set $\{I_i\}_1^n$ and the startpoint of another one, i.e., $s \in I_k \cap I_{k+1}$ for some $k \in \{1, \dots, n-1\}$, then $\{I_i\}_1^k \prec (-\infty, s] \cap T$ and $\{I_i\}_{k+1}^n \prec [s, t] \cap T$. Hence in accordance with property (8) we obtain $k \le n_s$ and $n - k \le n_{s,t}$ such that $n \le n_s + n_{s,t}$. Finally, if s is located inside some segment I_k, $k \in \{1, \dots, n\}$, then $\{I_i\}_1^{k-1} \prec (-\infty, s] \cap T$ and $\{I_i\}_{k+1}^n \prec [s, t] \cap T$, where $\{I_i\}_1^0 = \emptyset = \{I_i\}_{n+1}^n$. Hence in view of property (8) we find $k - 1 \le n_s$ and $n - k \le n_{s,t}$, and therefore $n \le n_s + n_{s,t} + 1$. □

In order to prove equality (4) we need the notion of a stepwise function. Recall that a function $g : [a, b] \to X$ is called *stepwise*, if there exist a partition $a = c_0 < c_1 < \cdots < c_{m-1} < c_m = b$ of the segment $[a, b]$ and elements $x_1, \ldots, x_m \in X$ (dependent on g) such that $g(t) = x_i$ for all $t \in (c_{i-1}, c_i)$, $i = 1, \ldots, m$. For such a function g we have

$$N_p(\varepsilon, g, [a, b]) \le 2m < \infty \text{ for all } p \in \mathcal{P} \text{ and } \varepsilon > 0. \tag{11}$$

Proof of equality (4). *Inclusion* "$\supset$". Let a point $\tau \in (a, b]$ be arbitrary (considerations for $\tau \in [a, b)$ are analogous). Let us prove that for any $p \in \mathcal{P}$ and $\varepsilon > 0$ one can find $\delta(\varepsilon, p) \in (0, \tau - a)$ such that $d_p(f(s), f(t)) \le \varepsilon$ for all $s, t \in S \cap [\tau - \delta(\varepsilon, p), \tau)$. Let us prove this assertion ad contrario. Let $p_0 \in \mathcal{P}$ and $\varepsilon_0 > 0$ violate the above assertion. Then for arbitrary $\delta_1 \in (0, \tau - a)$ one can find points $s_1, t_1 \in S \cap [\tau - \delta_1, \tau)$, $s_1 < t_1$, such that $d_{p_0}(f(s_1), f(t_1)) > \varepsilon_0$. Further, by induction, if $i \in \mathbb{N}$, $i \ge 2$, and $\delta_{i-1} \in (0, \tau - a)$ and points $s_{i-1}, t_{i-1} \in S \cap [\tau - \delta_{i-1}, \tau)$, $s_{i-1} < t_{i-1}$, are already chosen, then we put $\delta_i = \tau - t_{i-1}$ and find points $s_i, t_i \in S \cap [\tau - \delta_i, \tau) = S \cap [t_{i-1}, \tau)$, $s_i < t_i$, such that $d_{p_0}(f(s_i), f(t_i)) > \varepsilon_0$. Let $n \in \mathbb{N}$ and $I_i = [s_i, t_i]$, $i = 1, \ldots, n$. Then by construction $\{I_i\}_1^n \prec S \cap (a, \tau) \subset S$ and $|f(I_i)|_{p_0} > \varepsilon_0$ for all $i = 1, \ldots, n$. Due to the arbitrariness of n and property (7) this means that $N_{p_0}(\varepsilon_0, f, S) = \infty$, which contradicts the condition.

Inclusion "$\subset$". Let $f \in U_S([a, b]; X)$. In accordance with [18] (§ 4, lemma 4) for any $p \in \mathcal{P}$ there exists a sequence of stepwise functions $\{f_j\} \subset X^{[a,b]}$ (dependent on p) such that $\lim_{j\to\infty} \sup_{t \in S} d_p(f_j(t), f(t)) = 0$. Fix $p \in \mathcal{P}$ arbitrarily. Then taking into account inequality (11), the value $N_p(\varepsilon, f_j, [a, b])$ is finite for all $j \in \mathbb{N}$ and $\varepsilon > 0$. Since $S \subset [a, b]$, in view of proposition (b) of the lemma $N_p(\varepsilon, f_j, S) < \infty$ for all $j \in \mathbb{N}$ and $\varepsilon > 0$. Let us prove that $N_p(\varepsilon, f, S) < \infty$ for all $\varepsilon > 0$.

Fix $\varepsilon > 0$ arbitrarily. Assume that for some $n \in \mathbb{N}$ there exists a set $\{I_i\}_1^n \prec S$ such that $|f(I_i)|_p > \varepsilon$ for all $i = 1, \ldots, n$ (otherwise $N_p(\varepsilon, f, S) = 0$ and the assertion is evident). Similarly to the proof of proposition (c) of the lemma, we choose $\varepsilon' = \varepsilon'(n, p) > 0$ so that $\min_{1 \le i \le n} |f(I_i)|_p > \varepsilon' > \varepsilon$. Then due to the uniform convergence of f_j to f on S one can find a number $j_0 = j_0(\varepsilon', \varepsilon) \in \mathbb{N}$ such that $d_p(f_j(s), f(s)) \le (\varepsilon' - \varepsilon)/2$ for all $j \ge j_0$ and $s \in S$. Therefore, correlation (10) is valid for all $j \ge j_0$ and $i = 1, \ldots, n$. In particular, with $j = j_0$ we get $|f_{j_0}(I_i)|_p > \varepsilon$ for all $i = 1, \ldots, n$ or, in accordance with property (8), $n \le N_p(\varepsilon, f_{j_0}, S) < \infty$. Thus, due to the arbitrariness of n definition (1) implies that the value $N_p(\varepsilon, f, S)$ is finite for all $\varepsilon > 0$. □

Equality (4) describes the set $U_S([a, b]; X)$ in terms of the value $\{N_p(\varepsilon, f, T)\}_{p \in \mathcal{P}}$. The set $U_S([a, b]; \mathbb{R})$ was first considered in [23]; in other terms the set $U_S([a, b]; X)$ was described in [18] and [24–27].

4. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 1. Let $\mathrm{Mon}(T; \mathbb{N})$ stand for the set of all nondecreasing bounded functions that map T in $\mathbb{N}$. Note that for $p \in \mathcal{P}$ and given $\varepsilon > 0$ in view of proposition (b) of the lemma the function $t \mapsto N_p(\varepsilon, f_j, (-\infty, t] \cap T)$ is nondecreasing in $t \in T$ for each $j \in \mathbb{N}$.

Without loss of generality in the proof we assume that $\mathcal{P} = \mathbb{N}$.

1. Let us prove that there exist a subsequence in $\{f_j\}$ (we also denote it by $\{f_j\}$) and a function $n_{k,p} \in \mathrm{Mon}(T; \mathbb{N})$ for any $k \in \mathbb{N}$ and $p \in \mathcal{P}$ such that

$$\lim_{j\to\infty} N_p(1/k, f_j, (-\infty, t] \cap T) = n_{k,p}(t) \text{ for all } t \in T. \tag{12}$$

To this end we several times apply the Cantor diagonal process.

Condition (5) with $p = 1$ implies that for any $\varepsilon > 0$ there exist numbers $J_1(\varepsilon), M_1(\varepsilon) \in \mathbb{N}$ such that $N_1(\varepsilon, f_j, T) \le M_1(\varepsilon) < \infty$ for all $j \ge J_1(\varepsilon)$. Then in view of proposition (b) of the lemma the sequence of nondecreasing functions $\{t \mapsto N_1(1, f_j, (-\infty, t] \cap T)\}_{j=J_1(1)}^\infty$ is uniformly bounded on T by the constant $M_1(1)$. Consequently, by the Helly selection principle there exist a subsequence $\{f_{J_1^1(j)}\}_{j=1}^\infty$

in $\{f_j\}_{j=J_1(1)}^{\infty}$, where $J_1^1 : \mathbb{N} \to \mathbb{N}$ is a strictly increasing subsequence in $\{J_1(1)+j-1\}_{j=1}^{\infty}$, and a function $n_{1,1} \in \mathrm{Mon}(T;\mathbb{N})$ such that

$$N_1(1, f_{J_1^1(j)}, (-\infty, t] \cap T) \to n_{1,1}(t) \text{ as } j \to \infty \text{ for all } t \in T.$$

Choose the least number $j_1^1 \in \mathbb{N}$ such that $J_1^1(j_1^1) \geq J_1(1/2)$.

Further, by induction, let $k \geq 2$ and let a subsequence $\{f_{J_{k-1}^1(j)}\}_{j=1}^{\infty}$ in the initial sequence $\{f_j\}$ and a number $j_{k-1}^1 \in \mathbb{N}$ such that $J_{k-1}^1(j_{k-1}^1) \geq J_1(1/k)$ be already chosen. Then by applying the Helly theorem to the sequence of nondecreasing functions $\{t \mapsto N_1(1/k, f_{J_{k-1}^1(j)}, (-\infty, t] \cap T)\}_{j=j_{k-1}^1}^{\infty}$ uniformly bounded on T by the constant $M_1(1/k)$, we obtain that there exist a subsequence $\{f_{J_k^1(j)}\}_{j=1}^{\infty}$ in $\{f_{J_{k-1}^1(j)}\}_{j=j_{k-1}^1}^{\infty}$ and a function $n_{k,1} \in \mathrm{Mon}(T;\mathbb{N})$ such that

$$N_1(1/k, f_{J_k^1(j)}, (-\infty, t] \cap T) \to n_{k,1}(t) \text{ as } j \to \infty \text{ for all } t \in T.$$

Then the diagonal sequence $\{f_{J_j^1(j)}\}_{j=1}^{\infty}$ (we denote it by $\{f_j^1\} \equiv \{f_j^1\}_{j=1}^{\infty}$) satisfies the condition

$$\lim_{j\to\infty} N_1(1/k, f_j^1, (-\infty, t] \cap T) = n_{k,1}(t) \text{ for all } k \in \mathbb{N} \text{ and } t \in T. \tag{13}$$

Using equality (13), we again apply the method of induction: If $p \in \mathcal{P}$, $p \geq 2$, and a subsequence $\{f_j^{p-1}\} \equiv \{f_j^{p-1}\}_{j=1}^{\infty}$ of the sequence $\{f_j^1\}$ is already chosen, then in view of condition (5) for each $\varepsilon > 0$ one can find numbers $J_p(\varepsilon), M_p(\varepsilon) \in \mathbb{N}$ such that $N_p(\varepsilon, f_j^{p-1}, T) \leq M_p(\varepsilon) < \infty$ for all $j \geq J_p(\varepsilon)$. Then in accordance with proposition (b) of the lemma the sequence of nondecreasing functions $\{t \mapsto N_p(1, f_j^{p-1}, (-\infty, t] \cap T)\}_{j=J_p(1)}^{\infty}$ is uniformly bounded on T by the constant $M_p(1)$. Hence by the Helly theorem we deduce that there exist a subsequence $\{f_{J_1^p(j)}\}_{j=1}^{\infty}$ in $\{f_j^{p-1}\}_{j=J_p(1)}^{\infty}$, where $J_1^p : \mathbb{N} \to \mathbb{N}$ is a strictly increasing subsequence in $\{J_p(1)+j-1\}_{j=1}^{\infty}$, and a function $n_{1,p} \in \mathrm{Mon}(T;\mathbb{N})$ such that

$$N_p(1, f_{J_1^p(j)}, (-\infty, t] \cap T) \to n_{1,p}(t) \text{ as } j \to \infty \text{ for all } t \in T.$$

Choose the least number $j_1^p \in \mathbb{N}$ such that $J_1^p(j_1^p) \geq J_p(1/2)$.

