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THE AUTOMORPHISM GROUP OF A VARIETY WITH TORUS ACTION OF COMPLEXITY ONE

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Abstract. We consider a normal complete rational variety with a torus action of complexity one. In the main results, we determine the roots of the automorphism group and give an explicit description of the root system of its semisimple part. The results are applied to the study of almost homogeneous varieties. For example, we describe all almost homogeneous (possibly singular) del Pezzo $\mathbb{K}^*$-surfaces of Picard number one and all almost homogeneous (possibly singular) Fano threefolds of Picard number one having a reductive automorphism group with two-dimensional maximal torus.

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Contents

1. Introduction

For any complete rational algebraic variety X, the unit component $\mathrm{Aut}(X)^0$ of its automorphism group is linear algebraic and it is a natural desire to understand the structure of this group. Essential insight is provided by the roots, i.e., the eigenvalues of the adjoint representation of a maximal torus on the Lie algebra. Recall

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that if a linear algebraic group is reductive, then its set of roots forms a so-called root system and, up to coverings, determines the group. In the general case, the group is generated by its maximal torus and the additive one-parameter subgroups corresponding to the roots. Seminal work on the structure of automorphism groups has been done by Demazure [7] for the case of smooth complete toric varieties X. Here, the acting torus T of X is as well a maximal torus of $\mathrm{Aut}(X)^0$ and Demazure described the roots of $\mathrm{Aut}(X)^0$ with respect to T in terms of the defining fan of X; see [6], [5], [17], [19] for further development in this direction. Cox [6] presented an approach to the automorphism group of a toric variety via the homogeneous coordinate ring and thereby generalized Demazure's results to the simplicial case; see [4] for an application of homogeneous coordinates to the study of automorphism groups in the more general case of spherical varieties.

In the present paper, we go beyond the toric case in the sense that we consider normal complete rational varieties X coming with an effective torus action $T \times X \to X$ of complexity one, i.e., the dimension of T is one less than that of X; the simplest nontrivial examples are $\mathbb{K}^*$-surfaces, see [20], [21]. Our approach is based on the Cox ring $\mathcal{R}(X)$ and the starting point is the explicit description of $\mathcal{R}(X)$ in the complexity one case provided by [12], [13]; see also Section 3 for details. Generators and relations of $\mathcal{R}(X)$ as well as the grading by the divisor class group $\mathrm{Cl}(X)$ can be encoded in a sequence $A = a_0, \ldots, a_r$ of pairwise linearly independent vectors in $\mathbb{K}^2$ and an integral matrix

$$P = \begin{bmatrix} -l_0 & l_1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -l_0 & 0 & \ldots & l_r & 0 \\ d_0 & d_1 & \ldots & d_r & d' \end{bmatrix}$$

of size $(n+m) \times (r+s)$, where l_i are nonnegative integral vectors of length n_i, the d_i are $s \times n_i$ blocks, d' is an $s \times m$ block and the columns of P are pairwise different primitive vectors generating the column space $\mathbb{Q}^{r+s}$ as a convex cone. Conversely, the data A, P always define a Cox ring $\mathcal{R}(X) = R(A, P)$ of a complexity one T-variety X. The dimension of X equals $s+1$ and the acting torus T has $\mathbb{Z}^s$ as its character lattice. The matrix P determines the grading and the exponents occuring in the relations, whereas A is responsible for continuous aspects, i.e., coefficients in the relations.

The crucial concept for the investigation of the automorphism group $\mathrm{Aut}(X)$ are the *Demazure P-roots*, which we introduce in Definition 5.2. Roughly speaking, these are finitely many integral linear forms u on $\mathbb{Z}^{r+s}$ satisfying a couple of linear inequalities on the columns of P. In particular, given P, the Demazure P-roots can be easily determined. In contrast with the toric case, the Demazure P-roots are divided into two types. Firstly, there are "vertical" ones corresponding to root subgroups whose orbits are contained in the closures of generic torus orbits. Such Demazure P-roots are defined by free generators of the Cox ring and their description is analogous to the toric case. Secondly, there are "horizontal" Demazure P-roots corresponding to root subgroups whose orbits are transversal to generic torus orbits. Dealing with this type heavily involves the relations among generators of the Cox ring. Our first main result expresses the roots of $\mathrm{Aut}(X)^0$ and,

moreover, the approach shows how to obtain the corresponding root subgroups, see Theorem 5.5 and Corollary 5.11 for the precise formulation:

Theorem. *Let X be a nontoric normal complete rational variety with an effective torus action $T \times X \to X$ of complexity one. Then $\mathrm{Aut}(X)$ is a linear algebraic group having T as a maximal torus and the roots of $\mathrm{Aut}(X)$ with respect to T are precisely the $\mathbb{Z}^s$-parts of the Demazure P-roots.*

The basic idea of the proof is to relate the group $\mathrm{Aut}(X)$ to the group of graded automorphisms of the Cox ring. This is done in Section 2 more generally for arbitrary Mori dream spaces, i.e., normal complete varieties with a finitely generated Cox ring $\mathcal{R}(X)$. In this setting, the grading by the divisor class group $\mathrm{Cl}(X)$ defines an action of the characteristic quasitorus $H_X = \mathrm{Spec}\,\mathbb{K}[\mathrm{Cl}(X)]$ on the total coordinate space $\overline{X} = \mathrm{Spec}\,\mathcal{R}(X)$ and X is the quotient of an open subset $\widehat{X} \subseteq \overline{X}$ by the action of H_X. The group of $\mathrm{Cl}(X)$-graded automorphisms of $\mathcal{R}(X)$ is isomorphic to the group $\mathrm{Aut}(\overline{X},\, H_X)$ of H_X-equivariant automorphisms of $\overline{X}$. Moreover, the group $\mathrm{Bir}_2(X)$ of birational automorphisms of X defined on an open subset of X having complement of codimension at least two plays a role. Theorem 2.1 brings all groups together:

Theorem. *Let X be a (not necessarily rational) Mori dream space. Then there exists a commutative diagram of morphisms of linear algebraic groups where the rows are exact sequences and the upwards inclusions are of finite index:*

$$\begin{tikzcd}
1 \arrow[r] & H_X \arrow[r] \arrow[d, equal] & \mathrm{Aut}(\overline{X},\, H_X) \arrow[r] & \mathrm{Bir}_2(X) \arrow[r] & 1 \\
1 \arrow[r] & H_X \arrow[r] & \mathrm{Aut}(\widehat{X},\, H_X) \arrow[r] \arrow[u, hook] & \mathrm{Aut}(X) \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd}$$

This means in particular that the unit component of $\mathrm{Aut}(X)$ coincides with that of $\mathrm{Bir}_2(X)$, which in turn is determined by $\mathrm{Aut}(\overline{X},\, H_X)$, the group of graded automorphisms of the Cox ring. Coming back to rational varieties X with torus action of complexity one, the task then is a detailed study of the graded automorphism group of the rings $\mathcal{R}(X) = R(A,\, P)$. This is done in a purely algebraic way. The basic concepts are provided in Section 3. The key result is the description of the "primitive homogeneous locally nilpotent derivations" on $R(A,\, P)$ given in Theorem 4.4. The proof of the first main theorem in Section 5 then relates the Demazure P-roots via these derivations to the roots of the automorphism group $\mathrm{Aut}(X)$.

In Section 6 we apply our results to the study of almost homogeneous rational $\mathbb{K}^*$-surfaces X of Picard number one; here, almost homogeneous means that $\mathrm{Aut}(X)$ has an open orbit in X. It turns out that these surfaces are always (possibly singular) del Pezzo surfaces and, up to isomorphism, there are countably many of them, see Corollary 6.3. Finally in the case that X is log terminal with only one singularity, we give classifications for fixed Gorenstein index.

In Section 7, we investigate the semisimple part $\mathrm{Aut}(X)^{\mathrm{ss}} \subseteq \mathrm{Aut}(X)$ of the automorphism group; recall that the semisimple part of a linear algebraic group is a maximal connected semisimple subgroup. In the case of a toric variety, by Demazure's results, the semisimple part of the automorphism group has a root

system composed of systems A_i. Here comes a summarizing version of our second main result, which settles the complexity one case; see Theorem 7.2 for the detailed description.

Theorem. *Let X be a nontoric normal complete rational variety with an effective torus action $T \times X \to X$ of complexity one. The root system Φ of the semisimple part splits as $\Phi = \Phi^{\mathrm{vert}} \oplus \Phi^{\mathrm{hor}}$ with*

$$\Phi^{\mathrm{vert}} = \bigoplus_{\mathrm{Cl}(X)} A_{m_D - 1}, \quad \Phi^{\mathrm{hor}} \in \{\varnothing,\, A_1,\, A_2,\, A_3,\, A_1 \oplus A_1,\, B_2\},$$

where m_D is the number of invariant prime divisors in X with infinite T-isotropy that represent a given class $D \in \mathrm{Cl}(X)$. The number m_D as well as the possibilities for Φ^{hor} can be read off from the defining matrix P.

Examples and applications of this result are discussed in Section 8. The main results concern varieties of dimension three which are almost homogeneous under the action of a reductive group and additionally admit an effective action of a two-dimensional torus. In Proposition 8.4, we explicitly describe the Cox rings of these varieties. Moreover, in Proposition 8.6, we list all those having Picard number one and a reductive automorphism group; it turns out that any such variety is a Fano variety.

2. The Automorphism Group of a Mori Dream Space

Let X be a normal complete variety defined over an algebraically closed field $\mathbb{K}$ of characteristic zero with finitely generated divisor class group $\mathrm{Cl}(X)$ and Cox sheaf $\mathcal{R}$; we recall the definition below. If X is a Mori dream space, i.e., the Cox ring $\mathcal{R}(X) = \Gamma(X, \mathcal{R})$ is finitely generated as a $\mathbb{K}$-algebra, then we obtain the following picture

$$\begin{array}{c} \mathrm{Spec}_X \mathcal{R} = \widehat{X} \subseteq \overline{X} = \mathrm{Spec}\, \mathcal{R}(X), \\ {\scriptstyle /\!/ H_X} \big\downarrow \\ X \end{array}$$

where the *total coordinate space* $\overline{X}$ comes with an action of the *characteristic quasitorus* $H_X := \mathrm{Spec}\, \mathbb{K}[\mathrm{Cl}(X)]$, the *characteristic space* $\widehat{X}$, i.e. the relative spectrum of the Cox sheaf, occurs as an open H_X-invariant subset of $\overline{X}$ and the map $p_X \colon \widehat{X} \to X$ is a good quotient for the action of H_X.

We study automorphisms of X in terms of automorphisms of $\overline{X}$ and $\widehat{X}$. By an *H_X-equivariant automorphism* of $\overline{X}$ we mean a pair $(\varphi, \widetilde{\varphi})$, where $\varphi \colon \overline{X} \to \overline{X}$ is an automorphism of varieties and $\widetilde{\varphi} \colon H_X \to H_X$ is an automorphism of linear algebraic groups satisfying

$$\varphi(t \cdot x) = \widetilde{\varphi}(t) \cdot \varphi(x) \quad \text{for all } x \in \overline{X},\ \ t \in H_X.$$

We denote the group of H_X-equivariant automorphisms of $\overline{X}$ by $\mathrm{Aut}(\overline{X}, H_X)$. Analogously, one defines the group $\mathrm{Aut}(\widehat{X}, H_X)$ of H_X-equivariant automorphisms of $\widehat{X}$. A *weak automorphism* of X is a birational map $\varphi \colon X \to X$ which defines

an isomorphism of big open subsets, i.e., there are open subsets $U_1, U_2 \subseteq X$ with complement $X \setminus U_i$ of codimension at least two in X such that $\varphi_{|U_1} \colon U_1 \to U_2$ is a regular isomorphism. We denote the group of weak automorphisms of X by $\mathrm{Bir}_2(X)$.

Theorem 2.1. *Let X be a Mori dream space. Then there exists a commutative diagram of morphisms of linear algebraic groups where the rows are exact sequences and the upwards inclusions are of finite index:*

$$\begin{tikzcd}
1 \arrow[r] & H_X \arrow[r] \arrow[d, equal] & \mathrm{Aut}(\overline{X},\, H_X) \arrow[r] & \mathrm{Bir}_2(X) \arrow[r] & 1 \\
1 \arrow[r] & H_X \arrow[r] & \mathrm{Aut}(\widehat{X},\, H_X) \arrow[r] \arrow[u, hook] & \mathrm{Aut}(X) \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

Moreover, there is a big open subset $U \subseteq X$ with $\mathrm{Aut}(U) = \mathrm{Bir}_2(X)$ and the groups $\mathrm{Aut}(\overline{X},\, H_X)$, $\mathrm{Bir}_2(X)$, $\mathrm{Aut}(\widehat{X},\, H_X)$, $\mathrm{Aut}(X)$ act morphically on $\overline{X}$, U, $\widehat{X}$, X, respectively.

Our proof uses some ingredients from algebra, which we develop first. Let K be a finitely generated abelian group and consider a finitely generated integral $\mathbb{K}$-algebra

$$R \;=\; \bigoplus_{w \in K} R_w.$$

The *weight monoid* of R is the submonoid $S \subseteq K$ consisting of the elements $w \in K$ with $R_w \neq 0$. The *weight cone* of R is the convex cone $\omega \subseteq K_{\mathbb{Q}}$ in the rational vector space $K_{\mathbb{Q}} = K \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the weight monoid $S \subseteq K$. We say that the K-grading of R is *pointed* if the weight cone $\omega \subseteq K_{\mathbb{Q}}$ contains no line and $R_0 = \mathbb{K}$ holds. By an *automorphism* of the K-graded algebra R we mean a pair $(\psi,\, F)$, where $\psi \colon R \to R$ is an isomorphism of $\mathbb{K}$-algebras and $F \colon K \to K$ is an isomorphism such that $\psi(R_w) = R_{F(w)}$ holds for all $w \in K$. We denote the group of such automorphisms of R by $\mathrm{Aut}(R,\, K)$.

Proposition 2.2. *Let K be a finitely generated abelian group and $R = \bigoplus_{w \in K} R_w$ a finitely generated integral $\mathbb{K}$-algebra with $R^* = \mathbb{K}^*$. Suppose that the grading is pointed. Then $\mathrm{Aut}(R,\, K)$ is a linear algebraic group over $\mathbb{K}$ and R is a rational $\mathrm{Aut}(R,\, K)$-module.*

Proof. The idea is to represent the automorphism group $\mathrm{Aut}(R,\, K)$ as a closed subgroup of the linear automophism group of a suitable finite dimensional vector subspace $V^0 \subseteq R$. In the subsequent construction of V^0, we may assume that the weight cone ω generates $K_{\mathbb{Q}}$ as a vector space.

Consider the subgroup $\Gamma \subseteq \mathrm{Aut}(K)$ of $\mathbb{Z}$-module automorphisms $K \to K$ such that the induced linear isomorphism $K_{\mathbb{Q}} \to K_{\mathbb{Q}}$ leaves the weight cone $\omega \subseteq K_{\mathbb{Q}}$ invariant. By finite generation of R, the cone ω is polyhedral and thus Γ is finite. Let $f_1, \ldots, f_r \in R$ be homogeneous generators and denote by $w_i := \deg(f_i) \in K$ their degrees. Define a finite Γ-invariant subset and a vector subspace

$$S^0 := \Gamma \cdot \{w_1, \ldots, w_r\} \subseteq K, \quad V^0 := \bigoplus_{w \in S^0} R_w \subseteq R.$$

For every automorphism (ψ, F) of the graded algebra R, we have $F(S^0) = S^0$ and thus $\psi(V^0) = V^0$. Moreover, (ψ, F) is uniquely determined by its restriction on V^0. Consequently, we may regard the automorphism group $H := \mathrm{Aut}(R, K)$ as a subgroup of the general linear group $\mathrm{GL}(V^0)$. Note that every $g \in H$

(i) permutes the components R_w of the decomposition $V^0 = \bigoplus_{w \in S^0} R_w$,
(ii) satisfies $\sum_\nu a_\nu g(f_1)^{\nu_1} \cdots g(f_r)^{\nu_r} = 0$ for any relation $\sum_\nu a_\nu f_1^{\nu_1} \cdots f_r^{\nu_r} = 0$.

Obviously, these are algebraic conditions. Moreover, every $g \in \mathrm{GL}(V^0)$ satisfying the above conditions can be extended uniquely to an element of $\mathrm{Aut}(R, K)$ via

$$g\Big(\sum_\nu a_\nu f_1^{\nu_1} \cdots f_r^{\nu_r}\Big) := \sum_\nu a_\nu g(f_1)^{\nu_1} \cdots g(f_r)^{\nu_r}.$$

Thus, we see that $H \subseteq \mathrm{GL}(V^0)$ is precisely the closed subgroup defined by the above conditions (i) and (ii). In particular $H = \mathrm{Aut}(R, K)$ is linear algebraic. Moreover, the symmetric algebra SV^0 is a rational $\mathrm{GL}(V^0)$-module, hence SV^0 is a rational H-module for the algebraic subgroup H of $\mathrm{GL}(V^0)$, and so is its factor module R. □

Corollary 2.3. *Let K be a finitely generated abelian group and $R = \bigoplus_{w \in K} R_w$ a finitely generated integral $\mathbb{K}$-algebra with $R^* = \mathbb{K}^*$. Consider the corresponding action of $H := \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X} := \operatorname{Spec} R$. Then we have a canonical isomorphism*

$$\mathrm{Aut}(\overline{X}, H) \to \mathrm{Aut}(R, K), \quad (\varphi, \widetilde{\varphi}) \mapsto (\varphi^*, \widetilde{\varphi}^*),$$

where φ^* is the pullback of regular functions and $\widetilde{\varphi}^*$ the pullback of characters. If the K-grading is pointed, then $\mathrm{Aut}(\overline{X}, H)$ is a linear algebraic group acting morphically on $\overline{X}$.

We will also need details of the construction of the Cox sheaf $\mathcal{R}$ on X, which we briefly recall now. Denote by $c\colon \mathrm{WDiv}(X) \to \mathrm{Cl}(X)$ the map sending the Weil divisors to their classes, let $\mathrm{PDiv}(X) = \ker(c)$ denote the group of principal divisors and choose a character, i.e., a group homomorphism $\chi\colon \mathrm{PDiv}(X) \to \mathbb{K}(X)^*$ with

$$\mathrm{div}(\chi(E)) = E, \quad \text{for all } E \in \mathrm{PDiv}(X).$$

This can be done by prescribing χ suitably on a $\mathbb{Z}$-basis of $\mathrm{PDiv}(X)$. Consider the associated sheaf of divisorial algebras

$$\mathcal{S} := \bigoplus_{\mathrm{WDiv}(X)} \mathcal{S}_D, \quad \mathcal{S}_D := \mathcal{O}_X(D).$$

Denote by $\mathcal{I}$ the sheaf of ideals of $\mathcal{S}$ locally generated by the sections $1 - \chi(E)$, where 1 is homogeneous of degree zero, E runs through $\mathrm{PDiv}(X)$ and $\chi(E)$ is homogeneous of degree $-E$. The *Cox sheaf* associated to K and χ is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}(X)$-grading

$$\mathcal{R} = \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \quad \mathcal{R}_{[D]} := \pi\Big(\bigoplus_{D' \in c^{-1}([D])} \mathcal{S}_{D'}\Big),$$

where $\pi\colon \mathcal{S} \to \mathcal{R}$ denotes the projection. The Cox sheaf $\mathcal{R}$ is a quasicoherent sheaf of $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebras. The *Cox ring* is the ring $\mathcal{R}(X)$ of global sections of the Cox sheaf.

