

A CR twistor space of a G_2 -manifold

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Abstract

Let M be a G_2 -manifold. We consider an almost CR-structure on the sphere bundle of unit tangent vectors on M , called **the CR twistor space**. This CR-structure is integrable if and only if M is a holonomy G_2 manifold. We interpret G_2 -instanton bundles as CR-holomorphic bundles on its twistor space.

Contents

1	Introduction	1
1.1	A CR twistor space of a G_2 -manifold	1
1.2	Applications of twistor geometry	4
2	Differential forms on $\text{Tot}(\Lambda^k M)$	6
3	The G_2-action on $\Lambda^2(\mathbb{R}^7)$ and $SU(3)$-action on $\Lambda^2(\mathbb{R}^6)$	8
3.1	Octonion algebra and quaternions	8
3.2	G_2 -action and $SU(3)$ -action	9
4	Integrability of the twistor CR-structure	11

1 Introduction

1.1 A CR twistor space of a G_2 -manifold

This note is inspired by the Claude LeBrun's paper [LB]. In his work, LeBrun constructed a twistor space for a 3-manifold, which happens to be a CR-manifold of real dimension 5. The geometry of G_2 -manifolds is remarkably similar to the geometry of 3-manifolds, and it is not surprising that an analogue of LeBrun's construction can be obtained.

Definition 1.1: Let M be a smooth manifold, $B \subset TM$ a subbundle in its tangent bundle, and $I \in \text{End } B$ its automorphism, $I^2 = -\text{Id}_B$. Consider the (1,0) and (0,1)-bundles $B^{1,0}, B^{0,1} \subset B \otimes \mathbb{C}$, which are the eigenspaces

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of I corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. The sub-bundle $B^{1,0} \subset TM \otimes \mathbb{C}$ is called a **CR-structure on M** if it is *involutive*, that is, satisfies $[B^{1,0}, B^{1,0}] \subset B^{1,0}$.

Example 1.2: If $B = TM$, CR-structures are the same as complex structures. For any codimension 1 real submanifold M in a complex manifold (X, I) , the intersection $B := TX \cap I(TX)$ is a complex subbundle of codimension 1 in TX , and the restriction $I|_B$ defines a CR-structure.

G_2 -manifolds originally appeared in Berger's classification of holonomy ([Ber], [Bes]). The first examples of G_2 -manifolds were obtained by R. Bryant and S. Salamon ([BS]). The compact examples of G_2 -manifolds were constructed by D. Joyce ([J1], [J2]). In this introduction we follow the approach to G_2 -geometry which is due to N. Hitchin (see [Hi]).

Definition 1.3: Let $\rho \in \Lambda^3 \mathbb{R}^7$ be a 3-form on \mathbb{R}^7 . We say that ρ is **non-degenerate** if the dimension of its stabilizer is maximal:

$$\dim St_{GL(7)}\rho = \dim GL(7) - \dim \Lambda^3(\mathbb{R}^7) = 49 - 35 = 14.$$

In this case, $St(\rho)$ is one of two real forms of a 14-dimensional Lie group $G_2(\mathbb{C})$. We say that ρ is **non-split** if it satisfies $St(\rho|_x) \cong G_2$, where G_2 denotes the compact real form of $G_2(\mathbb{C})$. A **G_2 -structure** on a 7-manifold is a 3-form $\rho \in \Lambda^3(M)$, which is non-degenerate and non-split at each point $x \in M$. We shall always consider a G_2 -manifold as a Riemannian manifold, with the Riemannian structure induced by the G_2 -structure as indicated below.

Remark 1.4: Such a form defines a $\Lambda^7 M$ -valued metric on M :

$$g(x, y) = (\rho]x) \wedge (\rho]y) \wedge \rho \tag{1.1}$$

(we denote by $\rho]x$ the contraction of ρ with a vector field x). The Riemannian volume form associated with this metric gives a section of $\Lambda^7 M \otimes (\Lambda^7 M)^{7/2}$. Squaring and taking the 9-th degree root, we obtain a trivialization of the volume. Then (1.1) defines a metric g on M , by construction G_2 -invariant.

