Functional Analysis and Its Applications, *Vol.* 45, *No.* 2, *pp.* 81–98, 2011
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, *Vol.* 45, *No.* 2, *pp.* 1–22, 2011

Mixed Problems in a Lipschitz Domain for Strongly Elliptic Second-Order Systems*

M. S. Agranovich

Received December 16, 2010

Abstract. We consider mixed problems for strongly elliptic second-order systems in a bounded domain with Lipschitz boundary in the space $\mathbb{R}^n$. For such problems, equivalent equations on the boundary in the simplest L_2-spaces H^s of Sobolev type are derived, which permits one to represent the solutions via surface potentials. We prove a result on the regularity of solutions in the slightly more general spaces H_p^s of Bessel potentials and Besov spaces B_p^s. Problems with spectral parameter in the system or in the condition on a part of the boundary are considered, and the spectral properties of the corresponding operators, including the eigenvalue asymptotics, are discussed.

Key words: strongly elliptic system, mixed problem, potential type operator, spectral problem, eigenvalue asymptotics.

1. Introduction

1.1. The content of the paper. Let Ω be a bounded domain with Lipschitz boundary Γ in the space $\mathbb{R}^n$, $n \geqslant 2$. Consider a strongly elliptic second-order system $Lu = f$ in Ω written in the divergence form (see (1.1) below). The smoothness assumptions imposed on the coefficients of the system are minimized to some extent. The main "energy" form $\Phi_\Omega(u, v)$ is assumed to be coercive for $u = v$ in the strengthened sense on the space $H^1(\Omega) = W_2^1(\Omega)$; this condition implies the unique solvability of the Dirichlet and Neumann problems.

In the most part of the paper (Sections 1–6), we assume that the boundary Γ is divided into two domains Γ_1 and Γ_2 by a closed $(n-1)$-dimensional Lipschitz surface $\partial\Gamma_j$ without self-intersections; the Dirichlet condition is posed on Γ_1, and the Neumann condition is posed on Γ_2.

There is a wide literature devoted to mixed problems. This is because they occur in numerous applications; e.g., see the books [16] and [41]. (They also provide involved explicit solution formulas for a number of specific mixed problems.)

In many papers, additional smoothness assumptions are imposed on Γ_1 and Γ_2, or Γ, and on $\partial\Gamma_j$ (e.g., see [39], [38], and the bibliography therein, as well as [42], [43], and [30]). In a number of papers, the boundary geometry is subjected to a certain condition; roughly speaking, one usually requires that the angle between the normals to Γ_1 and Γ_2 at the points of $\partial\Gamma_j$ be less than π (creased domains; see [10] for the precise definition; cf. also [46], [27], and [11]). There are numerous papers where specific equations or systems are considered. Most often, these are the Laplace–Helmholtz equations or elasticity systems; in addition to the above-mentioned papers, see [19] and [35]. These studies are largely aimed at analyzing the regularity of the solution and especially the solution behavior near $\partial\Gamma_j$. One can readily construct examples (like $r^\alpha \sin(\alpha\theta)$ in the polar coordinates (r, θ) for the Laplace equation in a planar domain) showing that even if the Γ_j (or even Γ) and $\partial\Gamma_j$ are infinitely smooth, the solution may have a singularity near $\partial\Gamma_j$. The asymptotics of the solution near $\partial\Gamma_j$ in the smooth case was obtained in [42], [43], and [30] by the Wiener–Hopf method in the form proposed in [15].

There are considerably fewer papers concerned with general strongly elliptic equations or systems *without* any additional smoothness conditions on Γ_j and $\partial\Gamma_j$ or geometric conditions. Here we

*Supported by RFBR grant no. 11-01-00277-a.

mention the monograph [24] and the papers [34] (about a scalar equation with formally self-adjoint principal part), [18], and [33, Section 3] (about a scalar formally self-adjoint equation without lower-order terms). The present paper (at least in Sections 1–5) belongs just to this direction.

We give a general theorem on the existence and uniqueness of a variational solution in the spaces H^s and compare various approaches to its proof. Since there are some subtle issues to be touched upon, we start by recalling the properties of the spaces H^s (Section 1.3), the statement of the Dirichlet and Neumann problems (Section 2.1), and a straightforward proof of the unique solvability of the mixed problem on the basis of the Lax–Milgram lemma (Section 2.2). Then, in Section 3, we give two different derivations of *equations on* Γ *with operators of surface potential type on a part of the boundary* equivalent to the problem and uniquely solvable. These operators are of interest in themselves. In [24], the equations were obtained under the assumption that the leading part of the system is formally self-adjoint. We do not need this assumption and use the transparent approach in [30] to the anisotropic elasticity system.

In Section 4, we generalize the unique solvability theorem to the spaces H_p^s of Bessel potentials and Besov spaces B_p^s, close to H^s, by using *Shneiberg's theorem* [40] on the extrapolation of invertibility of operators in interpolation scales; cf., in particular, [27] (where very general unique solvability theorems are obtained for the Laplace equation under an additional geometric assumption) and also [11]. Note that the authors of these and some other papers go beyond the function spaces used in the present paper.

In Sections 5–6, we consider spectral problems. We indicate results on the basis property of eigenfunctions for self-adjoint problems and give conditions ensuring the completeness of systems of root functions and the Abel–Lidskii summability of Fourier series in these systems for non-self-adjoint problems and problems in non-Hilbert spaces considered here (cf. [3]–[5]).

If the system (or at least its principal part) is formally self-adjoint, then the eigenvalue asymptotics is of interest. For problems with spectral parameter in the system, the result follows from a theorem in [25], where a strong remainder estimate was also obtained. Problems with spectral parameter in the condition on a part of the boundary are spectral Poincaré–Steklov problems. Here we use the *variational approach* going back to Courant–Hilbert [13] under the condition that the boundary is *almost smooth* (i.e., smooth outside a closed subset Γ_{sing} of measure zero; this definition was given in [6]). We also supplement the results on spectral asymptotics obtained in [4] and [5].

In Section 7, we briefly discuss some generalizations.

1.2. Refinement of the setting of the problem. We write out the system in the form

$$Lu := -\sum \partial_j a_{j,k}(x)\partial_k u(x) + \sum b_j(x)\partial_j u(x) + c(x)u(x) = f(x). \tag{1.1}$$

Here and below, $\partial_k = \partial/\partial x_k$; u is a vector-valued function, a column of height m, so that the coefficients are $m \times m$ matrices. Their entries, as well as the components of u, are complex-valued functions. We shall assume that $a_{j,k} \in C^1(\overline{\Omega})$, $b_j \in C^{0,1}(\overline{\Omega})$ (Lipschitz functions), and $c \in L_\infty(\Omega)$. Sometimes, one can assume less. The strong ellipticity condition is the requirement of uniform positive definiteness of the real part $\frac{1}{2}(a + a^*)$ of the principal symbol (i.e., of the matrix $a(x,\xi) = \sum a_{j,k}(x)\xi_j\xi_k$) for real ξ with $|\xi| = 1$. First, we seek a solution in $H^1(\Omega)$. The space for f will be indicated in Section 2.2.

Now write out the boundary conditions:

$$u^+ = g \quad \text{on } \Gamma_1, \quad T^+u = h \quad \text{on } \Gamma_2. \tag{1.2}$$

Here $u^+ = g \in H^{1/2}(\Gamma_1)$ is the trace of u, which will also be denoted by γ^+u, and $T^+u = h$ is the conormal derivative of u, whose definition will be recalled and briefly discussed in Section 2.1; it belongs to $H^{-1/2}(\Gamma_2)$.

Let $\Phi_\Omega(u,v)$ be the sesquilinear form

$$\int_\Omega \Big[\sum a_{j,k}(x)\partial_k u(x) \cdot \partial_j \overline{v}(x) + \sum b_j(x)u(x) \cdot \overline{v}(x) + c(x)u(x) \cdot \overline{v}(x)\Big]dx \tag{1.3}$$

corresponding to the system; it is defined on functions $u, v \in H^1(\Omega)$. We require the (strengthened) coercivity of Φ_Ω, which means that the Gårding inequality holds in the form

$$\|u\|^2_{H^1(\Omega)} \leqslant C_0 \operatorname{Re} \Phi_\Omega(u, u), \tag{1.4}$$

without the additional term $\|u\|^2_{L_2(\Omega)}$ on the right-hand side. If $u^+ = 0$ on Γ, then this inequality follows from the strong ellipticity alone provided that

$$\operatorname{Re}(cu, u)_\Omega \geqslant C_1 \|u\|^2_{L_2(\Omega)} \tag{1.5}$$

with a sufficiently large constant C_1, which we assume to be true. Here $(\cdot, \cdot)_\Omega$ is the standard inner product in $L_2(\Omega)$. Inequality (1.4) for $u \in H^1(\Omega)$ holds for a scalar equation with real symmetric matrix of leading coefficients. For this inequality to be true in the matrix case, the coefficients of L are subjected to certain (sufficient) conditions (e.g., see [24] and [5]); in particular, these conditions are satisfied for the generalized elasticity systems (e.g., see [32]). Here the form $\operatorname{Re}(cu, u)_\Omega$ is again assumed to be sufficiently large.

1.3. Spaces $\boldsymbol{H^s}$. For example, see [24]. The spaces $H^s(\mathbb{R}^n)$ of Bessel potentials ($s \in \mathbb{R}$) are introduced by the formula

$$H^s(\mathbb{R}^n) = J^{-s} L_2(\mathbb{R}^n), \qquad J^{-s} = F^{-1}(1 + |\xi|)^{-s} F, \tag{1.6}$$

where F is the Fourier transform in the sense of distributions; one sets $\|u\|_{H^s(\mathbb{R}^n)} = \|J^s u\|_{L_2(\mathbb{R}^n)}$. For $s \geqslant 0$, these are the spaces $W^s_2(\mathbb{R}^n)$, the Sobolev spaces for integer s and the Slobodetskii spaces for noninteger s. The spaces $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$ are dual with respect to the extension of the standard inner product in $L_2(\mathbb{R}^n)$ to their direct product.

The space $H^s(\Omega)$ is defined as consisting of the restrictions of elements in $H^s(\mathbb{R}^n)$ to Ω in the sense of distributions with the usual inf-norm. There exists a universal bounded extension operator $\mathscr{E} \colon H^s(\Omega) \to H^s(\mathbb{R}^n)$ independent of s [37].

The space $\widetilde{H}^s(\Omega)$ is defined as the subspace of $H^s(\mathbb{R}^n)$ formed by the elements supported in $\overline{\Omega}$. The norm is inherited from $H^s(\mathbb{R}^n)$. The space $\widetilde{H}^s(\Omega)$ can be identified with the completion in $H^s(\mathbb{R}^n)$ of the linear manifold $C^\infty_0(\Omega)$, the functions in the latter being extended by zero outside Ω. For $-1/2 < s < 3/2$, $s \neq 1/2$, the space $\widetilde{H}^s(\Omega)$ can be identified with the completion $\mathring{H}^s(\Omega)$ of $C^\infty_0(\Omega)$ in $H^s(\Omega)$. All identifications are understood up to norm equivalence.