Proof of Theorem 2.1. We set $G_{\overline{X}} := \mathrm{Aut}(\overline{X}, H_X)$ for short. According to Corollary 2.3, the group $G_{\overline{X}}$ is linear algebraic and acts morphically on $\overline{X}$. Looking at the representations of H_X and $G_{\overline{X}}$ on $\Gamma(\overline{X}, \mathcal{O}) = \mathcal{R}(X)$ defined by the respective actions, we see that the canonical inclusion $H_X \to G_{\overline{X}}$ is a morphism of linear algebraic groups.

Next we construct the subset $U \subseteq X$ from the last part of the statement. Consider the translates $g \cdot \widehat{X}$, where $g \in G_{\overline{X}}$. Each of them admits a good quotient with a complete quotient space:

$$p_{X,g} \colon g \cdot \widehat{X} \to (g \cdot \widehat{X}) /\!\!/ H_X.$$

By [3], there are only finitely many open subsets of $\overline{X}$ with such a good quotient. In particular, the number of translates $g \cdot \widehat{X}$ is finite.

Let $W \subseteq X$ denote the maximal open subset such that the restricted quotient $\widehat{W} \to W$, where $\widehat{W} := p_X^{-1}(W)$, is geometric, i.e., has the H_X-orbits as its fibers. Then, for any $g \in G_{\overline{X}}$, the translate $g \cdot \widehat{W} \subseteq g \cdot \widehat{X}$ is the (unique) maximal open subset which is saturated with respect to the quotient map $p_{X,g}$ and defines a geometric quotient. Consider

$$\widehat{U} := \bigcap_{g \in G_{\overline{X}}} g \cdot \widehat{W} \subseteq \widehat{X}.$$

By the preceding considerations $\widehat{U}$ is open, and by construction it is $G_{\overline{X}}$-invariant and saturated with respect to p_X. By [1, Prop. 6.1.6] the set $\widehat{W}$ is big in $\overline{X}$. Consequently, also $\widehat{U}$ is big in $\overline{X}$. Thus, the (open) set $U := p_X(\widehat{U})$ is big in X. By the universal property of the geometric quotient, there is a unique morphical action of $G_{\overline{X}}$ on U making $p_X \colon \widehat{U} \to U$ equivariant. Thus, we have homomorphism of groups

$$G_{\overline{X}} \to \mathrm{Aut}(U) \subseteq \mathrm{Bir}_2(X).$$

We show that $\pi \colon G_{\overline{X}} \to \mathrm{Bir}_2(X)$ is surjective. Consider a weak automorphism $\varphi \colon X \to X$. The pullback defines an automorphism of the group of Weil divisors

$$\varphi^* \colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(X), \quad D \mapsto \varphi^* D.$$

As in the construction of the Cox sheaf, consider the sheaf of divisorial algebras $\mathcal{S} = \bigoplus \mathcal{S}_D$ associated to $\mathrm{WDiv}(X)$ and fix a character $\chi \colon \mathrm{PDiv}(X) \to \mathbb{K}(X)^*$ with $\mathrm{div}(\chi(E)) = E$ for any $E \in \mathrm{PDiv}(X)$. Then we obtain a homomorphism

$$\alpha \colon \mathrm{PDiv}(X) \to \mathbb{K}^*, \quad E \mapsto \frac{\varphi^*(\chi(E))}{\chi(\varphi^*(E))}.$$

We extend this to a homomorphism $\alpha \colon \mathrm{WDiv}(X) \to \mathbb{K}^*$ as follows. Write $\mathrm{Cl}(X)$ as a direct sum of a free part and cyclic groups $\Gamma_1, \ldots, \Gamma_s$ of order n_i. Take $D_1, \ldots, D_r \in \mathrm{WDiv}(X)$ such that the classes of $D_1, \ldots, D_s$ are generators for $\Gamma_1, \ldots, \Gamma_s$ and the remaining ones define a basis of the free part. Set

$$\alpha(D_i) := \sqrt[n_i]{\alpha(n_i D_i)} \text{ for } 1 \leqslant i \leqslant s, \quad \alpha(D_i) := 1 \text{ for } s+1 \leqslant i \leqslant r.$$

Then one directly checks that this extends α to a homomorphism $\mathrm{WDiv}(X) \to \mathbb{K}^*$. Using $\alpha(E)$ as a "correction term", we define an automorphism of the graded sheaf

$\mathcal{S}$ of divisorial algebras: for any open set $V \subseteq X$ we set

$$\varphi^*\colon \Gamma(V, \mathcal{S}_D) \to \Gamma(\varphi^{-1}(V), \mathcal{S}_{\varphi^*(D)}), \quad f \mapsto \alpha(D) f \circ \varphi.$$

By construction φ^* sends the ideal $\mathcal{I}$ arising from the character χ to itself. Consequently, φ^* descends to an automorphism (ψ, F) of the (graded) Cox sheaf $\mathcal{R}$; note that F is the pullback of divisor classes via φ. The degree zero part of ψ equals the usual pullback of regular functions on X via φ. Thus, the element in $\mathrm{Aut}(\overline{X}, H_X)$ defined by $\operatorname{Spec}\psi\colon \widehat{U} \to \widehat{U}$ maps to φ.

Clearly, H_X lies in the kernel of $\pi\colon G_{\overline{X}} \to \mathrm{Bir}_2(X)$. For the reverse inclusion, consider an element $g \in \ker(\pi)$. Then g is a pair $(\varphi, \widetilde{\varphi})$ and, by the construction of π, we have a commutative diagram

$$\begin{tikzcd} \widehat{U} \arrow[r, "\varphi"] \arrow[d, "p_X"'] & \widehat{U} \arrow[d, "p_X"] \\ U \arrow[r, "\mathrm{id}"'] & U. \end{tikzcd}$$

In particular, φ stabilizes all H_X-invariant divisors. It follows that the pullback φ^* on $\Gamma(\widehat{U}, \mathcal{O}) = \mathcal{R}(X)$ stabilizes the homogeneous components. Thus, for any homogeneous f of degree w, we have $\varphi^*(f) = \lambda(w) f$ with a homomorphism $\lambda\colon K \to \mathbb{K}^*$. Consequently $\varphi(x) = h \cdot x$ holds with an element $h \in H_X$. The statements concerning the upper sequence are verified.

Now, consider the lower sequence. Since $\widehat{X}$ is big in $\overline{X}$, every automorphism of $\widehat{X}$ extends to an automorphism of $\overline{X}$. We conclude that $\mathrm{Aut}(\widehat{X}, H_X)$ is the (closed) subgroup of $G_{\overline{X}}$ leaving the complement $\overline{X} \setminus \widehat{X}$ invariant. As seen before, the collection of translates $G_{\overline{X}} \cdot \widehat{X}$ is finite and thus the subgroup $\mathrm{Aut}(\widehat{X}, H_X)$ of $G_{\overline{X}}$ is of finite index. Moreover, lifting $\varphi \in \mathrm{Aut}(X)$ as before gives an element of $\mathrm{Aut}(\overline{X}, H_X)$ leaving $\widehat{X}$ invariant. Thus, $\mathrm{Aut}(\widehat{X}, H_X) \to \mathrm{Aut}(X)$ is surjective with kernel H_X. By the universal property of the qood quotient $\widehat{X} \to X$, the action of $\mathrm{Aut}(X)$ on X is morphical. □

Corollary 2.4. *The automorphism group* $\mathrm{Aut}(X)$ *of a Mori dream space* X *is linear algebraic and acts morphically on* X*.*

Corollary 2.5. *If two Mori dream spaces* X_1, X_2 *admit open subsets* $U_i \subseteq X_i$ *such that* $X_i \setminus U_i$ *is of codimension at least two in* X_i *and* U_1 *is isormorphic to* U_2*, then the unit components of* $\mathrm{Aut}(X_1)$ *and* $\mathrm{Aut}(X_2)$ *are isomorphic to each other.*

Let $\mathrm{CAut}(\overline{X}, H_X)$ denote the centralizer of H_X in the automorphism group $\mathrm{Aut}(\overline{X})$. Then $\mathrm{CAut}(\overline{X}, H_X)$ consists of all automorphisms $\varphi\colon \overline{X} \to \overline{X}$ satisfying

$$\varphi(t \cdot x) = t \cdot \varphi(x) \quad \text{for all } x \in \overline{X},\ t \in H_X.$$

In particular, we have $\mathrm{CAut}(\overline{X}, H_X) \subseteq \mathrm{Aut}(\overline{X}, H_X)$. The group $\mathrm{CAut}(\overline{X}, H_X)$ may be used to detect the unit component $\mathrm{Aut}(X)^0$ of the automorphism group of X.

Corollary 2.6. *Let X be a Mori dream space. Then there is an exact sequence of linear algebraic groups*

$$1 \longrightarrow H_X \longrightarrow \mathrm{CAut}(\overline{X},\, H_X)^0 \longrightarrow \mathrm{Aut}(X)^0 \longrightarrow 1.$$

Proof. According to [24, Cor. 2.3], the group $\mathrm{CAut}(\overline{X},\, H_X)^0$ leaves $\widehat{X}$ invariant. Thus, we have $\mathrm{CAut}(\overline{X},\, H_X)^0 \subseteq \mathrm{Aut}(\widehat{X},\, H_X)$ and the sequence is well defined. Moreover, for any $\varphi \in \mathrm{Aut}(X)^0$, the pullback $\varphi^*\colon \mathrm{Cl}(X) \to \mathrm{Cl}(X)$ is the identity. Consequently, φ lifts to an element of $\mathrm{CAut}(\overline{X},\, H_X)$. Exactness of the sequence thus follows by dimension reasons. □

Corollary 2.7. *Let X be a Mori dream space. Then, for any closed subgroup $F \subseteq \mathrm{Aut}(X)^0$, there is a closed subgroup $F' \subseteq \mathrm{CAut}(\overline{X},\, H_X)^0$ such that the induced map $F' \to F$ is an epimorphism with finite kernel.*

Corollary 2.8. *Let X be a Mori dream space such that the group $\mathrm{CAut}(\overline{X},\, H_X)$ is connected, e.g. a toric variety. Then there is an exact sequence of linear algebraic groups*

$$1 \longrightarrow H_X \longrightarrow \mathrm{CAut}(\overline{X},\, H_X) \longrightarrow \mathrm{Aut}(X)^0 \longrightarrow 1.$$

Example 2.9. Consider the nondegenerate quadric X in the projective space $\mathbb{P}_{n+1}$, where $n \geqslant 4$ is even. Then the Cox ring of X is the $\mathbb{Z}$-graded ring

$$\mathcal{R}(X) = \mathbb{K}[T_0,\, \ldots,\, T_{n+1}] \,/\, \langle T_0^2 + \ldots + T_{n+1}^2 \rangle, \quad \deg(T_0) = \ldots = \deg(T_{n+1}) = 1.$$

The characteristic quasitorus is $H_X = \mathbb{K}^*$. Moreover, for the equivariant automorphisms and the centralizer of H_X we obtain

$$\mathrm{Aut}(\overline{X},\, H_X) = \mathrm{CAut}(\overline{X},\, H_X) = \mathbb{K}^* E_{n+2} \cdot \mathrm{O}_{n+2}.$$

Thus, $\mathrm{CAut}(\overline{X},\, H_X)$ has two connected components. Note that for $n = 4$, the quadric X comes with a torus action of complexity one.

3. Rings with a Factorial Grading of Complexity One

Here we recall the necessary constructions and results on factorially graded rings of complexity one and Cox rings of varieties with a torus action of complexity one from [12]. The main result of this section is Proposition 3.5, which describes the dimension of the homogeneous components in terms of the (common) degree of the relations. As before, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Let K be an abelian group and $R = \bigoplus_K R_w$ a K-graded algebra. The grading is called *effective* if the weight monoid S of R generates K as a group. Moreover, we say that the grading is of *complexity one*, if it is effective and $\dim(K_{\mathbb{Q}})$ equals $\dim(R) - 1$. By a *K-prime* element of R we mean a homogeneous nonzero nonunit $f \in R$ such that $f \,|\, gh$ with homogeneous $g,\, h \in R$ implies $f \,|\, g$ or $f \,|\, h$. We say that R is *factorially K-graded* if every nonzero homogeneous nonunit of R is a product of K-primes.

Construction 3.1. See [12, Section 1]. Fix $r \in \mathbb{Z}_{\geqslant 1}$, a sequence $n_0,\, \ldots,\, n_r \in \mathbb{Z}_{\geqslant 1}$, set $n := n_0 + \ldots + n_r$ and let $m \in \mathbb{Z}_{\geqslant 0}$. The input data are

- a matrix $A := [a_0, \ldots, a_r]$ with pairwise linearly independent column vectors $a_0, \ldots, a_r \in \mathbb{K}^2$,
- an integral $r \times (n+m)$ block matrix $P_0 = (L_0, 0)$, where L_0 is a $r \times n$ matrix build from tuples $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geqslant 1}$ as follows

$$L_0 = \begin{bmatrix} -l_0 & l_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ -l_0 & 0 & \ldots & l_r \end{bmatrix}.$$

Consider the polynomial ring $\mathbb{K}[T_{ij}, S_k]$ in the variables T_{ij}, where $0 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n_i$ and S_k, where $1 \leqslant k \leqslant m$. For every $0 \leqslant i \leqslant r$, define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \leqslant i_1 < i_2 < i_3 \leqslant r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}.$$

Let P_0^* denote the transpose of P_0. We introduce a grading on $\mathbb{K}[T_{ij}, S_k]$ by the factor group $K_0 := \mathbb{Z}^{n+m}/\mathrm{im}(\mathrm{P}_0^*)$. Let $Q_0 \colon \mathbb{Z}^{n+m} \to K_0$ be the projection and set

$$\deg(T_{ij}) := w_{ij} := Q_0(e_{ij}), \quad \deg(S_k) := w_k := Q_0(e_k),$$

where $e_{ij} \in \mathbb{Z}^{n+m}$, for $0 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n_i$, and $e_k \in \mathbb{Z}^{n+m}$, for $1 \leqslant k \leqslant m$, are the canonical basis vectors. Note that all the g_I are K_0-homogeneous of degree

$$\mu := l_{01}w_{01} + \ldots + l_{0n_0}w_{0n_0} = \ldots = l_{r1}w_{r1} + \ldots + l_{rn_r}w_{rn_r} \in K_0.$$

In particular, the trinomials g_I generate a K_0-homogeneous ideal and thus we obtain a K_0-graded factor algebra

$$R(A, P_0) := \mathbb{K}[T_{ij}, S_k] \,/\, \langle g_I;\ I \in \mathfrak{I} \rangle.$$

Theorem 3.2 (See [12, Theorems 1.1 and 1.3])**.** *With the notation of Construction 3.1, the following statements hold.*

(i) *The K_0-grading of ring $R(A, P_0)$ is effective, pointed, factorial and of complexity one.*
(ii) *The variables T_{ij} and S_k define a system of pairwise nonassociated K_0-prime generators of $R(A, P_0)$.*
(iii) *Every finitely generated normal $\mathbb{K}$-algebra with an effective, pointed, factorial grading of complexity one is isomorphic to some $R(A, P_0)$.*

Note that in the case $r = 1$, there are no relations and the theorem thus treats the effective, pointed gradings of complexity one of the polynomial ring.

Example 3.3 (The E_6-singular cubic I)**.** Let $r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$, $m = 0$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad P_0 = L_0 = \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \end{bmatrix}.$$

Then we have exactly one triple in $\mathfrak{I}$, namely $I = (0, 1, 2)$, and, as a ring, $R(A, P_0)$ is given by

$$R(A, P_0) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] \,/\, \langle T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2 \rangle.$$

The grading group $K_0 = \mathbb{Z}^4/\mathrm{im}(P_0^*)$ is isomorphic to $\mathbb{Z}^2$ and the grading can be given explicitly via

$$\deg(T_{01}) = \begin{pmatrix} -3 \\ 3 \end{pmatrix}, \quad \deg(T_{02}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \deg(T_{11}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \deg(T_{21}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Recall that for any integral ring $R = \bigoplus_K R_w$ graded by an abelian group K, one has the subfield of degree zero fractions inside the field of fractions:

$$Q(R)_0 = \left\{ \frac{f}{g};\ f, g \in R \text{ homogeneous}, g \neq 0, \deg(f) = \deg(g) \right\} \subseteq Q(R).$$

Proposition 3.4. *Take any i, j with $i \neq j$ and $0 \leqslant i, j \leqslant r$. Then the field of degree zero fractions of the ring $R(A, P_0)$ is the rational function field*

$$Q(R(A, P_0))_0 = \mathbb{K}\left(\frac{T_j^{l_j}}{T_i^{l_i}}\right).$$

Proof. It suffices to treat the case $m = 0$. Let $F = \prod T_{ij}$ be the product of all variables. Then $\mathbb{T}^n = \mathbb{K}_F^n$ is the n-torus and P_0 defines an epimorphism having the quasitorus $H_0 := \operatorname{Spec} \mathbb{K}[K_0]$ as its kernel

$$\pi \colon \mathbb{T}^n \to \mathbb{T}^r, \quad (t_{ij}) \mapsto \left(\frac{t_1^{l_1}}{t_0^{l_0}}, \ldots, \frac{t_r^{l_r}}{t_0^{l_0}} \right).$$

Set $\overline{X} := \operatorname{Spec} R(A, P_0)$. Then $\pi(\overline{X}_F) = \overline{X}_F/H_0$ is a curve defined by affine linear equations in the coordinates of $\mathbb{T}^r$ and thus rational. The assertion follows. □

The following observation shows that the common degree $\mu = \deg(g_I)$ of the relations generalizes the "remarkable weight" introduced by Panyushev [22] in the factorial case. Recall that the weight monoid $S_0 \subseteq K_0$ consists of all $w \in K_0$ admitting a nonzero homogeneous element.

Proposition 3.5. *Consider the K_0-graded ring $R := R(A, P_0)$ and the degree $\mu = \deg(g_I)$ of the relations as defined in Construction 3.1. For $w \in S_0$ let $s_w \in \mathbb{Z}_{\geqslant 0}$ be the unique number with $w - s_w\mu \in S_0$ and $w - (s_w + 1)\mu \notin S_0$. Then we have*

$$\dim(R_w) = s_w + 1 \quad \textit{for all } w \in S_0.$$

The element $\mu \in K_0$ is uniquely determined by this property. We have $\dim(R_\mu) = 2$ and any two nonproportional elements in R_μ are coprime. Moreover, any $w \in S_0$ with $w - \mu \notin S_0$ satisfies $\dim(R_w) = 1$.

