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Mathematical Modeling of Transport-Growth Processes in Multiphase Biological Continua

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Abstract—The model of a growing medium consisting of two phases, liquid and solid, is developed. Growth is treated as a combination of the irreversible deformation of the solid phase and its mass increment due to mass exchange with the liquid phase. The inelastic strain rate of the solid phase depends on the stresses in it, which are determined by the forces both external with respect to the medium and exerted by the liquid phase. In the liquid phase the pressure develops due to the presence of a chemical component whose displacement is hampered by its interaction with the solid phase. The approach developed makes it possible to waive many problems discussed in the theory of growing continua. Possible generalizations are considered.

Keywords: multiphase continua, biological growth, deformation theory, mass transfer, osmosis, mathematical models.

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It has long been known that the mechanical loads affect the growth of biological tissues (see reviews [1–3]). Attempts to describe this effect within the framework of continuum mechanics were first made also more than forty years ago. Most of approaches now developed treat the growing tissue as a solid. Nevertheless, it is impossible to adequately describe the deformation and mass transfer in the growing tissue without account for the presence in it of liquid phases that provide the delivery of a "building material". The general method of such description was proposed in [4], where it was first demonstrated that the internal stresses (at the phase level) necessarily participate in the growth processes. In the present study, which continues [4], an approach based on the systematic application to the modeling of a growing tissue of the methods of multiphase continuum mechanics is developed and the model of a medium in which the growth deformation is related with the stresses in the solid skeleton is proposed. Such stresses arise due to the presence of the pressure in the liquid phase, which develops with osmotic processes involved, and can change under the action of external forces. Possible assumptions, which lead to models of different levels of complexity, including single-phase models, are discussed.

1. BIOLOGICAL BACKGROUND AND BASIC ASSUMPTIONS

Growth is one of the main components of biological development, along with shaping (morphogenesis) and the appearance of new cell species (differentiation). There is no accurate definition of biological growth: biologists usually speak of irreversible changes in mass and dimensions [5]. In the language of mechanics, we can say that growth is a combination of deformation and mass influx.

The effect of external mechanical loads on the rate and orientation of growth has been known and used since antiquity. The role of mechanical forces in the growth of objects unloaded from outside has started to be discussed comparatively recently. The necessity of loading at the cellular level was first understood by plant physiologists. At present, it is commonly agreed that the growth elongation of the cell wall occurs

under the action of the intracellular pressure (up to several atmospheres) [6]. Substantial pressures develop in soft tissues [7]. Tissue growth during cell division, determined by the intense increase in the cell volume after division, is related with increase in pressure inside such cells [8]. Numerous facts point that without development of mechanical stresses (at least at the microlevel) growth is impossible [4].

During biological growth, the formation of new mass may be distributed over a volume (volumetric growth) or occur on a surface (surface growth). In biological objects, the latter implies the volumetric growth in a thin layer (at least one cell thick). This differs biological growth from surface material deposition that takes place in many technological and natural processes [9]. Internal growth, material deposition (resorption) on the surface of pores, is also possible [1].

Summing numerous data briefly mentioned above we can formulate the general hypothesis that in normal growth (in the absence of external loading) the inelastic deformation is realized due to the same mechanisms as the inelastic deformation in growth under the action of external loads. Thus, the growth deformation is always associated with stress development (at least at the microlevel). This basic hypothesis was first formulated explicitly in [4].

The main mechanism that leads to the development of the stresses necessary for growth is osmosis. The hampering of mobility of species dissolved in biological fluids due to the interaction of these species with non-fluid components of biological tissues favors the appearance of considerable gradients of their concentrations and, as a result, to the development of forces acting on the fluid. The action of these forces leads to swelling and, if swelling is restricted, to internal pressures. Non-osmotic swelling mechanisms are also possible, for example, the stress development in the tissue due to a change in its composition [7].