The spaces $\widetilde{H}^{-s}(\Omega)$ and $H^s(\Omega)$ are dual with respect to the extension of the standard inner product in $L_2(\Omega)$ to their direct product. This extension has the form

$$(f, v)_\Omega = (f, \mathscr{E} v)_{\mathbb{R}^n}, \tag{1.7}$$

where the extension of the inner product in $L_2(\mathbb{R}^n)$ is used on the right-hand side. Obviously, the functions before and after the comma in these forms can be simultaneously interchanged. The spaces $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$ are identified for $|s| < 1/2$.

The spaces $H^s(\Gamma)$ on a Lipschitz surface are defined for $|s| \leqslant 1$ with the use of a partition of unity and norms in $H^s(\mathbb{R}^{n-1})$. The trace operator $\gamma^+ v = v^+$ is bounded from $H^{s+1/2}(\Omega)$ to $H^s(\Gamma)$ for $0 < s < 1$ and has a bounded right inverse. The spaces $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ are dual with respect to the extension of the standard inner product in $L_2(\Gamma)$ (with respect to the naturally defined measure, the surface area on Γ) to their direct product.

The space $\widetilde{H}^{-s}(\Omega)$, $s > 1/2$, contains elements of $H^{-s}(\mathbb{R}^n)$ supported in Γ. For $1/2 < s < 3/2$, they have the form $(w, v^+)_\Gamma$, $v \in H^s(\Omega)$ and $w \in H^{-s+1/2}(\Gamma)$ [26].

If Γ_0 is a domain with Lipschitz boundary on Γ, then the space $H^s(\Gamma_0)$ is defined for $|s| \leqslant 1$ as consisting of the restrictions to Γ_0 of elements in $H^s(\Gamma)$ and is equipped with the inf-norm. There exists a bounded extension operator from $H^s(\Gamma_0)$ to $H^s(\Gamma)$ independent of s. The space $\widetilde{H}^s(\Gamma_0)$ is defined as the subspace of $H^s(\Gamma)$ formed by the elements supported in $\overline{\Gamma}_0$. The spaces $\widetilde{H}^{-s}(\Gamma_0)$ and $H^s(\Gamma_0)$ are dual with respect to the extension of the standard inner product in $L_2(\Gamma_0)$ to their direct product. This extension is constructed similarly to (1.7). The spaces $H^s(\Gamma_0)$ and $\widetilde{H}^s(\Gamma_0)$ are identified for $|s| < 1/2$.

The operator of multiplication by the characteristic function of Γ_0 is a multiplier in $H^s(\Gamma)$ for $|s| < 1/2$ and is not a multiplier in $H^{\pm 1/2}(\Gamma)$. However, it is a multiplier in $\widetilde{H}^s(\Gamma_0)$ for $|s| \leqslant 1/2$; moreover, the multiplication by this function does not change the elements of these spaces.

2. Variational Approach

2.1. Dirichlet and Neumann problems. Recall their variational statement. Let us write the Green formula

$$\Phi_\Omega(u, v) = (Lu, v)_\Omega + (T^+u, v^+)_\Gamma. \tag{2.1}$$

Here v is an arbitrary test function.

We start with the Neumann problem ($\Gamma = \Gamma_2$). In this case, $u, v \in H^1(\Omega)$, so that the trace $\gamma^+ v = v^+$ belongs to $H^{1/2}(\Gamma)$. In accordance with the choice of the space for v, the right-hand side of the system $Lu = f$ is taken in $\widetilde{H}^{-1}(\Omega)$. We use the corresponding dualities on the right-hand side in (2.1).

The conormal derivative is defined on smooth functions by the formula $T^+u(x) = \sum \nu_j(x)\partial_j u(x)$ at all boundary points x where there exists a normal to the boundary. (These are almost all points.) Here the $\nu_j(x)$ are the coefficients of the unit outward normal. In the general case, given a function $u \in H^1(\Omega)$, the distribution $Lu(x)$ is uniquely determined only in the interior of Ω, while $f \in \widetilde{H}^{-1}(\Omega)$ may also contain a term in $H^{-1}(\mathbb{R}^n)$ supported in Γ. Therefore, formula (2.1) is *postulated* to be true, and it is taken for the definition of the conormal derivative $T^+u \in H^{-1/2}(\Gamma)$ (for given u and $Lu = f$) as well as for the definition of a solution of the Neumann problem (for given $f = Lu$ and $h = T^+u$). One can always assume that $h = 0$, changing f by a term supported in Γ if necessary.

Inequality (1.4) implies the unique solvability of this problem by virtue of the following Lax–Milgram lemma on weak solutions of the abstract equation $Lu = f$, where L is the bounded operator defined by Eq. (2.4) below (e.g., cf. [24, p. 43]).

Lemma 2.1. *Let H be a Hilbert space, and let H^* be the dual space with respect to the form (f, v), $v \in H$, $f \in H^*$. Let $f \in H^*$. Suppose that a continuous sesquilinear form $\Phi(u, v)$ on H satisfies the inequality*

$$\|u\|_H^2 \leqslant C \operatorname{Re} \Phi_\Omega(u, u). \tag{2.2}$$

Then there exists a unique element $u \in H$ such that

$$\Phi(u, v) = (f, v) \tag{2.3}$$

for all $v \in H$. Moreover, the operator $L^{-1}\colon f \mapsto u$ is bounded.

Let us proceed to the Dirichlet problem ($\Gamma = \Gamma_1$). In this case, $u \in H^1(\Omega)$ and $v \in \widetilde{H}^1(\Omega)$. Accordingly, $f \in H^{-1}(\Omega)$. The Green formula becomes

$$\Phi_\Omega(u, v) = (Lu, v)_\Omega. \tag{2.4}$$

This formula does not contain u^+ explicitly, but it is assumed that $u^+ = g$.

Let $u_0 \in H^1(\Omega)$ be a function with $u_0^+ = g$. Define $f_0 = Lu_0 \in H^{-1}(\Omega)$ by the Green formula $\Phi_\Omega(u_0, v) = (Lu_0, v)_\Omega$. The difference $u - u_0$ belongs to $\widetilde{H}^1(\Omega) = \mathring{H}^1(\Omega)$, and if we redenote it by u, then we again obtain the Green formula (2.4), where now both functions u and v belong to $\widetilde{H}^1(\Omega)$, while $f = Lu$ belongs to the dual space $H^{-1}(\Omega)$. This is the standard setting of the Dirichlet problem with homogeneous boundary condition. The form $\Phi_\Omega(u, u)$ is coercive on $\widetilde{H}^1(\Omega)$, and so the Dirichlet problem is uniquely solvable.

Remark. Assume (within this remark) that the operator L is formally self-adjoint. We see from the Green formula that then the space $H^1(\Omega)$ is the orthogonal sum (with respect to the inner product $\Phi_\Omega(u, v)$) of the subspace $\mathring{H}^1(\Omega) = \widetilde{H}^1(\Omega)$ and the subspace formed by the solutions of the system $Lu = 0$. The latter subspace can be parametrized by the Dirichlet data in $H^{1/2}(\Gamma)$, and hence linear continuous functionals on it are in isomorphic correspondence with functionals on $H^{1/2}(\Gamma)$, i.e., with elements of $H^{-1/2}(\Gamma)$. Thus, the space $H^{-1}(\Omega)$ dual to $\widetilde{H}^1(S)$ can be identified with the quotient of the space $\widetilde{H}^{-1}(\Omega)$ dual to $H^1(\Omega)$ by the subspace of functionals supported in Γ.

2.2. Let us proceed to the mixed problem. Its solutions u are sought in $H^1(\Omega)$ as well. Here the crucial point is the choice of the space of test functions v: it is the (closed) subspace $H^1(\Omega, \Gamma_1) \subset H^1(\Omega)$ of functions with zero trace on Γ_1. (That is, the trace, which belongs to $H^{1/2}(\Gamma)$, is zero on Γ_1.)

As is seen from [27], the space $H^1(\Omega, \Gamma_1)$ can also be defined as (i) the space of restrictions to Ω of functions belonging to the completion of the linear manifold $C_0^\infty(\mathbb{R}^n \setminus \overline{\Gamma}_1)$ in $H^1(\mathbb{R}^n)$; (ii) the completion in $H^1(\Omega)$ of the set of restrictions to Ω of functions belonging to $C_0^\infty(\mathbb{R}^n \setminus \overline{\Gamma}_1)$.

The Green formula defining the solution of the problem becomes

$$\Phi_\Omega(u, v) = (Lu, v)_\Omega + (T^+u, v^+)_{\Gamma_2}, \tag{2.5}$$

where $Lu = f$ and $T^+u = h$. The choice of the space for v determines the space of right-hand sides of the system: it is the space

$$\widetilde{H}^{-1}(\Omega, \Gamma_1) := [H^1(\Omega, \Gamma_1)]^* \tag{2.6}$$

dual to $H^1(\Omega, \Gamma_1)$ with respect to the extension of the form $(f, v)_\Omega$ to their direct product. By analogy with the remark at the end of Section 2.1, we see that the space (2.6) can be identified with the quotient space of $\widetilde{H}^{-1}(\Omega)$ by the subspace of functionals supported in $\overline{\Gamma}_1$. Then every f in (2.6) is an element of $\widetilde{H}^{-1}(\Omega)$ defined modulo addition of an arbitrary element supported in $\overline{\Gamma}_1$. One can simply treat f as an element of $\widetilde{H}^{-1}(\Omega)$ (by extending the corresponding functional to the entire $H^1(\Omega)$ by the Hahn–Banach theorem); this is actually done in [24].

The equality $g = u^+$ is again assumed. Take a function $u_0 \in H^1(\Omega)$ such that $u_0^+ = g$ on Γ_1. (For this, extend g to a function in $H^{1/2}(\Gamma)$ and take u_0 such that $u_0^+ = g$.) Define Lu_0 by the formula $\Phi_\Omega(u_0, v) = (Lu_0, v)_\Omega$, taking the corresponding conormal derivative to be zero on Γ_2. Redenote the difference $u - u_0$ by u. For this difference, we obtain the Green formula (2.5) in which both functions u and v belong to the space $H^1(\Omega, \Gamma_1)$ and $Lu = f$ belongs to the dual space (2.6). One can take $h = 0$ if desired.

All this is consistent with what was said above about the Dirichlet and Neumann problems.

Now we use the coercivity of the form $\Phi_\Omega(u, u)$ on $H^1(\Omega, \Gamma_1)$ (just now it is sufficient) to obtain the following well-known theorem with the help of the Lax–Milgram lemma (cf. [24]).