Proof. According to Proposition 3.4, the field $Q(R)_0$ of degree zero fractions is the field of rational functions in p_1/p_0, where $p_0 := T_0^{l_0}$ and $p_1 := T_1^{l_1}$ are coprime and of degree μ. Moreover, by the structure of the relations g_I, we have $\dim(R_\mu) = 2$.

Now, consider $w \in S_0$. If we have $\dim(R_w) = 1$, then $\dim(R_\mu) = 2$ implies $s_w = 0$ and the assertion follows in this case. Suppose that we have $\dim(R_w) > 1$. Then

we find two nonproportional elements f_0, $f_1 \in R_w$ and two coprime homogeneous polynomials F_0, F_1 of a common degree $s > 0$ such that

$$\frac{f_1}{f_0} = \frac{F_1(p_0, p_1)}{F_0(p_0, p_1)}.$$

Observe that $F_1(p_0, p_1)$ must divide f_1. This implies $w - s\mu \in S_0$. Repeating the procedure with $w - s\mu$ and so on, we finally arrive at a weight $\widetilde{w} = w - s_w\mu$ with $\dim(R_{\widetilde{w}}) = 1$. Moreover, by the procedure, any element of R_w is of the form $hF(p_0, p_1)$ with $0 \neq h \in R_{\widetilde{w}}$ and a homogeneous polynomial F of degree s_w. The assertion follows. □

Corollary 3.6. *Assume that we have $r \geqslant 2$ and that $l_{i1} + \ldots + l_{in_i} \geqslant 2$ holds for all i.*

(i) *The K_0-homogeneous components $R(A, P_0)_{w_{ij}}$ and $R(A, P_0)_{w_k}$ of the generators T_{ij} and S_k are all of dimension one.*
(ii) *Consider $w = w_{i_1j_1} + \ldots + w_{i_tj_t} \in K_0$, where $1 \leqslant t \leqslant r$. If $l_{i_kj_k} = 1$ holds for $1 \leqslant k \leqslant t-1$, then $R(A, P_0)_w$ is of dimension one.*

Proof. According to Proposition 3.5, we have to show that the shifts of the weights w_{ij}, w_k and w by $-\mu$ do not belong to the weight monoid S_0. For w_k this is clear. For w_{ij}, the assumption gives

$$w_{ij} - \mu = -(l_{ij} - 1)w_{ij} - \sum_{b \neq j} l_{ib}w_{ib} \notin S_0.$$

Let us consider the weight w of (ii). Since $t \leqslant r$ holds, there is an index $0 \leqslant i_0 \leqslant r$ with $i_0 \neq i_k$ for $k = 1, \ldots, t$. We have $w - \mu = Q_0(e)$ for

$$e := e_{i_1j_1} + \ldots + e_{i_tj_t} - (l_{i_01}e_{i_01} + \ldots + l_{i_0n_{i_0}}e_{i_0n_{i_0}}) \in \mathbb{Z}^{n+m}.$$

By the assumptions, we find $1 \leqslant c_i \leqslant n_i$, where $0 \leqslant i \leqslant r$, such that $c_{i_k} \neq j_k$ holds for $1 \leqslant k \leqslant t-1$ and $c_{i_t} \neq j_t$ or $l_{i_tc_{i_t}} \geqslant 2$. Then the linear form

$$l_{0c_0}^{-1}e_{0c_0}^* + \ldots + l_{rc_r}^{-1}e_{rc_r}^*$$

vanishes along the kernel of $Q_0\colon \mathbb{Q}^{n+m} \to (K_0)_{\mathbb{Q}}$ and thus induces a linear form on $(K_0)_{\mathbb{Q}}$ which separates $w - \mu = Q_0(e)$ from the weight cone. □

We turn to Cox rings of varieties with a complexity one torus action. They are obtained by suitably downgrading the rings $R(A, P_0)$ as follows.

Construction 3.7. Fix $r \in \mathbb{Z}_{\geqslant 1}$, a sequence $n_0, \ldots, n_r \in \mathbb{Z}_{\geqslant 1}$, set $n := n_0 + \ldots + n_r$, and fix integers $m \in \mathbb{Z}_{\geqslant 0}$ and $0 < s < n + m - r$. The input data are

- a matrix $A := [a_0, \ldots, a_r]$ with pairwise linearly independent column vectors $a_0, \ldots, a_r \in \mathbb{K}^2$,
- an *integral block matrix* P of size $(r+s) \times (n+m)$ the columns of which are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a cone:

$$P = \begin{pmatrix} L_0 & 0 \\ d & d' \end{pmatrix},$$

  where d is an $(s \times n)$-matrix, d' an $(s \times m)$-matrix and L_0 an $(r \times n)$-matrix build from tuples $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}_{\geqslant 1}^{n_i}$ as in Construction 3.1.

Let P^* denote the transpose of P, consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(\mathrm{P}^*)$ and the projection $Q\colon \mathbb{Z}^{n+m} \to K$. We define a K-grading on $\mathbb{K}[T_{ij},\, S_k]$ by setting

$$\deg(T_{ij}) := Q(e_{ij}), \quad \deg(S_k) := Q(e_k).$$

The trinomials g_I of Construction 3.1 are K-homogeneous, all of the same degree. In particular, we obtain a K-graded factor ring

$$R(A,\, P) := \mathbb{K}[T_{ij},\, S_k;\ 0 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n_i,\, 1 \leqslant k \leqslant m] \,/\, \langle g_I;\ I \in \mathfrak{I}\rangle.$$

Theorem 3.8 (See [12, Theorem 1.4])**.** *With the notation of Construction 3.7, the following statements hold.*

(i) *The K-grading of the ring $R(A,\, P)$ is factorial, pointed and almost free, i.e., K is generated by any $n+m-1$ of the* $\deg(T_{ij})$, $\deg(S_k)$.

(ii) *The variables T_{ij} and S_k define a system of pairwise nonassociated K-prime generators of $R(A,\, P)$.*

Remark 3.9. As rings $R(A,\, P_0)$ and $R(A,\, P)$ coincide but the K_0-grading is finer than the K-grading. The downgrading map $K_0 \to K$ fits into the following commutative diagram built from exact sequences

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathbb{Z}^s \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}^r \arrow[r, "P_0^*"] \arrow[d] & \mathbb{Z}^{n+m} \arrow[r, "Q_0"] \arrow[d, equal] & K_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}^{r+s} \arrow[r, "P^*"'] \arrow[d] & \mathbb{Z}^{n+m} \arrow[r, "Q"'] & K \arrow[r] \arrow[d] & 0. \\
 & \mathbb{Z}^s \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

The snake lemma [15, Sec. III.9] allows us to identify the direct factor $\mathbb{Z}^s$ of $\mathbb{Z}^{r+s}$ with the kernel of the downgrading map $K_0 \to K$. Note that for the quasitori T, H_0 and H associated to abelian groups $\mathbb{Z}^s$, K_0 and K we have $T = H_0/H$.

Construction 3.10. Consider a ring $R(A,\, P)$ with its K-grading and the finer K_0-grading. Then the quasitori $H = \operatorname{Spec}\mathbb{K}[K]$ and $H_0 := \operatorname{Spec}\mathbb{K}[K_0]$ act on $\overline{X} := \operatorname{Spec} R(A,\, P)$. Let $\widehat{X} \subseteq \overline{X}$ be a big H_0-invariant open subset with a good quotient

$$p\colon \widehat{X} \to X = \widehat{X}/\!\!/H$$

such that X is complete and for some open set $U \subseteq X$, the inverse image $p^{-1}(U) \subseteq \overline{X}$ is big and H acts freely on U. Then X is a Mori dream space of dimension $s+1$ with divisor class group $\mathrm{Cl}(X) \cong K$ and Cox ring $\mathcal{R}(X) \cong R(A,\, P)$. Moreover, X comes with an induced effective action of the s-dimensional torus $T := H_0/H$.

Remark 3.11. Let X be a T-variety arising from data A and P via Construction 3.10. Then every $T_{ij} \in R(A,\, P)$ defines an invariant prime divisor $D_{ij} = \overline{T \cdot x_{ij}}$ in X such that the isotropy group $T_{x_{ij}}$ is cyclic of order l_{ij} and the gcd of the entries of $d_{ij} \in \mathbb{Z}^s$ represents the nonzero weight of the cotangent presentation of $T_{x_{ij}}$ at x_{ij}. Moreover, each S_k defines an invariant prime divisor $E_k \subseteq X$ such that the one-parameter subgroup $\mathbb{K}^* \to T$ corresponding to $d'_k \subseteq \mathbb{Z}^s$ acts trivially on E_k.

Theorem 3.12. *Let X be an n-dimensional complete normal rational variety with an effective action of an $(n-1)$-dimensional torus S. Then X is equivariantly isomorphic to a T-variety arising from data $(A,\, P)$ as in Construction 3.10.*

Proof. We may assume that X is not a toric variety. According to [12, Theorem 1.5], the $\mathrm{Cl}(X)$-graded Cox ring of X is isomorphic to a K-graded ring $R(A,\, P)$. Thus, in the notation of Construction 3.10, there is a big H-invariant open subset $\widehat{X}$ of $\overline{X}$ with $X \cong \widehat{X}/\!\!/H$. Applying [24, Cor. 2.3] to a subtorus $T_0 \subseteq H_0$ projecting onto $T = H_0/H$, we see that $\widehat{X}$ is even invariant under H_0. Thus, T acts on X. Since the T-action is conjugate in $\mathrm{Aut}(X)$ to the given S-action on X, the assertion follows. □

Example 3.13 (The E_6-singular cubic II)**.** Let $r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$, $m = 0$, $s = 1$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix}.$$

Then, as remarked before, we have exactly one triple $I = (0,\, 1,\, 2)$ and, as a ring, $R(A,\, P)$ is given by

$$R(A,\, P) = \mathbb{K}[T_{01},\, T_{02},\, T_{11},\, T_{21}] \,/\, \langle T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2 \rangle.$$

The grading group $K = \mathbb{Z}^4/\mathrm{im}(P^*)$ is isomorphic to $\mathbb{Z}$ and the grading can be given explicitly via

$$\deg(T_{01}) = 3, \quad \deg(T_{02}) = 1, \quad \deg(T_{11}) = 2, \quad \deg(T_{21}) = 3.$$

As shown for example in [10], see also [11, Example 3.7], the ring $R(A,\, P)$ is the Cox ring of the E_6-singular cubic surface in the projective space given by

$$X = V(z_1 z_2^2 + z_2 z_0^2 + z_3^3) \subseteq \mathbb{P}_3.$$

4. Primitive Locally Nilpotent Derivations

Here, we investigate the homogeneous locally nilpotent derivations of the K_0-graded algebra $R(A,\, P_0)$. The description of the "primitive" ones given in Theorem 4.4 is the central algebraic tool for our study of automorphism groups. As before, $\mathbb{K}$ is an algebraically closed field of characteristic zero.

Let us briefly recall the necessary background. We consider derivations on an integral $\mathbb{K}$-algebra R, that is, $\mathbb{K}$-linear maps $\delta\colon R \to R$ satisfying the Leibniz rule

$$\delta(fg) = \delta(f)g + f\delta(g).$$

Any such $\delta\colon R \to R$ extends uniquely to a derivation $\delta\colon Q(R) \to Q(R)$ of the quotient field. Recall that a derivation $\delta\colon R \to R$ is said to be *locally nilpotent* if for every $f \in R$ there is an $n \in \mathbb{N}$ with $\delta^n(f) = 0$. Now suppose that R is graded by a finitely generated abelian group:

$$R = \bigoplus_{w \in K} R_w.$$

A derivation $\delta\colon R \to R$ is called *homogeneous* if for every $w \in K$ there is a $w' \in K$ with $\delta(R_w) \subseteq R_{w'}$. Any homogeneous derivation $\delta\colon R \to R$ has a *degree* $\deg(\delta) \in K$ satisfying $\delta(R_w) \subseteq R_{w+\deg(\delta)}$ for all $w \in K$.

Definition 4.1. Let K be a finitely generated abelian group, $R = \bigoplus_K R_w$ a K-graded $\mathbb{K}$-algebra and $Q(R)_0 \subseteq Q(R)$ the subfield of all fractions f/g of homogeneous elements $f, g \in R$ with $\deg(f) = \deg(g)$.

(i) We call a homogeneous derivation $\delta\colon R \to R$ *primitive* if $\deg(\delta)$ does not lie in the weight cone $\omega \subseteq K_{\mathbb{Q}}$ of R.

(ii) We say that a homogeneous derivation $\delta\colon R \to R$ is of *vertical type* if $\delta(Q(R)_0) = 0$ holds and of *horizontal type* otherwise.

Remark 4.2. Every primitive homogeneous derivation is locally nilpotent. Indeed, for any weight w of the weight monoid S_0 there exists a positive integer k such that $w + k\deg(\delta) \notin S_0$, hence $\delta(R_w) = 0$.

Construction 4.3. Notation as in Construction 3.1. We define derivations of the K_0-graded algebra $R(A, P_0)$ constructed there. The input data are

- a sequence $C = (c_0, \ldots, c_r)$ with $1 \leqslant c_i \leqslant n_i$,
- a vector $\beta \in \mathbb{K}^{r+1}$ lying in the row space of the matrix $[a_0, \ldots, a_r]$.

Note that for $0 \neq \beta$ as above either all entries differ from zero or there is a unique i_0 with $\beta_{i_0} = 0$. According to these cases, we put further conditions and define:

(i) if all entries $\beta_0, \ldots, \beta_r$ differ from zero and there is at most one i_1 with $l_{i_1 c_{i_1}} > 1$, then we set

$$\delta_{C,\beta}(T_{ij}) := \begin{cases} \beta_i \prod\limits_{k \neq i} \dfrac{\partial T_k^{l_k}}{\partial T_{kc_k}}, & j = c_i, \\ 0, & j \neq c_i, \end{cases}$$

$$\delta_{C,\beta}(S_k) := 0 \quad \text{for } k = 1, \ldots, m,$$

(ii) if $\beta_{i_0} = 0$ is the unique zero entry of β and there is at most one i_1 with $i_1 \neq i_0$ and $l_{i_1 c_{i_1}} > 1$, then we set

$$\delta_{C,\beta}(T_{ij}) := \begin{cases} \beta_i \prod\limits_{k \neq i, i_0} \dfrac{\partial T_k^{l_k}}{\partial T_{kc_k}}, & j = c_i, \\ 0, & j \neq c_i, \end{cases}$$

$$\delta_{C,\beta}(S_k) := 0 \quad \text{for } k = 1, \ldots, m.$$

These assignments define K_0-homogeneous primitive derivations $\delta_{C,\beta}\colon R(A,\, P_0) \to R(A,\, P_0)$ of degree

$$\deg(\delta_{C,\beta}) = \begin{cases} r\mu - \sum_k \deg(T_{kc_k}), & \text{in case (i)}, \\ (r-1)\mu - \sum_{k \neq i_0} \deg(T_{kc_k}), & \text{in case (ii)}. \end{cases}$$

Proof. The assignments (i) and (ii) on the variables define a priori derivations of the polynomial ring $\mathbb{K}[T_{ij},\, S_k]$. Recall from Construction 3.1 that $R(A,\, P_0)$ is the quotient of $\mathbb{K}[T_{ij},\, S_k]$ by the ideal generated by all

$$g_I = \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix},$$

where $I = (i_1,\, i_2,\, i_3)$. Since the vector β lies in the row space of $[a_0,\, \ldots,\, a_r]$, we see that $\delta_{C,\beta}$ sends every trinomial g_I to zero and thus descends to a well defined derivation of $R(A,\, P_0)$.

We check that $\delta_{C,\beta}$ is homogeneous. Obviously, every $\delta_{C,\beta}(T_{ij})$ is a K_0-homogeneous element of $\mathbb{K}[T_{ij}]$. Moreover, with the degree μ of the relations g_I, we have

$$\deg(\delta_{C,\beta}(T_{ij})) - \deg(T_{ij}) = \begin{cases} r\mu - \sum_k \deg(T_{kc_k}), & \text{in case (i)}, \\ (r-1)\mu - \sum_{k \neq i_0} \deg(T_{kc_k}), & \text{in case (ii)}. \end{cases}$$

In particular, the left hand side does not depend on $(i,\, j)$. We conclude that $\delta_{C,\beta}$ is homogeneous of degree $\deg(\delta_{C,\beta}(T_{ij})) - \deg(T_{ij})$.

For primitivity, we have to show that the degree of $\delta_{C,\beta}$ does not lie in the weight cone of $R(A,\, P_0)$. We exemplarily treat case (i), where we may assume that $i_1 = 0$ holds. As seen before, the degree of $\delta_{C,\beta}$ is represented by the vector

$$v_{C,\beta} := -e_{0c_0} + \sum_{j \neq c_1} l_{1j} e_{1j} + \ldots + \sum_{j \neq c_r} l_{rj} e_{rj} \in \mathbb{Z}^{n+m}.$$

Thus, we look for a linear form on $\mathbb{Q}^{n+m}$ separating this vector from the orthant $\operatorname{cone}(e_{ij},\, e_k)$ and vanishing along the kernel of $\mathbb{Q}^{n+m} \to (K_0)_{\mathbb{Q}}$, i.e., the linear subspace spanned by the columns of P_0^*. For example, we may take

$$l_{0c_0}^{-1} e_{0c_0}^* + l_{1c_1}^{-1} e_{1c_1}^* + \ldots + l_{rc_r}^{-1} e_{rc_r}^*. \qquad \square$$

Theorem 4.4. *Let $\delta\colon R(A,\, P_0) \to R(A,\, P_0)$ be a nontrivial primitive K_0-homogeneous derivation.*

(i) *If δ is of vertical type, then $\delta(T_{ij}) = 0$ holds for all $i,\, j$ and there is a k_0 such that $\delta(S_{k_0})$ does not depend on S_{k_0} and $\delta(S_k) = 0$ holds for all $k \neq k_0$.*

(ii) *If δ is of horizontal type, then we have $\delta = h\delta_{C,\beta}$, where $\delta_{C,\beta}$ is as in Construction 4.3 and h is K_0-homogeneous with $h \in \ker(\delta_{C,\beta})$.*

In the proof of this theorem we will make frequently use of the following facts; the statements of the first Lemma occur in Freudenburg's book, see [8, Principles 1, 5 and 7, Corollary 1.20].