Definition 1.5: An G_2 -structure is called a **integrable G_2 -structure**, if ρ is preserved by the corresponding Levi-Civita connection. An integrable

G_2 -manifold is a manifold equipped with an integrable G_2 -structure. Holonomy group of such a manifold clearly lies in G_2 ; for this reason, the integrable G_2 -manifolds are often called **holonomy G_2 -manifolds**.

Remark 1.6: In the literature, “the G_2 -manifold” often means a “holonomy G_2 -manifold”, and “ G_2 -structure” “an integrable G_2 -structure”. A G_2 -structure which is not necessarily integrable is called “an almost G_2 -structure”, taking analogy from almost complex structures.

Remark 1.7: As shown in [FG], integrability of a G_2 -structure induced by a 3-form ρ is equivalent to $d\rho = d(*\rho) = 0$. For this reason the 4-form $*\rho$ is called a **fundamental 4-form of a G_2 -manifold**, and ρ **the fundamental 3-form**.

Remark 1.8: Let $V = \mathbb{R}^7$ be a 7-dimensional real space equipped with a non-degenerate 3-form ρ with $St_{GL(7)}(\rho) = G_2$. As in Remark 1.4, one can easily see that V has a natural G_2 -invariant metric. For each vector $x \in V$, $|x| = 1$, its stabilizer $St_{G_2}(x)$ in G_2 is isomorphic to $SU(3)$. Indeed, the orthogonal complement x^\perp is equipped with a symplectic form $\rho|_x$, which gives a complex structure $g^{-1} \circ (\rho|_x)$ as usual. This gives an embedding $St_{G_2}(x) \hookrightarrow U(3)$. Since the space of such x is S^6 , and the action of G_2 in S^6 is transitive, one has $\dim St_{G_2}(x) = \dim G_2 - \dim S^6 = 8 = \dim U(3) - 1$. To see that $St_{G_2}(x) = SU(3) \subset U(3)$ and not some other codimension 1 subgroup, one should notice that $St_{G_2}(x)$ preserves two 3-forms $\rho|_{x^\perp}$ and $\rho^*|_x|_{x^\perp}$, where $\rho^* = *\rho$ is the fundamental 4-form of V . A simple linear-algebraic calculation implies that $\rho|_{x^\perp} + \sqrt{-1}\rho^*|_x|_{x^\perp}$ is a holomorphic volume form on x^\perp , which is clearly preserved by $St_{G_2}(x)$. Therefore, the natural embedding $St_{G_2}(x) \hookrightarrow U(3)$ lands $St_{G_2}(x)$ to $SU(3)$. Using the dimension count $\dim St_{G_2}(x) = \dim SU(3)$ (see above), we show that the embedding $St_{G_2}(x) \hookrightarrow SU(3)$ is also surjective.

Let now M be an almost G_2 -manifold. From Remark 1.8 it follows that with every vector $x \in TM$, $|x| = 1$, one can associate a complex Hermitian structure on its orthogonal complement x^\perp . The easiest way to define this structure is to notice that x^\perp is equipped with a symplectic structure $\rho|_x$ and a metric $g|_{x^\perp}$, which can be considered as a real and imaginary parts of a complex-valued semilinear Hermitian product. Then the complex structure is obtained as usual, as $I := (\rho|_x) \circ g^{-1}$.

Definition 1.9: Consider now the unit sphere bundle S^6M over M , with the fiber S^6 , and let $T_{\text{hor}}S^6M \subset TS^6M$ be the horizontal sub-bundle corresponding to the Levi-Civita connection. This sub-bundle has a natural section θ ; at each point $(x, m) \in S^6M$, $m \in M$, $x \in T_mM$, $|x| = 1$, we take $\theta|_{(x,m)} = x$, using the standard identification $T_{\text{hor}}S^6M|_{(x,m)} = T_mM$. Denote by $B \subset T_{\text{hor}}S^6M$ the orthogonal complement to θ in $T_{\text{hor}}S^6M$. Since at each point $(x, m) \in S^6M$, the restriction $B|_{(x,m)}$ is identified with $x^\perp \subset T_mM$, this bundle is equipped with a natural complex structure, that is, an operator $I \in \text{End } B$, $I^2 = -\text{Id}_B$.

The main result of the present paper is the following theorem.