Further, by induction, let $k \geq 2$ and let a subsequence $\{f_{J_{k-1}^p(j)}\}_{j=1}^{\infty}$ in $\{f_j^{p-1}\}$ and a number $j_{k-1}^p \in \mathbb{N}$ such that $J_{k-1}^p(j_{k-1}^p) \geq J_p(1/k)$ be already chosen. Then by the Helly theorem for the sequence of nondecreasing functions $\{t \mapsto N_p(1/k, f_{J_{k-1}^p(j)}, (-\infty, t] \cap T)\}_{j=j_{k-1}^p}^{\infty}$ uniformly bounded on T by the constant $M_p(1/k)$ one can find a subsequence $\{f_{J_k^p(j)}\}_{j=1}^{\infty}$ in $\{f_{J_{k-1}^p(j)}\}_{j=j_{k-1}^p}^{\infty}$ and a function $n_{k,p} \in \mathrm{Mon}(T;\mathbb{N})$ such that

$$N_p(1/k, f_{J_k^p(j)}, (-\infty, t] \cap T) \to n_{k,p}(t) \text{ as } j \to \infty \text{ for all } t \in T.$$

Then the diagonal sequence $\{f_{J_j^p(j)}\}_{j=1}^{\infty}$ (we denote it by $\{f_j^p\} \equiv \{f_j^p\}_{j=1}^{\infty}$) possesses the following property:

$$\lim_{j\to\infty} N_p(1/k, f_j^p, (-\infty, t] \cap T) = n_{k,p}(t) \text{ for all } k \in \mathbb{N} \text{ and } t \in T.$$

According to the above reasoning, the diagonal sequence $\{f_j^j\}_{j=1}^{\infty}$ (we again denote it by $\{f_j\}$) satisfies condition (12).

2. Let Q stand for an at most countable everywhere dense subset of T, i.e., $Q \subset T \subset \overline{Q}$; here $\overline{Q}$ is the closure of the set Q in $\mathbb{R}$. Note that any point $t \in T$ different from a limit one for T belongs to Q ([18], the proof of theorem 1). With any $k \in \mathbb{N}$ and $p \in \mathcal{P}$ the function $n_{k,p}$ is monotone on T, therefore the set $S_{k,p} \subset T$ of its discontinuity points (all of them are of the first kind) is at most countable. We put $S = Q \cup \bigcup_{k\in\mathbb{N}} \bigcup_{p\in\mathcal{P}} S_{k,p}$. Then S is an at most countable everywhere dense subset of T, and if $T \setminus S \neq \emptyset$, then

$$\text{the function } n_{k,p} \text{ is continuous at points } t \in T \setminus S \text{ for all } k \in \mathbb{N} \text{ and } p \in \mathcal{P}. \tag{14}$$

Since the sequence $\{f_j\}$ is pointwise relatively sequentially compact and the set $S \subset T$ is at most countable, without loss of generality we assume (if necessary, we proceed to a subsequence in $\{f_j\}$ with the help of the standard diagonal process) that with each $s \in S$ the sequence $\{f_j(s)\}$ converges in X to some element $f(s) \in X$ such that $\lim_{j\to\infty} d_p(f_j(s), f(s)) = 0$ for all $p \in \mathcal{P}$.

If $S = T$, then the proof is complete. Let $S \neq T$. Then we prove that for any point $t \in T \setminus S$ the sequence $\{f_j(t)\}$ is a Cauchy sequence in X:

$$\lim_{j,l\to\infty} d_p(f_j(t), f_l(t)) = 0 \text{ for all } p \in \mathcal{P}.$$

Let $p \in \mathcal{P}$ and $\varepsilon > 0$ be arbitrary. Choose and fix a number $k = k(\varepsilon) \in \mathbb{N}$ such that $1/k \le \varepsilon/3$. Since S is everywhere dense in T and (due to property (14)) t is a continuity point of the function $n_{k,p}$, there exists a point $s = s(k,p,t) \in S$ dependent only on ε such that $n_{k,p}(t) = n_{k,p}(s)$. Let for definiteness $s < t$ (the case $t < s$ is analogous). Using correlation (12), we choose numbers $J_1 = J_1(k,p,t)$, $J_2 = J_2(k,p,s) \in \mathbb{N}$ that also depend only on ε such that $N_p(1/k, f_j, (-\infty, t] \cap T) = n_{k,p}(t)$ for all $j \ge J_1$ and $N_p(1/k, f_j, (-\infty, s] \cap T) = n_{k,p}(s)$ for all $j \ge J_2$. Then proposition (d) of the lemma implies that

$$\begin{aligned} N_p(1/k, f_j, [s,t] \cap T) &\le N_p(1/k, f_j, (-\infty, t] \cap T) - N_p(1/k, f_j, (-\infty, s] \cap T) \\ &= n_{k,p}(t) - n_{k,p}(s) = 0 \;\text{ for all } j \ge \max\{J_1, J_2\}, \end{aligned}$$

whence we get $N_p(1/k, f_j, [s,t] \cap T) = 0$, and then in view of property (6)

$$d_p(f_j(s), f_j(t)) \le 1/k \le \varepsilon/3.$$

Since the sequence $\{f_j(s)\}$ converges in the uniform space X, it is a Cauchy sequence ([21], P. 252, theorem 21). Therefore there exists a number $J_3 = J_3(\varepsilon, p, s) \in \mathbb{N}$ such that

$$d_p(f_j(s), f_l(s)) \le \varepsilon/3 \;\text{ for all }\; j, l \ge J_3.$$

Then the number $J = \max\{J_1, J_2, J_3\}$ depends only on ε and p, and for all $j, l \ge J$ we get

$$d_p(f_j(t), f_l(t)) \le d_p(f_j(t), f_j(s)) + d_p(f_j(s), f_l(s)) + d_p(f_l(s), f_l(t)) \le \varepsilon.$$