Lemma 4.5. *Let R be an integral $\mathbb{K}$-algebra, $\delta\colon R \to R$ a locally nilpotent derivation and let $f, g \in R$.*

(i) *If $fg \in \ker(\delta)$ holds, then $f, g \in \ker(\delta)$ holds.*
(ii) *If $\delta(f) = fg$ holds, then $\delta(f) = 0$ holds.*
(iii) *The derivation $f\delta$ is locally nilpotent if and only if $f \in \ker(\delta)$ holds.*
(iv) *If $g \,|\, \delta(f)$ and $f \,|\, \delta(g)$, then $\delta(f) = 0$ or $\delta(g) = 0$.*

Lemma 4.6. *Let $\delta\colon R(A, P_0) \to R(A, P_0)$ be a primitive K_0-homogeneous derivation induced by a K_0-homogeneous derivation $\widehat{\delta}\colon \mathbb{K}[T_{ij}, S_k] \to \mathbb{K}[T_{ij}, S_k]$. Then $\widehat{\delta}(g_I) = 0$ holds for all relations g_I.*

Proof. Clearly, we have $\widehat{\delta}(\mathfrak{a}) \subseteq \mathfrak{a}$ for the ideal $\mathfrak{a} \subseteq \mathbb{K}[T_{ij}, S_k]$ generated by the g_I. Recall that all g_I are of the same degree μ. By primitivity, $\deg(\delta) = \deg(\widehat{\delta})$ is not in the weight cone. Thus, $\mathbb{K}[T_{ij}, S_k]_{\mu+\deg(\widehat{\delta})} \cap \mathfrak{a} = \{0\}$ holds. This implies $\widehat{\delta}(g_I) = 0$. □

Proof of Theorem 4.4. Suppose that δ is of vertical type. Then $\delta(T_i^{l_i}/T_s^{l_s}) = 0$ holds for any two $0 \leqslant i < s \leqslant r$. By the Leibniz rule, this implies

$$\delta(T_i^{l_i})T_s^{l_s} = T_i^{l_i}\delta(T_s^{l_s}).$$

We conclude that $T_i^{l_i}$ divides $\delta(T_i^{l_i})$ and $T_s^{l_s}$ divides $\delta(T_s^{l_s})$. By Lemma 4.5 (ii), this implies $\delta(T_i^{l_i}) = \delta(T_s^{l_s}) = 0$. Using Lemma 4.5 (i), we obtain $\delta(T_{ij}) = 0$ for all variables T_{ij}. Since δ is nontrivial, we should have $\delta(S_{k_0}) \neq 0$ at least for one k_0. Consider the basis $e_k = \deg(S_k)$ of $\mathbb{Z}^m$, where $k = 1, \ldots, m$, and write

$$\deg(\delta) = w' + \sum_{k=1}^{m} b_k e_k, \quad \text{where } w' \in K_0 \text{ and } b_k \in \mathbb{Z}.$$

Then $\deg(\delta(S_{k_0})) = w' + \sum_{k\neq k_0} b_k e_k + (b_{k_0} + 1)e_{k_0}$. By Lemma 4.5, the variable S_{k_0} does not divide $\delta(S_{k_0})$. This and the condition $\delta(S_{k_0}) \neq 0$ imply $b_{k_0} = -1$ and $b_k \geqslant 0$ for $k \neq k_0$. This proves that $\delta(S_k) = 0$ for all $k \neq k_0$ and $\delta(S_{k_0})$ is K_0-homogeneous and does not depend on S_{k_0}.

Now suppose that δ is of horizontal type. Then there exists a variable T_{ij} with $\delta(T_{ij}) \neq 0$. Write

$$\deg(\delta(T_{ij})) = \deg(T_{ij}) + w' + \sum_{k=1}^{m} b_k e_k.$$

Then all coefficients b_k are nonnegative and consequently we obtain $\delta(S_k) = 0$ for $k = 1, \ldots, m$.

We show that for any $T_i^{l_i}$ there is at most one variable T_{ij} with $\delta(T_{ij}) \neq 0$. Assume that we find two different j, k with $\delta(T_{ij}) \neq 0$ and $\delta(T_{ik}) \neq 0$. Note that we have

$$\frac{\partial T_i^{l_i}}{\partial T_{ij}}\delta(T_{ij}),\ \frac{\partial T_i^{l_i}}{\partial T_{ik}}\delta(T_{ik}) \in R(A, P_0)_{\mu+\deg(\delta)}.$$

By Proposition 3.5, the component of degree $\mu + \deg(\delta)$ is of dimension one. Thus, the above two terms differ by a nonzero scalar and we see that $T_{ik}^{l_{ik}}$ divides the second term. Consequently, T_{ik} must divide $\delta(T_{ik})$, which contradicts Lemma 4.5 (ii).

A second step is to see that for any two variables T_{ij} and T_{ks} with $\delta(T_{ij}) \neq 0$ and $\delta(T_{ks}) \neq 0$ we must have $l_{ij} = 1$ or $l_{ks} = 1$. Otherwise, we see as before that $\partial T_i^{l_i}/\partial T_{ij}\, \delta(T_{ij})$ and $\partial T_k^{l_k}/\partial T_{ks}\, \delta(T_{ks})$ differ by a nonzero scalar. Thus, we conclude $\delta(T_{ij}) = fT_{ks}$ and $\delta(T_{ks}) = hT_{ij}$, a contradiction to Lemma 4.5 (iv).

Finally, we prove the assertion. As already seen, for every $0 \leqslant k \leqslant r$ there is at most one c_k with $\delta(T_{kc_k}) \neq 0$. Let $\mathfrak{K} \subseteq \{0, \ldots, r\}$ denote the set of all k admitting such a c_k. From Proposition 3.5 we infer $R(A, P_0)_{\mu+\deg(\delta)} = \mathbb{K}f$ with some nonzero element f. We claim that

$$f = h \prod_{k \in \mathfrak{K}} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}}, \qquad \delta(T_{ic_i}) = \beta_i h \prod_{k \in \mathfrak{K} \setminus \{i\}} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}},$$

hold with a homogeneous element $h \in R(A, P_0)$ and scalars $\beta_0, \ldots, \beta_r \in \mathbb{K}$. Indeed, similar to the previous arguments, the first equation follows from fact that all $\partial T_k^{l_k}/\partial T_{kc_k}\, \delta(T_{kc_k})$ are nonzero elements of the same degree as f and hence each $\partial T_k^{l_k}/\partial T_{kc_k}$ must divide f. The second equation is clear then.

The vector $\beta := (\beta_0, \ldots, \beta_r)$ lies in the row space of the matrix A. To see this, consider the lift of δ to $\mathbb{K}[T_{ij}, S_k]$ defined by the second equation and apply Lemma 4.6. Now let $C = (c_0, \ldots, c_r)$ be any sequence completing the c_k, where $k \in \mathfrak{K}$. Then we have $\delta = h\delta_{C,\beta}$. The fact that h belongs to the kernel of $\delta_{C,\beta}$ follows from Lemma 4.5. □

Example 4.7 (The E_6-singular cubic III)**.** Situation as in Example 3.3. The primitive homogeneous derivations of $R(A, P_0)$ of the form $\delta_{C,\beta}$ are the following

(i) $C = (1, 1, 1)$ and $\beta = (\beta_0, 0, -\beta_0)$. Here we have $\deg(\delta_{C,\beta}) = (3, 0)$ and

$$\delta_{C,\beta}(T_{01}) = 2\beta_0 T_{21}, \quad \delta_{C,\beta}(T_{21}) = -\beta_0 T_{02}^3, \quad \delta_{C,\beta}(T_{02}) = \delta_{C,\beta}(T_{11}) = 0.$$

(ii) $C = (1, 1, 1)$ and $\beta = (\beta_0, -\beta_0, 0)$. Here we have $\deg(\delta_{C,\beta}) = (3, 1)$ and

$$\delta_{C,\beta}(T_{01}) = 3\beta_0 T_{11}^2, \quad \delta_{C,\beta}(T_{11}) = -\beta_0 T_{02}^3, \quad \delta_{C,\beta}(T_{02}) = \delta_{C,\beta}(T_{21}) = 0.$$

The general primitive homogeneous derivation δ of $R(A, P_0)$ has the form $h\delta_{C,\beta}$ with $h \in \ker(\delta_{C,\beta})$, and

$$\deg(\delta) = \deg(h) + \deg(\delta_{C,\beta}) \notin \omega.$$

In the above case (i), the only possibilities for $\deg(h)$ are $\deg(h) = (k, k)$ or $\deg(h) = (k, k) + (0, 2)$ and thus we have

$$\delta = T_{02}^k \delta_{C,\beta} \quad \text{or} \quad \delta = T_{02}^k T_{11} \delta_{C,\beta}.$$

In the above case (ii), the only possibility for $\deg(h)$ is $\deg(h) = (k, k)$ and thus we obtain

$$\delta = T_{02}^k \delta_{C,\beta}.$$

5. Demazure Roots

Here we present and prove the main result, Theorem 5.5. It describes the root system of the automorphism group of a rational complete normal variety X coming with an effective torus action $T \times X \to X$ of complexity one in terms of the defining matrix P of the Cox ring $\mathcal{R}(X) = R(A, P)$, see Construction 3.10 and Theorem 3.12.

Definition 5.1. Let A, P be as in Construction 3.7. We say that $R(A, P)$ is *minimally presented* if $r \geqslant 2$ holds and for every $0 \leqslant i \leqslant r$ we have $l_{i1} + \ldots + l_{in_i} \geqslant 2$

The assumption that $R(A, P)$ is minimally presented means that the resulting variety is nontoric and there occur no linear monomials in the defining relations g_I; the latter can always be achieved by omitting redundant generators.

Definition 5.2. Let P be a matrix as in Construction 3.7. Denote by $v_{ij}, v_k \in N = \mathbb{Z}^{r+s}$ the columns of P and by M the dual lattice of N.

(i) A *vertical Demazure P-root* is a tuple (u, k_0) with a linear form $u \in M$ and an index $1 \leqslant k_0 \leqslant m$ satisfying

$$\begin{aligned} \langle u, v_{ij} \rangle &\geqslant 0 \quad \text{for all } i, j, \\ \langle u, v_k \rangle &\geqslant 0 \quad \text{for all } k \neq k_0, \\ \langle u, v_{k_0} \rangle &= -1. \end{aligned}$$

(ii) A *horizontal Demazure P-root* is a tuple (u, i_0, i_1, C), where $u \in M$ is a linear form, $i_0 \neq i_1$ are indices with $0 \leqslant i_0, i_1 \leqslant r$, and $C = (c_0, \ldots, c_r)$ is a sequence with $1 \leqslant c_i \leqslant n_i$ such that

$$\begin{aligned} l_{ic_i} &= 1 \quad \text{for all } i \neq i_0, i_1, \\ \langle u, v_{ic_i} \rangle &= \begin{cases} 0, & i \neq i_0, i_1, \\ -1, & i = i_1, \end{cases} \\ \langle u, v_{ij} \rangle &\geqslant \begin{cases} l_{ij}, & i \neq i_0, i_1, \ \ j \neq c_i, \\ 0, & i = i_0, i_1, \ \ j \neq c_i, \\ 0, & i = i_0, \ \ j = c_i, \end{cases} \\ \langle u, v_k \rangle &\geqslant 0 \quad \text{for all } k. \end{aligned}$$

(iii) The $\mathbb{Z}^s$*-part* of a Demazure P-root $\kappa = (u, k_0)$ or $\kappa = (u, i_0, i_1, C)$ is the tuple α_κ of the last s coordinates of the linear form $u \in M = \mathbb{Z}^{r+s}$. We call α_κ also a *P-root*.

Note that in the minimally presented case, the P-roots are by their defining conditions always nonzero.

Example 5.3 (The E_6-singular cubic IV)**.** As earlier, let $r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$, $m = 0$, $s = 1$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix}.$$

There are no vertical Demazure P-roots because of $m = 0$. There is a horizontal Demazure P-root $\kappa = (u, i_0, i_1, C)$ given by

$$u = (-1, -2, 3), \quad i_0 = 1, \ i_1 = 2, \quad C = (1, 1, 1).$$

A direct computation shows that this is the only one. The $\mathbb{Z}^s$-part of κ is the third coordinate of the linear form u, i.e., it is $u_3 = 3 \in \mathbb{Z} = \mathbb{Z}^s$.

Note that the Demazure P-roots are certain Demazure roots [7, Section 3.1] of the fan with the rays through the columns of P as its maximal cones. In particular, there are only finitely many Demazure P-roots. For computing them explicitly, the following presentation is helpful.

Remark 5.4. The Demazure P-roots are the lattice points of certain polytopes in $M_{\mathbb{Q}}$. For an explicit description, we encode the defining conditions as a lattice vector $\zeta \in \mathbb{Z}^{n+m}$ and an affine subspace $\eta \subseteq M_{\mathbb{Q}}$:

(i) For any index $1 \leqslant k_0 \leqslant m$ define a lattice vector $\zeta = (\zeta_{ij}, \zeta_k) \in \mathbb{Z}^{n+m}$ and an affine subspace $\eta \subseteq M_{\mathbb{Q}}$ by

$$\zeta_{ij} := 0 \text{ for all } i, j, \quad \zeta_k := 0 \text{ for all } k \neq k_0, \quad \zeta_{k_0} := -1,$$
$$\eta := \{u' \in M_{\mathbb{Q}};\ \langle u', v_{k_0} \rangle = -1\} \subseteq M_{\mathbb{Q}}.$$

Then the vertical Demazure P-roots $\kappa = (u, k_0)$ are given by the lattice points u of the polytope

$$B(k_0) := \{u' \in \eta;\ P^* u' \geqslant \zeta\} \subseteq M_{\mathbb{Q}}.$$

(ii) Given $i_0 \neq i_1$ with $0 \leqslant i_0, i_1 \leqslant r$ and $C = (c_0, \ldots, c_r)$ with $1 \leqslant c_i \leqslant n_i$ such that $l_{ic_i} = 1$ holds for all $i \neq i_0, i_1$, set

$$\zeta_{ij} := \begin{cases} l_{ij}, & i \neq i_0, i_1,\ j \neq c_i, \\ -1, & i = i_1,\ j = c_{i_1}, \\ 0 & \text{else}, \end{cases} \qquad \zeta_k = 0 \text{ for } 1 \leqslant l \leqslant m.$$
$$\eta := \{u' \in M_{\mathbb{Q}};\ \langle u', v_{ic_i} \rangle = 0 \text{ for } i \neq i_0, i_1,\ \langle u', v_{i_1 c_{i_1}} \rangle = -1\}.$$

Then the horizontal Demazure P-roots $\kappa = (u, i_0, i_1, C)$ are given by the lattice points u of the polytope

$$B(i_0, i_1, C) := \{u' \in \eta;\ P^* u' \geqslant \zeta\} \subseteq M_{\mathbb{Q}}.$$

In order to state and prove the main result, let us briefly recall the necessary concepts from the theory of linear algebraic groups G. One considers the adjoint representation of the torus T on the Lie algebra $\mathrm{Lie}(G)$, i.e., the tangent representation at e_G of the T-action on G given by conjugation $(t, g) \mapsto tgt^{-1}$. There is a unique T-invariant splitting $\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \mathfrak{n}$, where $\mathfrak{n}$ is spanned by nilpotent vectors, and one has a bijection

$$\text{1-PASG}_T(G) \to \{T\text{-eigenvectors of } \mathfrak{n}\}, \quad \lambda \mapsto \dot{\lambda}(0).$$

Here $\text{1-PASG}_T(G)$ denotes the set of one parameter additive subgroups $\lambda\colon \mathbb{G}_a \to G$ normalized by T and $\dot{\lambda}$ denotes the differential. A root of G with respect to T is an eigenvalue of the T-representation on $\mathfrak{n}$, that is, a character $\chi \in \mathbb{X}(T)$ with $t \cdot v = \chi(t) v$ for some T-eigenvector $0 \neq v \in \mathfrak{n}$.

Theorem 5.5. *Let A, P be as in Construction 3.7 such that $R(A, P)$ is minimally presented and let X be a (nontoric) variety with a complexity one torus action $T \times X \to X$ arising from A, P according to Construction 3.10.*

(i) *The automorphism group $\mathrm{Aut}(X)$ is a linear algebraic group with maximal torus T.*

(ii) *Under the canonical identification $\mathbb{X}(T) = \mathbb{Z}^s$, the roots of $\mathrm{Aut}(X)$ with respect to T are precisely the P-roots.*

The rest of the section is devoted to the proof. We will have to deal with the K_0- and K-degrees of functions and derivations. It might be helpful to recall the relations between the gradings from Remark 3.9. The following simple facts will be frequently used.

Lemma 5.6. *In the setting of Constructions 3.1 and 3.7, consider the polynomial ring $\mathbb{K}[T_{ij}, S_k]$ with the K_0-grading and the coarser K-grading.*

(i) *For a monomial $h = \prod T_{ij}^{e_{ij}} \prod S_k^{e_k}$ with exponent vector $e = (e_{ij}, e_k)$, the K_0- and K-degrees are given as*

$$\deg_{K_0}(h) = Q_0(e), \quad \deg_K(h) = Q(e).$$

(ii) *A monomial $h \in \mathbb{K}[T_{ij}^{\pm 1}, S_k^{\pm 1}]$ is of K-degree zero if and only if there is an $u \in M$ with*

$$h = h^u := \prod T_{ij}^{P^*(u)_{ij}} \prod S_k^{P^*(u)_k} = \prod T_{ij}^{\langle u, v_{ij}\rangle} \prod S_k^{\langle u, v_k\rangle}.$$

(iii) *Let δ be a derivation on $\mathbb{K}[T_{ij}, S_k]$ sending the generators T_{ij}, S_k to monomials. Then δ is K-homogeneous of K-degree zero if and only if*

$$\deg_K(T_{ij}^{-1}\delta(T_{ij})) = \deg_K(S_k^{-1}\delta(S_k)) = 0 \quad \textit{holds for all } i, j, k.$$

If $0 \neq \delta$ is K_0-homogeneous, then $\deg_K(\delta) = 0$ holds if and only if one of the $T_{ij}^{-1}\delta(T_{ij})$ and $S_k^{-1}\delta(S_k)$ is nontrivial of K-degree zero.

As a first step towards the roots of the automorphism group $\mathrm{Aut}(X)$, we now associate K_0-homogeneous locally nilpotent derivations of $R(A, P)$ to the Demazure P-roots.