In what follows we consider the mathematical modeling of volumetric growth within the framework of continuum mechanics. In order to model growth adequately, it is necessary to take into account, firstly, the delivery of species spent for building the tissue and, secondly, the stresses that arise at the microlevel. This is only possible within the framework of the model which treats the tissue as a multiphase medium that includes at least one solid and one liquid phase. In this approach, two growth components, irreversible deformation and mass transfer, can be separated in a natural way.

2. TWO-PHASE MODEL OF A GROWING MEDIUM

We consider a minimal model of the growing medium, which includes the deformable porous skeleton (solid phase) with volume concentration α and the liquid phase with volume concentration $\beta = 1 - \alpha$ that flows along a system of interconnected pores. All fundamental problems that arise in constructing the models of growing media can be considered with reference to this model. In further sections, possible simplifications and generalizations of this model will be considered.

The apparent densities of the phases are equal to $\rho_1 = \rho_1^* \alpha$ and $\rho_2 = \rho_2^* \beta$, respectively, where the asterisk denotes the true densities of these phases. We also characterize the structure of the medium by the set of tensors $\mathbf{L}$ determining its anisotropy. The latter includes both the anisotropy of the solid phase and the anisotropy of the shape and distribution of the pores. The form of the relations can also be affected by the composition of the phases. We assume that in the liquid phase a generalized chemical component with concentration c is distributed.

For obtaining the equations (excluding the purely heuristic method) in mechanics of multiphase media two main methods are used: the method based on thermodynamics of irreversible processes and the averaging of microscopic equations. We apply the averaging method assuming the solid phase to be a viscoelastic body of the Maxwellian type and the liquid phase to be a linearly viscous, incompressible fluid. In the case considered, this well-known procedure [10, 11] based on certain standard assumptions mainly yields known relations. Therefore, we do not reproduce the calculations here and only dwell in more detail on certain non-standard terms. Since rapid processes are not considered, we omit the inertial terms and write down all relations in the quasi-static form.

Kinematics and continuity equations. In order to describe the deformations and displacements of the solid phase particles, we introduce the observer's coordinate system with Eulerian coordinates x^i and the comoving coordinate system with Lagrangian coordinates ξ^s, assuming the both systems to coincide at the actual moment. The law of medium motion has the form:

$$x^i = x^i(t, \xi^s). \tag{2.1}$$

It is natural to identify the law of solid phase motion with the law of motion for the medium as a whole.

In a standard way, from the law of motion (2.1) the fields of medium motion and deformation characteristics can be constructed, in particular, the velocity vector v^i and the strain rate tensor e_{ij}:

$$v^i = \frac{dx^i}{dt}, \qquad e_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right), \qquad \frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_{\xi^s = \text{const}}. \tag{2.2}$$

The equality determining the components e_{ij} is formulated in an arbitrary coordinate system. Of course, we can introduce an initial state and the strain tensor with respect to this state in accordance with a certain rule. For small strains all tensors so introduced coincide. The definition of the strain rate tensor (2.2) is independent of these definitions.

The liquid flow is characterized by the velocity vector with components v_2^i. The following continuity equations hold:

$$\frac{\partial \rho_1}{\partial t} + \nabla_i\left(\rho_1 v^i\right) = \Theta, \tag{2.3}$$

$$\frac{\partial \rho_2}{\partial t} + \nabla_i\left(\rho_2 v_2^i\right) = -\Theta, \tag{2.4}$$

where Θ is the interphase mass exchange rate (positive for proper growth). In (2.3), (2.4), and formulas that follow the summation convention applies.

For the generalized component with concentration in the liquid phase c, we have the continuity equation

$$\frac{\partial(\rho_2 c)}{\partial t} + \nabla_i\left(\rho_2 c v_c^i\right) = -\theta, \tag{2.5}$$

where θ is the rate of consumption of the component distributed in the liquid phase for the solid phase formation and v_c^i are its velocity components.

