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SHORT COMMUNICATIONS

# On the Existence of Solutions of the First Boundary Value Problem for Elliptic Equations on Unbounded Domains

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**Abstract.** The problem mentioned in the title is studied.

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## 1. INTRODUCTION

Let $\Omega$ be an unbounded open subset of $\mathbb{R}^n$, $n \geqslant 2$. Denote by $B_\rho^x$ the open ball in $\mathbb{R}^n$ of radius $\rho > 0$ centered at the point $x$. If $x = 0$, we write $B_\rho$ instead of $B_\rho^x$. As is customary, by $W_{2,\mathrm{loc}}^1(\Omega)$ we mean the set of functions in $\mathcal{D}'(\Omega)$ that belong to the spaces $W_2^1(\Omega \cap B_\rho)$ for any $\rho > 0$ [3]. In this case, denote by $\overset{o}{W}{}_{2,\,\mathrm{loc}}^1(\Omega)$ the subset of $W_{2,\mathrm{loc}}^1(\mathbb{R}^n)$ which is the closure of $C_0^\infty(\Omega)$ in the system of seminorms $\|u\|_{W_2^1(\Omega\cap B_\rho)}$, $\rho > 0$. Further, following [4, Subsec. 1.1], denote by $L_2^1(\Omega)$ the space of distributions ("generalized functions") whose first derivatives belong to $L_2(\Omega)$; in other words,

$$L_2^1(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega)\colon\ \int_\Omega |\nabla f|^2\, dx < \infty \Big\}.$$

Let $\omega \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{K} \subset \omega$ be a compact set. Denote by $\Phi_\varphi(\mathcal{K}, \omega)$ the set of functions $\psi \in C_0^\infty(\omega)$ such that $\psi = \varphi$ in a neighborhood of $\mathcal{K}$, or, in other words, $\psi - \varphi \in \overset{o}{W}{}_{2,\mathrm{loc}}^1(\mathbb{R}^n \setminus \mathcal{K})$. Write $\Psi(\mathcal{K}, \omega) = \{\psi \in C_0^\infty(\omega) : \psi = 1 \text{ in a neighborhood of } \mathcal{K}\}$. The quantity $\operatorname{cap}_\varphi(\mathcal{K}, \omega) = \inf_{\psi\in\Phi_\varphi(\mathcal{K},\,\omega)} \int_\omega |\nabla\psi|^2\, dx$ is referred to as the capacity of the compact set $\mathcal{K}$ with respect to an open set $\omega$. The capacity of an arbitrary closed subset $E \subset \omega$ of $\mathbb{R}^n$ is defined by the rule $\operatorname{cap}_\varphi(E, \omega) = \sup_{\mathcal{K}} \operatorname{cap}_\varphi(\mathcal{K}, \omega)$, where the supremum on the right-hand side is taken over all compacta $\mathcal{K} \subset E$. If $\omega = \mathbb{R}^n$, then we write $\operatorname{cap}_\varphi(E)$ instead of $\operatorname{cap}_\varphi(E, \mathbb{R}^n)$. We also need the following capacity [4, Subsec. 9.1]:

$$\operatorname{Cap}(\mathcal{K}, W_2^1(\omega)) = \inf_{\psi\in\Psi(\mathcal{K},\,\omega)} \left( \int_\omega |\nabla\psi|^2 dx + \int_\omega |\psi|^2\, dx \right).$$

As above, the capacity of an arbitrary set $E \subset \omega$ closed in $\mathbb{R}^n$ is given by the rule $\operatorname{Cap}(E, W_2^1(\omega)) = \sup_{\mathcal{K}} \operatorname{Cap}(\mathcal{K}, W_2^1(\omega))$, where the supremum on the right-hand side is taken over all compacta $\mathcal{K} \subset E$.

Finally, denote by $W_2^{-1}$ the space of continuous linear functionals on $W_2^1$. A set $E \subset \mathbb{R}^n$ is said to be $(2,1)$-*polar* if the only element of $W_2^{-1}$ supported by $E$ is zero [4, Subsec. 9.2].

The problems treated in the present note were studied earlier in [1, 2].