Since $p \in \mathcal{P}$ and $\varepsilon > 0$ are arbitrary, this means that $\{f_j(t)\}$ is a Cauchy sequence in X. By the assumption of the theorem the sequence $\{f_j(t)\}$ is relatively sequentially compact, consequently, it has a limit point; denote it by $f(t) \in X$. In accordance with [21] (P. 252, theorem 21) any Cauchy sequence in a uniform space converges to its limit point, therefore $\lim_{j\to\infty} d_p(f_j(t), f(t)) = 0$ for all $p \in \mathcal{P}$.

3. Since the space X is Hausdorff, the one-valued function $f : T = S \cup (T \setminus S) \to X$ is correctly defined on T and is a pointwise limit on T of the sequence $\{f_j\}$; by construction the latter is a subsequence of the initial sequence. By applying proposition (c) of the lemma, we obtain

$$N_p(\varepsilon, f, T) \le \liminf_{j\to\infty} N_p(\varepsilon, f_j, T) \le \limsup_{j\to\infty} N_p(\varepsilon, f_j, T) \le N_p(\varepsilon)$$

for all $p \in \mathcal{P}$ and $\varepsilon > 0$. □

Remark 1. A local variant of Theorem 1 is also valid. Namely, if condition (5) in this theorem is replaced with that

$$\limsup_{j\to\infty} N_p(\varepsilon, f_j, T \cap [a,b]) < \infty \;\text{ for all } a, b \in T,\, a < b,\, p \in \mathcal{P}, \text{ and } \varepsilon > 0,$$

then some subsequence $\{f_j\}$ converges pointwise on T to a function $f \in X^T$ such that $N_p(\varepsilon, f, T \cap [a,b]) < \infty$ for all $a, b \in T$, $a < b$, $p \in \mathcal{P}$, and $\varepsilon > 0$. One can obtain this assertion immediately by applying (on the base of Theorem 1) the diagonal process with respect to extending segments.

Remark 2. Without assumption (5) Theorem 1 is not valid. Let us demonstrate this in a simplest case, when $X = \mathbb{R}$. As is known, no subsequence of the sequence of functions $\{f_j\} \subset \mathbb{R}^{[0,2\pi]}$, where

$f_j(t) = |\sin(jt)|$, converges everywhere on $[0, 2\pi]$. According to definition (1), $N(\varepsilon, f_j, [0, 2\pi]) = 0$ with all $\varepsilon \geq 1$. Let us prove that for any number $j \in \mathbb{N}$,

$$4j \leq N(\varepsilon, f_j, [0, 2\pi]) \leq \frac{4j}{\varepsilon} \text{ for all } 0 < \varepsilon < 1, \tag{15}$$

whence $\limsup_{j\to\infty} N(\varepsilon, f_j, [0, 2\pi]) = \infty$ for all $0 < \varepsilon < 1$.

It is well-known that $V(f_j, [0, 2\pi]) = 4j$ ($j \in \mathbb{N}$), where $V(f, T)$ stands for the usual Jordan variation of the function f on T. Taking into account this fact, as well as the estimate $N(\varepsilon, f, T) \leq V(f, T)/\varepsilon$ that is valid for a function $f \in \mathbb{R}^T$ of bounded variation, we get the right-hand inequality in (15). If for any $j \in \mathbb{N}$ we put $t_i^j = \pi i/(2j)$ and $I_i = [t_{i-1}^j, t_i^j]$ for all $i = 0, 1, \ldots, 4j$, then we find

$$|f_j(I_i)| = \left|\sin\left(\frac{\pi i}{2}\right) - \sin\left(\frac{\pi(i-1)}{2}\right)\right| = 1 \text{ for all } i = 1, \ldots, 4j.$$

Hence in view of definition (1) we get the left-hand inequality in (15).

Let us return to studying the Hausdorff uniform space $(X, \mathcal{U})$. Recall that an *oscillation of a function* $f \in X^T$ *with respect to a pseudometric* d_p is the value $\operatorname{osc}_p(f, T) = \sup_{I \prec T} |f(I)|_p \equiv \sup_{s,t \in T} d_p(f(s), f(t))$. The *modulus of variation of a function* $f \in X^T$ *with respect to a pseudometric* d_p is a sequence $\{\nu_p(n, f, T)\}_{n=1}^\infty \subset [0, \infty]$ defined for given $p \in \mathcal{P}$ and $n \in \mathbb{N}$ by the following rule ([18, 24]):

$$\nu_p(n, f, T) = \sup\left\{\sum_{i=1}^n |f(I_i)|_p \;\middle|\; \{I_i\}_1^n \prec T\right\}. \tag{16}$$

More precisely, equality (16) is used with $n_T = \sup\{n \in \mathbb{N} \mid \exists \{I_i\}_1^n \prec T\} = \infty$, and if $n_T < \infty$, then $\nu_p(n, f, T)$ is defined by formula (16) with $n \leq n_T$, and $\nu_p(n, f, T) = \nu_p(n_T, f, T)$ with $n > n_T$. Definition (16) implies that $\nu_p(1, f, T) = \operatorname{osc}_p(f, T)$. In addition, for $\emptyset \neq E \subset T$ we put $\nu_p(n, f, E) = \nu_p(n, f|_E, E)$.