Theorem 2.2. *For any $f \in \widetilde{H}^{-1}(\Omega, \Gamma_1)$, $g \in H^{1/2}(\Gamma_1)$, and $h \in H^{-1/2}(\Gamma_2)$, problem* (1.1)–(1.2) *has a unique variational solution $u \in H^1(\Omega)$.*

3. Reduction of the Mixed Problem to Equations on Γ

3.1. To save space, we assume that the domain $\Omega = \Omega^+$ lies on the standard torus $\mathbb{T}^n$ with periodic coordinates and the surface Γ divides it into two domains $\Omega^\pm$. The normal to Γ, where it exists, is assumed to point into Ω^-. Now $\widetilde{H}^s(\Omega^+)$ is a subspace of $H^s(\mathbb{T})$. We assume that the coefficients of the system are extended to the torus, the assumptions about their "smoothness" and strong ellipticity being preserved. The form $\Phi_\mathbb{T}(u, u)$ is assumed to be coercive (in the strengthened sense) on $H^1(\mathbb{T})$ (this does not require additional assumptions on the principal part of the system), and the forms $\Phi_{\Omega^\pm}(u, u)$ are assumed to be coercive on $H^1(\Omega^\pm)$. Then the Dirichlet and Neumann problems in $\Omega^\pm$ are uniquely solvable. This permits us to use the results already obtained in [3] (where we followed the ideas of [31], [12], and [24]). We continue the analysis of the mixed problem in $\Omega = \Omega^+$, though one can simultaneously consider it in Ω^- as well.

System (1.1) is uniquely solvable in $H^1(\mathbb{T})$ for $f \in H^{-1}(\mathbb{T})$. (This permits us to assume that $f = 0$ in what follows.) The inverse operator L^{-1} is an integral operator,

$$L^{-1}f(x) = \int_\mathbb{T} \mathscr{E}(x, y) f(y)\, dy. \tag{3.1}$$

It is known as the Newtonian potential; its kernel $\mathscr{E}$ is a fundamental solution for L. With this kernel at hand, we introduce the *single layer potential*

$$\mathscr{A}\psi(x) = \int_\Gamma \mathscr{E}(x,y)\psi(y)\,dS_y \tag{3.2}$$

and the *double layer potential*; the latter is defined in the general case by the formula

$$\mathscr{B}\varphi(x) = \int_\Gamma (\widetilde{T}_y^+ \mathscr{E}^*(x,y))^* \varphi(y)\,dS_y \qquad (x \notin \Gamma), \tag{3.3}$$

where $\widetilde{T}^+(\,\cdot\,)$ is the conormal derivative corresponding to the operator $\widetilde{L}$ in Ω formally adjoint to L (see [24]). If $L = -\Delta$, the asterisks and the tilde are omitted. Recall the properties of these operators (see [24] and [3]).

Under our assumptions, the operator $\mathscr{A}$ can be extended to an operator that is bounded from $H^{-1/2}(\Gamma)$ to $H^1(\mathbb{T})$ and hence to $H^1(\Omega^\pm)$. For $\psi \in H^{-1/2}(\Gamma)$, the function $u = \mathscr{A}\psi$ satisfies the homogeneous system $Lu = 0$ in $\Omega^\pm$ and has coinciding traces $\gamma^\pm u$ on Γ, which are denoted by $A\psi$; this is a bounded invertible operator from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$. This permits one to construct the solution of the Dirichlet problem in Ω^+ for the system $Lu = 0$ by the formula

$$u = \mathscr{A} A^{-1} u^+. \tag{3.4}$$

The operator $\mathscr{B}$ is bounded from $H^{1/2}(\Gamma)$ to $H^1(\Omega^\pm)$, and if $\varphi \in H^{1/2}(\Gamma)$, then the function $u = \mathscr{B}\varphi$ satisfies the homogeneous system in $\Omega^\pm$ as well. It has traces $\gamma^\pm\mathscr{B}\varphi$ on Γ, and these are bounded operators in $H^{1/2}(\Gamma)$. As in [24], we set $B = \frac{1}{2}(\gamma^+\mathscr{B} + \gamma^-\mathscr{B})$. This is known as the *direct value of the double layer potential.* The jump $[u] = u^- - u^+$ of the function $u = \mathscr{B}\varphi$ is equal to φ, and hence $\gamma^+\mathscr{B} = -\frac{1}{2}I + B$. The operator on the right-hand side is bounded and invertible in $H^{1/2}(\Gamma)$. This permits one to write the solution of the Dirichlet problem in Ω^+ also in the form

$$u = \mathscr{B}(-\tfrac{1}{2}I + B)^{-1} u^+. \tag{3.5}$$

The operators $T^\pm$ can be applied to the functions $\mathscr{A}\psi$ and $\mathscr{B}\varphi$. Moreover, $T^+\mathscr{B} = T^-\mathscr{B}$. The operator $H = -T^\pm\mathscr{B}$ is called the *hypersingular operator*. It acts boundedly from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ and is invertible. Hence the solution of the Neumann problem for the system $Lu = 0$ in Ω^+ can be constructed by the formula

$$u = -\mathscr{B} H^{-1} T^+ u. \tag{3.6}$$

Next, the $T^\pm\mathscr{A}$ are bounded operators in $H^{-1}(\Gamma)$. As in [24], set $\widehat{B} = \frac{1}{2}(T^+\mathscr{A} + T^-\mathscr{A})$. The jump $[T\mathscr{A}\varphi] = T^-\mathscr{A}\varphi - T^+\mathscr{A}\varphi$ is equal to $-\varphi$, and hence $T^+\mathscr{A} = \frac{1}{2}I + \widehat{B}$. This is a bounded invertible operator in $H^{-1/2}(\Gamma)$, and the solution of the Neumann problem in Ω^+ can also be constructed by the formula

$$u = \mathscr{A}(\tfrac{1}{2}I + \widehat{B})^{-1} T^+ u. \tag{3.7}$$

We also need the operator N that maps the Neumann data T^+u into the Dirichlet data u^+ (the Neumann-to-Dirichlet operator) and the inverse operator D (the Dirichlet-to-Neumann operator). The operator N acts boundedly from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$, and D is bounded in the opposite direction. They satisfy the Gårding type inequalities

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} \leqslant C_1 \operatorname{Re}(D\varphi,\varphi)_\Gamma, \qquad \|\psi\|^2_{H^{-1/2}(\Gamma)} \leqslant C_2 \operatorname{Re}(N\psi,\psi)_\Gamma, \tag{3.8}$$

which follow from the Green formula (2.1) and the original coercivity assumption. The invertibility of D and N follows from the assumption on the unique solvability of the Dirichlet and Neumann problems. It also follows from these inequalities by virtue of the Lax–Milgram lemma. Next, one knows that $BA = A\widehat{B}$, and hence it follows from the preceding formulas that

$$N = A(\tfrac{1}{2}I + \widehat{B})^{-1} = (\tfrac{1}{2}I + B)^{-1} A. \tag{3.9}$$

If Γ and the coefficients of L are infinitely smooth, then A and N are (strongly elliptic) ΨDO (pseudodifferential operators) of order -1, and H and D are ΨDO of order 1.

3.2. Now we give two versions of composing the equations on Γ for the mixed problem. In the first version, we follow the paper [30]; see also [42] and [43].

This version consists in the following. We extend g to a function in $H^{1/2}(\Gamma)$, the extension being still denoted by g. Now, for the solution u of our problem, we have $u^+ = g + g_0$, where $g_0 \in \widetilde{H}^{1/2}(\Gamma_2)$. Once we find g_0, we can compute u as the solution of the Dirichlet problem. On the other hand, $h = (D(g+g_0))|_{\Gamma_2}$. *Let us introduce the operator*

$$D_{\Gamma_2}\varphi = (D\varphi)|_{\Gamma_2} \tag{3.10}$$

from $\widetilde{H}^{1/2}(\Gamma_2)$ to $H^{-1/2}(\Gamma_2)$. It is obtained from the operator D by narrowing its domain and by restricting the resulting functions to Γ_2. We obtain the following equation for g_0:

$$D_{\Gamma_2} g_0 = h_0, \tag{3.11}$$

where $h_0 = h - (Dg)|_{\Gamma_2}$ is a known function.

Theorem 3.1. *The operator D_{Γ_2} is invertible.*

Proof. The first inequality in (3.8) is 'inherited" by the operator D_{Γ_2}:

$$\|\varphi\|^2_{\widetilde{H}^{1/2}(\Gamma_2)} \leqslant C_1 \operatorname{Re}(D_{\Gamma_2}\varphi, \varphi)_{\Gamma_2}. \tag{3.12}$$

Here the functions in the form on the right-hand side lie in the dual spaces $H^{-1/2}(\Gamma_2)$ and $\widetilde{H}^{1/2}(\Gamma_2)$. It suffices to use the Lax–Milgram lemma. □

Thus, the first approach is essentially to solve Eq. (3.11) and then, say, the equation $A\psi = g+g_0$, after which u can be determined by the formula $u = \mathscr{A}\psi$. As is seen from (3.5), one can also construct the solution in the form of a double layer potential.

The second approach is similar to the first except that the Neumann problem is used rather than the Dirichlet problem. Let us extend h to an element of $H^{-1/2}(\Gamma)$. After this, we have $T^+u = h+h_0$, where $h_0 \in \widetilde{H}^{-1/2}(\Gamma_1)$. *Let us introduce the operator*

$$N_{\Gamma_1}\psi = (N\psi)|_{\Gamma_1} \tag{3.13}$$

from $\widetilde{H}^{-1/2}(\Gamma_1)$ to $H^{1/2}(\Gamma_1)$. For h_0, we obtain the equation

$$N_{\Gamma_1} h_0 = g_1, \tag{3.14}$$

where g_1 is the known function $g - (Nh)|_{\Gamma_1}$.

Theorem 3.2. *The operator N_{Γ_1} is invertible.*

Proof. The second inequality in (3.8) for N is inherited by the operator N_{Γ_1}:

$$\|\psi\|^2_{\widetilde{H}^{-1/2}(\Gamma_1)} \leqslant C_2 \operatorname{Re}(N_{\Gamma_1}\psi, \psi)_{\Gamma_1}. \tag{3.15}$$

The functions on the right-hand side belong to the dual spaces $H^{1/2}(\Gamma_1)$ and $\widetilde{H}^{-1/2}(\Gamma_1)$, and we again apply the Lax–Milgram lemma. □

Thus, in the second approach we solve first Eq. (3.14) and then, say, the equation $H\varphi = h+h_0$, after which the solution is found in the form $u = -\mathscr{B}\varphi$. One can also construct the solution in the form of a single layer potential (see (3.7)).