Construction 5.7. Let A and P be as in Construction 3.7. For $u \in M$ and the lattice vector $\zeta \in \mathbb{Z}^{n+m}$ of Remark 5.4 consider the monomials

$$h^u = \prod_{i,j} T_{ij}^{\langle u, v_{ij}\rangle} \prod_k S_k^{\langle u, v_k\rangle}, \quad h^\zeta := \prod_{i,j} T_{ij}^{\zeta_{ij}} \prod_k S_k^{\zeta_k}.$$

We associate to any Demazure P-root κ a locally nilpotent derivation δ_κ of $R(A, P)$. If $\kappa = (u, k_0)$ is vertical, then we define a δ_κ of vertical type by

$$\delta_\kappa(T_{ij}) := 0 \text{ for all } i, j, \quad \delta_\kappa(S_k) := \begin{cases} S_{k_0} h^u, & k = k_0, \\ 0, & k \neq k_0. \end{cases}$$

If $\kappa = (u, i_0, i_1, C)$ is horizontal, then there is a unique vector β in the row space of A with $\beta_{i_0} = 0$, $\beta_{i_1} = 1$ and we define a δ_κ of horizonal type by

$$\delta_\kappa := \frac{h^u}{h^\zeta} \delta_{C,\beta}.$$

In all cases, the derivation δ_κ is K_0-homogeneous; its K_0-degree is the $\mathbb{Z}^s$-part of κ and the K-degree is zero:

$$\deg_{K_0}(\delta_\kappa) = Q_0(P^*(u)), \quad \deg_K(\delta_\kappa) = 0.$$

Proof. In the vertical case $\delta_\kappa(S_{k_0})$ does not depend on S_{k_0} and in the horizontal case the factors before $\delta_{C,\beta}$ in the definitions of δ_κ are contained in $\ker(\delta_{C,\beta})$. Thus, the derivations δ_κ are locally nilpotent. Clearly, the δ_κ are K_0-homogeneous. By Lemma 5.6, the monomial h^u is of K_0-degree $Q_0(P^*(u))$. In the vertical case, this implies directly that δ_κ is of K_0-degree $Q_0(P^*(u))$. In the horizontal case, we use Lemma 5.6 and the degree computation of Construction 4.3 to see that h^ζ and $\delta_{C,\beta}$ have the same K_0-degree. Thus δ_κ is of K_0-degree $Q_0(P^*(u))$. Since $P^*(u) \in \ker(Q)$ holds, we obtain that all δ_κ are of K-degree $Q(P^*(u)) = 0$. □

Proposition 5.8. *Consider a minimally presented algebra $R(A, P)$ with its fine K_0-grading and the coarser K-grading and let δ be a K_0-homogeneous locally nilpotent derivation of K-degree zero on $R(A, P)$.*

(i) *If δ is of vertical type, then there is an index $1 \leqslant k_0 \leqslant m$ such that δ is a linear combination of derivations δ_{κ_t} with Demazure P-roots $\kappa_t = (u_t, k_0)$.*

(ii) *If δ is of horizontal type, then are indices $0 \leqslant i_0, i_1 \leqslant r$ and a sequence $C = (c_0, \ldots, c_r)$ such that δ is a linear combination of derivations δ_{κ_t} with Demazure P-roots $\kappa_t = (u_t, i_0, i_1, C)$.*

Lemma 5.9. *Let δ be a nontrivial K_0-homogeneous locally nilpotent derivation on a minimally presented algebra $R(A, P)$ and let $r \geqslant 2$. If δ is of K-degree zero, then δ is primitive with respect to the K_0-grading.*

Proof. We have to show that the K_0-degree w of δ does not lie in the weight cone of the K_0-grading. First observe that $w \neq 0$ holds: otherwise Corollary 3.6 yields that δ annihilates all generators T_{ij} and S_k, a contradiction to $\delta \neq 0$. Now assume that w lies in the weight cone of the K_0-grading. Then, for some $d > 0$, we find a nonzero $f \in R(A, P)_{dw}$. The K-degree of f equals zero and thus f is constant, a contradiction. □

Proof of Proposition 5.8. First assume that δ is vertical. Lemma 5.9 tells us that δ is primitive with respect to the K_0-grading. According to Theorem 4.4, there is an index $1 \leqslant k_0 \leqslant m$ and an element $h \in R(A, P)$ represented by a polynomial only depending on variables from $\ker(\delta)$ such that we have

$$\delta(T_{ij}) = 0 \text{ for all } i, j, \quad \delta(S_k) = 0 \text{ for all } k \neq k_0, \quad \delta(S_{k_0}) = h.$$

Clearly, $hS_{k_0}^{-1}$ is K_0-homogeneous of K-degree zero. Lemma 5.6 shows that the monomials of $hS_{k_0}^{-1}$ are of the form h^u with $u \in M$. The facts that the monomials $h^u S_{k_0}$ do not depend on S_{k_0} and have nonnegative exponents yield the inequalities of a vertical Demazure P-root for each (u, k_0). Consequently, δ is a linear combination of deriviations arising from vertical Demazure P-roots.

We turn to the case that δ is horizontal. Again by Lemma 5.9, our δ is primitive with respect to the K_0-grading and by Theorem 4.4 it has the form $h\delta_{C,\beta}$ for some K_0-homogeneous $h \in \ker(\delta_{C,\beta})$. By construction, $\delta_{C,\beta}$ is induced by a homogeneous derivation of $\mathbb{K}[T_{ij}, S_k]$ having the same K_0- and K-degrees; we denote this lifted

derivation again by $\delta_{C,\beta}$. Similarly, h is represented by a polynomial in $\mathbb{K}[T_{ij},\, S_k]$, which we again denote by h.

We show that any monomial of h depends only on variables from $\ker(\delta_{C,\beta})$. Indeed, suppose that there occurs a monomial $T_{ij}h'$ with $\delta_{C,\beta}(T_{ij}) \neq 0$ in h. Then, using the fact that δ is of K-degree zero, we obtain

$$\deg(T_{ij}) = \deg(\delta(T_{ij})) = \deg(T_{ij}) + \deg(h') + \deg(\delta_{C,\beta}(T_{ij})).$$

This implies $\deg(h') + \deg(\delta_{C,\beta}(T_{ij})) = 0$; a contradiction to the fact that the weight cone of the K-grading contains no lines. This proves the claim. Thus, we may assume that the polynomial h is a monomial.

The next step is to see that it is sufficient to take derivations $\delta_{C,\beta}$ with a vector β in the row space having one zero coordinate. Consider a general β, which means one with only nonvanishing coordinates. By construction, the row space of A contains unique vectors β^0 and β^1 with $\beta^0_0 = \beta^1_1 = 0$ and $\beta = \beta^0 + \beta^1$. With these vectors, we have

$$h\delta_{C,\beta} = h\frac{\partial T_0^{l_0}}{\partial T_{0c_0}}\delta_{C,\beta^0} + h\frac{\partial T_1^{l_1}}{\partial T_{1c_1}}\delta_{C,\beta^1}.$$

By Construction 4.3, the K_0-degrees and thus the K-degrees of the left hand side and of the summands coincide. Moreover, h is a monomial in generators from $\ker(\delta_{C,\beta})$ and any such generator is annihilated by δ_{C,β^0} and by δ_{C,β^1} too.

Let $e = (e_{ij},\, e_k)$ denote the exponent vector of the monomial h. According to Lemma 5.6, the condition that the (K_0-homogeneous) derivation δ has K-degree zero is equivalent to the fact that the monomial

$$T_{i_1c_{i_1}}^{-1} h\delta_{C,\beta}(T_{i_1c_{i_1}}) = T_{i_1c_{i_1}}^{-1} T_{i_0c_{i_0}}^{e_{i_0c_{i_0}}} \prod_{\substack{i\\ j\neq c_i}} T_{ij}^{e_{ij}} \prod_k S_k^{e_k}\beta_{i_1} \prod_{i\neq i_0,i_1} \frac{\partial T_i^{l_i}}{\partial T_{ic_i}}$$

has the form h^u for some linear form $u \in M$. Taking into account that the exponents e_{ij} and e_k are nonnegative, we see that these conditions are equivalent to equalities and inequalities in the definition of a horizontal Demazure P-root. □

We recall the correspondence between locally nilpotent derivations and one parameter additive subgroups. Consider any integral affine $\mathbb{K}$-algebra R, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. Every locally nilpotent derivation $\delta\colon R \to R$ gives rise to a rational representation $\varrho_\delta\colon \mathbb{G}_a \to \mathrm{Aut}(R)$ of the additive group $\mathbb{G}_a$ of the field $\mathbb{K}$ via

$$\varrho_\delta(t)(f) := \exp(t\delta)(f) := \sum_{d=0}^{\infty} \frac{t^d}{d!}\delta^d(f).$$

This sets up a bijection between the locally nilpotent derivations of R and the rational representations of $\mathbb{G}_a$ by automorphisms of R. The representation associated to a locally nilpotent derivation $\delta\colon R \to R$ gives rise to a one parameter additive subgroup (1-PASG) of the automorphism group of $\overline{X} := \mathrm{Spec}\, R$:

$$\lambda_\delta\colon \mathbb{G}_a \to \mathrm{Aut}(\overline{X}), \quad t \mapsto \mathrm{Spec}(\varrho_\delta(t)).$$

Now suppose that R is graded by some finitely generated abelian group K_0 and consider the associated action of $H_0 := \operatorname{Spec} \mathbb{K}[K_0]$ on $\overline{X} = \operatorname{Spec} R$. We relate homogeneity of locally nilpotent derivation δ to properties of the associated subgroup $\overline{U}_\delta := \lambda_\delta(\mathbb{G}_a)$ of $\operatorname{Aut}(\overline{X})$.

Lemma 5.10. *In the above setting, let δ be a locally nilpotent derivation on R. The following statements are equivalent.*

(i) *The derivation δ is K_0-homogeneous.*
(ii) *One has $h\overline{U}_\delta h^{-1} = \overline{U}_\delta$ for all $h \in H_0$.*

Moreover, if one of these two statements holds, then the degree $w := \deg(\delta) \in K_0$ is uniquely determined by the property

$$h\varrho_\delta(t)h^{-1} = \varrho_\delta(\chi^w(h)t) \text{ for all } h \in H_0.$$

Proof of Theorem 5.5. Assertion (i) is clear by Corollary 2.4 and the fact that X is nontoric. We prove (ii). Consider $R(A, P)$ with its fine K_0-grading and the coarser K-grading. The quasitori $H_0 := \operatorname{Spec} \mathbb{K}[K_0]$ and $H := \operatorname{Spec} \mathbb{K}[K]$ act effectively on $\overline{X} = \operatorname{Spec} R(A, P)$. We view H_0 and H as subgroups of $\operatorname{Aut}(\overline{X})$. For any locally nilpotent deriviation δ on $R(A, P)$ and $\overline{U}_\delta = \lambda_\delta(\mathbb{G}_a)$, Lemma 5.10 gives

$$\delta \text{ is } K_0\text{-homogeneous} \iff h\overline{U}_\delta h^{-1} = \overline{U}_\delta \text{ for all } h \in H_0,$$
$$\delta \text{ is } K\text{-homogeneous of degree } 0 \iff huh^{-1} = u \text{ for all } h \in H,\ u \in \overline{U}_\delta.$$

Recall that X arises as $X = \widehat{X}/\!\!/H$ for an open H_0-invariant set $\widehat{X} \subseteq \overline{X}$. Moreover, the action of $T = H_0/H$ on X is the induced one, i.e. it makes the quotient map $p\colon \widehat{X} \to X$ equivariant. Set for short

$$\overline{G} := \operatorname{CAut}(\overline{X}, H)^0, \quad G := \operatorname{Aut}(X)^0.$$

Denote by $\text{1-PASG}_{H_0}(\overline{G})$ and $\text{1-PASG}_T(G)$ the one parameter additive subgroups normalized by H_0 and T respectively. Moreover, let $\operatorname{LND}(R(A, P))_0$ denote the set of K-homogeneous locally nilpotent derivations of K-degree zero and $\operatorname{LND}_{K_0}(R(A, P))_0$ the subset of K_0-homogeneous ones. Then we arrive at a commutative diagram

$$\begin{array}{ccc}
\operatorname{LND}_{K_0}(R(A, P))_0 & \subseteq & \operatorname{LND}(R(A, P))_0 \\
\cong \updownarrow & & \updownarrow \cong \\
\text{1-PASG}_{H_0}(\overline{G}) & \subseteq & \text{1-PASG}(\overline{G}) \\
p_* \downarrow & & \downarrow p_* \\
\text{1-PASG}_T(G) & \subseteq & \text{1-PASG}(G).
\end{array}$$

Construction 5.7 associates an element $\delta_\kappa \in \operatorname{LND}_{K_0}(R(A, P))_0$ to any Demazure P-root κ. Going downwards the left hand side of the above diagram, the latter turns into an element $\lambda_\kappa \in \text{1-PASG}_T(G)$. Differentiation gives the T-eigenvector $\dot{\lambda}_\kappa(0) \in \operatorname{Lie}(G)$ having as its associated root the unique character χ of T satisfying

$$t\lambda_\kappa(z)t^{-1} = \lambda_\kappa(\chi(t)z) \quad \text{for all } t \in T,\ z \in \mathbb{K}.$$

Remark 3.9 and Lemma 5.10 show that under the identification $\mathbb{X}(T) = \mathbb{Z}^s$ the character χ is just the $\mathbb{Z}^s$-part of the Demazure P-root κ. Proposition 5.8 tells us that any element of $\mathrm{LND}_{K_0}(R(A, P))_0$ is a linear combination of derivations δ_κ arising from Demazure P-roots. Moreover, by Corollary 2.7, the push forward p_* maps $1\text{-PASG}_{H_0}(\overline{G})$ onto $1\text{-PASG}_T(G)$. We conclude that $\mathrm{Lie}(G)$ is spanned as a $\mathbb{K}$-vector space by $\mathrm{Lie}(T)$ and $\dot{\lambda}_\kappa(0)$, where κ runs through the Demazure P-roots. Assertion (ii) follows. □

Corollary 5.11 (of proof)**.** *Let X be a nontoric normal complete rational variety with a torus action $T \times X \to X$ of complexity one arising as a good quotient $p\colon \widehat{X} \to X$ from $R(A, P)$ according to Construction 3.10.*

(i) *Every Demazure P-root κ induces an additive one parameter subgroup $\lambda_\kappa = p_*\lambda_{\delta_\kappa} : \mathbb{G}_a \to \mathrm{Aut}(X)$.*
(ii) *The Demazure P-root κ is vertical if and only if the general orbit of λ_κ is contained in some T-orbit closure.*
(iii) *The Demazure P-root κ is horizontal if and only if the general orbit of λ_κ is not contained in any T-orbit closure.*
(iv) *The unit component $\mathrm{Aut}(X)^0$ of the automorphism group is generated by T and the images $\lambda_\kappa(\mathbb{G}_a)$.*

Proof. Assertions (i) and (iv) are clear by the proof of Theorem 5.5. For (ii) and (iii) recall that κ is vertical (horizontal) if and only if δ_κ is of vertical (horizontal) type. The latter is equivalent to saying that $\lambda_\kappa(\mathbb{G}_a)$ acts trivially (non-trivially) on the field of T-invariant rational functions. □

Example 5.12 (The E_6-singular cubic V)**.** Let A and P as in Example 5.3. From there we infer that $R(A, P)$ admits precisely one horizontal Demazure P-root. For the automorphism group of the corresponding surface X this means that $\mathrm{Aut}(X)^0$ is the semidirect product of $\mathbb{K}^*$ and $\mathbb{G}_a$ twisted via the weight 3, see again Example 5.3. In particular, the surface X is almost homogeneous. Moreover, in this case, one can show directly that the group of graded automorphisms of $R(A, P)$ is connected. Thus, Theorem 2.1 yields that $\mathrm{Aut}(X)$ is the semidirect product of $\mathbb{K}^*$ and $\mathbb{G}_a$. This is in accordance with [23]; we would like to thank Antonio Laface for mentioning this reference to us.

6. Almost Homogeneous Surfaces

A variety is *almost homogeneous* if its automorphism group acts with an open orbit. We take a closer look to this case with a special emphasis on almost homogeneous rational $\mathbb{K}^*$-surfaces of Picard number one. The first statement characterizes the almost homogeneous varieties coming with a torus action of complexity one in arbitrary dimension.

Theorem 6.1. *Let X be a nontoric normal complete rational variety with a torus action $T \times X \to X$ of complexity one and Cox ring $\mathcal{R}(X) = R(A, P)$. Then the following statements are equivalent.*

(i) *The variety X is almost homogeneous.*
(ii) *There exists a horizontal Demazure P-root.*

Moreover, if one of these statements holds, and $R(A,\,P)$ is minimally presented, then the number $r-1$ of relations of $R(A,\,P)$ is bounded by

$$r-1 \leqslant \dim(X) + \mathrm{rk}(\mathrm{Cl}(X)) - m - 2.$$

Proof. If (i) holds, then $\mathrm{Aut}(X)$ acts with an open orbit on X and by Corollary 5.11, there must be a horizontal Demazure P-root κ. Conversely, if (ii) holds, then there is a horizontal Demazure P-root κ and Corollary 5.11 says that for $U = p_*(\delta_\kappa(\mathbb{G}_a))$, the group $T \ltimes U$ acts with an open orbit on X.

For the supplement, recall first that $R(A,\,P)$ is a complete intersection with $r-1$ necessary relations and thus we have

$$n + m - (r-1) = \dim(R(A,\,P)) = \dim(X) + \mathrm{rk}(\mathrm{Cl}(X)).$$

Now observe that any relation g_I involving only three variables prevents existence of a horizontal Demazure P-root. Consequently, by suitably arranging the relations, we have $n_0,\, n_1 \geqslant 1$ and $n_i \geqslant 2$ for all $i \geqslant 2$. Thus, $n \geqslant 2 + 2(r-1)$ holds and the assertion follows. □

We specialize to dimension two. Any normal complete rational $\mathbb{K}^*$-surface X is determined by its Cox ring and thus is given up to isomorphism by the defining data A and P of the ring $\mathcal{R}(X) = R(A,\,P)$; we also say that the $\mathbb{K}^*$-surface X arises from A and P and refer to [11, Sec. 3.3] for more background. A first step towards the almost homogeneous X is to determine possible horizontal Demazure P-roots in the following setting.