In [18] (theorem 1) one obtains the following pointwise selection principle: If in assumptions of Theorem 1 condition (5) is replaced with the following requirement:

$$\mu_p(n) \equiv \limsup_{j\to\infty} \nu_p(n, f_j, T) = o(n) \text{ for all } p \in \mathcal{P}, \tag{17}$$

then the assertion of Theorem 1 remains valid, and the limit function of the extracted subsequence $f \in X^T$ satisfies the condition $\nu_p(n, f, T) \leq \mu_p(n)$ for all $n \in \mathbb{N}$ and $p \in \mathcal{P}$. In (17) the denotation $\nu_p(n, f, T) = o(n)$ means that $\lim_{n\to\infty} \nu_p(n, f, T)/n = 0$.

This result follows from Theorem 1, taking into account that (17) is equivalent to two conditions, namely, (5) and $\limsup_{j\to\infty} \operatorname{osc}_p(f_j, T) < \infty$ for all $p \in \mathcal{P}$. This fact is established in the next theorem.

Theorem 4. *Let* $\{f_j\} \subset X^T$. *Then the following propositions are valid*:

(a) *for some* $p \in \mathcal{P}$ *the condition* $\limsup_{j\to\infty} \nu_p(n, f_j, T) = o(n)$ *is fulfilled if and only if* $\limsup_{j\to\infty} \operatorname{osc}_p(f_j, T) < \infty$ *and* $\limsup_{j\to\infty} N_p(\varepsilon, f_j, T) < \infty$ *for all* $\varepsilon > 0$;

(b) *the condition* $\limsup_{j\to\infty} \nu_p(n, f_j, T) = o(n)$ *with any* $p \in \mathcal{P}$ *is equivalent to the following ones*: $\limsup_{j\to\infty} \operatorname{osc}_p(f_j, T) < \infty$ *for all* $p \in \mathcal{P}$ *and* $\limsup_{j\to\infty} N_p(\varepsilon, f_j, T) < \infty$ *for all* $p \in \mathcal{P}$ *and* $\varepsilon > 0$.

Proof. Propositions (a) and (b) can be proved analogously, so for definiteness let us prove (b).

(b) *Necessity.* Fix arbitrarily $\varepsilon > 0$. In accordance with the restriction imposed on the modulus of variation, for any $p \in \mathcal{P}$ there exists a number $n_0(\varepsilon, p) \in \mathbb{N}$ such that $\lim_{k\to\infty} \sup_{j\ge k} \nu_p(n_0(\varepsilon, p), f_j, T) < \varepsilon n_0(\varepsilon, p)$. Consequently, one can find a number $j_0 = j_0(\varepsilon, p) \in \mathbb{N}$ such that $\nu_p(n_0(\varepsilon, p), f_j, T) < \varepsilon n_0(\varepsilon, p)$ for all $j \ge j_0(\varepsilon, p)$. Since

$$\operatorname{osc}_p(f_j, T) = \nu_p(1, f_j, T) \le \nu_p(n_0(\varepsilon, p), f_j, T) < \varepsilon n_0(\varepsilon, p) \text{ for all } j \ge j_0(\varepsilon, p), \tag{18}$$

we get $\limsup_{j\to\infty} \operatorname{osc}_p(f_j, T) \le \sup_{j\ge j_0(\varepsilon,p)} \operatorname{osc}_p(f_j, T) \le \varepsilon n_0(\varepsilon, p) < \infty$.

In view of property (18) for any $j \ge j_0(\varepsilon, p)$ the function f_j is bounded, therefore the sequence $\{\nu_p(n, f_j, T)/n\}_{n=1}^{\infty}$ is nonincreasing for all $j \ge j_0(\varepsilon, p)$ ([18], the remark to lemma 2). Hence we find

$$\nu_p(n, f_j, T)/n \le \nu_p(n_0(\varepsilon, p), f_j, T)/n_0(\varepsilon, p) < \varepsilon \text{ for all } n \ge n_0(\varepsilon, p) \text{ and } j \ge j_0(\varepsilon, p),$$

which means that $\sup_{j\ge j_0(\varepsilon,p)} \nu_p(n, f_j, T) < \varepsilon n$ for all $n \ge n_0(\varepsilon, p)$. Let us prove that then

$$\sup_{j\ge j_0(\varepsilon,p)} N_p(\varepsilon, f_j, T) \le n_0(\varepsilon, p).$$

Really, let a number $k \in \mathbb{N}$, $k \ge j_0(\varepsilon, p)$, be such that $N_p(\varepsilon, f_k, T) > 0$. Then for any natural n and a set $\{I_i\}_1^n \prec T$ such that $|f_k(I_i)|_p > \varepsilon$ for all $i = 1, \dots, n$ the inequality $n \le n_0(\varepsilon, p)$ is fulfilled; otherwise, if $n > n_0(\varepsilon, p)$, then $\sup_{j\ge j_0(\varepsilon,p)} \nu_p(n, f_j, T) \ge \nu_p(n, f_k, T) \ge \sum_{i=1}^{n} |f_k(I_i)|_p > n\varepsilon$, which leads to a contradiction. Thus, in accordance with definition (1) the value $N_p(\varepsilon, f_j, T) \le n_0(\varepsilon, p)$ for all $j \ge j_0(\varepsilon, p)$, whence we finally obtain

$$\limsup_{j\to\infty} N_p(\varepsilon, f_j, T) \le \sup_{j\ge j_0(\varepsilon,p)} N_p(\varepsilon, f_j, T) \le n_0(\varepsilon, p) < \infty.$$

