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Reflection of Long Internal Waves of Small Amplitudes from an Underwater Slope

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Abstract—The dynamics of long waves in the vicinity of a transition point of a two-layer flow into a single-layer one is studied within the linear theory of shallow water. The analogy between this problem and the classical problem of surface wave runup on the shore is shown. Conditions for breaking internal waves on a slope are discussed.

Keywords: internal waves, pycnocline, wave runup, wave breakdown

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1. INTRODUCTION

High-energy internal waves are observed in different regions of the World Ocean, mainly in shelf zones [1–3]. Different physico-mathematical theories are used for a numerical simulation of the dynamics of internal waves on the shelf, from weakly nonlinear on the basis of the equation of Korteweg–de Vries to so-called primitive hydrodynamic equations [1, 3–6]. Internal waves do not usually approach the coast, being reflected from a slope or breaking on them. This process was not previously considered in the literature; the existing approximate analytical equations have been derived for internal waves in the case of a slowly changing bottom and slowly varying vertical density and horizontal flow stratifications [1, 7]. In the region of internal wave reflection from the slope, the approximation of slowness of variations in the medium parameters is inapplicable, as are well-developed asymptotic methods like geometrical optics or acoustic methods.

In this work we consider a simple analytical model of transformation and reflection of internal waves in a variable-depth two-layer flow of liquids of different densities. We assume that the double-layer is transformed into a single-layer one at a certain point. The internal waves evidently cannot exist in the single-layer flow, and this zone plays the role of the shore line of surface waves. This analogy turns out to be quite efficient; it allows a strict statement of the problem of wave reflection from a slope. As a result, well known results for a description of sea wave runup on a plane shore (see, e.g., [8]) can be used for a description of internal waves in a zone where a two-layer flow transforms into a single-layer one. This analogy is discussed in Section 2. The equations for calculating parameters of an internal wave in the transition zone are given in Section 3. Breakdown conditions of internal waves are discussed in Section 4. The results are summarized in Conclusions.

2. ANALOGY BETWEEN PROBLEMS OF RUNUP OF SURFACE AND INTERNAL WAVES

Let us consider the propagation of internal waves in a double-layer ocean of a variable depth where a two-layer flow transforms into a single-layer one (Fig. 1). If the amplitude of long internal waves and the density gradient are small (the so-called Boussinesq approximation, common in oceanology), then the following linearized equations of shallow water are valid for internal waves [9, 10]:

$$h_1 u_1 + h_2(x) u_2 = 0, \tag{1}$$

$$\frac{\partial (u_2 - u_1)}{\partial t} + g' \frac{\partial \eta}{\partial x} = 0, \tag{2}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[h_2(x) u_2\right] = 0, \tag{3}$$

where we also used the "rigid rid" approximation, which excludes surface waves. Here η is the displacement of the interface between liquids of different densities, u_1 and u_2 are the averaged speeds of flows in the upper and bottom layers, h_1 and $h_2(x)$ are the depths of the upper and bottom layers, and $g' = g(\rho_2 - \rho_1)/\rho_1$ is the reduced acceleration of gravity. The depth of the upper layer h_1 is assumed to be constant; only the lower layer depth varies and, hence, the total length $H(x) = h_1 + h_2$.

Excluding the flow speed in the upper layer from Eq. (1) and applying cross differentiation to Eqs. (2) and (3), we derive the wave equation for the interface shift:

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[c^2(x)\frac{\partial \eta}{\partial x}\right] = 0, \quad (4)$$

where

$$c(x) = \sqrt{g'\frac{h_1 h_2(x)}{h_1 + h_2(x)}} \quad (5)$$

defines the propagation speed of long waves in the double-layer ocean.

Wave equation (4) written in such a general form can be used to describe many wave motions in inhomogeneous media [11]; hence, it is possible to transfer some well-known results of the general wave theory to internal waves. Here we pay attention to the analogy with surface waves in a homogeneous liquid, for which Eq. (4) describes a water surface displacement, and the speed of wave propagation

$$c_s(x) = \sqrt{gh(x)}, \quad (6)$$

where $h(x)$ is the basin depth. In the case of a basin bounded with the shore, $h(x)$ begins with zero and increases with the distance to the shore. The study of wave parameters near point $h = 0$ (the shore line) allows us to describe the wave runup on the shore in a linear approximation. The plane slope model

$$h(x) = \alpha x, \quad (7)$$

is the most common. Here α is the slope ratio, which can be often identified with the slope angle due to its smallness. It is evident that Eq. (7) can be considered a natural asymptotics of the bottom profile near the shore; it is the first term of the Taylor expansion of the function $h(x)$. However, in practice, singular asymptotics $h(x) \sim x^b$ is often implementable; b can be both lower than unity (the well-known Dean equilibrium profile with $b = 2/3$) and higher than unity (reflectionless beach with $b = 4/3$), though the case where $b = 1$ is also often implementable (see, e.g., [12, 13]).