Remarks. 1. The operators N_{Γ_1} and D_{Γ_2} for parts of a closed Lipschitz surface are analogs of the operators N and D for the entire surface and inherit the properties of the latter (the Gårding inequalities and the invertibility). Operators A_S and H_S with similar properties were considered in [4]. In special cases, they occurred, e.g., in [42], [43], and [30].

2. The term "Neumann-to-Dirichlet operator" can rightly be used not only for the operator N_{Γ_1} but also for the operator $D^{-1}_{\Gamma_1}$ acting from $H^{-1/2}(\Gamma_1)$ to $\widetilde{H}^{-1/2}(\Gamma_1)$. These are operators with distinct domains and distinct ranges. There are two operators, N_{Γ_2} and $D^{-1}_{\Gamma_2}$, on Γ_2 as well.

There are a lot of applications of the operators N, D, N_{Γ_1}, and D_{Γ_2} (e.g., see [23], [48], and the bibliography therein).

4. Regularity of Solutions

4.1. Spaces $\boldsymbol{H_p^s}$ of Bessel potentials and the Besov spaces $\boldsymbol{B_p^s}$. (For example, see [47], [21], [29], and [2].) We agree to assume that

$$s \in \mathbb{R}, \quad 1 < p < \infty, \quad p + p' = pp'. \tag{4.1}$$

The space $H_p^s(\mathbb{R}^n)$ is defined by the formula

$$H_p^s(\mathbb{R}^n) = J^{-s} L_p(\mathbb{R}^n), \tag{4.2}$$

where J^{-s} is the same operator as in (1.6). For integer $s \geqslant 0$, this is the Sobolev space $W_p^s(\mathbb{R}^n)$.

The Slobodetskii spaces $W_p^s(\mathbb{R}^n)$ are defined for noninteger $s > 0$. If $0 < s < 1$, then

$$\|u\|^p_{W_p^s(\mathbb{R}^n)} = \|u\|^p_{L_p(\mathbb{R}^n)} + \iint \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}}\, dx\, dy. \tag{4.3}$$

The Besov space $B_p^s(\mathbb{R}^n) = B_{p,p}^s(\mathbb{R}^n)$ is defined by the formula

$$B_p^s(\mathbb{R}^n) = J^{\sigma - s} W_p^\sigma(\mathbb{R}^n), \qquad 0 < \sigma < 1; \tag{4.4}$$

if we change $\sigma \in (0,1)$, the norm is replaced by an equivalent norm. For noninteger $s > 0$, these spaces coincide with the Slobodetskii spaces.

For $p = 2$, the spaces H_p^s and B_p^s coincide with H^s.

In what follows in this section, the letter H can be replaced by B.

The spaces $H_p^s(\mathbb{R}^n)$ and $H_{p'}^{-s}(\mathbb{R}^n)$ are dual with respect to the extension of the standard inner product in $L_2(\mathbb{R}^n)$ to their direct product.

The space $H_p^s(\Omega)$ consists of the restrictions of elements in $H_p^s(\mathbb{R}^n)$ to Ω and is equipped with the inf-norm. The operator $\mathscr{E}$ (the same as before) is a bounded operator of extension of elements in $H_p^s(\Omega)$ to elements in $H_p^s(\mathbb{R}^n)$ [37].

The space $\widetilde{H}_p^s(\Omega)$ is defined as the subspace of $H_p^s(\mathbb{R}^n)$ formed by the elements supported in $\overline{\Omega}$. For $-1/p' < s < 1 + 1/p$, $s \neq 1/p$, it can be identified with the completion $\mathring{H}_p^s(\Omega)$ of the linear manifold $C_0^\infty(\Omega)$ in $H_p^s(\Omega)$.

The spaces $\widetilde{H}_{p'}^{-s}(\Omega)$ and $H_p^s(\Omega)$ are dual with respect to the form (1.7). The spaces $H_p^s(\Omega)$ and $\widetilde{H}_p^s(\Omega)$ are identified for $-1/p' < s < 1/p$.

The spaces $B_p^s(\Gamma)$, $|s| \leqslant 1$, are introduced with the help of a partition of unity on Γ and the norms in $B_p^s(\mathbb{R}^{n-1})$. The spaces $B_p^s(\Gamma)$ and $B_{p'}^{-s}(\Gamma)$ are dual with respect to the extension of the inner product in $L_2(\Gamma)$ to their direct product.

The trace operator acts boundedly from $H_p^{s+1/p}(\Omega)$ and from $B_p^{s+1/p}(\Omega)$ to $B_p^s(\Gamma)$ for $0 < s < 1$. These two operators have a common right inverse [22].

Let Γ_0 be a domain on Γ. The space $B_p^s(\Gamma_0)$ is defined as consisting of the restrictions to Γ_0 of elements in $B_p^s(\Gamma)$ and is equipped with the inf-norm. There exists a bounded operator, independent of s and p, of extension of elements in $B_p^s(\Gamma_0)$ to elements in $B_p^s(\Gamma)$. The space $\widetilde{B}_p^s(\Gamma_0)$ is defined as the subspace of $B_p^s(\Gamma)$ formed by the elements supported in $\overline{\Gamma}_0$. The spaces $\widetilde{B}_{p'}^{-s}(\Gamma_0)$ and $B_p^s(\Gamma_0)$ are dual with respect to the extension of the standard inner product in $L_2(\Gamma_0)$ to their direct product. The spaces $B_p^s(\Gamma_0)$ and $\widetilde{B}_p^s(\Gamma_0)$ are identified for $-1/p' < s < 1/p$.

4.2. Now the solutions of the mixed problem are sought in $H_p^{1/2+s+1/p}(\Omega)$, where $|s|$ is necessarily less than $1/2$: in (1.2), we have

$$g \in B_p^{1/2+s}(\Gamma_1), \quad h \in B_p^{-1/2+s}(\Gamma_2). \tag{4.5}$$

The variational (weak) setting of the mixed problem preserves the form (2.5). The test functions v belong to the subspace $H_{p'}^{1/2-s+1/p'}(\Omega, \Gamma_1) \subset H_{p'}^{1/2-s+1/p'}(\Omega)$ of functions with zero trace on Γ_1 (cf. [27]).

The right-hand side f of the system belongs to the dual space

$$\widetilde{H}_p^{-1/2+s-1/p'}(\Omega,\Gamma_1) := [H_{p'}^{1/2-s+1/p'}(\Omega,\Gamma_1)]^* \tag{4.6}$$

with respect to the extension of the form (1.7).

Here the H-spaces can be replaced by the B-spaces.

The admissible points (s,t), $t = 1/p$, form the square

$$Q = \{(s,t) : |s| < 1/2,\ 0 < t < 1\}. \tag{4.7}$$

The Dirichlet and Neumann problems are special cases of the mixed problem and are known [3] to be uniquely solvable for $|s| < \varepsilon$ and $|t - 1/2| < \delta$ with sufficiently small ε and δ. We wish to obtain a similar result for the mixed problem. To this end, as in [4], we consider potential type operators in scales of spaces on the boundary (cf. [17], [27], and [11]).

First, note that the operators

$$\begin{aligned} A, N &\colon B_p^{-1/2+s}(\Gamma) \to B_p^{1/2+s}(\Gamma), & H, D &\colon B_p^{1/2+s}(\Gamma) \to B_p^{-1/2+s}(\Gamma),\\ \tfrac{1}{2}I \pm B &\colon B_p^{1/2+s}(\Gamma) \to B_p^{1/2+s}(\Gamma), & \tfrac{1}{2}I \pm \widehat{B} &\colon B_p^{-1/2+s}(\Gamma) \to B_p^{-1/2+s}(\Gamma) \end{aligned} \tag{4.8}$$

are bounded and invertible for the same (s,t) [3]. Moreover, $N = D^{-1}$, and relations (3.9) are preserved.

The operator N_{Γ_1} acts boundedly from $\widetilde{B}_p^{-1/2+s}(\Gamma_1)$ to $B_p^{1/2+s}(\Gamma_1)$ (for the same (s,t)). *Each of these two families of spaces is an interpolation scale* with respect to the complex interpolation method in each of the indices. We explained this in [4] when considering the operators A_S and H_S. Our operator is invertible at the point $(s,t) = (0, 1/2)$. Hence the Shneiberg theorem [40] applies, and we obtain the following assertion.

Theorem 4.1. *The operator $N_{\Gamma_1} \colon \widetilde{B}_p^{-1/2+s}(\Gamma_1) \to B_p^{1/2+s}(\Gamma_1)$ remains invertible for $|s| < \varepsilon$ and $|t - 1/2| < \delta$ with sufficiently small ε and δ.*

Now we shall establish the unique solvability of the mixed problem for these (s,t). First, we verify the uniqueness. Let $g = 0$ and $h = 0$. Then $T^+u = h_0 \in \widetilde{B}_p^{-1/2+s}(\Gamma_1)$. Consequently, $Nh_0 = u^+$ on Γ, and hence $N_{\Gamma_1}h_0 = 0$. But the operator N_{Γ_1} is invertible, hence $h_0 = 0$ and $T^+u = 0$ on Γ. It remains to use the uniqueness for the Neumann problem. We have used the second version of the approach to the mixed problem (see Section 3.2). With this version, the existence can obviously be proved as well. This gives the main result of the present section.

Theorem 4.2. *The mixed problem* (1.1)–(1.2) *with* $f = 0$, $g \in B_p^{1/2+s}(\Gamma_1)$, *and* $h \in B_p^{-1/2+s}(\Gamma_2)$ *remains uniquely solvable for $|s| < \varepsilon$ and $|t - 1/2| < \delta$ with sufficiently small ε and δ.*

For these s and t, the "better" the right-hand sides, the "better" the solution itself is. This is the promised regularity result. It follows automatically from Theorem 4.2.

Remarks. 1. Instead of N_{Γ_1}, one can use the operator $D_{\Gamma_2} \colon \widetilde{B}_p^{1/2+s}(\Gamma_2) \to B_p^{-1/2+s}(\Gamma_2)$; *for this operator, one also has the boundedness and invertibility for $|s| < \varepsilon$ and $|t - 1/2| < \delta$* with sufficiently small ε and δ. Then we use the first approach to the mixed problem (Section 3.2).

2. We see that *both versions of reducing the mixed problem to equivalent equations on the boundary and solving these equations with the help of potential type operators are preserved* in these more general spaces.

3. *Theorem* 4.2 *can be generalized to the case of a nonzero right-hand side f in the equation* $Lu = f$. Indeed, let $f \in \widetilde{H}_p^{-1/2+s-1/p'}(\Omega,\Gamma_1)$. Then we can treat f as an element of the space $\widetilde{H}_p^{-1/2+s-1/p'}(\Omega)$. (The letter H can be replaced by B.) Let u_0 be the solution of the Neumann problem for the equation $Lu_0 = f$ with zero conormal derivative. By subtracting u_0 from u, we obtain a problem to which Theorem 4.2 applies.