Sufficiency. Let $\limsup_{j\to\infty} \operatorname{osc}_p(f_j, T) < M_p < \infty$ for all $p \in \mathcal{P}$. Then for $p \in \mathcal{P}$ there exists a number $j_1(p) \in \mathbb{N}$ such that $\sup_{j\ge j_1(p)} \operatorname{osc}_p(f_j, T) < M_p$. Let $\varepsilon > 0$ be arbitrary. By condition $\limsup_{j\to\infty} N_p(\varepsilon, f_j, T) < \infty$ for all $p \in \mathcal{P}$, therefore for $p \in \mathcal{P}$ one can find a number $j_2(\varepsilon, p) \in \mathbb{N}$ such that $N(\varepsilon, p) \equiv \sup_{j\ge j_2(\varepsilon,p)} N_p(\varepsilon, f_j, T) < \infty$. We put $j_3 = \max\{j_1(p), j_2(\varepsilon, p)\}$. In view of property (9) for arbitrary numbers $j \ge j_3$, $n \ge N_p(\varepsilon, f_j, T) + 1$ and a set $\{I_i\}_1^n \prec T$ one can find at least $n - N_p(\varepsilon, f_j, T)$ segments I_{i_k} (all $i_k \in \{1, \dots, n\}$ are different) such that $|f_j(I_{i_k})|_p \le \varepsilon$ for all $k = 1, \dots, n - N_p(\varepsilon, f_j, T)$. Hence

$$\sum_{i=1}^{n} |f_j(I_i)|_p = \sum_{m=1}^{N_p(\varepsilon, f_j, T)} |f_j(I_{i_m})|_p + \sum_{k=1}^{n-N_p(\varepsilon, f_j, T)} |f_j(I_{i_k})|_p \le N_p(\varepsilon, f_j, T) M_p + n\varepsilon \ \text{ for all } \ j \ge j_3,$$

where $\{i_m\}_{m=1}^{N_p(\varepsilon, f_j, T)} = \{i\}_{i=1}^{n} \setminus \{i_k\}_{k=1}^{n-N_p(\varepsilon, f_j, T)}$. Proceeding in the latter expression to supremum for all $\{I_i\}_1^n \prec T$, we get $\nu_p(n, f_j, T) \le N_p(\varepsilon, f_j, T) M_p + n\varepsilon$ for all $j \ge j_3$. Consequently, $\sup_{j\ge j_3} \nu_p(n, f_j, T) \le N(\varepsilon, p) M_p + n\varepsilon$, therefore

$$\limsup_{j\to\infty} \nu_p(n, f_j, T) \le \sup_{j\ge j_3} \nu_p(n, f_j, T) \le N(\varepsilon, p) M_p + n\varepsilon \le 2n\varepsilon$$

with $n \ge \max\{N(\varepsilon, p) + 1, N(\varepsilon, p) M_p/\varepsilon\}$, which means that $\limsup_{j\to\infty} \nu_p(n, f_j, T) = o(n)$. □

Let us adduce an example of a pointwise convergent on $[0, 1]$ sequence of functions $\{f_j\}$ that satisfies assumptions of Theorem 1, but the existence conditions for the pointwise convergence of a subsequence stated in the selection principle proposed in [17, 18] are violated. To this end we consider the Banach space of summable sequences $l_1 = \left\{x = \{x_n\}_{n=1}^{\infty} \subset \mathbb{R};\ \|x\| = \sum\limits_{n=1}^{\infty} |x_n| < \infty\right\}$. Denote by $e_i \in l_1$ a unit basis vector $e_i = \{x_n\}_{n=1}^{\infty}$, for which $x_n = 0$ with $n \neq i$ and $x_i = 1$. Let us define functions $f_j : [0,1] \to l_1$ as follows: $f_j(t) = je_j$ with $t = 1/(j+1)$ and $f_j(t) = 0 \in l_1$ with $t \neq 1/(j+1)$. Then $\{f_j\}$ everywhere converges to $f(t) \equiv 0 \in l_1$. Moreover, by definition $\{f_j(t)\} = \{0\} \in l_1$ with $t \notin \{1/(n+1)\}_{n\in\mathbb{N}}$ and $\{f_j(t)\} = \{0, ne_n\} \in l_1$ with $t = 1/(n+1)$, $n \in \mathbb{N}$, and, as a consequence, $\{f_j\}$ is pointwise compact. Further, $\operatorname{osc}(f_j, [0,1]) = j$ (tends to infinity as $j \to \infty$); $N(\varepsilon, f_j, [0,1]) = 2$ with $0 < \varepsilon < j$ and $N(\varepsilon, f_j, [0,1]) = 0$ with $\varepsilon \geq j$. Therefore, $\limsup\limits_{j\to\infty} N(\varepsilon, f_j, [0,1]) = 2$ for all $\varepsilon > 0$, while $\limsup\limits_{j\to\infty} \nu(n, f_j, [0,1]) = \infty$.

Proof of Theorem 2. Let $p \in \mathcal{P}$ and $\varepsilon > 0$ be arbitrary. Without loss of generality we assume that for all $j \in \mathbb{N}$ the value $N_p(\varepsilon, f_j, T) > 0$. Owing to the uniform convergence of $\{f_j\}$ to f for any δ, $0 < \delta < \varepsilon$, there exists a number $j_0(\varepsilon, \delta) \in \mathbb{N}$ such that for all $j \geq j_0(\varepsilon, \delta)$ the following inequality is valid:

$$d_p(f_j(t), f(t)) < (\varepsilon - \delta)/2 \ \text{ for all } \ t \in T. \tag{19}$$

Definition (1) with $n_j \equiv N_p(\varepsilon, f_j, T) > 0$ and $j \geq j_0(\varepsilon, \delta)$ implies that there exists a set $\{I_i\}_1^{n_j} \prec T$ such that $|f_j(I_i)|_p > \varepsilon$ for all $i = 1, \ldots, n_j$. Then in view of the triangle inequality and correlation (19) for all segments I_i from $\{I_i\}_1^{n_j}$ we get

$$\varepsilon < |f_j(I_i)|_p \leq d_p(f_j(s_i), f(s_i)) + d_p(f(s_i), f(t_i)) + d_p(f(t_i), f_j(t_i)) \leq \varepsilon - \delta + |f(I_i)|_p.$$