Profile (7) is also of special interest in the theory of sea wave runup on the shore. It provides for an exact solution of the nonlinear theory of shallow water and a detailed description of sea-wave runup on the shore (see, e.g., [8]). The possibility of using the linear wave equation for calculations of extreme parameters of runup (maximum runup height and back sweep depth; maximal flow speeds during runup and back sweep) turned out an important and unexpected conclusion of the nonlinear theory, since extreme parameters calculated within the nonlinear and linear theories coincide. Moreover, the criterion of wave breakdown, following from the nonlinear theory as a condition of gradient catastrophe, can be derived from a linear theory solution [8, 14]. That is why the linear theory of sea wave runup on the shore, which have been developed quite long ago [8], remains popular.

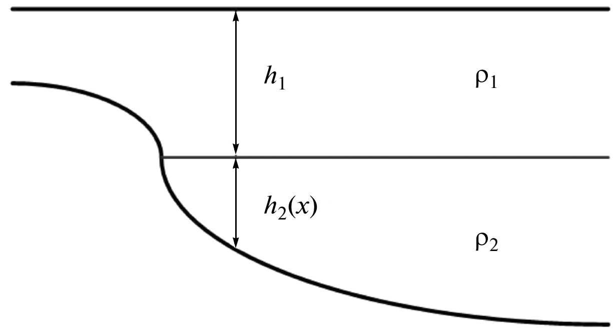

Fig. 1. Double-layer variable-depth flow geometry.

Due to the complete analogy between the linear wave equation for surface waves and waves at the interface, one may assume that the results for surface waves are true for internal waves. This naturally requires corresponding scaling of depths and, hence, similar variations in the propagation speed. Comparing Eqs. (5)–(7), we get the condition for a change in the bottom layer depth with the distance

$$h_2(x) = \frac{\beta x}{1 - \beta x/h_1} \quad (8)$$

with an arbitrary value of the coefficient β, which determines the bottom slope at point $x = 0$, where the two-layer flow transforms into a single-layer one. It is natural to call this point the "shore" line of internal waves. The main difference between profile (8) and profile (7) is that the former exists only at finite distances from the shore line ($x < 1/\beta h_1$). However, this restriction is not significant, since it allows sewing together a slope with any zone of constant depth (correspondingly, the slope angle changes, like for surface waves). Fixing the slope width L and the depth of the bottom layer at its edge h_0 (or the total depth $H_0 = h_1 + h_0$), one may define the slope angle

$$\beta = \frac{h_1 h_0}{L H_0}. \quad (9)$$

The corresponding profile of the bottom-layer depth is shown schematically in Fig. 2. We do not show here the region of single-layer flow ($x < 0$), since we intuitively assume that the depth h_2 becomes negative there (i.e., the depth h_1 of the upper layer starts to decrease) for an internal wave to "physically" run up on the slope. However, motion in this region is significantly nonlinear and cannot be considered within the linear wave equation. Like in similar problems of sur-

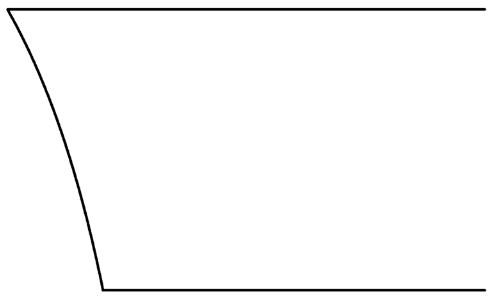

Fig. 2. Profile of the bottom layer of a double-layer flow. The left part shows the zone of a decrease in the bottom layer depth down to zero (shore line).