Equilibrium equations for the phases and components. After the averaging, the quasi-static momentum equations take the form:

$$\nabla_j\left(\alpha T^{ij}\right) - R^i + F_1^i = 0, \tag{2.6}$$

$$-\nabla_i(\beta p) + R^i + F_2^i = 0. \tag{2.7}$$

Here, T^{ij} is the stress tensor in the solid phase, p is the pressure in the liquid phase, F_1^i and F_2^i are the components of the external volume force that acts on the solid and liquid phases, respectively, and R^i is the interphase force that arises as a result of the averaging of the microscopic liquid-phase momentum equations.

Summing (2.6) and (2.7), we can write down the equilibrium equation for the medium as a whole

$$\nabla_j \sigma^{ij} + F^i = 0, \tag{2.8}$$

in which the total stress in the medium σ^{ij} and the external volume force F^i are defined by the relations

$$\sigma^{ij} = \alpha T^{ij} - \beta p g^{ij}, \qquad F^i = F_1^i + F_2^i, \tag{2.9}$$

where g^{ij} are the components of the metric tensor.

The interphase force has the following structure:

$$R^i = k^{ij}(v_j - v_{2j}) + p\nabla^i\beta + H^i. \tag{2.10}$$

In expression (2.10), the first term is the viscous interphase force $R^i_v = k^{ij}(v_j - v_{2j})$ that arises from the averaging of the viscous stresses, where k^{ij} is the tensor coefficient of hydraulic permeability. This tensor is generally anisotropic even for the viscous fluid to be isotropic at the microlevel due to the anisotropy of the pores. The second term appears from the averaging of the pressure gradient with account for porosity inhomogeneity (an Archimedes-like force).

The osmotic effects can only be realized if the mobility of the component dissolved in the liquid phase is hampered by its interaction, separately from the liquid phase, with the solid phase. This component cannot freely diffuse in the liquid as in the absence of the solid phase. This may be related with either the presence of distributed semipermeable membranes or the direct force interaction of the component with the solid phase. Therefore, in order to establish the structure of the third term H^i, we first have to consider the liquid flow separately from this component with account for the force it exerts. The separate equilibrium equation for the liquid in the mixture with a component dissolved in it has the form [4, 7]:

$$-\rho_2\nabla^i\left(\frac{p}{\rho_2^*} + \mu\right) + R^i_{v0} + R^i_{vc} + F^i_{20} = 0. \tag{2.11}$$

Here, R^i_{v0} and R^i_{vc} are the forces of resistance to water motion exerted by the solid phase and the dissolved component, respectively, F^i_{20} is that part of the volume force external with respect to the liquid which is applied to the liquid itself (without the component dissolved), and μ is the chemical potential of the solvent (water). The chemical potential μ is a function of the concentration c and possibly other parameters. Due to the smallness of the concentration c, it is lawful to identify the force R^i_{v0} with R^i_v and the force F^i_{20} with F^i_2 and to neglect the force R^i_{vc} as compared with R^i_v. Then, substituting (2.11) in (2.10), with account for (2.7) and the constancy of the true liquid density, we obtain the expression for the third (osmotic) component of the interphase force

$$H^i = -\rho_2\nabla^i\mu. \tag{2.12}$$

For the component distributed in the liquid phase, the equilibrium equation has the form:

$$-\rho_c\nabla^i\mu_c + R^i_{sc} - R^i_{vc} + F^i_c = 0. \tag{2.13}$$

Here, R^i_{sc} is the force of resistance to the component motion exerted by the solid phase, F^i_c is the external volume force acting on the component, and μ_c is the chemical potential of the dissolved species.

The chemical potential μ_c is a function of the component concentration c, the solid phase concentration α, and possibly other parameters. This is the presence of the solid phase concentration among the arguments of the potential μ_c that leads to the formation in the growth zone of the region in which the component concentration is increased and, as a result, to liquid flow into this region and the pressure development in the liquid.