Theorem 1.10: Let M be an almost G_2 -manifold, $S^6M \subset TM$ its unit sphere bundle, and $B \subset TS^6M$ a sub-bundle of its tangent bundle constructed above, and equipped with the complex structure I as above. Then $B^{0,1} \subset B \otimes \mathbb{C} \subset TS^6M \otimes \mathbb{C}$ is involutive if and only if M is a holonomy G_2 -manifold.

We prove Theorem 1.10 in Section 4.

Definition 1.11: Let M be a holonomy G_2 -manifold, and

$$\text{Tw}(M) := (S^6M, B, I)$$

the CR-manifold constructed in Theorem 1.10. Then $\text{Tw}(M)$ is called a **CR-twistor space of M** .

1.2 Applications of twistor geometry

G_2 instanton bundles were introduced in [DT], and much studied since then. This notion is a special case of a more general notion of an instanton on a calibrated manifold, which is already well developed. Many estimates known for 4-dimensional manifolds (such as Uhlenbeck's compactness theorem) can be generalized to the calibrated case ([T], [TT]).

Recently, G_2 -instantons became a focus of much activity because of attempts to construct a higher-dimensional topological quantum field theory, associated with G_2 and 3-dimensional Calabi-Yau manifolds ([DS]).

Definition 1.12: Let M be a G_2 -manifold, and $\Lambda^2 M = \Lambda_7^2(M) \oplus \Lambda_{14}^2(M)$ the irreducible decomposition of the bundle of 2-forms $\Lambda^2(M)$ associated with the G_2 -action (Subsection 3.2). A vector bundle (B, ∇) with connection is called a **G_2 -instanton** if its curvature lies in $\Lambda_{14}^2(M) \otimes \text{End}(B)$.

Remark 1.13: A tangent bundle and all its tensor powers are obviously G_2 -instantons (see Step 1 in the proof of Theorem 1.10 in Section 4).

The definition of CR-holomorphic bundles is a straightforward generalization of a usual differential-geometric notion of a holomorphic bundle as a bundle equipped with a Dolbeault differential $\bar{\partial}$ which satisfies $\bar{\partial}^2 = 0$.

Definition 1.14: Let $(M, B, I \in \text{End } B)$ be a CR-manifold, and $B^{0,1} \subset B \otimes \mathbb{C}$ the $\sqrt{-1}$ -eigenspace of B . Using the Cartan's formula, we define the CR Dolbeault differential

$$\Lambda^k(B^{0,1})^* \xrightarrow{\bar{\partial}_B} \Lambda^{k+1}(B^{0,1})^*$$

as usual,

$$\begin{aligned} (\bar{\partial}_B \alpha)(b_1, \dots, b_{k+1}) &:= \sum_i (-1)^i \text{Lie}_{b_i} \alpha(b_1, \dots, \check{b}_i, \dots, b_{k+1}) \\ &+ \sum_{i < j} (-1)^{i+j+1} \alpha([b_i, b_j], b_1, \dots, \check{b}_i, \dots, \check{b}_j, \dots, b_{k+1}). \end{aligned}$$

Let E be a vector bundle, and $\bar{\partial}_E : E \rightarrow E \otimes (B^{0,1})^*$ an operator which satisfies the Leibniz rule

$$\bar{\partial}_E(f\xi) = \xi \otimes \bar{\partial}_B f + f \bar{\partial}_E \xi.$$

We extend $\bar{\partial}_E$ to

$$E \otimes \Lambda^k(B^{0,1})^* \xrightarrow{\bar{\partial}_E} E \otimes \Lambda^{k+1}(B^{0,1})^*$$

using the same Leibniz formula,

$$\bar{\partial}_E(\xi \otimes \alpha) = \xi \otimes \bar{\partial}_B \alpha + \bar{\partial}_E \xi \otimes \alpha.$$

for any $\xi \in E, \alpha \in \Lambda^k(B^{0,1})^*$. A bundle $(E, \bar{\partial}_E)$ is called **CR-holomorphic**, if $\bar{\partial}_E^2 = 0$. In this case, $\bar{\partial}_E$ is called **an operator of CR-holomorphic structure**.