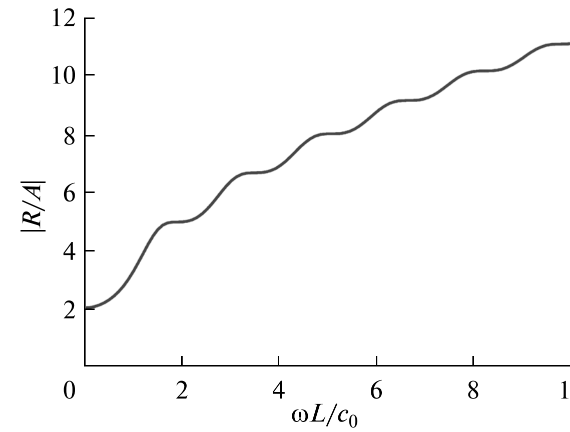

Fig. 3. Runup height as a function of the internal wave frequency.

face waves, we restrict the consideration to region $x > 0$, where there are liquids of different densities.

Let us make another important remark. A comparison of Eqs. (5)–(8) implies that

$$\alpha = \beta \frac{g'}{g} = \frac{\Delta\rho}{\rho}\beta, \tag{10}$$

and, hence, similar wave motion in surface and internal waves occur at significantly different, by three orders of magnitude, slope angles.

3. INTERNAL WAVE RUNUP ON THE SLOPE

Let us now consider a solution of linear wave equation (4) in the case of a variation in the bottom layer depth shown in Fig. 2. In region $x > L$, where the basin depth is constant, the field is a superposition of two monochromatic waves (incident and reflected):

$$\eta(x,t) = A\exp[i\omega(t + x/c_0)] + A_{\text{ref}}\exp[i\omega(t - x/c_0)], \tag{11}$$

where

$$c_0 = \sqrt{g'\frac{h_1 h_0}{h_1 + h_0}} \tag{12}$$

is the speed of internal wave propagation in the basin of a constant depth, A is the amplitude of the incident wave, and A_{ref} is the amplitude of the reflected wave. In the zone of variable depth (on the shelf, $0 < x < L$), the bounded solution of wave equation (4) is expressed via the Bessel function:

$$\eta(x,t) = RJ_0\left[\sqrt{\frac{4\omega^2 x}{g'\beta}}\right]\exp(i\omega t), \tag{13}$$

where R determines the height of the interface displacement at the shore line ($x = 0$). Sewing solutions (12) and (13) at the shelf edge ($x = L$) with the help of ordinary boundary conditions of continuity of the depth (pressure) and water flow in the upper layer

$$RJ_0\sqrt{\frac{4\omega^2 L}{g'\beta}} = A\exp\left(\frac{i\omega L}{c_0}\right) + A_{\text{ref}}\exp\left(-\frac{i\omega L}{c_0}\right), \tag{14}$$

$$iRJ_1\left[\sqrt{\frac{4\omega^2 L}{g'\beta}}\right] = A\exp\left(\frac{i\omega L}{c_0}\right) - A_{\text{ref}}\exp\left(-\frac{i\omega L}{c_0}\right), \tag{15}$$

we can find the generally unknown complex constants R and A_{ref}. In particular, the runup height (liquid interface displacement at the point $x = 0$)

$$\left|\frac{R}{A}\right| = \frac{2}{\sqrt{J_0^2\left(\frac{2\omega L}{c_0}\right) + J_1^2\left(\frac{2\omega L}{c_0}\right)}}. \tag{16}$$

Equation (16) in this form is universal and can be used for the runup height of both surface waves on a linear slope (7) and internal waves on a nonlinear slope (8). The runup height as a function of the dimensionless parameter $\omega L/c_0$, which is equal to the ratio of slope length to internal wavelength, is shown in Fig. 3. As was expected, if a slope is sufficiently steep, the wave is reflected from it like from a vertical wall and the wave height doubles on the wall. In the case of a smooth slope, a wave intensifies on the shelf and the runup height of the internal wave increases.

The oscillation phase of the shore line differs from the incident wave phase by magnitude

$$\varphi = -\arctan\frac{J_1\left(\frac{2\omega L}{c_0}\right)}{J_0\left(\frac{2\omega L}{c_0}\right)}. \tag{17}$$

The phase is small for steep slopes and tends to $-\pi/2$ for smooth slopes. The frequency dependence of the shore line oscillation amplitude and phase results in a complex wave transformation at the shore line in the case of a single wave approaching, which is beyond the scope of this work.