For the force R^i_{vc} we assume the linear dependence on the component velocity relative to the liquid $R^i_{vc} = L^{ij}\rho_c(v_{ci} - v_i)$. The substitution of relation (2.13) in the component continuity equation (2.5) then yields the equation

$$\frac{\partial(\rho_2 c)}{\partial t} + \nabla_i(\rho_2 c v^i_2) + \nabla_i\left[D^{ij}(-\rho_2 c\nabla_j\mu_c + R_{scj} + F_{cj})\right] = -\theta, \tag{2.14}$$

where D^{ij} is the tensor inverse to L^{ij}. If $R_{scj} = 0$, Eq. (2.14) transforms into the usual diffusion law with a tensor diffusion coefficient.

This is the presence of the force $\mathbf{R}_{sc}$ that provides the maintenance of the concentration gradients necessary for the realization of the osmotic effects. The relations determining this force should take into account

the physics of processes specific for the tissue considered. The simplest form of the constitutive relation for $\mathbf{R}_{sc}$ is the linear dependence on the relative velocity of the component with respect to the solid phase

$$R^i_{sc} = M^{ij}\rho_2 c\big(v_j - v_{cj}\big). \tag{2.15}$$

Such a relation is, for example, lawful if the solid phase contains semipermeable membranes.

If we assume the linear relationship (2.15) and represent the relative velocity in the form $v_{ci} - v_i = (v_{ci} - v_{2i}) + (v_{2i} - v_i)$, Eq. (2.14) can be so transformed as to retain the traditional form of the diffusion equation:

$$\frac{\partial(\rho_2 c)}{\partial t} + \nabla_i\big(\rho_2 c v_2^i\big) + \nabla_i\big[-D_1^{ij}\big(\rho_2 c\nabla_j\mu_c + F_{cj}\big) + \rho_2 c M^{ij}\big(v_j - v_{2j}\big)\big] = -\theta \tag{2.16}$$

with the difference that the tensor D_1^{ij} is now inverse to the sum $L^{ij} + M^{ij}$ and there is an additional term related with the relative phase velocity, irreducible to diffusion.

Deformation of the solid phase. Consider the solid phase at a certain (actual) moment of time. In processes rapid as compared with growth it should behave as a porous solid. As sufficient for most problems, we assume the medium to behave elastically in rapid processes. At each point, the medium can locally be unloaded, which means that the stresses in the solid phase vanish: $T_{ij} = 0$. We can locally introduce the metric tensor at the actual moment $g_{ij} = g'_{ij}$ and in the unloaded state g^*_{ij}. By prime we denote components in the Lagrangian coordinate system. We introduce the elastic strain tensor by the formula [12]

$$\varepsilon^{(e)\prime}_{ij} = \frac{1}{2}\big(g'_{ij} - g^*_{ij}\big). \tag{2.17}$$

In processes rapid as compared with growth the medium displays the elastic behavior and the strains are determined by the stresses in the solid phase

$$\varepsilon^{(e)}_{ij} = \varepsilon^{(e)}_{ij}(T^{kl},\ \alpha,\ \mathbf{L}), \qquad \varepsilon^{(e)}_{ij}(0,\ \alpha,\ \mathbf{L}) = 0. \tag{2.18}$$

The deformation process is determined by the strain rate tensor e_{ij} for which the following formula holds [12]:

$$e'_{ij} = \frac{1}{2}\frac{dg'_{ij}}{dt} = \frac{d\varepsilon^{(e)\prime}_{ij}}{dt} + \frac{1}{2}\frac{dg^*_{ij}}{dt}. \tag{2.19}$$

When differentiating, we must consider time moments close to the actual time, at which the components g'_{ij} no longer coincide with g_{ij}. It is natural to call the tensor $e^{(p)}_{ij}$ whose components are determined in the Lagrangian coordinate system by the formula

$$e^{(p)\prime}_{ij} = \frac{1}{2}\frac{dg^*_{ij}}{dt} \tag{2.20}$$

the inelastic strain rate tensor. This tensor characterizes the strain rate of the unloaded medium element. It is natural to anticipate that it is this characteristic that obeys the law of irreversible deformation determining biological growth.