Definition 1.15: Let $(M, B, I \in \text{End } B)$ be a CR-manifold, and (E, ∇) a complex vector bundle on M with connection. Denote by

$$\Pi : \Lambda^i M \longrightarrow \Lambda^i(B^{0,1})^*$$

the restriction map, and let

$$\bar{\partial}_E := \nabla \circ \Pi : E \longrightarrow E \otimes (B^{0,1})^*$$

be the $B^{0,1}$ -part of the connection. We say that (E, ∇) is **CR-holomorphic** if $\bar{\partial}_E$ is an operator of CR-holomorphic structure on E .

Theorem 1.16: Let M be a holonomy G_2 -manifold, (B, ∇) a bundle with connection, and $\text{Tw}(M) \xrightarrow{\pi} M$ its twistor space defined as in Definition 1.11. Then the following assertions are equivalent.

- (i) The pullback $(\pi^*E, \pi^*\nabla)$ is a CR-holomorphic bundle on $\text{Tw}(M)$
- (ii) (E, ∇) is a G_2 -instanton.

Proof: Let $\nabla^2 R \in \Lambda^2 M \otimes \text{End } E$ be the curvature form of E , π^*R the curvature of $(\pi^*E, \pi^*\nabla)$, and $\Pi(\pi^*R) \in \Lambda^2(B^{0,1})^* \otimes \text{End } E$ its restriction to $B^{0,1}$. Clearly, $\Pi(\pi^*R) = 0$ if and only if $(\pi^*E, \pi^*\nabla)$ is CR-holomorphic. However, $\Pi(\pi^*R) = 0$ if and only if π^*R is of Hodge type $(1, 1)$ on B , and this is equivalent to $R \in \Lambda_{14}^2 \otimes \text{End } E$, as follows from Proposition 3.2. ■

Remark 1.17: Following the same approach as in the paper [Le1], it is possible to deduce from Theorem 1.10 that the space of knots (non-parametrized loops) in a holonomy G_2 manifold is formally Kaehler (see the forthcoming paper [V]). For 3-manifolds this theorem is due to J.-L. Brylinski ([Br]).

2 Differential forms on $\text{Tot}(\Lambda^k M)$

In this section, we extend the standard results about the Hamiltonian 2-form on T^*M to the total space $\text{Tot}(\Lambda^k(M))$ of the bundle of k -forms. These generalizations are elementary, but used further on in this paper.

Let M be a smooth manifold, and $X = \text{Tot}(\Lambda^k M)$ a total space of a bundle of k -forms. On X , a pair of forms is defined: a k -form Θ , which is an analogue of the 1-form $\sum q_i dp_i$ on T^*M , and a $(k+1)$ -form $\Xi = d\Theta$, which is an analogue of the symplectic form.

Let $\pi : X \rightarrow M$ be a standard projection. At a point $(\lambda, m) \in X$, with $m \in M$ and $\lambda \in \Lambda^k(M)|_m$, we define

$$\Theta(x_1, \dots, x_k) := \lambda(D\pi(x_1), D\pi(x_2), \dots, D\pi(x_k)), \quad (2.1)$$

where $D\pi : TX \rightarrow TM$ is the differential of π .

Let $\Xi := d\Theta$. In local coordinates p_1, \dots, p_n on M ,

$$q_{i_1, \dots, i_k} := dp_{i_1} \wedge dp_{i_2} \wedge \dots \wedge dp_{i_k}$$

on the fibers of π , Ξ can be written as as sum

$$\Xi = \sum_{i_1 < i_2 < \dots < i_k} dq_{i_1, \dots, i_k} \wedge dp_{i_1} \wedge dp_{i_2} \wedge \dots \wedge dp_{i_k} \quad (2.2)$$

(this is clear from the same argument as used to obtain a more familiar relation $d(\sum q_i dp_i) = \sum dq_i \wedge dp_i$). Further on, we shall need the following elementary lemma.

Lemma 2.1: Let M be a manifold equipped with a torsion-free connection ∇ , $X = \text{Tot}(\Lambda^k M)$ the total space of the bundle of k -forms, and $\Xi \in \Lambda^{k+1}(X)$ the fundamental $(k+1)$ -form constructed above. Using the connection ∇ , we split the tangent bundle onto horizontal and vertical components, $TX = T_{\text{hor}}X \oplus T_{\text{vert}}X$. Then $\Xi|_{T_{\text{hor}}X} = 0$.