Proposition 6.2. *Consider integers $l_{02} \geqslant 1$, $l_{11} \geqslant l_{21} \geqslant 2$ and $d_{01},\, d_{02},\, d_{11},\, d_{21}$ such that the following matrix has pairwise different primitive columns generating $\mathbb{Q}^3$ as a convex cone:*

$$P := \begin{bmatrix} -1 & -l_{02} & l_{11} & 0 \\ -1 & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}.$$

Moreover, assume that P is positive in the sense that $\det(P_{01}) > 0$ holds, where P_{01} is the 3×3 matrix obtained from P by deleting the first column. Then the possible horizontal Demazure P-roots are

(i) *$\kappa = (u,\,1,\,2,\,(1,\,1,\,1))$, where $u = \Big(d_{01}\alpha + \dfrac{d_{21}\alpha+1}{l_{21}},\, -\dfrac{d_{21}\alpha+1}{l_{21}},\, \alpha\Big)$ with an integer α satisfying*

$$l_{21} \,|\, d_{21}\alpha + 1, \qquad \frac{l_{02}}{d_{02} - l_{02}d_{01}} \leqslant \alpha \leqslant -\frac{l_{11}}{l_{21}d_{11} + l_{11}d_{21} + d_{01}l_{11}l_{21}},$$

(ii) *if $l_{02} = 1$: $\kappa = (u,\,1,\,2,\,(2,\,1,\,1))$, where $u = \Big(d_{02}\alpha + \dfrac{d_{21}\alpha+1}{l_{21}},\, -\dfrac{d_{21}\alpha+1}{l_{21}},\, \alpha\Big)$ with an integer α satisfying*

$$l_{21} \,|\, d_{21}\alpha + 1, \qquad -\frac{l_{11}}{l_{21}d_{11} + l_{11}d_{21} + d_{02}l_{11}l_{21}} \leqslant \alpha \leqslant \frac{1}{d_{01} - d_{02}},$$

(iii) $\kappa = (u,\, 2,\, 1,\, (1,\, 1,\, 1))$, *where* $u = \Big(-\dfrac{d_{11}\alpha+1}{l_{11}},\, d_{01}\alpha + \dfrac{d_{11}\alpha+1}{l_{11}},\, \alpha\Big)$ *with an integer* α *satisfying*

$$l_{11}\,|\,d_{11}\alpha+1, \qquad \frac{l_{02}}{d_{02}-l_{02}d_{01}} \leqslant \alpha \leqslant -\frac{l_{21}}{l_{21}d_{11}+l_{11}d_{21}+d_{01}l_{11}l_{21}},$$

(iv) *if* $l_{02}=1$: $\kappa = (u,\, 2,\, 1,\, (2,\, 1,\, 1))$, *where* $u = \Big(-\dfrac{d_{11}\alpha+1}{l_{11}},\, d_{02}\alpha+\dfrac{d_{11}\alpha+1}{l_{11}},\, \alpha\Big)$ *with an integer* α *satisfying*

$$l_{11}\,|\,d_{11}\alpha+1, \qquad -\frac{l_{21}}{l_{21}d_{11}+l_{11}d_{21}+d_{02}l_{11}l_{21}} \leqslant \alpha \leqslant \frac{1}{d_{01}-d_{02}}.$$

Proof. In the situation of (i), evaluating the general linear form $u = (u_1,\, u_2,\, u_3)$ on the columns of P gives the following conditions for a Demazure P-root:

$$-u_1 - u_2 + u_3 d_{01} = 0, \qquad u_2 l_{21} + u_3 d_{21} = -1,$$
$$-u_1 l_{02} - u_2 l_{02} + u_3 d_{02} \geqslant l_{02}, \qquad u_1 l_{11} + u_3 d_{11} \geqslant 0.$$

Resolving the equations for u_1, u_2 and plugging the result into the inequalities gives the desired roots with $\alpha := u_3$. The other cases are treated analogously. □

Corollary 6.3. *The nontoric almost homogeneous normal complete rational* $\mathbb{K}^*$*-surfaces* X *of Picard number one are precisely the ones arising from data*

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} -1 & -l_{02} & l_{11} & 0 \\ -1 & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}$$

as in Proposition 6.2 allowing an integer α *according to one of the Conditions 6.2* (i) *to* (iv). *In particular, the Cox ring of* X *is given as*

$$\mathcal{R}(X) = \mathbb{K}[T_{01},\, T_{02},\, T_{11},\, T_{21}]\,/\,\langle T_{01}T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}\rangle$$

with the grading by $\mathbb{Z}^4/\mathrm{im}(P^*)$. *Moreover, the anticanonical divisor of* X *is ample, i.e.,* X *is a del Pezzo surface.*

Proof. As any surface with finitely generated Cox ring, X is $\mathbb{Q}$-factorial. Since X has Picard number one, the divisor class group $\mathrm{Cl}(X)$ is of rank one. Now take a minimal presentation $\mathcal{R}(X) = R(A,\, P)$ of the Cox ring. Then, according to Theorem 6.1, we have $m = 0$ and there is exactly one relation in $R(A,\, P)$. Thus P is a 3×4 matrix. Moreover, Theorem 6.1 says that there is a horizontal Demazure P-root. Consequently, one of the exponents l_{01} and l_{02} must equal one, say l_{01}. Fixing a suitable order for the last two variables we ensure $l_{11} \geqslant l_{21}$. Passing to the $\mathbb{K}^*$-action $t^{-1} \cdot x$ instead of $t \cdot x$ if necessary, we achieve that P is positive in the sense of Proposition 6.2.

Let us see why X is a del Pezzo surface. Denote by P_{ij} the matrix obtained from P by deleting the column v_{ij}. Then, in $\mathrm{Cl}(X)^0 = \mathbb{Z}$, the factor group of $\mathrm{Cl}(X)$ by the torsion part, the weights w_{ij}^0 of T_{ij} are given up to a factor α as

$$(w_{01}^0,\, w_{02}^0,\, w_{11}^0,\, w_{21}^0) = \alpha(\det(P_{01}),\, -\det(P_{02}),\, \det(P_{11}),\, -\det(P_{21})).$$

According to [1, Prop. III.3.4.1], the class of the anticanonical divisor in $\mathrm{Cl}(X)^0$ is given as the sum over all w_{ij}^0 minus the degree of the relation. The inequalities on

the l_{ij}, d_{ij} implied by the existence of an integer α as in Proposition 6.2 (i) to (iv) show that the anticanonical class is positive (note that α rules out). $\square$

We turn to the case of precisely one singular point. In that case, the integer α in clauses (i) to (iv) of Proposition 6.2 satisfies the divisibility conditions automatically.

Construction 6.4 ($\mathbb{K}^*$-surfaces with one singularity)**.** Consider a triple (l_0, l_1, l_2) of integers satisfying the following conditions:

$$l_0 \geqslant 1, \quad l_1 \geqslant l_2 \geqslant 2, \quad l_0 < l_1 l_2, \quad \gcd(l_1, l_2) = 1.$$

Let (d_1, d_2) be the (unique) pair of integers with $d_1 l_2 + d_2 l_1 = -1$ and $0 \leqslant d_2 < l_2$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -l_0 & l_1 & 0 \\ -1 & -l_0 & 0 & l_2 \\ 0 & 1 & d_1 & d_2 \end{bmatrix}$$

Then the associated ring $R(l_0, l_1, l_2) := R(A, P)$ is graded by $\mathbb{Z}^4/\mathrm{im}(P^*) \cong \mathbb{Z}$, and is explicitly given by

$$R(l_0, l_1, l_2) = \mathbb{K}[T_1, T_2, T_3, T_4]/\langle T_1 T_2^{l_0} + T_3^{l_1} + T_4^{l_2} \rangle,$$
$$\deg(T_1) = l_1 l_2 - l_0, \quad \deg(T_2) = 1, \quad \deg(T_3) = l_2, \quad \deg(T_4) = l_1.$$

Proposition 6.5. *For the $\mathbb{K}^*$-surface $X = X(l_0, l_1, l_2)$ with Cox ring $R(l_0, l_1, l_2)$, the following statements hold*:

(i) *X is nontoric and we have $\mathrm{Cl}(X) = \mathbb{Z}$,*
(ii) *X comes with precisely one singularity,*
(iii) *X is a del Pezzo surface if and only if $l_0 < l_1 + l_2 + 1$ holds,*
(iv) *X is almost homogeneous if and only if $l_0 \leqslant l_1$ holds.*

Moreover, any normal complete rational nontoric $\mathbb{K}^*$-surface of Picard number one with precisely one singularity is isomorphic to some $X(l_0, l_1, l_2)$.

Proof. First note that $X = X(l_0, l_1, l_2)$ is obtained as in Construction 3.10: the group $H_X = \mathbb{K}^*$ acts on $\mathbb{K}^4$ by

$$t \cdot z = (t^{l_1 l_2 - l_0} z_1, tz_2, t^{l_2} z_3, t^{l_1} z_4),$$

the total coordinate space $\overline{X} := V(T_1 T_2^{l_0} + T_3^{l_1} + T_4^{l_2})$ is invariant under this action and we have

$$\widehat{X} = \overline{X} \setminus \{0\}, \quad X = \widehat{X}/\mathbb{K}^*.$$

Thus, $\mathrm{Cl}(X) = \mathbb{Z}$ holds and, since the Cox ring $\mathcal{R}(X) = R(l_0, l_1, l_2)$ is not a polynomial ring, X is nontoric.

Using [1, Prop. III.3.1.5], we show that the set of singular points of X consists of the image $x_0 \in X$ of the point $(1, 0, 0, 0) \in \widehat{X}$ under the quotient map $\widehat{X} \to X$. If $l_1 l_2 - l_0 > 1$ holds, then the local divisor class group

$$\mathrm{Cl}(X, x_0) = \mathbb{Z}/(l_1 l_2 - l_0)\mathbb{Z}$$

is nontrivial and thus $x_0 \in X$ singular. If $l_1 l_2 - l_0 = 1$ holds, then we have $l_0 > 1$ and therefore $(1, 0, 0, 0) \in \widehat{X}$ and hence $x_0 \in X$ is singular. Since all other local

divisor class groups of X are trivial and, moreover, all singular points of $\widehat{X}$ lie in the orbit $\mathbb{K}^* \cdot (1, 0, 0, 0)$, we conclude that $x_0 \in X$ is the only singular point.

According to [1, Prop. III.3.4.1], the anticanonical class of X is $l_1 + l_2 + 1 - l_0$. This proves (iii). Finally, for (iv), we infer from Proposition 6.2 that the existence of a horizontal Demazure P-root is equivalent to the existence of an integer α with $l_0 \leqslant \alpha \leqslant l_1$, which in turn is equivalent to $l_0 \leqslant l_1$.

We come to the supplement. The surface X arises from a ring $R(A, P)$, where we may assume that $R(A, P)$ is minimally presented. The first task is to show that $n = 4$, $m = 0$ and $r = 2$ holds. We have

$$n + m - (r - 1) = \dim(X) + \mathrm{rk}(\mathrm{Cl}(X)) = 3.$$

Any relation g_I involving only three variables gives rise to a singularity in the source and a singularity in the sink of the $\mathbb{K}^*$-action. We conclude that at most two of the monomials occuring in the relations may depend only on one variable. Thus, the above equation shows that $n = 4$, $m = 0$ and $r = 2$ hold.

We may assume that the defining equation is of the form $T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}$. Again, since one of the two elliptic fixed points must be smooth, we can conclude that one of l_{0j} equals one, say l_{01}. Now it is a direct consequence of the description of the local divisor class groups given in [1, Prop. III.3.1.5] that a $\mathbb{K}^*$-surface with precisely one singularity arises from a matrix P as in the assertion. □

Now we look at the log terminal ones of the $X(l_0, l_1, l_2)$; recall that a singularity is *log terminal* if all its resolutions have discrepancies bigger than -1. Over $\mathbb{C}$, the log terminal surface singularities are precisely the quotient singularities by subgroups of $\mathrm{GL}_2(\mathbb{C})$, see for example [18, Sec. 4.6]. The *Gorenstein index* of X is the minimal positive integer $\imath(X)$ such that $\imath(X)$ times the canonical divisor $\mathcal{K}_X$ is Cartier.

Corollary 6.6. *Assume that $X = X(l_1, l_2, l_3)$ is log terminal. Then we have the following three cases*:

(i) *the surface X is almost homogeneous,*
(ii) *the singularity of X is of type E_7,*
(iii) *the singularity of X is of type E_8.*

Moreover, for the almost homogenoeus surfaces $X = X(l_1, l_2, l_3)$ of Gorenstein index $\imath(X) = a$, we have

(i) $(l_0, l_1, l_2) = (1, l_1, l_2)$ *with the bounds* $l_2 \leqslant l_1 \leqslant \frac{8}{3}a^2 + \frac{4}{3}a$,
(ii) $(l_0, l_1, l_2) = (2, l_1, 2)$ *with the bound* $l_1 \leqslant 4a$,
(iii) $(l_0, l_1, l_2) = (3, 3, 2), (2, 4, 3), (2, 5, 3), (3, 5, 2)$.

Proof. The condition that X is log terminal means that the number $l_0 l_1 l_2$ is bounded by $l_0 l_1 + l_0 l_2 + l_1 l_2$; this can be seen by explicitly performing the canonical resolution of singularities of $X(l_0, l_1, l_2)$, see [11, Sec. 3]. Thus, the allowed (l_0, l_1, l_2) must be platonic triples and we are left with

$$(1, l_1, l_2),\ (2, l_1, 2),\ (3, 3, 2),\ (2, 4, 3),\ (2, 5, 3),\ (3, 5, 2),\ (4, 3, 2),\ (5, 3, 2).$$

The last two give the surfaces with singularities E_7, E_8 and in all other cases, the resulting surface is almost homogeneous by Proposition 6.5. The Gorenstein

condition says that $a\mathcal{K}_X$ lies in the Picard group. According to [1, Cor. III.3.1.6], this is equivalent to the fact that $l_1 l_2 - l_0$ divides $a \cdot (l_1 + l_2 + 1 - l_0)$. The bounds then follow by direct estimations, see [14, Sec. 7.2] for details. □

Corollary 6.7. *The following tables list the triples* (l_0, l_1, l_2) *together with roots of* $\mathrm{Aut}(X)$ *for the log terminal almost homogeneous complete rational* $\mathbb{K}^*$*-surfaces* $X = X(l_0, l_1, l_2)$ *with precisely one singularity up to Gorenstein index* $\imath(X) = 5$.

$\imath(X) = 1$	$\imath(X) = 2$	$\imath(X) = 3$
(1, 3, 2): {1, 2, 3}	(1, 7, 3): {1, 3, 4, 7}	(2, 7, 2): {2, 3, 5, 7}
(2, 3, 2): {2, 3}		(1, 13, 4): {1, 4, 5, 9, 13}
(3, 3, 2): {3}		(1, 8, 5): {3, 5, 8}

$\imath(X) = 4$	$\imath(X) = 5$
(2, 5, 2): {2, 3, 5}	(2, 11, 2): {2, 3, 5, 7, 9, 11}
(1, 21, 5): {1, 5, 6, 11, 16, 21}	(1, 13, 7): {2, 6, 13}
	(2, 4, 3): {3, 4}
	(1, 17, 3): {2, 3, 5, 8, 11, 14, 17}
	(1, 31, 6): {1, 6, 7, 13, 19, 25, 31}
	(1, 18, 7): {4, 7, 11, 18}

7. Structure of the Semisimple Part

We describe the root system of the semisimple part of the automorphism group of a nontoric normal complete rational variety admitting a torus action of complexity one. Let us first recall the necessary background on semisimple groups and their root systems.

A connected linear algebraic group G is semisimple if it has only trivial closed connected commutative normal subgroups. Any linear algebraic group G admits a maximal connected semisimple subgroup $G^{\mathrm{ss}} \subseteq G$ called a semisimple part. The semisimple part is unique up to conjugation by elements from the unipotent radical. If G is semisimple, then the set $\Phi_G \subseteq \mathbb{X}_{\mathbb{R}}(T)$ of roots with respect to a given maximal torus $T \subseteq G$ is a root system, i.e., for every $\alpha \in \Phi_G$ one has

$$\Phi_G \cap \mathbb{R}\alpha = \{\pm\alpha\}, \quad s_\alpha(\Phi_G) = \Phi_G,$$

where $s_\alpha \colon \mathbb{X}_{\mathbb{R}}(T) \to \mathbb{X}_{\mathbb{R}}(T)$ denotes the reflection with fixed hyperplane $\alpha^\perp$ with respect to a given scalar product on $\mathbb{X}_{\mathbb{R}}(T)$. The possible root systems are elementarily classified; for us, the following types (always realized with the standard scalar product) will be important:

$$A_n := \{e_i - e_j;\ 1 \leqslant i, j \leqslant n+1,\ i \neq j\} \subseteq \mathbb{R}^{n+1},$$
$$B_2 := \{\pm e_1,\ \pm e_2,\ \pm(e_1 + e_2),\ \pm(e_1 - e_2)\} \subseteq \mathbb{R}^2.$$

The root system of a connected semisimple linear group G determines G up to coverings; for example, A_n belongs to the (simply connected) special linear group SL_{n+1} and B_2 to the (simply connected) symplectic group Sp_4.

We turn to varieties with a complexity one torus action. Consider data A, P as in Construction 3.7 and the resulting ring $R(A, P)$ with its fine K_0-grading and the coarser K-grading. Recall that the fine grading group K_0 splits as

$$K_0 = K_0^{\mathrm{vert}} \oplus K_0^{\mathrm{hor}}, \quad \text{where } K_0^{\mathrm{vert}} := \langle \deg_{K_0}(S_k) \rangle,\ K_0^{\mathrm{hor}} := \langle \deg_{K_0}(T_{ij}) \rangle.$$

Note that $K_0^{\mathrm{vert}} \cong \mathbb{Z}^m$ is freely generated by $\deg_{K_0}(S_1), \ldots, \deg_{K_0}(S_m)$. Moreover, by Remark 3.9, the direct factor $\mathbb{Z}^s$ of the column space $\mathbb{Z}^{r+s}$ of P is identified via $Q_0 \circ P^*$ with the kernel of the downgrading map $K_0 \to K$.

Definition 7.1. Let A, P be as in Construction 3.7 such that the associated ring $R(A, P)$ is minimally presented and write α_κ for the P-root, i.e., the $\mathbb{Z}^s$-part, associated to Demazure P-root κ.

(i) We call a P-root α_κ *semisimple* if $-\alpha_\kappa = \alpha_{\kappa'}$ holds for some Demazure P-root κ'.
(ii) We call a semisimple P-root α_κ *vertical* if $\alpha_\kappa \in K_0^{\mathrm{vert}}$ and *horizontal* if $\alpha_\kappa \in K_0^{\mathrm{hor}}$ holds.
(iii) We write Φ_P^{ss}, Φ_P^{vert} and Φ_P^{hor} for the set of semisimple, vertical semisimple and horizontal semisimple P-roots in $\mathbb{R}^s$ respectively.