In accordance with the basic hypothesis, the set of arguments of the function determining the strain rate tensor should necessarily include the solid skeleton stress tensor $\mathbf{T}$. The minimal set of arguments is given by the relation

$$e^{(p)}_{ij} = e^{(p)}_{ij}(T^{kl},\ \alpha,\ \mathbf{L}), \qquad e^{(p)}_{ij}(0,\ \alpha,\ \mathbf{L}) = 0. \tag{2.21}$$

In view of the first equation in (2.9), the argument $\mathbf{T}$ can be replaced by the stress in the medium as a whole $\boldsymbol{\sigma}$, which leads to the appearance in the list of arguments of the pressure p in the liquid phase. Generally, dependence (2.21) may contain among arguments the liquid phase pressure p directly. Such

dependence may be related with the effect of the liquid phase pressure on the microstresses in the solid phase that disappear in averaging or the effect of this pressure on the rate of chemical reactions.

Taking into account that relation (2.18) is written in the observer's system, whereas in (2.20) the differentiation is performed in the Lagrangian system, we obtain the following relation for the strain rate tensor components in the observer's system:

$$e_{ij} = \frac{\partial \xi^r}{\partial x^i}\frac{\partial \xi^s}{\partial x^j}\frac{d}{dt}\left(\varepsilon^{(e)}_{kl}\frac{\partial x^k}{\partial \xi^r}\frac{\partial x^l}{\partial \xi^s}\right) + e^{(p)}_{ij} = \frac{d\varepsilon^{(e)}_{ij}}{dt} + \varepsilon^{(e)}_{ik}\nabla_j v^k + \varepsilon^{(e)}_{kj}\nabla_i v^k + e^{(p)}_{ij}. \tag{2.22}$$

Relation (2.22) completely determines the deformation law in the differential form for finite elastic strains. The total strain tensor that would link the actual state with a certain state initial for the whole process is not introduced. The problem of definition of such a tensor is thus abolished. This problem arises if there is a need to include the strain tensor in the list of arguments of functions (2.18) or (2.21).

If the elastic strains are small, relation (2.22) can be simplified. In this case, we can omit the products of small quantities and assume

$$e_{ij} = \frac{d\varepsilon^{(e)}_{ij}}{dt} + e^{(p)}_{ij}. \tag{2.23}$$

Possible methods of closing the system of equations. In order to close the system of equations, we need relations determining the kinetics of mass transfer from the liquid to the solid phase, as well as the changing of the porosity β and the anisotropy tensors $\mathbf{L}$ during inelastic deformation. Among the tensors $\mathbf{L}$ there are parameters determining the shape and orientation of the pores. This aspect was discussed in the general form in [4].

As a simplest hypothesis, the hypothesis of structure conservation [4, 13] can be formulated. It assumes that in the new-born material the structure, that is, the porosity and the anisotropy-determining parameters, remains unchangeable. Then the system becomes closed if we specify the functions and coefficients it contains, including the relationship between the dissolved component consumption rate θ and the solid phase production rate Θ.

The model is constructed not using the energy equation. Generally speaking, this equation could be written but the presence of the difficult-to-estimate consumption of the chemical (metabolic) energy makes it ineffective for solving specific problems. The methods of thermodynamics of irreversible processes were used to justify the relations of growth mechanics and to obtain restrictions on the constitutive functions [14, 15]. In so doing, the obvious fact was often ignored that the presence of chemical reactions, never taken into account explicitly, casts some doubt on the thermodynamic consequences obtained.

On the formulation of problems. Residual stresses. The boundary and initial conditions are determined by the specific problem. The system of equations presented does not need additional boundary conditions for the liquid phase and the component distributed in it as compared with those that should be assigned for the liquid with a diffusing admixture which flows through a solid porous medium.