Proof: Choose coordinates p_1, \dots, p_n on M , and let $p_1, \dots, p_n, q_{i_1, \dots, i_k}$, $1 \leq i_1 < i_2 < \dots < i_k \leq n$ be the coordinates on X constructed as above. This coordinate system induces a flat torsion-free connection ∇_0 on TM and $\Lambda^k M$. For an arbitrary connection ∇ on TM , one obtains a splitting $TX = T_{\text{vert}} \oplus T_{\text{hor}}(\nabla)$. Write $\nabla = \nabla_0 + A$, where $A \in \Lambda^1 M \otimes \text{End}(TM)$ is an $\text{End}(TM)$ -valued 1-form on M . The difference between $T_{\text{hor}}(\nabla)$ and $T_{\text{hor}}(\nabla_0)$ can be expressed as a section of $\text{Hom}(\pi^* TM, T_{\text{vert}}X)$, at (λ, m) giving $A_k(\lambda) \in \Lambda^1(M) \otimes T_{\text{vert}}X$, where $A_k \in \Lambda^1(M) \otimes \text{End}(\Lambda^k M)$ is the connection form on $\Lambda^k(M)$ induced from ∇ , $A_k = \nabla - \nabla_0$. Denote $A_k(\lambda)$ by $E : \pi^* \Lambda^1 M \rightarrow T_{\text{vert}}X = \pi^* \Lambda^k M$. Using the formula for Ξ written in coordinates as in (2.2), we obtain

$$\Xi(v'_1, v'_2, \dots, v'_{i_{k+1}}) = \Xi(v_{i_1} + E(v_{i_1}), v_{i_2} + E(v_{i_2}), \dots, v_{i_{k+1}} + E(v_{i_{k+1}})), \quad (2.3)$$

where $v_i := \frac{d}{dp_i}$ are coordinate vector fields on X , and $v'_i = v_i + E(v_i)$ the corresponding sections of $T_{\text{hor}}(\nabla)$. As follows from (2.2), the latter

expression gives

$$\Xi(v'_{i_1}, v'_{i_2}, \dots, v'_{i_{k+1}}) = \frac{1}{(k+1)!} \sum (-1)^{|\sigma|} E(v_{\sigma(i_1)}) \left(v_{\sigma(i_2)}, v_{\sigma(i_3)}, \dots, v_{\sigma(i_{k+1})} \right) \quad (2.4)$$

where the sum is taken over all permutations $\sigma \in S_{k+1}$. The operator $E : \pi^* \Lambda^1 M \rightarrow \pi^* \Lambda^k M$ can be expressed in terms of the connection form $A \in \Lambda^1 M \otimes \text{End}(TM)$ as follows: at $(\lambda, m) \in X$ one has

$$\begin{aligned} E \Big|_{(\lambda, m)} (v)(v_{i_1}, \dots, v_{i_k}) \\ = \lambda(A_v(v_{i_1}), v_{i_2}, \dots, v_{i_k}) + \lambda(v_{i_1}, A_v(v_{i_2}), \dots, v_{i_k}) \\ + \dots + \lambda(v_{i_1}, v_{i_2}, \dots, A_v(v_{i_k})), \end{aligned} \quad (2.5)$$

where A_v denotes $A(v)$. Comparing (2.5) and (2.4), we obtain

$$\begin{aligned} \Xi(v'_{i_1}, v'_{i_2}, \dots, v'_{i_{k+1}}) \Big|_{(\lambda, m)} \\ = \frac{1}{(k+1)!} \sum (-1)^{|\sigma|} \lambda \left(A_{v_{\sigma(i_1)}}(v_{\sigma(i_2)}), v_{\sigma(i_3)}, \dots, v_{\sigma(i_{k+1})} \right). \end{aligned} \quad (2.6)$$

Since ∇ and ∇_0 are torsion-free, one has $A_{v_{\sigma(i_1)}}(v_{\sigma(i_2)}) = A_{v_{\sigma(i_2)}}(v_{\sigma(i_1)})$. Therefore, the alternating sum (2.6) vanishes. ■

Remark 2.2: This argument repeats a more familiar argument showing that the restriction of the Hamiltonian symplectic form $\omega = \sum_i dp_i \wedge dq_i$ on T^*M to a horizontal space of a torsion-free connection is always zero.