## 2. STATEMENT OF THE PROBLEM

Here and below, $L$ stands for the divergence operator of the form $L = \sum_{i,j=1}^n \partial/\partial x_i \big(a_{ij}(x)\partial/\partial x_j\big)$ with measurable bounded coefficients satisfying the uniform ellipticity condition

$$c_1|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\,\xi_i\,\xi_j \leqslant c_2|\xi|^2, \quad \xi \in \mathbb{R}^n, \quad c_1, c_2 > 0.$$

By a solution of the Dirichlet problem

$$Lu = 0 \quad \text{on} \quad \Omega, \qquad u|_{\partial\Omega} = \varphi, \tag{1}$$

where $\varphi \in W^1_{2,\,\mathrm{loc}}(\mathbb{R}^n)$, we mean a function $u \in W^1_{2,\mathrm{loc}}(\Omega)$ such that

1. $u - \varphi \in \overset{\circ}{W}{}^1_{2,\mathrm{loc}}(\Omega)$, i.e., $(u - \varphi)\eta \in \overset{\circ}{W}{}^1_2(\Omega)$ for any function $\eta \in C_0^\infty(\mathbb{R}^n)$;
2. the function $u$ has the bounded Dirichlet integral $\int_\Omega |\nabla u|^2 dx < \infty$;
3. $\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\partial u/\partial x_j \, \partial\psi/\partial x_i \, dx = 0$ for any function $\psi \in C_0^\infty(\Omega)$.

## 3. MAIN RESULTS

**Theorem 1.** *Let* $\mathrm{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$ *for some* $c \in \mathbb{R}$. *Then problem* (1) *has a solution.*

**Theorem 2.** *Let problem* (1) *have a solution, and let* $\int_{\mathbb{R}^n\setminus\Omega} |\nabla\varphi|^2 dx < \infty$. *Then there is a* $c \in \mathbb{R}$ *such that* $\mathrm{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$.

**Theorem 3.** *For any function* $\psi \in W^1_{2,\mathrm{loc}}(\mathbb{R}^n)$, *the condition* $\mathrm{cap}_\psi(\mathbb{R}^n \setminus \Omega) < \infty$ *is equivalent to the inequality* $\sum_{k=1}^\infty \mathrm{cap}_\psi((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \,\mathrm{cap}(\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) < \infty$, *where* $r_k = 2^k$ *if* $n \geqslant 3$ *and* $r_k = 2^{2^k}$ *if* $n = 2$.

Let $\omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $\mu$ be a measure on $\omega$ such that

$$\sup_{x\in\mathbb{R}^n,\,\rho>0} \rho^{1-n}\mu(B_\rho^x \cap \omega) < \infty. \tag{2}$$

In this case, for any function $v \in W_2^1(\omega)$, there is a $c \in \mathbb{R}$ such that

$$\sigma(\omega,\mu)\|v - c\|_{L_2(\omega,\mu)} \leqslant \|\nabla v\|_{L_2(\omega)}, \tag{3}$$

where the constant $\sigma(\omega,\mu) > 0$ does not depend on $v$ [4, Subsec. 1.4.5].

**Theorem 4.** *Let problem* (1) *have a solution, and let* $\mu_k$ *be a family of measures on* $\omega_k$, *where* $\omega_k$, $k = 1, 2, \dots$, *are pairwise disjoint Lipschitz domains in* $\mathbb{R}^n$ *such that*

$$\sup_{x\in\mathbb{R}^n,\,\rho>0} \rho^{1-n}\mu_k(B_\rho^x \cap \omega_k) < \infty, \ \sum_{k=1}^\infty \int_{\omega_k\setminus\Omega} |\nabla\varphi|^2 dx < \infty.$$

*Write* $m_k(\varphi) = \inf_{c\in\mathbb{R}} \|\varphi - c\|_{L_2(\omega_k\setminus\Omega,\mu_k)}$. *Then* $\sum_{k=1}^\infty \sigma^2(\omega_k,\mu_k)m_k^2(\varphi) < \infty$, *where* $\sigma(\omega_k,\mu_k)$ *stands for the coefficient in inequality* (2).

**Corollary 1.** *Let* $\Omega = \{(x', x_n) \in \mathbb{R}^n | x_n \geqslant 0\}$ *and* $\varphi(x) = (1 + |x|)^\alpha$. *In this case, problem* (1) *has a solution if and only if either* $\alpha < -1/2$ *or* $\alpha = 0$.

**Corollary 2.** *Let* $n \geqslant 3$, *let* $\Omega$ *be the complement to the set* $\{(x', x_n) \in \mathbb{R}^n | x_n \geqslant 1, |x'| \leqslant x_n^\beta\}$, *where* $\beta < 0$, *and let* $\varphi(x) = (1 + |x|)^\alpha$. *In this case, problem* (1) *has a solution if and only if either* $\alpha < -(1 + \beta(n-3))/2$ *or* $\alpha = 0$.

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