For the solid phase, due to the absence of additional spatial derivatives as compared with theory of elasticity, the formulation of the boundary-value problem is the same as in this theory. The important characteristic of the growth problems is the presence of residual stresses in the medium. Such stresses arise as a result of inconsistency of the constitutive equation for the inelastic strain rate with the compatibility equations [1]. Since in any realistic problem we consider an object formed as a result of the previous growth, the field of residual stresses (or elastic strains) corresponding to the complete unloading of the object at the initial moment should be given. The complete unloading is taken to mean the equality to zero of all external (volume and surface) forces acting on the solid phase. Among such forces there also are the forces, internal for the medium as a whole, exerted by the liquid phase, including the component dissolved in it. However, in most cases the solid phase equilibrium equations contain the only important force of this group, the liquid pressure force. If the pressure p is equal to zero, as can be seen from (2.9), the tensors $\boldsymbol{\sigma}$

and $\mathbf{T}$ differ only in the scalar multiplier α. The residual stress field is thus any symmetric tensor field σ_0^{ij} satisfying in the volume the equation

$$\nabla_j \sigma_0^{ij} = 0$$

and on the boundary the condition of equality to zero of the stress vector $\sigma_0^{ij} n_j = 0$, where n_j are the component of the normal to the boundary. In this field, stress tensor discontinuity surfaces, on which the stress vector remains continuous, may be present.

A real object maximally approaching the two-phase model developed is the growing cartilage. In this case, the solid phase is the cartilage matrix and the liquid phase is the tissue fluid. The matrix-producing cells occupy a small part of the volume and can be included in the solid phase.

3. SINGLE-PHASE APPROXIMATION

From the two-phase model considered, the model of a growing single-phase medium can be obtained if we assume that all the structural parameters and the pressure remain unchangeable. The stresses in the solid phase and in the medium as a whole are then linked by the linear inhomogeneous relation with constant coefficients

$$T^{ij} = \frac{1}{\alpha}\sigma^{ij} + \frac{\beta}{\alpha} p g^{ij}. \tag{3.1}$$

In the absence of stresses in the medium as a whole

$$T^{ij} = T_0^{ij} = \frac{\beta}{\alpha} p g^{ij}. \tag{3.2}$$

We can now assume that

$$e_{ij}^{(p)} = e_{ij}^{(p)}\left(\sigma^{kl}\right), \qquad \varepsilon_{ij}^{(e)} = \varepsilon_{ij}^{(e)}\left(\sigma^{kl}\right) \tag{3.3}$$

and the relation (2.22) determining the medium strain rate (or in the case of small strains (2.23)) retains its form.

The important difference from the two-phase model consists in the fact that the equality to zero of the only stress present in the model σ^{ij} does not lead to the vanishing of functions (3.3) since for $\sigma^{ij} = 0$ the stress in the solid phase $T^{ij} = T_0^{ij} \neq 0$. The second function (3.3) can easily be redefined by taking for the unloaded state the state in which $\sigma^{ij} = 0$. Then the equality $\varepsilon_{ij}^{(e)}(0) = 0$ remains valid. The fact that the tensor $A_{ij} = e_{ij}^{(p)}(0)$ differs from zero is important and means that growth deformation is possible in the absence of stresses. The quantity A_{ij} is called the proper growth strain rate [1].

If the assumption of constancy of the structural parameters is fairly natural (the above-mentioned structure conservation hypothesis), the assumption of a constant liquid phase pressure seems voluntary. Nevertheless, it has a clear biological sense and means that due to certain undiscussed (regulatory) mechanisms a stable composition of the liquid phase is maintained and it is this stability that provides a constant, necessary for growth, internal pressure. For example, for certain growing plant tissues it is known that the intracellular pressure remains constant [16]. Of course, this assumption (as the single-phase model itself) cannot be used in problems for which such stability may be violated. Another possible situation for which the assumption formulated is lawful is the case of growth possible only in the presence of external forces, so that in their absence growth can be neglected (for example, in distraction osteogenesis) [13, 17].