3 The G_2 -action on $\Lambda^2(\mathbb{R}^7)$ and $SU(3)$ -action on $\Lambda^2(\mathbb{R}^6)$

3.1 Octonion algebra and quaternions

Let $V = \mathbb{R}^7$ be a 7-dimensional space equipped with a non-degenerate, non-split 3-form ρ inducing a G_2 -action on V . Then V is equipped with the vector product, defined as follows: $x \star y = \rho(x, y, \cdot)^\sharp$. Here $\rho(x, y, \cdot)$ is a 1-form obtained by contraction, and $\rho(x, y, \cdot)^\sharp$ its dual vector field. It is not hard to see that (V, \star) becomes isomorphic to the imaginary part of the octonion algebra, with \star corresponding to half of the commutant. In fact, this is one of a definitions of an octonion algebra. The whole octonion algebra is defined $\mathbb{O} := V \oplus \mathbb{R}$, with the product given by

$$(x, t)(y, t') = (ty + t'x + x \star y, g(x, y) + tt')$$

Here, x, y and $ty + t'x + x \star y$ are vectors in V , and $t, t', g(x, y) + tt' \in \mathbb{R}$.

Given two non-collinear vectors in V , they generate a quaternion subalgebra in octonions. When these two vectors satisfy $|v| = |v'| = 1$, $v \perp v'$, the standard basis I, J, K in imaginary quaternions can be given by a triple $v, v', v \star v' \in V$.

A 3-dimensional subspace $A \subset V$ is called **associative** if it is closed under the vector product. The set of associative subspaces is in bijective correspondence with the set of quaternionic subalgebras in octonions.

The following well-known lemma is used further on in this section.

Lemma 3.1: Let (V, ρ) be a 7-dimensional space with a G_2 -structure, $v \in V$ a unit vector, and $W = v^\perp \subset V$ its orthogonal complement, equipped with a Hermitian structure as explained in Remark 1.8. Consider another unit vector $v_1 \in V$, $v_1 \perp v$, and let $W_1 := v_1^\perp$, with its own complex structure. Denote by $\omega_W \in \Lambda^2 W$ the Hermitian form on W . Then the restriction $\omega_W|_{W_1}$ is of Hodge type $(2, 0) + (0, 2)$.

Proof: Let $A \subset V$ be a 3-dimensional associative space generated by v and v_1 , and $H = A^\perp$. The intersection $W \cap W_1$ is a 5-dimensional space generated by H and $v \star v_1$, where \star denotes the vector product. The restriction $\omega_W|_{W_1}$ vanishes on $v \star v_1$, because the complex structure on W maps $v \star v_1$ to $-v_1$. Therefore, $\omega_W|_{W_1}$ is an image of $\omega_W|_H$ under the standard orthogonal embedding $\Lambda^2 H \rightarrow \Lambda^2 W_1$.

Three complex structures $v, v_1, v \star v_1$ define a quaternionic structure on H , and the corresponding three Hermitian forms give a hyperkaehler triple of symplectic structures on H . Since $\omega_W|_H$ is a real part of the standard $(2, 0)$ -form on H associated with v_1 , it is a $(2, 0) + (0, 2)$ -form on W_1 . ■

3.2 G_2 -action and $SU(3)$ -action

Let $V = \mathbb{R}^7$ be a 7-dimensional space equipped with a non-degenerate, non-split 3-form ρ inducing a G_2 -action on V as above. The space $\Lambda^2 V$ is a 21-dimensional representation of G_2 . The Lie algebra \mathfrak{g}_2 can be considered as a subspace in $\Lambda^2 V$, because one has an embedding $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(V) = \Lambda^2 V$. This gives a 14-dimensional G_2 -invariant subspace in $\Lambda^2 V$. There is also a 7-dimensional subspace given by an embedding $V \rightarrow \Lambda^2 V$, $v \mapsto \rho \lrcorner v$. Since $\Lambda^2 V$ is 21-dimensional, this gives a decomposition $\Lambda^2 V = \Lambda_7^2 V \oplus \Lambda_{14}^2 V$ of $\Lambda^2 V$ onto irreducible 7-dimensional and 14-dimensional G_2 -subrepresenta-

tions.