Theorem 7.2. *Let A, P be as in Construction 3.7 such that $R(A, P)$ is minimally presented and let X be a (nontoric) variety with a complexity one torus action $T \times X \to X$ arising from A, P according to Construction 3.10. Then the following statements hold.*

(i) *Φ_P^{vert}, Φ_P^{hor} and Φ_P^{ss} are root systems, we have $\Phi_P^{\mathrm{ss}} = \Phi_P^{\mathrm{vert}} \oplus \Phi_P^{\mathrm{hor}}$ and Φ_P^{ss} is the root system with respect to T of the semisimple part $\mathrm{Aut}(X)^{\mathrm{ss}}$.*
(ii) *For $p \in K$ denote by m_p the number of variables S_k with $\deg_K(S_k) = p$. Then we have*

$$\Phi_P^{\mathrm{vert}} \cong \bigoplus_{p \in K} A_{m_p - 1}, \qquad \sum_{p \in K} (m_p - 1) < \dim(X) - 1.$$

(iii) *Suppose $\Phi_P^{\mathrm{hor}} \neq \varnothing$. Then $r = 2$ holds, and, after suitably renumbering the variables one has*
  (a) *$T_{01}T_{02} + T_{11}T_{12} + T_2^{l_2}$, $\overline{w}_{01} = \overline{w}_{11}$ and $\overline{w}_{02} = \overline{w}_{12}$,*
  (b) *$T_{01}T_{02} + T_{11}^2 + T_2^{l_2}$, and $\overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11}$*
  for the defining relation of $R(A, P)$ and the degrees $\overline{w}_{ij} = \deg_K(T_{ij})$ of the variables.
(iv) *In the above case* (iii a), *we obtain the following possibilities for the root system Φ_P^{hor}:*
  - *If $l_{21} + \ldots + l_{2n_2} \geqslant 3$ holds, then one has*

$$\Phi_P^{\mathrm{hor}} = \begin{cases} A_1 \oplus A_1, & \overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11} = \overline{w}_{12}, \\ A_1, & \textit{otherwise.} \end{cases}$$

- *If $n_2 = 2$ and $l_{21} = l_{22} = 1$ hold, then one has*

$$\Phi_P^{\mathrm{hor}} = \begin{cases} A_3, & \overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11} = \overline{w}_{12} = \overline{w}_{21} = \overline{w}_{22}, \\ A_2, & \overline{w}_{01} = \overline{w}_{11} = \overline{w}_{21}, \overline{w}_{02} = \overline{w}_{12} = \overline{w}_{22}, \overline{w}_{01} \neq \overline{w}_{02}, \\ A_1 \oplus A_1, & \overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11} = \overline{w}_{21}, \overline{w}_{01} \neq \overline{w}_{21}, \overline{w}_{01} \neq \overline{w}_{22}, \\ A_1, & \textit{otherwise}. \end{cases}$$

(v) *In the above case* (iii b), *we obtain the following possibilities for the root system Φ_P^{hor}:*

- *If $l_{21} + \ldots + l_{2n_2} \geqslant 3$ holds, then one has*

$$\Phi_P^{\mathrm{hor}} = A_1.$$

- *If $n_2 = 1$ and $l_{21} = 2$ hold, then one has*

$$\Phi_P^{\mathrm{hor}} = \begin{cases} A_1 \oplus A_1, & \overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11} = \overline{w}_{21}, \\ A_1, & \textit{otherwise}. \end{cases}$$

- *If $n_2 = 2$ and $l_{21} = l_{22} = 1$ hold, then one has*

$$\Phi_P^{\mathrm{hor}} = \begin{cases} B_2, & \overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11} = \overline{w}_{21} = \overline{w}_{22}, \\ A_1, & \textit{otherwise}. \end{cases}$$

The rest of the section is devoted to the proof of this theorem. Some of the steps are needed later on and therefore formulated as Lemmas. We fix A, P as in Construction 3.7 and assume that $R(A, P)$ is minimally presented.

Lemma 7.3. *Let δ be a nonzero primitive K_0-homogeneous derivation on $R(A, P)$, decompose $\deg_{K_0}(\delta) = w^{\mathrm{vert}} + w^{\mathrm{hor}}$ according to $K_0 = K_0^{\mathrm{vert}} \oplus K_0^{\mathrm{hor}}$ and write $w^{\mathrm{vert}} = \sum b_k w_k$ with $w_k = \deg_{K_0}(S_k)$ and $b_k \in \mathbb{Z}$.*

(i) *If δ is of vertical type, then there is a k_0 with $b_{k_0} = -1$ and $b_k \geqslant 0$ for all $k \neq k_0$. Moreover, w^{hor} belongs to the weight monoid of $R(A, P)$.*

(ii) *If δ is of horizontal type, then $b_k \geqslant 0$ holds for all k. Moreover, $-w^{\mathrm{hor}}$ does not belong to the weight monoid of $R(A, P)$.*

Proof. If δ is nonzero of vertical type, then Theorem 4.4 (i) directly yields the assertion. If δ is nonzero of horizontal type, then it is of the form $\delta = h\delta_{C,\beta}$ as in Theorem 4.4 (ii). For the degree of δ, we have

$$\deg_{K_0}(\delta) = w^{\mathrm{vert}} + w^{\mathrm{hor}} = \deg_{K_0}(h)^{\mathrm{vert}} + \deg_{K_0}(h)^{\mathrm{hor}} + \deg_{K_0}(\delta_{C,\beta}),$$

where $\deg_{K_0}(\delta_{C,\beta})$ lies in K_0^{hor} by Construction 4.3. Since $\deg_{K_0}(h)^{\mathrm{vert}}$ belongs to the weight monoid S and, due to primitivity, $\deg_{K_0}(\delta)$ lies outside the weight cone, we must have $w^{\mathrm{hor}} \neq 0$. Moreover, for a T_{ij} with $\delta(T_{ij}) \neq 0$ and $w_{ij} := \deg_{K_0}(T_{ij})$, the degree computation of Construction 4.3 shows

$$0 \neq \deg_{K_0}(h)^{\mathrm{hor}} + \deg_{K_0}(\delta_{C,\beta}) + w_{ij} = \deg_{K_0}(\delta(T_{ij}))^{\mathrm{hor}}.$$

Thus, $w^{\mathrm{hor}} + w_{ij}$ is a nonzero element in S. If also $-w^{\mathrm{hor}}$ belongs to S, then we can take $0 \neq f, g \in R(A, P)$ homogeneous of degree $w^{\mathrm{hor}} + w_{ij}$ and $-w^{\mathrm{hor}}$ respectively. The product fg is of degree w_{ij}. By Corollary 3.6, this means $fg = cT_{ij}$ with $c \in \mathbb{K}$. A contradiction to the fact that T_{ij} is K_0-prime. $\square$

By definition, the semisimple roots occur in pairs α_+, α_- with $\alpha_+ + \alpha_- = 0$. Given a pair κ_+, κ_- of Demazure P-roots defining α_+, α_-, write δ_+, δ_- for the derivations arising from κ_+, κ_- via Construction 5.7. We call the pairs κ_+, κ_- and δ_+, δ_- *associated* to α_+, α_-. Note that associated pairs κ_+, κ_- or δ_+, δ_- are in general not uniquely determined by α_+, α_-.

Lemma 7.4. *Let δ_+, δ_- be a pair of primitive K_0-homogeneous derivations associated to a pair α_+, α_- of semisimple roots.*

(i) *The roots α_+, α_- are the K_0-degrees of the derivations δ_+, δ_-. In particular, we have*
$$\deg_{K_0}(\delta_+) + \deg_{K_0}(\delta_-) = \alpha_+ + \alpha_- = 0.$$
(ii) *If δ_+ is of vertical type, then also δ_- is of vertical type and α_+, α_- are both vertical.*

(iii) *If δ_+ is of horizontal type, then also δ_- is of horizontal type and α_+, α_- are both horizontal.*

Proof. The first assertion is clear by Construction 5.7. Using the decomposition $K_0 = K_0^{\mathrm{vert}} \oplus K_0^{\mathrm{hor}}$, we obtain
$$\alpha_+^{\mathrm{vert}} + \alpha_-^{\mathrm{vert}} = 0, \quad \alpha_+^{\mathrm{hor}} + \alpha_-^{\mathrm{hor}} = 0.$$
If δ_+ is of horizontal type, then Lemma 7.3 (ii) shows that $-\alpha_+^{\mathrm{hor}} = \alpha_-^{\mathrm{hor}}$ does not belong to the weight monoid. Thus Lemma 7.3 (i) says that δ_- must be of horizontal type. Moreover, Lemma 7.3 (ii) shows that α_+^{vert} and α_-^{vert} vanish and thus α_+, α_- are horizontal.

If δ_+ is of vertical type, then, by the preceeding consideration, also δ_- must be vertical. Moreover, Lemma 7.3 (i) tells us that α_+^{hor} and α_-^{hor} both belong to the weight monoid. As seen above they have opposite signs. Since the weight cone is pointed, we obtain $\alpha_+^{\mathrm{hor}} = \alpha_-^{\mathrm{hor}} = 0$, which means that α_+, α_- are vertical. □

For the subsequent study, we will perform certain elementary column and row operations with the matrix P, which we will call *admissible*:

(I) swap two columns inside a block $v_{ij_1}, \ldots, v_{ij_{n_i}}$,

(II) swap two whole column blocks $v_{ij_1}, \ldots, v_{ij_{n_i}}$ and $v_{i'j_1}, \ldots, v_{i'j_{n_{i'}}}$,

(III) add multiples of the upper r rows to one of the last s rows,

(IV) any elementary row operation among the last s rows.

(V) swap two columns inside the d' block.

The operations of type (III) and (IV) do not change the ring $R(A, P)$ whereas the types (I), (II), (V) correspond to certain renumberings of the variables of $R(A, P)$ keeping the (graded) isomorphy type.

For a Demazure P-root $\kappa = (u, k_0)$ of vertical type, the index k_0 is uniquely determined by the $\mathbb{Z}^s$-part of u. Thus, we may speak of the *distinguished* index of a vertical P-root. Note that for any pair $\alpha^\pm$ of vertical semisimple P-roots, the distinguished indices satisfy $k_0^+ \neq k_0^-$.

Lemma 7.5. *Let $1 \leqslant k_0^+ < k_0^- \leqslant m$ be given and denote by $f \in \mathbb{Z}^{n+m}$ the vector with $f_{k_0^\pm} = \mp 1$ and all other entries zero. Then the following statements are equivalent.*

(i) *There exists a pair* $\alpha^\pm$ *of vertical semisimple roots with distinguished indices* $k_0^\pm$.
(ii) *The vector* f *can be realized by admissible operations of type* (III) *and* (IV) *as the* $(r+1)$*th row of* P.
(iii) *The variables* $S_{k_0^+}$ *and* $S_{k_0^-}$ *have the same* K*-degree.*

Proof. Suppose that (i) holds. Let $\kappa^\pm = (u^\pm, k_0^\pm)$ be a pair of Demazure P-roots associated to $\alpha^\pm$. Then $u := u^+ + u^-$ satisfies $\langle u, v_{ij}\rangle \geqslant 0$ for all i, j and $\langle u, v_k\rangle \geqslant 0$ for all k. Since the columns of P generate $\mathbb{Q}^{r+s}$ as a cone, we obtain $u = 0$. Consequently $u^- = -u^+$ holds and we conclude

$$\langle u^+, v_{ij}\rangle = 0 \text{ for all } i, j, \quad \langle u^+, v_k\rangle = 0 \text{ for all } k \neq k_0^\pm,$$
$$\langle u^+, v_{k_0^+}\rangle = -1, \quad \langle u^+, v_{k_0^-}\rangle = 1.$$

Now write $u^+ = (u_1^+, \alpha^+)$ with the $\mathbb{Z}^s$-part α^+ and let σ be an $(s-1)\times s$ matrix complementing the (primitive) row α^+ to a unimodular matrix. Consider the block matrix

$$\begin{bmatrix} E_r & 0 \\ u_1^+ & \alpha^+ \\ 0 & \sigma \end{bmatrix}.$$

Applying this matrix from the left to P describes admissible operations of type (III) and (IV) realizing the vector f as the $(r+1)$th row of P. Thus, (i) implies (ii).

To see that (ii) implies (i), we may assume that f is already the $(r+1)$th row of P. Consider $u^\pm \in \mathbb{Z}^{r+s}$ having $u^\pm_{r+1} = \pm 1$ as the only nonzero coordinate. Then the $\mathbb{Z}^s$-parts $\alpha^\pm$ of the vertical Demazure P-roots $\kappa^\pm = (u^\pm, k_0^\pm)$ are as wanted.

Clearly, (ii) implies (iii). Conversely, the implication "(iii) $\Rightarrow$ (ii)" is obtained by similar arguments as "(i) $\Rightarrow$ (ii)". □

Lemma 7.6. *Let* $\mathbb{E}_q$ *denote the* $q \times (q+1)$ *block matrix* $[-\mathbf{1}, E_q]$, *where* $\mathbf{1}$ *is a column with all entries equal one and* E_q *is the* $q \times q$ *unit matrix. After admissible operations of type* (III), (IV) *and* (V), *the* $[d, d']$ *block of* P *is of the form*

$$[d, d'] = \begin{bmatrix} 0 & \mathbb{E}_{m_{p_1}} & & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & \mathbb{E}_{m_{p_t}} & 0 \\ d^* & d_1'' & \dots & d_t'' & d'' \end{bmatrix}$$

where $p_1, \dots, p_t \in K$ *are the elements such that the number* m_{p_i} *of variables* S_k *of degree* p_i *is at least two and* d_i'' *is a block of length* m_{p_i} *with only the first column possibly nonzero. Moreover,* Φ_P^{vert} *is a root system and we have*

$$\Phi_P^{\mathrm{vert}} \cong \bigoplus_{p\in K} A_{m_p - 1}, \qquad \sum_{p\in K}(m_p - 1) < \dim(X) - 1.$$

Proof. This is a direct application of Lemma 7.5. □

Lemma 7.7. *If there exists a pair of semisimple roots* $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$, *then* $r = 2$, *and after suitably renumbering* l_0, l_1, l_2, *the following two cases remain.*

(i) *We have* $n_0 = n_1 = 2$ *and* $l_{01} = l_{02} = l_{11} = l_{12} = 1$ *and for any pair* $\delta_\pm$ *of derivations associated to* $\alpha_\pm$ *one has* $i_0^+ = i_0^- = 2$.

(ii) *We have* $n_0 = 1$, $l_{01} = 2$ *and* $n_1 = 2$, $l_{11} = l_{12} = 1$ *and for any pair* $\delta_\pm$ *of derivations associated to* $\alpha_\pm$ *one has* $i_0^+ = i_0^- = 2$.

Proof. Lemma 7.4 says that δ_+, δ_- are of horizontal type, by Construction 5.7 they are of the form $\delta_\pm = h_\pm \delta_{C^\pm,\beta^\pm}$ and the degree computation of Construction 4.3 gives

$$\deg_{K_0}(\delta_\pm) = (r-1)\mu - \sum_{k \neq i_0^\pm} w_{kc_k^\pm} + w_\pm,$$

where μ is the common K_0-degree of the relations, w_{ij} the K_0-degree of T_{ij}, the $i_0^\pm$-th component of $\beta_\pm$ vanishes and $w_\pm$ is the K_0-degree of $h_\pm$. Now fix two distinct i^+, i^- with $i^\pm \neq i_0^\pm$ and $l_{i^+c_{i^+}^+} = 1$. Then $\deg_{K_0}(\delta_+) + \deg_{K_0}(\delta_-) = 0$ leads to

$$w := w_+ + w_- + \sum_{k \neq i_0^+, i^+} (\mu - w_{kc_k^+}) + \sum_{k \neq i_0^-, i^-} (\mu - w_{kc_k^-}) = w_{i^+c_{i^+}^+} + w_{i^-c_{i^-}^-}.$$

Note that all summands are elements of the weight monoid of $R(A, P)$ and, except possibly $w_\pm$, all are nonzero. Moreover, Corollary 3.6 (ii) shows that the K_0-homogeneous component $R(A, P)_w$ is generated by f^+f^-, where

$$f^+ := T_{i^+c_{i^+}^+}, \quad f^- := T_{i^-c_{i^-}^-}.$$

Now, choosing suitable presentations of the μ, write the first presentation of w as the K_0-degree of a monomial f in the variables T_{ij} corresponding to the occuring w_{ij}. Then $f = f^+f^-$ holds. Since f^+ and f^- are K_0-prime, we conclude $w_\pm = 0$ and $r = 2$. Renumbering $i_0^+ \mapsto 2$, $i^+ \mapsto 1$, the above equation simplifies to

$$w = \mu - w_{0c_0^+} + \mu - w_{i_2^-c_{i_2^-}^-} = w_{1c_1^+} + w_{i^-c_{i^-}^-},$$

where i_2^- differs from i_0^- and i^-. We conclude further $i_2^- = 1$ and $\sum l_{0j} = \sum l_{1j} = 2$ and $i_0^- = i_0^+$. Since $l_{i^+c_{i^+}^+} = l_{1c_1^+}$ equals one, we arrive at the cases (i) and (ii). □

The above lemma shows that for a given pair $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$, all associated pairs of Demazure P-roots (or derivations) share the same $i_0 = i_0^+ = i_0^-$. This allows us to speak about the *distinguished index* i_0 *of* $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$.

Lemma 7.8. *Suppose* $n_0 = n_1 = 2$, $l_{01} = l_{02} = l_{11} = l_{12} = 1$. *If there exists a pair* $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$ *with distinguished index* $i_0 = 2$, *then* P *can be brought by admissible operations, without moving the* n_2*-block, into the form*

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_2 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d_{12}^* & d_2^* & d_*' \end{bmatrix}, \tag{7.8.1}$$

where the lower line is a matrix of size $(s-1) \times (n+m)$. *Conversely, if* P *is of the above shape, then* $\alpha_\pm = (\pm 1, 0) \in \Phi_P^{\mathrm{hor}}$ *has distinguished index* $i_0 = 2$. *Moreover, up to admissible operations of type* (III) *and* (IV), *situation* (7.8.1) *is equivalent to*

$$\deg_K(T_{01}) = \deg_K(T_{12}), \quad \deg_K(T_{02}) = \deg_K(T_{11}).$$

Proof. Fix an associated pair $\kappa_\pm = (u^\pm, 2, i_1^\pm, C_\pm)$ of Demazure P-roots. Renumbering the variables, we first achieve $i_1^+ = 1$ and $C_+ = (1, 1, 1)$. Adding suitable multiples of the top two rows of P to the lower s rows, brings P into the form

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_2 & 0 \\ 0 & d_{02} & 0 & d_{12} & d_2 & d' \end{bmatrix}$$

Now we explicitly go through the defining conditions of the Demazure P-root κ_+ with

$$u^+ = (u_1^+, u_2^+, \alpha_+), \text{ where } u_i^\pm \in \mathbb{Z}, \quad i_1^+ = 1, \quad C_+ = (1, 1, 1).$$

This gives in particular, $u_1^+ = -1$ and $u_1^- = 1$. Going through the conditions of a Demazure P-root $\kappa_- = (u^-, 2, i_1^-, C_-)$ leaves us with the two possibilities

$$u^- = (1, -1, -\alpha_+), \quad i_1^- = 0, \quad C_- = (2, 2, 1),$$
$$u^- = (0, -1, -\alpha_+), \quad i_1^- = 1, \quad C_- = (2, 2, 1).$$

In both cases, we obtain

$$\langle \alpha_+, d_{02} \rangle = \langle \alpha_+, d_{12} \rangle = 1,$$
$$\langle \alpha_+, d_{2j} \rangle = -l_{2j} \text{ for } j = 1, \ldots, n_2,$$
$$\langle \alpha_+, d'_k \rangle = 0 \text{ for } j = 1, \ldots, m.$$

Now choose any invertible $s \times s$ matrix with α_+ as its first row and apply it from the left to P. Then the third row of P looks as follows

$$\begin{bmatrix} 0 & 1 & 0 & 1 & -l_2 & 0 \end{bmatrix}.$$

Adding suitable multiples of the third row to the last $s-1$ rows and adding the second to the third row brings P into the desired form. The remaining statements are directly checked. See also [14, Section 7.3] for a detailed proof. □

Lemma 7.9. *Consider a pair $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$ with distinguished index $i_0 = 2$. There exists another pair $\tilde{\alpha}_\pm \in \Phi_P^{\mathrm{hor}}$ with distinguished index $\tilde{i}_0 = 2$ if and only if we have*

$$\deg_K(T_{01}) = \deg_K(T_{02}) = \deg_K(T_{11}) = \deg_K(T_{12}).$$

Moreover, if the latter holds, then $\alpha_\pm, \tilde{\alpha}_\pm$ are the only pairs of semisimple roots with distinguished index 2 and they form a root system isomorphic to $A_1 \oplus A_1$.