Thus, $|f(I_i)|_p > \delta$ for all $i = 1, \ldots, n_j$, which in accordance with property (8) means that

$$N_p(\varepsilon, f_j, T) \equiv n_j \leq N_p(\delta, f, T) \ \text{ for all } \ j \geq j_0(\varepsilon, \delta).$$

Hence we get

$$\limsup_{j\to\infty} N_p(\varepsilon, f_j, T) \leq \sup_{j \geq j_0(\varepsilon,\delta)} N_p(\varepsilon, f_j, T) \leq N_p(\delta, f, T) \ \text{ for all } \ 0 < \delta < \varepsilon,$$

and, taking into account the fact (proposition (a) of the lemma) that the function $\delta \mapsto N_p(\delta, f, T)$ is nondecreasing, it remains to proceed to the limit for $\delta \to \varepsilon - 0$. □

Proof of Theorem 3. Let us use the idea of the proof of theorem 6 in [26]. Let $T_0 \subset T$ be a zero measure set (possibly an empty one) such that the sequence $\{f_j(t)\}$ is relatively sequentially compact for all $t \in T \setminus T_0$. Let us apply Theorem 1 and the diagonal process. By assumption there exists a measurable set $E_1 \subset T$ of measure $\mathcal{L}(E_1) \leq 1$ such that $\limsup\limits_{j\to\infty} N_p(\varepsilon, f_j, T \setminus E_1) < \infty$ for all $p \in \mathcal{P}$ and $\varepsilon > 0$. The sequence $\{f_j\}$ is relatively sequentially compact on $T \setminus (T_0 \cup E_1)$ and in view of proposition (b) of the lemma we have

$$\limsup_{j\to\infty} N_p(\varepsilon, f_j, T \setminus (T_0 \cup E_1)) \leq \limsup_{j\to\infty} N_p(\varepsilon, f_j, T \setminus E_1) < \infty.$$

Then in accordance with Theorem 1 one can find a subsequence $\{f_j^{(1)}\}_{j=1}^{\infty}$ in $\{f_j\}$ and a function $f^1 : T \setminus (T_0 \cup E_1) \to X$ such that $f_j^{(1)} \to f^1$ on $T \setminus (T_0 \cup E_1)$ as $j \to \infty$ and $N_p(\varepsilon, f^1, T \setminus (T_0 \cup E_1)) < \infty$ for all $p \in \mathcal{P}$ and $\varepsilon > 0$.

Let $k \geq 2$ and let a subsequence $\{f_j^{(k-1)}\}_{j=1}^{\infty}$ in $\{f_j\}$ be already chosen. Then by assumption there exists a measurable set $E_k \subset T$ of measure $\mathcal{L}(E_k) \leq 1/k$ such that

$$\limsup_{j\to\infty} N_p(\varepsilon, f_j, T \setminus E_k) < \infty \ \text{ for all } \ p \in \mathcal{P} \text{ and } \varepsilon > 0.$$

The sequence $\{f_j^{(k-1)}\}_{j=1}^{\infty}$ with is relatively sequentially compact on the set $T \setminus (T_0 \cup E_k)$, therefore in view of proposition (b) of the lemma we obtain

$$\limsup_{j\to\infty} N_p(\varepsilon, f_j^{(k-1)}, T\setminus(T_0\cup E_k)) \le \limsup_{j\to\infty} N_p(\varepsilon, f_j^{(k-1)}, T\setminus E_k)$$
$$\le \limsup_{j\to\infty} N_p(\varepsilon, f_j, T\setminus E_k) < \infty.$$

In accordance with Theorem 1 one can find a subsequence $\{f_j^{(k)}\}_{j=1}^\infty$ in $\{f_j^{(k-1)}\}_{j=1}^\infty$ and a function $f^k : T\setminus(T_0\cup E_k)\to X$ such that $f_j^{(k)}\to f^k$ on $T\setminus(T_0\cup E_k)$ as $j\to\infty$ and $N_p(\varepsilon, f^k, T\setminus(T_0\cup E_k)) < \infty$ for all $p\in\mathcal{P}$ and $\varepsilon>0$.

We put $E = T_0\cup\bigcap_{k=1}^{\infty} E_k$, then the set E is measurable, its measure $\mathcal{L}(E)$ equals 0, and $T\setminus E = \bigcup_{k=1}^{\infty}\big(T\setminus(T_0\cup E_k)\big)$. Let us define a function $f : T\setminus E\to X$ as follows: For each $t\in T\setminus E$ there exists a number $k\in\mathbb{N}$ such that $t\in T\setminus(T_0\cup E_k)$, therefore we put $f(t) = f^k(t)$. The definition of the function f is correct, i.e., it is independent of the number k: Let a number $k_1\in\mathbb{N}$ be such that $t\in T\setminus(T_0\cup E_{k_1})$ and $k\le k_1$ (without loss of generality), then the sequence $\{f_j^{(k_1)}\}_{j=1}^\infty$ is a subsequence in $\{f_j^{(k)}\}_{j=1}^\infty$ such that

$$f^{k_1}(t) = \lim_{j\to\infty} f_j^{(k_1)}(t) = \lim_{j\to\infty} f_j^{(k)}(t) = f^k(t) \text{ in } X.$$

Let us prove that the diagonal sequence $f_j^{(j)}$ (that is a subsequence in $\{f_j\}$) converges pointwise to f on $T\setminus E$. Really, if $t\in T\setminus E$, then $t\in T\setminus(T_0\cup E_k)$ for some $k\in\mathbb{N}$ and $f(t) = f^k(t)$. Since $\{f_j^{(j)}\}_{j=k}^\infty$ is a subsequence in $\{f_j^{(k)}\}_{j=1}^\infty$, we have

$$\lim_{j\to\infty} f_j^{(j)}(t) = \lim_{j\to\infty} f_j^{(k)}(t) = f^k(t) = f(t) \text{ in } X.$$