Further on in this paper, we shall need the following linear-algebraic result.

Proposition 3.2: Let $V = \mathbb{R}^7$ be a 7-dimensional space equipped with a G_2 -action, and $\Lambda^2 V = \Lambda_7^2 V \oplus \Lambda_{14}^2 V$ the irreducible decomposition obtained as above. For any $v \in V$, consider the Hermitian structure on its orthogonal complement v^\perp (Remark 1.8). Let $\alpha \in \Lambda^2 V$ be a 2-form. Then the following conditions are equivalent.

- (i) α lies in $\Lambda_{14}^2 V$.
- (ii) For any non-zero $v \in V$, the restriction of α to v^\perp is of type $(1, 1)$ with respect to the complex structure on v^\perp , and orthogonal to the Hermitian form.
- (iii) For any non-zero $v \in V$, the restriction of α to v^\perp is of type $(1, 1)$ with respect to the complex structure on v^\perp .

Proof: Denote by $W \subset V$ the orthogonal complement to v , with the natural $SU(3)$ -structure constructed as in Remark 1.8. After the standard identification between $\mathfrak{so}(W)$ and $\Lambda^2 W$, the algebraic condition (ii) is translated to $\alpha|_W \in \mathfrak{su}(W)$. The restriction map $\Lambda^2 V \rightarrow \Lambda^2 W$ is dual to the standard embedding $\mathfrak{so}(W) \hookrightarrow \mathfrak{so}(V)$. Under this embedding, $\mathfrak{su}(W)$ is mapped to \mathfrak{g}_2 (Remark 1.8), hence the restriction $\Lambda_{14}^2(V)|_W$ lies in $\mathfrak{su}(W)$. We proved the implication (i) \Rightarrow (ii).

To obtain the converse implication, take some $\alpha \in \Lambda^2 V$ satisfying assumptions of (ii); since V is odd-dimensional, α has an annihilator $R \subset V$, which is at least one-dimensional. Let $v \in R$ be a non-zero vector. Then α , considered as an element of $\mathfrak{so}(V)$, preserves $W = v^\perp$ and acts trivially on v , hence it is an image of $\alpha_0 \in \Lambda^2(W)$ under the standard embedding $\mathfrak{so}(W) \hookrightarrow \mathfrak{so}(V)$. By our assumptions, $\alpha_0 \in \mathfrak{su}(W)$, hence α belongs to an image of the standard embedding $\mathfrak{su}(W) \hookrightarrow \mathfrak{g}_2 = \Lambda_{14}^2(V)$.

To finish the proof of Proposition 3.2, it suffices to prove that (iii) implies (i). Let $\alpha \in \Lambda^2 V$ be a 2-form satisfying (iii). Consider a codimension 1 subspace $W \subset V$ constructed as above, with α in the image of the standard orthogonal embedding $\Lambda^2 W \hookrightarrow \Lambda^2 V$. Consider a decomposition $\alpha = \alpha_0 + t\omega_W$ of α in $\mathfrak{u}(W) = \Lambda^{1,1}(W)$ onto its traceless part $\alpha_0 \in \mathfrak{su}(W)$ and the part $t\omega_W$ proportional to the Hermitian form. Since $\alpha_0 \in \mathfrak{su}(W)$, this form satisfies $\alpha_0 \in \Lambda_{14}^2(V)$, because $\mathfrak{su}(W)$ lies in \mathfrak{g}_2 . To prove that $\alpha \in \Lambda_{14}^2(V)$

it would suffice to show that $t\omega_W = 0$. However, since α_0 lies in $\Lambda_{14}^2(V)$, it satisfies the assumptions of Proposition 3.2 (ii), hence satisfies Proposition 3.2 (iii). Therefore, the same is true for $t\omega_W = \alpha - \alpha_0$: the restriction of $t\omega_W$ to any $W' = (v')^\perp$ of codimension 1 in V is of type $(1, 1)$ with respect to the standard complex structure on W' . This implies $t = 0$, because, as follows from Lemma 3.1, the restriction of ω_W to another 6-dimensional subspace W' is of type $(2, 0) + (0, 2)$. ■

4 Integrability of the twistor CR-structure

The following claim is well-known from the standard textbooks on differential geometry.