4. DISCUSSION

The consideration performed with reference to the two-pase model makes it possible to easily solve many problems violently discussed in mechanics of growing continuum, which often arise as a consequence of artificial approaches applied. The notion of proper growth strain rate acquires an exact physical sense: as the rate of growth deformation produced by external loads, this characteristic is different from zero only in the presence of nonzero tensile stresses, but now developing on small spatial scales. Since deformation

and mass transfer are separated in the model explicitly, the problem of distinction between "growth" and "non-growth" inelastic deformations (strains) is abolished: there is a single inelastic deformation (strain) accompanied or not accompanied by the deposition of a new solid phase mass.

We should especially dwell on the theory of deformation in a growing tissue. In this area there is extensive literature (see, for example, [15, 18, 19]). The power of the theory mainly attacks the problem of taking into account the total deformation from a certain initial to the actual state. The theory usually considers finite ("very large") strains. However, it is beyond reason to suggest that the tissue material "remembers" these strains. On small times it behaves as a usual solid body (in most problems it can be treated as elastic) with residual stresses. The local unloading does not return the tissue element to a bygone "initial" state. It is the mechanical characteristics at the actual moment that should determine the actual deformation. The elastic strains associated with the actual residual stresses are among such characteristics. It is difficult to imagine that the "program" of growth which is inevitably present in the tissue is related with such a nonphysical characteristic as the total strain tensor. To seek the real physical quantities responsible for this program is the actual problem. Among such characteristics there may be parameters related with the formation and breakup of chemical bonds [16]. Obviously, the law of motion and, hence, the finite strain tensor defined with respect to any state (real or conventional) can be calculated as a result of solution of the corresponding problem.

5. GENERALIZATIONS

There may be a lot of ways to generalize the proposed two-phase model. Obviously, the presence of a single generalized chemical component distributed in the liquid phase is insufficient even for abstract problems. At least two generalized components are needed: one responsible for osmotic effects and another for the building of the solid skeleton.

The liquid phase should also not necessarily be single. In [19], the model of a growing plant tissue, which takes into account in addition to the solid phase (cell walls) and the main liquid phase (intracellular fluid) a delivering liquid phase (extracellular fluid), is in detail considered. The internal pressure needed for growth deformation is generated in the intracellular fluid, whereas the extracellular fluid participates in the transport of the liquid and components dissolved in it. A separate problem is to take into account the liquid phases that move along specialized transport systems (vessels) distributed in the tissue (if such systems are present). Inside the vessels the liquid velocity is usually much higher than outside.

In many cases, it may be necessary to consider several solid phases obeying different deformation laws and connected by interphase elastic and viscoelastic forces [20, 21]. An example of such a system is the plant leaf in which ribs (veins) grow obeying other laws than the tissue that surrounds them, which leads to considerable deformations.

The development of growth mechanics will inevitably lead to explicitly taking into account additional parameters responsible for the physicochemical state of the tissue. One of few examples of effective usage of such parameters is the above-mentioned account for the chemical bonds in the cell walls for describing the plant root growth.

Summary. A maximally simple model of the volumetrically growing two-phase medium with account for the inelastic deformation of the solid phase is developed. Within the framework of this model, the inelastic deformation and mass formation of the solid phase, which together comprise growth, and the transport of the liquid and a generalized component dissolved in it can be considered. Deformation is provided by stresses in the solid phase due to both the pressure in the liquid and the external forces if present. The liquid pressure develops as a result of osmosis due to the presence of a component distributed in the liquid phase, whose displacement is restricted by its interaction with the solid phase. A theory of deformation which makes it possible not to introduce the total strain tensor is developed. The model of a growing single-phase medium can be obtained from the two-phase model as a limiting case for the structure unchangeable and the liquid pressure constant. For analyzing specific problems, the model can easily be generalized, including the introduction of additional phases and components.

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