5. Spectral Problems

5.1. Problem with spectral parameter in the system. Consider the problem

$$Lu = \lambda u \quad \text{in } \Omega, \quad u^+ = 0 \quad \text{on } \Gamma_1, \quad T^+ u = 0 \quad \text{on } \Gamma_2. \tag{5.1}$$

First, let the operator L be formally self-adjoint, $L = \widetilde{L}$. Assuming that the equation $Lu = f$ is uniquely solvable in $H^1(\Omega, \Gamma_1)$ for f in $\widetilde{H}^{-1}(\Omega, \Gamma_1)$, we equip the latter space with the inner product (cf. [34])

$$\langle f_1, f_2 \rangle_\Omega := (L^{-1} f_1, f_2)_\Omega. \tag{5.2}$$

The unbounded operator L in this space (with domain $H^1(\Omega, \Gamma_1)$) remains self-adjoint with respect to this inner product. The eigenvalues are positive. Just as in other problems considered earlier in [3] and [5], the eigenvalues and eigenfunctions coincide with those of the similar operator in $L_2(\Omega)$. Since the space $H^1(\Omega, \Gamma_1)$ contains $\widetilde{H}^1(\Omega) = \mathring{H}^1(\Omega)$ and is contained in $H^1(\Omega)$, we can apply Métivier's result [25] stating that the counting function $N_L(\lambda)$, i.e., the number (counting multiplicities) of eigenvalues that are less than λ, has the asymptotics

$$N_L(\lambda) = c_L \lambda^{n/2} + O(\lambda^{(n-1/2)/2}), \tag{5.3}$$

where the coefficient c_L is the same as for the Dirichlet and Neumann problems (see [5]).

Next, we have the same assertions about the eigenfunctions as in these problems [5]. The eigenfunctions form an orthonormal basis in $\widetilde{H}^{-1}(\Omega, \Gamma_1)$. They belong to the space $H^1(\Omega, \Gamma_1)$ and form there an orthonormal basis with respect to the inner product $(Lu, v)_\Omega$, which is equal to $\Phi_\Omega(u, v)$. This result extends to the intermediate spaces. Moreover, one can prolong the scale of these spaces to the left and right by ε. As to the spaces corresponding to the values of t with $|t - 1/2| < \delta$, $t \neq 1/2$, the completeness of eigenfunctions and the Abel–Lidskii summability of Fourier series in these functions remain valid in these spaces.

If only the principal part of the operator L is formally self-adjoint, then the eigenvalues, starting from some number, lie in an arbitrarily narrow sector with bisector $\mathbb{R}_+$ and have asymptotics with the same leading term. The assertions on the smoothness of root functions, their completeness, and summability remain valid.

If we drop the self-adjointness assumptions completely, then the estimate $|\lambda_j(L^{-1})| \leqslant C j^{-2/n}$ for the eigenvalues is preserved. The most general case in which we still can obtain assertions on the completeness and summability is the case in which all values of the form $\Phi_\Omega(u, u)$ lie in a sector with bisector $\mathbb{R}_+$ and opening angle less than $2\pi/n$ (cf. [5]).

5.2. Poincaré–Steklov problems with spectral parameter on a part of the boundary. Consider the following two problems.

$$\text{I.} \quad Lu = 0 \quad \text{in } \Omega, \quad T^+ u = 0 \quad \text{on } \Gamma_2, \quad \lambda T^+ u = u^+ \quad \text{on } \Gamma_1.$$

Here $T^+ u \in \widetilde{H}^{-1/2}(\Gamma_1)$ and (see Section 3.2) $N_{\Gamma_1} T^+ u = u^+ \in H^{1/2}(\Gamma_1)$. Hence for the *eigenfunctions* of the spectral problem I we obtain the equation $N_{\Gamma_1} \psi = \lambda \psi$, where $\psi = T^+ u$, and it is equivalent to this problem if $L = \widetilde{L}$.

$$\text{II.} \quad Lu = 0 \quad \text{in } \Omega, \quad u^+ = 0 \quad \text{on } \Gamma_2, \quad \lambda T^+ u = u^+ \quad \text{on } \Gamma_1.$$

Here $u^+ \in \widetilde{H}^{1/2}(\Gamma_1)$ and (see Section 3.2) $D_{\Gamma_1} u^+ = T^+ u \in H^{-1/2}(\Gamma_1)$. Hence for the *eigenfunctions* of the spectral problem II we obtain the equation $D_{\Gamma_1}^{-1} \psi = \lambda \psi$, where again $\psi = T^+ u$, and it is equivalent to this problem if $L = \widetilde{L}$.

We have interchanged Γ_1 and Γ_2 in the second problem as compared with Section 3.2 in order to compare the corresponding operators in what follows.

The spectral properties of the operators N_{Γ_1} and D_{Γ_1} are similar to those of the operators A_S and H_S (for $\Gamma_1 = S$), respectively, described in [4]. For $L = \widetilde{L}$, we define the inner product $\langle \psi_1, \psi_2 \rangle_{\Gamma_1}$ in the first case in $\widetilde{H}^{-1/2}(\Gamma_1)$ as $(N_{\Gamma_1} \psi_1, \psi_2)_{\Gamma_1}$ and in the second case in $H^{-1/2}(\Gamma_1)$ as $(D_{\Gamma_1}^{-1} \psi_1, \psi_2)_{\Gamma_1}$.

In particular, we mean the smoothness properties of the eigenfunctions and root functions, the results concerning an orthonormal basis of eigenfunctions for $p = 2$ and $L = \widetilde{L}$, and the results on the completeness of the root functions and the Abel–Lidskii summability of Fourier series in root functions in the other cases. If the operator L has formally self-adjoint leading part, then the eigenvalues, starting from some number, lie in an arbitrarily narrow sector with bisector $\mathbb{R}_+$. The most general case in which assertions on the completeness and summability can be obtained is the case in which all values of the quadratic forms $\Phi_{\Omega^\pm}(u, u)$ lie in a sector with bisector $\mathbb{R}_+$ and opening angle less than $\pi/(n-1)$.

Additional completeness results can be obtained as a consequence of (dense) embeddings of the spaces considered.

6. Spectral Asymptotics

Here we wish to obtain asymptotic formulas for the eigenvalues of the operators N_{Γ_1} and $D_{\Gamma_1}^{-1}$. Before doing this, we consider operators on Γ. First, let us accept the following assumptions:

1°. $L = \widetilde{L}$.

2°. The surface Γ is almost smooth (see Section 1.1).

3°. L is a scalar operator or a matrix operator whose principal part coincides with the Lamé operator.

It was shown in [5] that under these conditions one can derive the asymptotic formula

$$\lambda_j(N) \sim C_N j^{-1/(n-1)} \tag{6.1}$$

for the eigenvalues $\lambda_j(N)$ of the operator N (in the sense that the difference of the left- and right-hand sides is $o(j^{-1/(n-1)})$). This follows from the results of the paper [6] in conjunction with formula (3.9) above and the very deep known results on the invertibility of the operators $\frac{1}{2}I \pm B$ and $\frac{1}{2}I \pm \widehat{B}$ in $L_2(\Gamma)$ ([28], [14]). The eigenvalues are numbered in nonincreasing order, counting multiplicities.

Here we get rid of assumption 3° by using the variational approach to asymptotics. Moreover, we shall consider the operator bN, where the function b is assumed, first, to be a multiplier in $H^{\pm 1/2}(\Gamma)$ and, for simplicity, nonnegative. We assume that the Lipschitz surface Γ is almost smooth in a neighborhood of the support of b. The result is stated in Theorem 6.5 below (cf. [1] and especially [45], where piecewise smooth surfaces were considered).

Let T be a compact self-adjoint operator in a Hilbert space H with inner product $(\,\cdot\,,\,\cdot\,)$; for simplicity, we assume that T is nonnegative (has nonnegative eigenvalues). The counting function $N(\lambda) = N(\lambda, T)$ of its positive eigenvalues $\lambda_j(T)$ $(j = 1, 2, \dots)$ is the number of eigenvalues, counting multiplicities, greater than λ $(\lambda > 0)$. The asymptotics $N(\lambda) \sim \beta\lambda^{-\alpha}$ as $\lambda \to 0$ (where $\alpha > 0$ and $\beta > 0$) is equivalent to the asymptotics $\lambda_j \sim \beta' j^{-1/\alpha}$ as $j \to \infty$, where $\beta' = \beta^{1/\alpha}$.

The variational quotient for T has the form $R(x) = (Tx, x)/(x, x)$. The eigenvalues are the "subsequent maxima" of this quotient:

$$\lambda_{j+1}(T) = \min_{\operatorname{codim} X \leqslant j} \; \max_{0 \neq x \in X} R(x), \tag{6.2}$$

where X ranges over the subspaces of H.

We need some well-known lemmas, which we give in a simplified form (cf. [8, Supplement 1] or [45, Section 1]). The first of these lemmas permits one to compare the counting functions of operators acting in different spaces.

Lemma 6.1. *Let H_1 and H_2 be Hilbert spaces with inner products $(\,\cdot\,,\,\cdot\,)_1$ and $(\,\cdot\,,\,\cdot\,)_2$, respectively, let T_1 and T_2 be compact nonnegative operators in these spaces, and let S be a bounded operator from H_1 to H_2. Suppose that $(T_1x, x)_1 = 0$ for $Sx = 0$ and*

$$(T_1x, x)_1/(x, x)_1 \leqslant (T_2Sx, Sx)_2/(Sx, Sx)_2 \qquad (x \in H_1,\ Sx \neq 0). \tag{6.3}$$

Then $N(\lambda, T_1) \leqslant N(\lambda, T_2)$. In particular, this is true if $H_1 \subset H_2$, $(\,\cdot\,,\,\cdot\,)_1 = (\,\cdot\,,\,\cdot\,)_2$ on H_1, and S is the embedding operator; the denominators in (6.3) are not needed in this case.