Proof. We may assume that we are in the setting of Lemma 7.8. Then we just have to go through the possible cases $\tilde{i}_1^\pm$ and C_- and observe that the existence of $\tilde{\alpha}_\pm$ implies a special shape of P equivalent to the above degree condition. □

Lemma 7.10. *Suppose $n_0 = 1$, $l_{01} = 2$ and $n_2 = 2$, $l_{11} = l_{12} = 1$. Then there is at most one pair $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$ with distinguished index $i_0 = 2$. If there is one, then P can be brought by admissible operations, without moving the n_2-block, into the form*

$$P = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_2 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ d_{01}^* & 0 & 0 & d_2^* & d'_* \end{bmatrix}, \tag{7.10.1}$$

where the lower line is a matrix of size $(s-1)\times(n+m)$. *Conversely, if* P *is of the above shape, then* $\alpha_\pm = (\pm 1,\, 0) \in \Phi_P^{\mathrm{hor}}$ *has distinguished index* $i_0 = 2$. *Moreover, up to admissible operations of type* (III) *and* (IV), *situation* (7.10.1) *is equivalent to*

$$\deg_K(T_{01}) = \deg_K(T_{11}) = \deg_K(T_{12}).$$

Proof. This is a similar computation as in the previous lemma. Clearly, we may assume $C_+ = (1,\, 1,\, 1)$ and by suitable row operations, we bring P into the form

$$P = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_2 & 0 \\ d_{01} & 0 & d_{12} & d_2 & d' \end{bmatrix}$$

Now enter the defining conditions of a Demazure P-root κ_+ with $u^+ = (u_1^+,\, u_2^+,\, \alpha_+)$. It turns out that $C_- = (1,\, 2,\, 2)$ or $C_- = (1,\, 2,\, 1)$ must hold. We end up with $u_1^+ = 0$ and

$$\begin{aligned} \langle \alpha_+,\, d_{01}\rangle &= 2u_2^+ - 1, \\ \langle \alpha_+,\, d_{12}\rangle &= 1, \\ \langle \alpha_+,\, d_{2j}\rangle &= -u_2^+ l_{2j} \text{ for } j = 1,\, \ldots,\, n_2, \\ \langle \alpha_+,\, d'_k\rangle &= 0 \text{ for } j = 1,\, \ldots,\, m. \end{aligned}$$

As before, this enables us to bring P via suitable row operations into the desired form. The remaining statements are directly seen, we refer again to [14, Section 7.3] for the details. □

Proof of Theorem 7.2. Lemma 7.6 shows that Φ_P^{vert} is a root system and has the desired form. This proves (ii). Concerning Φ_P^{hor}, assertion (iii) as well as the cases $l_{21} + \ldots + l_{2n_2} \geqslant 3$ of assertions (iv) and (v) are proven by Lemmas 7.7, 7.8 and 7.9. The remaining cases of (iv) and (v) are deduced using the special shape of P guaranteed by Lemma 7.7. So, Φ_P^{vert} and Φ_P^{hor} are root systems of the desired shape. Lemma 7.4 and the decomposition of K_0 into a vertical and a horizontal part show that Φ_P^{ss} splits into the direct sum of Φ_P^{vert} and Φ_P^{hor}.

To conclude the proof, we have to verify that Φ_P^{ss} is in fact the root system of the semisimple part of $\mathrm{Aut}(X)$. For this, it suffices to realize the root system Φ_P^{ss} as the root system of some semisimple subgroup $G \subseteq \mathrm{Aut}(X)$ (which then necessarily is a semisimple part). Consider $\overline{X} := \operatorname{Spec} R(A,\, P)$ and the action of $H_X := \operatorname{Spec} \mathbb{K}[K]$. The group $G \subseteq \mathrm{Aut}(X)$ will be induced by a representation of a suitable semisimple group on $\mathbb{K}^{n+m}$ commuting with the action of H_X and leaving $\overline{X} := \operatorname{Spec} R(A,\, P)$ invariant.

We first care about Φ_P^{vert}. Consider the semisimple group $G^{\mathrm{vert}} := \bigtimes_p \mathrm{SL}_{m_p}$. It acts on $\mathbb{K}^{n+m}$: trivially on $\mathbb{K}^n$ and blockwise on $\mathbb{K}^m = \bigoplus_p \mathbb{K}^{m_p}$. This action commutes with the action of H_X and leaves $\overline{X}$ as well as $\widehat{X}$ invariant. Thus the G^{vert}-action descends to X. This realizes $G^{\mathrm{vert}} \subseteq \mathrm{Aut}(X)$ as a subgroup with root system Φ_P^{vert}.

Similarly, going through the cases, we realize Φ_P^{hor} as a root system of a semisimple group $G^{\mathrm{hor}} \subseteq \mathrm{Aut}(X)$. Recall that for the defining relation, we have the possibilities

$$\text{(a)} \quad T_{01}T_{02} + T_{11}T_{12} + T_2^{l_2}, \qquad \text{(b)} \quad T_{01}T_{02} + T_{11}^2 + T_2^{l_2}.$$

Consider case (a). If we have the equations $\overline{w}_{01} = \overline{w}_{11}$ and $\overline{w}_{02} = \overline{w}_{12}$, then $g \in \mathrm{SL}_2$ acts on the T_{01}, T_{11}-space as $(g^t)^{-1}$ and on the T_{02}, T_{12}-space as g. By trivial extension, we obtain an action on $\mathbb{K}^{n+m}$ commuting with the H-action, leaving the defining relation and thus $\overline{X}$ invariant. Thus, we have an induced effective action of a semisimple group G^{hor} with root system A_1 on X.

If we have $\overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11} = \overline{w}_{12}$, then the canonical action of the group SO_4 on the $T_{01}, T_{11}, T_{02}, T_{12}$-space extends trivially to $\mathbb{K}^{n+m}$, commutes with the action of H_X and leaves $\overline{X}$ invariant. Again, this gives an induced effective action of a semisimple group G^{hor} on X, this time the root system is $A_1 \oplus A_1$; recall that SO_4 has $\mathrm{SL}_2 \times \mathrm{SL}_2$ as its universal covering.

Now let $n_2 = 2$ and $l_{21} = l_{22} = 1$. Then we have $n = 6$. If all six K-degrees $\overline{w}_{ij}$ coincide, then take the action of SO_6 on $\mathbb{K}^n$ and extend it trivially to $\mathbb{K}^{n+m}$. This induces an action of a semisimple group G^{hor} on X. The root system is that of the universal covering SL_4, i.e. we obtain A_3. If all $\overline{w}_{i1}$ and all $\overline{w}_{i2}$ coincide but we have $\overline{w}_{01} \neq \overline{w}_{02}$, then consider the action of SL_3 given by $(g^t)^{-1}$ on the T_{i1}-space and by g on the T_{i2}-space. This leads to a G^{hor} with root system A_2.

Finally, consider case (b). If $\overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11}$ holds, then the canonical action of the group SO_3 on the T_{01}, T_{02}, T_{11}-space defines a semisimple subgroup G^{hor} of $\mathrm{Aut}(X)$ with root system A_1. If $n_2 = 1$ and $l_{21} = 2$ holds and we have $\overline{w}_{01} = \overline{w}_{02} = \overline{w}_{11} = \overline{w}_{21}$, then $n = 4$ holds and the canonical action of SO_4 on $\mathbb{K}^n$ induces a semisimple subgroup G^{hor} of $\mathrm{Aut}(X)$ with root system $A_1 \oplus A_1$. If we have $n_2 = 2$ and $l_{21} = l_{22} = 1$ and all degrees $\overline{w}_{ij}$ coincide, then the canonical action of SO_5 on $\mathbb{K}^5$ extends to $\mathbb{K}^{n+m}$ and induces a subgroup G^{hor} of $\mathrm{Aut}(X)$ with root system B_2.

Alltogether, we realized the root systems Φ_P^{vert} and Φ_P^{hor} by semisimple subgroups G^{vert} and G^{hor} of $\mathrm{Aut}(X)$. By construction, these groups commute and thus define a semisimple subgroup $G := G^{\mathrm{vert}}G^{\mathrm{hor}}$ of $\mathrm{Aut}(X)$ with the desired root system $\Phi_P^{\mathrm{ss}} = \Phi_P^{\mathrm{vert}} \oplus \Phi_P^{\mathrm{hor}}$. □

Corollary 7.11 (of proof)**.** *In the situation of Theorem 7.2, any pair of semisimple roots defines a subgroup of the automorphism group locally isomorphic to* SL_2*.*

8. Applications

In this section we present applications of Theorem 7.2. A first one concerns the automorphism group of arbitrary nontoric Mori dream surfaces (not necessarily admitting a $\mathbb{K}^*$-action).

Proposition 8.1. *Let X be a nontoric Mori dream surface. Then* $\mathrm{Aut}(X)^0$ *is solvable and the following cases can occur:*

(i) *X is a $\mathbb{K}^*$-surface,*
(ii) $\mathrm{Aut}(X)^0$ *is unipotent.*

Proof. Consider a maximal torus $T \subseteq \mathrm{Aut}(X)^0$. If T is trivial, then we are in case (ii). In particular, $\mathrm{Aut}(X)^0$ is solvable then. If T is one-dimensional, then we are in case (i) and the task is to show that the semisimple part of $\mathrm{Aut}(X)$ is trivial, i.e., $\mathrm{Aut}(X)$ has no semisimple roots. For this, we remark first that as a Mori dream surface with a nontrivial $\mathbb{K}^*$-action, X is rational. Thus, we may assume that X arises from A, P as in Construction 3.7 and that $R(A, P)$ is minimally presented. Note that we have $s = 1$.

The estimate of Theorem 7.2 (ii) forbids vertical semisimple P-roots. Let us see why there are no horizontal semisimple P-roots. Otherwise, Lemmas 7.8 and 7.10 show that we must have $m = 0$ and, up to admissible operations, the matrix P is one of the following:

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & l_2 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -2 & 1 & 1 & 0 \\ -2 & 0 & 0 & l_2 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

Since the columns of P are pairwise different primitive vectors, we obtain $n_2 = 1$ and $l_{21} = 1$; a contradiction to the assumption that the ring $R(A, P)$ is minimally presented. □

We take a brief look at q-dimensional varieties X coming with a nontrivial action of SL_q. They were classified (in the smooth case) by Mabuchi [16]. Let us see how to obtain the rational nontoric part of his list with the aid of Theorem 7.2.

Example 8.2. Let X a nontoric q-dimensional complete normal variety with a nontrivial action of SL_q. Since SL_q has only finite normal subgroups, X comes with a torus action $T \times X \to X$ of complexity one. Proposition 8.1 tells us that X is of dimension at least three. If SL_q acts with an open orbit, then X has a finitely generated divisor class group and, by existence of the T-action of complexity one, must be rational. So, we may assume that we are in the setting of Theorem 7.2, where we end up in the cases A_2, A_3 of second item of (iv), which amounts to the following two possibilities.

(i) We have $q = 3$ and X is the flag variety SL_3/B_3; in particular, $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ and

$$\mathcal{R}(X) \cong \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{22}] \,/\, \langle T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} \rangle$$

hold, where the $\mathbb{Z}^2$-grading is given by

$$\deg(T_{01}) = \deg(T_{11}) = \deg(T_{21}) = (1, 0),$$
$$\deg(T_{02}) = \deg(T_{12}) = \deg(T_{22}) = (0, 1).$$

(ii) We have $q = 4$ and X is the smooth quadric $V(T_0T_1 + T_2T_3 + T_4T_5)$ in $\mathbb{P}_5$, where SL_4 acts as the universal covering of SO_6.

Now assume that SL_q acts with orbits of dimension at most $q - 1$. Then this action gives a root system $A_{q-1} \subseteq \Phi_P^{\mathrm{vert}}$. If X were rational, then Theorem 7.2 (ii) would require $q < \dim(X)$, which is excluded by assumption. Thus, X must be nonrational. In particular, the rational nontoric part of Mabuchi's list is established, even for a priori singular varieties.

We turn to nontoric varieties with a torus action of complexity one which are almost homogeneous under some reductive group. Recall that an action of a reductive group G on X is spherical if some Borel subgroup of G acts with an open orbit on X.

Proposition 8.3. *Let X be a nontoric complete normal variety. Then the following statements are equivalent.*

(i) *X allows a torus action of complexity one and an almost homogeneous reductive group action.*
(ii) *X is spherical with respect to an action of a reductive group of semisimple rank one.*
(iii) *X is isomorphic to a variety as in Theorem 7.2* (iii).

Proof. The implication "(ii) $\Rightarrow$ (i)" is obvious. For the implication "(iii) $\Rightarrow$ (ii)", take a pair of semisimple horizontal P-roots α_+ and α_-. Then the acting torus T of X and the root subgroups $U_\pm$ associated to $\alpha_\pm$ generate a reductive group G of semisimple rank one in $\mathrm{Aut}(X)$ and X is spherical with respect to the action of G.

Assume that (i) holds. Then the open orbit of the acting reductive group G is unirational. Thus, X is unirational. By the existence of a torus action of complexity one, X contains an open subset of the form $T \times C$ with some affine curve C. We conclude that C and hence X are rational. Consequently, X is a Mori dream space. In particular, Corollary 2.4 yields that $\mathrm{Aut}(X)$ is linear algebraic. Moreover, we may assume that we are in the setting of Theorem 7.2. The image of G^0 in $\mathrm{Aut}(X)$ is contained in a maximal connected reductive subgroup G' of $\mathrm{Aut}(X)$. Suitably conjugating G', we may assume that the acting torus T of X is a maximal torus of G'. Since G' is generated by root subgroups, we infer from Corollary 5.11 that there must be a horizontal Demazure P-root. Since every root of G' is semisimple, we end up in Case 7.2 (iii). □

Specializing to dimension three, we obtain a quite precise picture of the possible matrices P in the above setting.

Proposition 8.4. *Let X be a three-dimensional nontoric complete normal rational variety. Suppose that X is almost homogeneous under an action of a reductive group and there is an effective action of a two-dimensional torus on X. Then the Cox ring of X is given as $\mathcal{R}(X) = R(A, P)$ with a matrix P according to the following cases*

$$\text{(i)}\quad P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_2 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d_{12}^* & d_2^* & d_*' \end{bmatrix},$$

$$\text{(ii)}\quad P = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_2 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ d_{01}^* & 0 & 0 & d_2^* & d_*' \end{bmatrix}.$$

In both cases, we have $m \leqslant 2$; that means that the d_*'-part can be either empty, equal to ± 1 or equal to $(\pm 1, \mp 1)$.

Proof. Clearly, we may assume that we are in the situation of Theorem 7.2. Since X is nontoric but almost homogeneous, there must be a semisimple horizontal P-root. Thus, Lemmas 7.8 and 7.10 show that after admissible operations, P is of the desired shape. □

As a direct consequence, one retrieves results of Haddad [9] on the Cox rings of three-dimensional varieties that are almost homogeneous under an SL_2-action and additionally come with an effective action of a two-dimensional torus T; compare also [2] for the affine case.

Corollary 8.5. *Let X be a three-dimensional nontoric complete normal rational variety. Suppose that X is almost homogeneous under an action of SL_2 and there is an effective action of a two-dimensional torus T on X. Then X has at most two T-invariant prime divisors with infinite T-isotropy, i.e., we have $m \leqslant 2$ and the Cox ring of X is given as*

$$\mathcal{R}(X) = \mathbb{K}[T_{ij},\, S_k]/\langle T_{01}T_{02} + T_{11}T_{12} + T_{21}^{l_{21}} \cdots T_{2n_2}^{l_{2n_2}} \rangle.$$

Proof. We are in the situation of Proposition 8.4. The assumption that SL_2 acts with an open orbit implies that we are in case (i). □

Finally, we consider almost homogeneous varieties with reductive automorphism group; see for example [19] for results on the toric case. Here, we list all almost homogeneous threefolds of Picard number one with a reductive automorphism group having a maximal torus of dimension two.

Proposition 8.6. *Let X be a $\mathbb{Q}$-factorial three-dimensional complete normal variety of Picard number one. Suppose that $\mathrm{Aut}(X)$ is reductive, has a maximal torus of dimension two and acts with an open orbit on X. Then X is a rational Fano variety and, up to isomorphy, arises from a matrix P from the following list*:

(i) $\begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & l_{21} \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_{12} & d_{21} \end{bmatrix}, \quad l_{21} > 1, \;\; d_{12} > 2, \;\; -\frac{d_{21}}{d_{12}-1} < l_{21} < -d_{21},$

(ii) $\begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_{21} & l_{22} \\ -1 & 0 & 1 & 0 & 0 \\ d_{01} & 0 & 0 & d_{21} & d_{22} \end{bmatrix}, \quad l_{21}, l_{22} > 1, \;\; 2d_{22} > -d_{01}l_{22}, \;\; -2d_{21} > d_{01}l_{21},$

(iii) $\begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & l_{22} \\ -1 & 0 & 1 & 0 & 0 \\ d_{01} & 0 & 0 & d_{21} & d_{22} \end{bmatrix}, \quad \begin{array}{l} l_{22} > 1, \; d_{22} > d_{21}l_{22} + l_{22}, \; 2d_{22} > -d_{01}l_{22}, \\ -2d_{21} > d_{01}, \\ \textit{or} \\ l_{22} > 1, \; 2d_{22} > -d_{01}l_{22}, \; 1 - 2d_{21} > d_{01}, \end{array}$

(iv) $\begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix},$

$$\text{(v)} \quad \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_{21} & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & d_{21} & 1 \end{bmatrix}, \quad 1 < l_{21} < -2d_{21} < 2l_{21}.$$

Conversely, each of the above listed matrices defines a $\mathbb{Q}$-factorial rational almost homogeneous Fano variety with reductive automorphism group having a two-dimensional maximal torus.

Proof. See [14, Section 7.4] for the detailed version. According to Proposition 8.3, we may assume that X arises from a matrix P. The fact that $\mathrm{Aut}(X)$ is reductive means that every root must be semisimple and the fact that it acts with an open orbit means that there exists at least one pair of horizontal semisimple roots. Now, suppose we have $m = 0$. If there is only one pair of horizontal semisimple P-roots, then reductivity of the automorphism group forbids further P-roots and we end up with the first three cases. If there are more then one pair of semisimple roots, then we end up in case four. Finally, if $m > 0$ holds, then $m = 1$ is the only possibility and one is left with case (v). □

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