Let us arbitrarily continue f from $T\setminus E$ onto all T and again denote the continuation by f. For given $\delta>0$ we choose $k\in\mathbb{N}$ such that $1/k\le\delta$ and put $E' = E'(\delta) = T_0\cup E_k$. Then $\mathcal{L}(E') = \mathcal{L}(E_k)\le 1/k\le\delta$, $f = f^k$ on $T\setminus(T_0\cup E_k) = T\setminus E'$ and $N_p(\varepsilon, f^k, T\setminus E') = N_p(\varepsilon, f^k, T\setminus(T_0\cup E_k)) < \infty$ for all $p\in\mathcal{P}$ and $\varepsilon>0$. □

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Note added in proof: Since the submission of this paper, works [28] and [29] have been published, where a pointwise principle of choice have been obtained for real-valued functions in [28] and for functions with values in a metric space in [29].

REFERENCES

1. E. Helly, "Über Lineare Functionaloperationen," Wien. Ber. **121**, 265–297 (1912).
2. I. P. Natanson, *Theory of Functions of Real Variable* (Nauka, Moscow, 1974) [in Russian].
3. V. V. Chistyakov, "Selections of Bounded Variation," J. Appl. Anal. **10** (1), 1–82 (2004).
4. J. Musielak and W. Orlicz, "On Generalized Variations. I," Stud. Math. **18** (1), 11–41 (1959).
5. D. Waterman, "On Λ-Bounded Variation," Stud. Math. **57** (1), 33–45 (1976).
6. S. Gniłka, "On the Generalized Helly's Theorem," Funct. Approx. Comment. Math. **4**, 109–112 (1976).
7. M. Schramm, "Functions of Φ-Bounded Variation and Riemann–Stieltjes Integration," Trans. Amer. Math. Soc. **287** (1), 49–63 (1985).
8. S. A. Belov and V. V. Chistyakov, "A Selection Principle for Mappings of Bounded Variation," J. Math. Anal. Appl. **249** (2), 351–366 (2000).
9. V. V. Chistyakov, "On Mappings of Bounded Variation," J. Dynam. Control Syst. **3** (2), 261–289 (1997).
10. V. V. Chistyakov, "On the Theory of Multivalued Mappings of Bounded Variation of One Real Variable," Matem. Sborn. **189** (5), 153–176 (1998).

11. V. V. Chistyakov, "Mappings of Bounded Variation with Values in a Metric Space: Generalizations," J. Math. Sci. **100** (6), 2700–2715 (2000).
12. V. V. Chistyakov, "Generalized Variation of Mappings with Applications to Composition Operators and Multifunctions," Positivity **5** (4), 323–358 (2001).
13. V. V. Chistyakov, "Metric Space-Valued Mappings of Bounded Variation," J. Math. Sci. **111** (2), 3387–3429 (2002).
14. V. V. Chistyakov, "Mappings of Generalized Variation and Composition Operators," J. Math. Sci. **110** (2), 2455–2466 (2002).
15. V. V. Chistyakov, "Multi-Valued Mappings of Bounded Generalized Variation," Matem. Zametki **71** (4), 611–632 (2002).
16. V. Barbu and Th. Precupanu, *Convexity and Optimization in Banach Spaces* (Reidel, Dordrecht, 1986).
17. V. V. Chistyakov, "A Selection Principle for Uniform Space-Valued Functions," Dokl. Phys. **409** (5), 591–593 (2006).
18. V. V. Chistyakov, "A Pointwise Selection Principle for Functions of a Single Variable with Values in a Uniform Space," Matem. Tr. **9** (1), 176–204 (2006).
19. Yu.V. Tret'yachenko, "Existence Conditions for a Pointwise Convergent Subsequence," Tr. Matem. Tsentra im. N. I. Lobachevskogo. Teoriya Funktsii, Ee Prilozheniya i Smezhnye Voprosy (Theory of Functions, Its Applications, and Related Issues), Kazan, **35**, 246–247 (2007).
20. Yu. V. Tret'yachenko, "A Selection Principle for Pointwise Bounded Sequences of Functions," Tr. Matem. Tsentra im. N. I. Lobachevskogo. Lobachevskie Chteniya, Kazan, **36**, 219–221 (2007).
21. J. Kelley, *General Topology* (Springer-Verlag, New York, 1975; Nauka, Moscow, 1981).
22. R. M. Dudley and R. Norvaiša, *Differentiability of Six Operators on Nonsmooth Functions and p-Variation* (Springer-Verlag, Berlin, 1999).
23. R. L. Jeffery, "Generalized Integrals with Respect to Functions of Bounded Variation," Canad. J. Math. **10** (4), 617–628 (1958).
24. Z. A. Chanturiya, "The Modulus of Variation of a Function and Its Application in the Theory of Fourier Series," Sov. Phjys. Dokl. **214** (1), 63–66 (1974).
25. V. V. Chistyakov, "A Selection Principle for Functions of a Real Variable," Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia **53** (1), 25–43 (2005).
26. V. V. Chistyakov, "The Optimal Form of Selection Principles for Functions of a Real Variable," J. Math. Anal. Appl. **310** (2), 609–625 (2005).
27. A. A. Tolstonogov, "Some Properties of the Space of Regular Functions," Matem. Zametki **35** (6), 803–812 (1984).
28. Yu. V. Tret'yachenko and V. V. Chistyakov, "Selection Principle for Pointwise Bounded Sequences of Functions," Matem. Zametki **84** (3), 428–439 (2008).
29. V. V. Chistyakov, C. Maniscalco, and Yu. V. Tret'yachenko, "Variants of a Selection Principle for Sequences of Regulated and Non-Regulated Functions," in: *Topics in Classical Analysis and Applications in Honor of Professor Daniel Waterman* (World Scientific, Singapore, 2008), pp. 45–72.

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