Lemma 6.2. *Let H be the orthogonal sum $H_1 \oplus H_2$ of subspaces H_1 and H_2 invariant with respect to a nonnegative operator T in H, and let T_j be the restrictions of this operator to H_j. Then $N(\lambda, T) = N(\lambda, T_1) + N(\lambda, T_2)$.*

Lemma 6.3 (M. Sh. Birman–M. Z. Solomyak). *Let a compact nonnegative operator T in a Hilbert space H admit a representation $T = T'_\varepsilon + T''_\varepsilon$ for each $\varepsilon > 0$, where T'_ε and T''_ε are compact operators, T'_ε is nonnegative, $\lambda_j(T'_\varepsilon) \sim C(T'_\varepsilon) j^{-\sigma}$ for some $\sigma > 0$, and* $\limsup |\lambda_j(T''_\varepsilon)| j^\sigma \leqslant \varepsilon$. *Then $C(T'_\varepsilon)$ has a finite limit $C(T)$ as $\varepsilon \to 0$, and $\lambda_j(T) \sim C(T) j^{-\sigma}$.*

6.1. Operator bN. First, assume that the coefficients of L are infinitely smooth. In $H^{-1/2}(\Gamma)$, we introduce the inner product

$$\langle \psi_1, \psi_2 \rangle_\Gamma = (N\psi_1, \psi_2)_\Gamma. \tag{6.4}$$

The operator bN remains compact and self-adjoint. Consider the variational quotient

$$R_b(\psi) = \langle bN\psi, \psi \rangle_\Gamma \big/ \langle \psi, \psi \rangle_\Gamma = (bN\psi, N\psi)_\Gamma / (N\psi, \psi)_\Gamma \qquad (\psi \in H^{-1/2}(\Gamma)). \tag{6.5}$$

Assuming that $\psi = T^+ u$ for a solution $u \in H^1(\Omega)$ of the system $Lu = 0$, we have $N\psi = u^+$. Hence the numerator is equal to $(bu^+, u^+)_\Gamma$, and the denominator is equal to $(T^+u, u^+)_\Gamma$, i.e., to $\Phi_\Omega(u, u)$ by the Green formula. The quotient (6.5) can be rewritten in the form

$$Q_b(u) = (bu^+, u^+)_\Gamma / \Phi_\Omega(u, u). \tag{6.6}$$

Now the numerator is the form corresponding to a compact operator in the subspace of $H^1(\Omega)$ formed by the solutions u of the system $Lu = 0$. We take $\Phi_\Omega(u, v)$ for the inner product in $H^1(\Omega)$. Then the orthogonal complement of the subspace of solutions is formed by functions with $u^+ = 0$; this was already noted in Section 2.1. Therefore (see Lemma 6.2), we shall consider the quotient (6.6) on all $u \in H^1(\Omega)$. (In fact, the passage from (6.5) to (6.6) is not necessarily needed.)

Now let us use the fact that, for a Lipschitz domain Ω, one can construct domains $\widetilde{\Omega}$ and $\widehat{\Omega}$ with infinitely smooth boundaries $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ such that $\widetilde{\Omega} \subset \Omega \subset \widehat{\Omega}$ and these boundaries are arbitrarily close to Γ. More precisely, there is a one-to-one correspondence between each of these smooth boundaries and Γ, and the distance between the corresponding points becomes uniformly arbitrarily small (e.g., see [49]).

In our case, we can fix a neighborhood U of arbitrarily small measure of the singular set Γ_{sing} and assume that $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ *coincide with* Γ *outside* U. Let θ_U be a smooth approximation to the characteristic function of the complement of U supported in that complement. Then the corresponding quotients $\widetilde{Q}_{b\theta_U}(u)$ and $\widehat{Q}_{b\theta_U}(u)$ satisfy

$$\widehat{Q}_{b\theta_U}(u) \leqslant Q_{b\theta_U}(u) \leqslant \widetilde{Q}_{b\theta_U}(u) \tag{6.7}$$

on functions in $H^1(\widehat{\Omega})$ and their restrictions to Ω and $\widetilde{\Omega}$. The left and right quotients correspond to smooth problems, and the eigenvalue asymptotics for the corresponding operators $b\theta_U \widehat{N}$ and $b\theta_U \widetilde{N}$ are known and coincide (we mean the leading term); see [9]. This implies a similar result for $b\theta_U N$ by Lemma 6.1.

Now consider the quotient $Q_c(u)$, where $c = b(1 - \theta_U)$. Using the boundedness of the trace operator, we have

$$Q_c(u) \leqslant C \int_\Gamma |c||u|^2 dS \Big/ \|u\|^2_{H^{1/2}(\Gamma)}, \tag{6.8}$$

and Γ can be replaced by $S = \operatorname{supp} c$ on the right-hand side. After that, the quotient on the right-hand side corresponds to a compact nonnegative operator in $H^{1/2}(S)$.

Lemma 6.4. *Let S be a domain on a bounded Lipschitz surface Γ of dimension $n-1$ with Lipschitz boundary, and let $c(x) \in L_r(S)$ be a nonnegative function on this surface. Then the counting function $N(\lambda, T)$ of the compact operator T in $H^s(S)$, $s > 0$, with the variational quotient*

$$\int_S c(x)|u(x)|^2 dS \Big/ \|u\|^2_{H^s(S)} \tag{6.9}$$

satisfies the inequality

$$N(\lambda, T) \leqslant C_1 \|c\|^{\tau}_{L_r(S)} \lambda^{-\tau} \tag{6.10}$$

for $r > 1$ *if* $n-1 = 2s$ *and for* $r = (n-1)/(2s)$ *if* $n-1 > 2s$, *with* $\tau = (n-1)/(2s)$.

Proof. For the case in which S is a domain in $\mathbb{R}^{n-1}$, this is Lemma 1.7 in [45]. As is indicated there, this assertion was obtained in [8, Theorem 4.1] for integer s, but the proof is preserved for noninteger s. The lemma was established for scalar functions but can readily be transferred to vector-valued functions.

To extend it to the case of a domain on a Lipschitz surface, one can assume that this domain is small and admits the representation $x_n = \phi(x')$ with a Lipschitz function $\phi(x')$ defined on the projection of this domain onto the x'-plane. It remains to take into account the simple fact that $dS = (1 + |\nabla\phi(x')|^2)^{1/2}\, dx'$, were the gradient is bounded. □

In our case, $s = 1/2$, and the lemma gives the estimate

$$|\lambda_j(T)| \leqslant C_2 \|c\|_{L_r(S)} \lambda^{-(n-1)}, \tag{6.11}$$

which implies the desired result. Indeed, the L_r-norm of the function $c = b(1-\theta_U)$ tends to zero as the neighborhood U shrinks to Γ_{sing}, because this function is bounded under our assumptions on b. Here one should use Lemma 6.3.

The assumption on the smoothness of the coefficients of L is removed by approximating the original coefficients by smooth ones (cf. [45, Section 5]). We arrive at the following assertion.

Theorem 6.5. *Let* $L = \widetilde{L}$, *let a nonnegative function* b *be a multiplier in* $H^{\pm 1/2}(\Gamma)$, *and suppose that the surface* Γ *is almost smooth in a neighborhood of the support of* b. *Then the asymptotics* (6.1) *holds for the eigenvalues of the operator* bN.

The coefficient C_{bN} is computed by the formula

$$C_{bN}^{n-1} = (2\pi)^{-(n-1)} \iint_{T^*\Gamma} b(x') n_\alpha(x', \xi')\, dx' d\xi' \tag{6.12}$$

in the case of smooth coefficients of L; here the singular set is deleted from Γ. By $\alpha(x', \xi')$ we denote the principal symbol of N, and $n_\alpha(x', \xi')$ is the number of its eigenvalues greater than 1. If the coefficients of L are nonsmooth and we approximate them by smooth ones, then the operators bN for these smooth problems converge to our operator in the operator norm. The desired coefficient is obtained by passage to the limit, which exists by the same Lemma 6.3 and can be computed via the principal symbol of L.

In view of space limitations, we do not present the generalization to the case in which b can change sign. However, note the following generalization.

Theorem 6.6. *The assertion of Theorem* 6.5 *remains valid if* $b \in L_r(\Gamma)$, *where* $r = n-1$ *for* $n > 2$ *and* $r > 1$ *for* $n = 2$, *provided that* bN *is defined as the operator with the variational quotient* (6.5) *or* (6.6).

When using, say, the quotient (6.6), one means the operator in $H^1(\Omega)$ with the form $(bu^+, v^+)_\Gamma$. This form is well defined by virtue of the embedding theorems and the Hölder inequality. In this case, b is not a multiplier in $H^{\pm 1/2}(\Gamma)$ in general.

6.2. Operator N_{Γ_1}. In this case, we would like to take the characteristic function θ of the domain Γ_1 for b. But it is not a multiplier in $H^{1/2}(\Gamma)$. However, this function *is* a multiplier in $\widetilde{H}^{-1/2}(\Gamma_1)$; moreover, the multiplication by it does not change the elements of this space. This permits one to rewrite the equation $N_{\Gamma_1}\psi = \lambda\psi$ in the form

$$\theta N \theta \psi = \lambda \theta \psi, \tag{6.13}$$

at least, assuming that $u^+ \in H^{1/2-\varepsilon}(\Gamma_1)$ with an arbitrarily small $\varepsilon > 0$. The corresponding variational quotient has the form

$$(\theta u^+, u^+)_\Gamma / \Phi_\Omega(u, u). \tag{6.14}$$

Here u belongs to the subspace of solutions of the system $Lu = 0$ in $H^1(\Omega)$ with $T^+u = 0$ outside Γ_1, and now we can assume that $u^+ \in H^{1/2}(\Gamma_1)$. By virtue of the Green formula

$$\Phi_\Omega(u, v) = (T^+u, v^+)_{\Gamma_1}, \tag{6.15}$$

the orthogonal complement of this subspace consists of functions u such that $u^+ = 0$ on Γ_1. The quotient (6.14) is zero on these functions and hence can be considered on all $u \in H^1(\Omega)$ with $Lu = 0$ and further on all $H^1(\Omega)$. Now Theorem 6.6 applies, and we obtain the desired result:

Theorem 6.7. *Let $L = \widetilde{L}$, and let the surface Γ be almost smooth in a neighborhood of the closure of the domain Γ_1. Then one has the asymptotics*

$$\lambda_j(N_{\Gamma_1}) \sim C_{N_{\Gamma_1}} j^{-1/(n-1)}. \tag{6.16}$$

Here $C_{N_{\Gamma_1}}$ is defined by formula (6.12) *with $b = 1$ and Γ_1 instead of Γ.*

6.3. Operator $D_{\Gamma_1}^{-1}$. The main result of this subsection, Theorem 6.8, was obtained by the author together with T. A. Suslina.

The operator $D_{\Gamma_1}^{-1}$ is the operator corresponding to the Dirichlet problem with $u^+ = 0$ outside Γ_1. For this operator, the variational quotient has the form

$$(u^+, u^+)_{\Gamma_1}/(Du^+, u^+)_{\Gamma_1} \qquad (Lu = 0 \text{ in } \Omega,\ u^+ \in \widetilde{H}^{1/2}(\Gamma_1)). \tag{6.17}$$

The asymptotics for smooth problems of this form was studied in [44]. In our case, the key point is to verify that *the quotient* (6.17) *for smooth Γ and $\partial\Gamma_j$ can be replaced by*

$$(\theta u^+, u^+)_\Gamma/(Du^+, u^+)_\Gamma \qquad (Lu = 0 \ \textit{in}\ \Omega,\ u^+ \in H^{1/2}(\Gamma)) \tag{6.18}$$

in the sense of coincidence of the asymptotics. It is relatively easy to do this in our case as follows. We use Lemma 6.1 twice.