The wave reflection coefficient can be found in a similar way:

$$A_{\text{ref}} = A\frac{J_0\left(\frac{2\omega L}{c_0}\right) - iJ_1\left(\frac{2\omega L}{c_0}\right)}{J_0\left(\frac{2\omega L}{c_0}\right) + iJ_1\left(\frac{2\omega L}{c_0}\right)}\exp\left(\frac{2i\omega L}{c_0}\right). \quad (18)$$

The amplitude of a reflected wave (modulus of A_{ref}) is equal to the amplitude of an incident wave, which is not a surprise, because we used the condition of total reflection at the shore line. At the same time, the phase of the reflected wave changes from zero for steep slopes to $\pi/2$ for smooth slopes. This naturally also affects the shape of the reflected wave in the case of a single internal wave approaching.

4. WAVE BREAKDOWN ON A SLOPE

Above we consider the solution to the linear problem of internal wave transformation in a zone where a double-layer flow changes to a single-layer one. It is evident that nonlinear effects should be significant near the shore line, since the oscillation amplitude equalizes and exceeds the bottom layer depth. The nonlinear problem has not been solved analytically so far. However, this does not mean that the linear problem solution is completely inapplicable to describing the wave process in the transition zone. As is mentioned above, the linear theory correctly predicts extreme characteristics, in particular, wave height at the shore line, of the runup in the similar problem of wave runup on a plane slope. Moreover, if the parameter

$$Br = \frac{\omega^2 R}{g\alpha^2} \quad (19)$$

is quite small, then the linear problem solution satisfactorily describes the form of water-level oscillations at the shore line [8, 14]. At the same time, the wave breaks down at the shore line if this parameter is equal to unity and in the sea, before approaching the shore, at higher values of the parameter. That is why this parameter has been called the breakdown parameter [8, 14]. Therefore, we can try to write an analogous breakdown parameter for an internal wave at a slope with accounting for Eq. (10):

$$Br_{\text{in}} = \frac{\omega^2 R}{g'\beta^2}, \quad (20)$$

though we cannot derive it rigorously from the nonlinear theory. If condition $Br_{\text{in}} = 1$ is taken as a breakdown condition, then one can derive the equation for the runup critical height under which the breakdown starts:

$$R_{\text{cr}} = \frac{g'\beta^2}{\omega^2}. \quad (21)$$

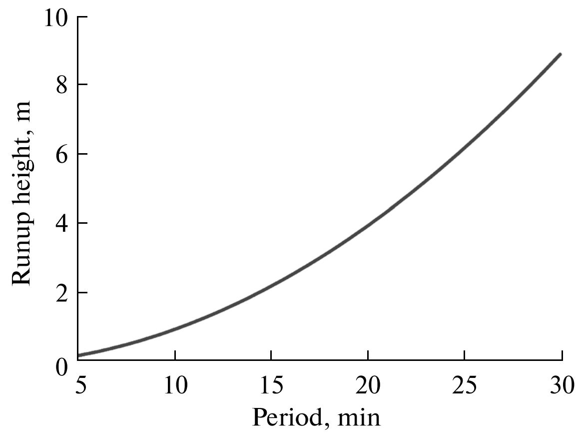

Fig. 4. Critical height of internal wave runup as a function of the wave period.

This equation includes a small parameter—small density gradient. However, it should be taken into account that the internal wave frequency is also usually significantly lower than the surface-wave frequency; hence, the internal wave heights are quite realistic. Dependence (21) is shown in Fig. 4 for a density gradient of 10^{-3} and a bottom slope of 0.1.

CONCLUSIONS

In this work, characteristics of internal waves in the transition zone from a two-layer flow to a single-layer one are calculated within the linear theory. This problem is mathematically equivalent to the similar problem of surface wave runup on a constant-gradient slope, thought the bottom profile is different in the problem about internal waves and the slope becomes plane only in a vicinity of the critical point (shore line). Let us note that the linear wave correctly predicts extreme runup parameters for surface waves, as well as the wave breakdown criterion at the shore. Therefore, the analogy between equations for surface and internal waves allows us to hope that the results are applicable for describing the runup of nonlinear internal waves on a slope, though we have not proven this fact rigorously yet.

Let us note that similar results can be obtained for the transformation and runup of surface and internal waves in the case of so-called reflectionless beaches with $h \sim x^{4/3}$ in the vicinity of the shore line. However, the corresponding results have been obtained only within the linear theory [10, 15] and have not been tested within nonlinear theory.

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