Let $\alpha, \beta \in C^\infty(\Gamma)$, $0 \leqslant \alpha, \beta \leqslant 1$, $\operatorname{supp}\alpha \subset \Gamma_1$, and $\alpha^2 + \beta^2 = 1$; α approximates the function θ. By setting $\varphi = u^+$, we obtain

$$(\alpha\varphi, \alpha\varphi)_\Gamma/[(D\alpha\varphi, \alpha\varphi)_\Gamma + (D\beta\varphi, \beta\varphi)_\Gamma] \leqslant (\alpha\varphi, \alpha\varphi)_\Gamma/(D\alpha\varphi, \alpha\varphi)_\Gamma. \tag{6.19}$$

Here $\varphi \in H^{1/2}(\Gamma)$. For S we take the mapping $\varphi \mapsto \alpha\varphi$ of $H^{1/2}(\Gamma)$ into $\widetilde{H}^{1/2}(\Gamma_1)$. In addition, note that the forms in the denominators in (6.18) and on the left-hand side in (6.19) coincide up to the addition of the form of a zero-order ΨDO, because the principal symbols coincide. It is known that such addition does not affect the asymptotics (see [8] or [45], Section 1).

Now let $\alpha_1 \in C^\infty(\Gamma)$ be a nonnegative function equal to 1 in a neighborhood of the closure of the domain Γ_1 and approximating θ. We have

$$(\varphi, \varphi)_\Gamma/(D\varphi, \varphi)_\Gamma = (\alpha_1\varphi, \alpha_1\varphi)_\Gamma/(D\varphi, \varphi)_\Gamma. \tag{6.20}$$

Here $\varphi \in \widetilde{H}^{1/2}(\Gamma_1)$ and S is the embedding of this space in $H^{1/2}(\Gamma)$.

Lemma 6.1 shows that the counting function for the operator with quotient (6.17) is enclosed between the counting functions for the operator with the quotient on the left-hand side in (6.19) and the operator with the quotient $(\alpha_1\varphi, \alpha_1\varphi)_\Gamma/(D\varphi, \varphi)_\Gamma$, $\varphi \in H^{1/2}(\Gamma)$. The asymptotics for operators with such variational quotients are known from [9]. Now the passage to the limit with the use of Lemma 6.3 proves the assertion highlighted above. After that, essentially, the variational quotient and hence the asymptotics are the same as in Section 6.2.

The result can be extended to the case of an almost smooth surface by an argument similar to that used in Section 6.1. We obtain

Theorem 6.8. *If $L = \widetilde{L}$ and the surface Γ is almost smooth in a neighborhood of the closure of Γ_1, then the eigenvalues of the operator $D_{\Gamma_1}^{-1}$ have the asymptotics* (6.1) *with the same coefficient as for N_{Γ_1}.*

Example (cf. [23, p. 50]). Consider the Laplace equation in the square $\{(x, y) : 0 < x < \pi,\ 0 < y < \pi\}$. Let Γ_1 be its left side, and let Γ_2 consist of the other three sides. Problem I has the solutions $\cosh k(\pi - x)\cos ky$ $(k = 0, 1, \dots)$. Problem II has the solutions $\sinh k(\pi - x)\sin ky$

($k = 1, 2, \dots$). The eigenfunctions are $\cos ky$ in Problem I and $\sin ky$ in Problem II. The eigenvalues are $(\tanh k\pi)^{-1}/k$ and $\tanh k\pi/k$, respectively. They are different in the two problems, but the asymptotics is the same.

6.4. Remarks to the papers [4] **and** [5]**.** The variational quotient for the operator H^{-1} has the form (cf. [5, Proposition 9.1])

$$(\varphi, \varphi)_\Gamma/(H\varphi, \varphi)_\Gamma \qquad (\varphi = [u] \in H^{1/2}(\Gamma),\ u = \mathscr{B}\varphi). \tag{6.21}$$

Here $[Tu] = 0$, and the solutions of the system $Lu = 0$ in $\Omega^\pm$ are determined by the jump $[u]$. The argument in Section 6.1 directly applies to (6.21) in the case of an almost smooth Γ. As compared with [5], *one can remove condition* 3°. Next, *one can obtain an analog of Theorem* 6.8 *for the operator* H_S^{-1} *essentially in the same way as in the preceding subsection.* The approach in [4] to H_S^{-1} requires some reconsideration, which is possible with the help of a variational argument, but this is no longer needed.

The variational approach to the operators A and A_S is also possible but does not give anything new compared with [36] and [4]. We only note that the variational quotient for A can be put in the form $(u^\pm, u^\pm)_\Gamma/\Phi_{\mathbb{T}}(u, u)$.

7. Some Generalizations

7.1. More general Poincaré–Steklov problems. Now we wish to consider the case in which the boundary Γ is divided into three domains Γ_1, Γ_2, and Γ_3 with the homogeneous Dirichlet condition in the first domain, the homogeneous Neumann condition in the second domain, and the Poincaré–Steklov spectral condition in the third domain. We assume that these domains are separated by two closed Lipschitz surfaces of dimension $n-2$ without self-intersections and without common points (cf. [34]).

For definiteness, consider the case in which Γ_3 lies between Γ_1 and Γ_2. We introduce the following notation. Let $\Gamma_{2,3}$ be the complement of $\overline{\Gamma}_1$, and let $\Gamma_{1,3}$ be the complement of $\overline{\Gamma}_2$. Let $\widehat{H}^{1/2}(\Gamma_3)$ be the space of restrictions to Γ_3 of functions in $\widetilde{H}^{1/2}(\Gamma_{2,3})$ with the inf-norm, and let $\widehat{H}^{-1/2}(\Gamma_3)$ be the space of restrictions to Γ_3 of elements of $\widetilde{H}^{-1/2}(\Gamma_{1,3})$ with the inf-norm. Next, by $\mathscr{E}_2$ we denote the operator of extension of functions in $\widehat{H}^{1/2}(\Gamma_3)$ through the boundary of Γ_2 to functions in $\widetilde{H}^{1/2}(\Gamma_{2,3})$, and by $\mathscr{E}_1$ we denote the operator of extension of elements of $\widehat{H}^{-1/2}(\Gamma_3)$ through the boundary of Γ_1 to elements of $\widetilde{H}^{-1/2}(\Gamma_{1,3})$. These two operators can be obtained by localization from the extension operators that we already know.

Proposition 7.1. *The spaces* $\widehat{H}^{-1/2}(\Gamma_3)$ *and* $\widehat{H}^{1/2}(\Gamma_3)$ *are dual with respect to the extension of the form* $(\varphi, \psi)_{\Gamma_3} = (\mathscr{E}_1\varphi, \mathscr{E}_2\psi)_\Gamma$ *to their direct product.*

To explain, we note that, at the boundary with Γ_1, the space $\widehat{H}^{1/2}(\Gamma_3)$ is locally "similar" to $\widetilde{H}^{1/2}(\Gamma_3)$ and the space $\widehat{H}^{-1/2}(\Gamma_3)$ is locally "similar" to $H^{-1/2}(\Gamma_3)$, while at the boundary with Γ_2, the space $\widehat{H}^{-1/2}(\Gamma_3)$ is locally "similar" to $\widetilde{H}^{-1/2}(\Gamma_3)$ and $\widehat{H}^{1/2}(\Gamma_3)$ is locally "similar" to $H^{1/2}(\Gamma_3)$. Hence the situation on both parts of the boundary of Γ_3 is standard from the viewpoint of duality, even though these spaces are formally of a somewhat new type.

Now the Green formula has the form $\Phi_\Omega(u, v) = (T^+u, v^+)_{\Gamma_3}$. It follows from our results in Sections 2–3 that the solution in the class in question is uniquely determined if one specifies the Dirichlet data in $\widehat{H}^{1/2}(\Gamma_3)$, and then the Neumann data on Γ_3 lie in $\widehat{H}^{-1/2}(\Gamma_3)$. By the same cause, the solution in our class is uniquely determined if one specifies the Neumann data in $\widehat{H}^{-1/2}(\Gamma_3)$, and then the Dirichlet data on Γ_3 lie in $\widehat{H}^{1/2}(\Gamma_3)$. Hence there are two well-defined invertible mutually inverse operators

$$D_{\Gamma_3} \colon \widehat{H}^{1/2}(\Gamma_3) \to \widehat{H}^{-1/2}(\Gamma_3) \quad \text{and} \quad N_{\Gamma_3} \colon \widehat{H}^{-1/2}(\Gamma_3) \to \widehat{H}^{1/2}(\Gamma_3) \tag{7.1}$$

that map the Dirichlet data on Γ_3 to the Neumann data on Γ_3 and vice versa. On these solutions, we have

$$\Phi_\Omega(u,u) = (T^+u, N_{\Gamma_3}T^+u)_{\Gamma_3} = (D_{\Gamma_3}u^+, u^+)_{\Gamma_3}. \tag{7.2}$$

One can write the Gårding type inequalities for our operators. If $L = \widetilde{L}$, then we can introduce the inner product $\langle\psi,\psi\rangle_{\Gamma_3} = (N_{\Gamma_3}\psi,\psi)_{\Gamma_3}$ in $\widehat{H}^{-1/2}(\Gamma_3)$ and treat N_{Γ_3} as a compact self-adjoint operator. One can obtain analogs of Theorems 4.1 and 6.7. We do not dwell on these details.

The other two cases of location of the domain Γ_3 are not more complicated, and we do not dwell on them either.

7.2. Third boundary condition instead of the Neumann condition. Let us return to problem (1.1)–(1.2) but replace the second condition in (1.2) by the following one:

$$T^+u + \sigma u^+ = h \quad \text{on } \Gamma_2. \tag{7.3}$$

Here $\sigma(x)$ is a given matrix function in $L_\infty(\Gamma_2)$. Similar boundary conditions were considered in many papers, e.g., in [24] (the case of $\Gamma_2 = \Gamma$) and [34] (a scalar equation). We shall not consider it in detail; this would be too lengthy. We restrict ourselves to the following considerations. By substituting $T^+u = h - \sigma u^+$ into the Green formula, we obtain the formula

$$\Phi_\Omega(u,v) + (\sigma u^+, v^+)_{\Gamma_2} = (Lu,v)_\Omega + (h,v^+)_{\Gamma_2}. \tag{7.4}$$

Now the left-hand side for $u = v$ is the quadratic form

$$\Phi_\Omega(u,u) + (\sigma u^+, u^+)_{\Gamma_2}, \tag{7.5}$$

and we can generalize our results if it is coercive in our sense on $H^1(\Omega)$. The case in which $\operatorname{Re}(\sigma u^+, u^+)_{\Gamma_2} \geqslant 0$ is the simplest one. If this condition is not satisfied, then one should have in mind that a lower-order term has been added in the left-hand side. Indeed, for arbitrarily small $\alpha > 0$ and $\beta > 0$ one has

$$|(\sigma u^+, u^+)_{\Gamma_2}| \leqslant C_1\|u^+\|^2_{L_2(\Gamma)} \leqslant C_1\|u^+\|^2_{H^\alpha(\Gamma)} \leqslant C_1\|u\|^2_{H^{1/2+\alpha}(\Omega)} \leqslant \beta\|u\|^2_{H^1(\Omega)} + C_\beta\|u\|^2_{L_2(\Omega)}.$$

Hence the desired coercivity again holds provided that the form $\operatorname{Re}(cu,u)_\Omega$ is sufficiently large.

7.3. The case when the boundary is divided into several domains. The results clearly can be generalized to the case in which the surface is divided into finitely many domains by a finite set of Lipschitz $(n-2)$-dimensional closed surfaces without self-intersections and without common points (we prefer to make this assumption to be careful) and either the Dirichlet condition, or the Neumann condition, or the spectral condition is posed in each of these domains (cf. [34], [23], and [50]).

The author is greatly indebted to N. D. Filonov and especially to T. A. Suslina for discussion and valuable advice. As was indicated in Section 6.3, Theorem 6.8 was obtained by the author and T. A. Suslina together.

Our bibliography on mixed problems is by no means complete. Many additional references can be found in the papers on our list.

References

[1] M. S. Agranovich, "On a mixed Poincaré–Steklov type spectral problem in a Lipschitz domain," Russian J. Math. Phys., **13**:3 (2006), 239–244.

[2] M. S. Agranovich, "Remarks on potential spaces and Besov spaces in a Lipschitz domain and on Whitney arrays on its boundary," Russian J. Math. Phys., **15**:2 (2008), 146–155.

[3] M. S. Agranovich, "Potential type operators and transmission problems for strongly elliptic second-order systems in Lipschitz domains," Funkts. Anal. Prilozhen., **43**:3 (2009), 3–25; English transl.: Functional Anal. Appl., **43**:3 (2009), 165–183.

[4] M. S. Agranovich, "Strongly elliptic second-order systems with boundary conditions on a nonclosed Lipschitz surface," Funkts. Anal. Prilozhen., **45**:1 (2011), 1–15; English transl.: Functional Anal. Appl., **45**:1 (2011), 1–12.

[5] M. S. Agranovich, "Spectral problems in Lipschitz domains," Sovremennaya Matematika. Fundamental'nye Napravleniya, **39** (2011) (to appear); English transl.: J. Math. Sci. (to appear).
[6] M. S. Agranovich and B. A. Amosov, "Estimates of s-numbers and spectral asymptotics for integral operators of potential type on nonsmooth surfaces," Funkts. Anal. Prilozhen., **30**:2 (1996), 1–18; English transl.: Functional Anal. Appl., **30**:2 (1996), 75–89.
[7] M. Sh. Birman and M. Z. Solomyak, "Asymptotic behavior of the spectrum of variational problems on solutions of elliptic equations," Sibirsk. Mat. Zh., **20**:1 (1979), 3–22; English transl.: Siberian Math. J., **20** (1979), 1–15.
[8] M. Sh. Birman and M. Z. Solomyak, Quantitative Analysis in Sobolev's Imbedding Theorems and Applications to Spectral Theory, Amer. Math. Soc. Transl., Ser. 2, vol. 114, Amer. Math. Soc., Providence, RI, 1980.
[9] M. Sh. Birman and M. Z. Solomyak, "Asymptotic behavior of the spectrum of pseudodifferential operators with anisotropically homogeneous symbols," Vestnik Leningrad. Univ., No. 13, issue 3 (1977), 13–21; English transl.: Vestnik Leningr. Univ., Math., **10** (1982), 237–247.
[10] R. M. Brown, "The mixed problem for Laplace's equation in a class of Lipschitz domains," Comm. Partial Differential Equations, **19**:7–8 (1994), 1217–1233.
[11] R. M. Brown and I. Mitrea, "The mixed problem for the Lamé system in a class of Lipschitz domains," J. Differential Equations, **246**:7 (2009), 2577–2589.
[12] M. Costabel, "Boundary integral operators in Lipschitz domains: elementary results," SIAM J. Math Anal., **19**:3 (1988), 613–626.
[13] R. Courant and D. Hilbert, Methods of Mathematical Physics. Vol. 1, Wiley, New York, 1989.
[14] B. E. J. Dahlberg, C. E. Kenig, and G. C. Verchota, "Boundary value problems for the system of elastostatics in Lipschitz domains," Duke Math. J., **57**:3 (1988), 795–818.
[15] G. I. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations, Amer. Math. Soc., Providence, RI, 1981.
[16] V. I. Fabrikant, Mixed Boundary Value Problems of Potential Theory and their Applications in Engineering, Kluwer, Dorderecht, 1991.
[17] J. A. Griepentrog, K. Gröger, H.-Chr. Kaiser, and J. Rehrberg, "Interpolation for function spaces related to mixed boundary value problems," Math. Nachr., **241** (2002), 110–120.
[18] J. A. Griepentrog, H-Chr. Kaiser, and J. Rehrberg, "Heat kernel and resolvent properties for second order elliptic differential operators with general boundary conditions on L_p," Adv. Math. Sci. Appl., **11** (2001), 87–112.
[19] K. Gröger, "A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations," Math. Ann., **283**:4 (1989), 679–687.
[20] G. C. Hsiao and W. L. Wendland, Boundary Integral Equations, Springer-Verlag, Berlin, 2008.
[21] D. Jerison and C. E. Kenig, "The inhomogeneous Dirichlet problem in Lipschitz domains," J. Funct. Anal., **130**:1 (1995), 164–219.
[22] A. Jonsson and H. Wallin, Function Spaces on Subsets of $\mathbb{R}^n$, Harwood Academic Publishers, 1984.
[23] V. I. Lebedev and V. I. Agoshkov, Poincaré–Steklov Operators and Their Applications in Analysis [in Russian], Akad. Nauk SSSR, Vychisl. Tsentr, Moscow, 1983.
[24] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge Univ. Press, Cambridge, 2000.
[25] G. Métivier, "Valeurs propres de problémes aux limites elliptiques irreguliers," Bull. Soc. Math. France, Mémoire **51–52** (1977), 125–219.
[26] S. E. Mikhailov, Traces, extensions, co-normal derivatives and solution regularity of elliptic systems with smooth and non-smooth coefficients, `http://arxiv.org/abs/0906.3875v1`.
[27] I. Mitrea and M. Mitrea, "The Poisson problem with mixed boundary conditions in Sobolev and Besov spaces in non-smooth domains," Trans. Amer. Math. Soc., **359**:9 (2007), 4143–4182.
[28] M. Mitrea and M. Taylor, "Boundary layer methods for Lipschitz domains in Riemannian manifolds," J. Funct. Anal., **163**:2 (1999), 181–251.
[29] M. Mitrea and M. Taylor, "Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev–Besov results and the Poisson problem," J. Funct. Anal., **176**:1 (2000), 1–79.

[30] D. G. Natroshvili, O. O. Chkadua, and E. M. Shargorodskii, “Mixed problems for homogeneous anisotropic elastic media,” Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy [in Russian], **39** (1990), 133–178 (abstract in English, 179–181).
[31] J. Nečhas, Les méthodes directes en théorie des équations elliptiques, Masson, Paris, 1967.
[32] O. A. Oleinik, A. S. Shamaev, and G. A. Yosifian, Mathematical Problems in Elasticity and Homogenization, North Holland, Amsterdam, 1992.
[33] K. A. Ott and R. M. Brown, The mixed problem for the Laplacian in Lipschitz domains `http://arxiv.org/abs/0909.0061v2`.
[34] B. V. Paltsev, “Mixed problems with non-homogeneous boundary conditions in Lipschitz domains for second-order elliptic equations with a parameter,” Mat. Sb., **187**:4 (1996), 59–116; English transl.: Sb. Math., **187**:4 (1996), 525–580.
[35] T. von Petersdorff, “Boundary integral equations for mixed Dirichlet, Neumann and transmission problems,” Math. Methods in Applied Sciences, **11**:2 (1989), 183–213.
[36] G. Rozenblum and G. Tashchiyan, “Eigenvalue asymptotics for potential type operators on Lipschitz surfaces,” Russian J. Math. Phys., **13**:3 (2006), 326–339.
[37] V. S. Rychkov, “On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains,” J. London Math. Soc. (2), **60**:1 (1999), 237–257.
[38] G. Savaré, “Regularity and perturbation results for mixed second order elliptic problems,” Comm. Partial Differential Equations, **22**:5–6 (1997), 869–899.
[39] E. Shamir, “Regularization of mixed second-order elliptic equations,” Israel J. Math., **6** (1968), 150–168.
[40] I. Ya. Shneiberg, “Spectral properties of linear operators in interpolation families of Banach spaces,” Mat. Issled. [in Russian], **9**:2 (1974), 214–227.
[41] I. N. Sneddon, Mixed Boundary Value Problems in Potential Theory, Elsevier, N.Y., 1966.
[42] E. Stephan, Boundary integral equations for mixed boundary value problems, screen and transmission problems in $\mathbb{R}^3$, Habilitationsschrift, Darmstadt, 1984 (THD-preprint 848).
[43] E. P. Stephan, “Boundary integral equations for mixed boundary value problems in $\mathbb{R}^3$,” Math. Nachr., **134** (1987), 21–53.
[44] T. A. Suslina, “Asymptotic behavior of the spectrum of variational problems on solutions of a homogeneous elliptic equation in the presence of constraints on part of the boundary,” in: Probl. Mat. Anal. [in Russian], vol. 9, Leningrad Univ., 1984, 84–97.
[45] T. A. Suslina, “Spectral asymptotics of variational problems with elliptic constraints in domains with piecewise smooth boundary,” Russian J. Math. Phys., **6**:2 (1999), 214–234.
[46] J. D. Sykes and R. M. Brown, “The mixed boundary problem in L^p an Hardy spaces for Laplace’s equation on a Lipschitz domain,” in: Contemporary Mathematics, vol. 227, Amer. Math. Soc., Providence, RI, 2001, 1–18.
[47] H. Triebel, “Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers,” Rev. Mat. Complut., **15** (2002), 475–524.
[48] G. Uhlmann, “Inverse boundary problems and applications,” Astérisque, **207** (1992), 153–207.
[49] G. Verchota, “Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains,” J. Funct. Analysis, **59**:3 (1984), 572–611.
[50] V. I. Voititsky, N. D. Kopachevsky, and P. A. Starkov, “Multicomponent transmission problems and auxiliary abstract boundary-value problems,” Sovremennaya Matematika. Fundamental’nye Napravleniya, **34** (2009), 5–44; English transl.: J. Math. Sci., **170**:2 (2010), 131–172.

Moscow Institute of Electronics and Mathematics
e-mail: magran@orc.ru

Translated by V. E. Nazaikinskii