Claim 4.1: ([KN]) Let M be a Riemannian n -manifold, $S^{n-1}M$ the sphere bundle consisting of all unit tangent vectors, and $T_{\text{hor}}S^{n-1}(M) \subset TS^{n-1}(M)$ the horizontal tangent bundle associated with the Levi-Civita connection. Denote by $R \in \text{Sym}^2 \Lambda^2 M$ the curvature of the Levi-Civita connection on M . Given a 2-form $\alpha \in \Lambda^2 M$, consider α as an element in $\mathfrak{so}(TM) = \Lambda^2 M$, acting on $S^{n-1}(M)$ by infinitesimal automorphisms, and let $\alpha^* \in T_{\text{vert}}S^{n-1}M$ be the corresponding vector field. Let $X, Y \in T_{\text{hor}}S^{n-1}(M)$ be horizontal vector fields. Then the vertical component of $[X, Y]$ is equal to $R(X, Y)^*$. ■

We are going to prove Theorem 1.10, which says that the bundle $B^{1,0} \subset T_{\text{hor}}X \otimes \mathbb{C}$ on a G_2 -manifold M is involutive, where $X = S^6M$. The strategy of the proof is the following. We consider the Frobenius bracket $[B^{1,0}, B^{1,0}] \xrightarrow{\Psi} TX \otimes \mathbb{C}/B^{1,0}$. First, we project $\Psi(X, Y)$, $X, Y \in B^{1,0}$ to $T_{\text{vert}}X$ along the horizontal component $T_{\text{hor}}X \otimes \mathbb{C} \supset B^{1,0}$, showing that this projection vanishes. This implies that Ψ is a map from $\Lambda^2 B^{1,0}$ to $T_{\text{hor}}X \otimes \mathbb{C}/B^{1,0}$:

$$[B^{1,0}, B^{1,0}] \xrightarrow{\Psi} T_{\text{hor}}X \otimes \mathbb{C}/B^{1,0}$$

Then we prove that the Frobenius bracket $[\theta^\perp, \theta^\perp] \longrightarrow TX/\theta^\perp$ vanishes at $B \subset TX$, where $\theta \in TX$ is a natural section of $T_{\text{hor}}X$ constructed as in Definition 1.9, and θ^\perp is its orthogonal complement. Since $B = T_{\text{hor}}X \cap \theta^\perp$, this implies that the Frobenius bracket $[B^{1,0}, B^{1,0}] \xrightarrow{\Psi} T_{\text{hor}}X \otimes \mathbb{C}/B^{1,0}$ lands in $\theta^\perp \cap T_{\text{hor}}X = B$. This brings us to the situation familiar from complex (and almost complex) geometry, where we have a bundle B with an endomorphism $I \in \text{End } B$, $I^2 = -\text{Id}_B$, and the commutator map $[B^{1,0}, B^{1,0}] \longrightarrow B \otimes$

\mathbb{C} determines whether I is integrable. To prove that

$$[B^{1,0}, B^{1,0}] \subset B^{1,0}, \quad (4.1)$$

we construct a closed $(0, 3)$ -form $\bar{\Omega}$ which vanishes at $B^{1,0}$ and is non-degenerate at $B^{0,1}$, and use this form to prove (4.1).

Proof of Theorem 1.10. Step 1. By Claim 4.1, the commutator bracket

$$[T_{\text{hor}}X, T_{\text{hor}}X] \xrightarrow{\Psi_1} TX/T_{\text{hor}}X$$

is expressed through the curvature R of the Levi-Civita connection on M , as follows:

$$\Psi_1(x, y) \Big|_{(z, m) \in X} = R(x, y, z).$$

Consider $R_z := R(\cdot, \cdot, z)$ as a TM -valued 2-form on M . If M is holonomy G_2 -manifold, this form lies in $\Lambda_{14}^2 \otimes TM$, because the curvature of a holonomy G_2 manifold lies in $\Lambda_{14}^2 \otimes \mathfrak{g}_2$. Therefore, R_z of type $(1, 1)$ on B , as follows from Proposition 3.2. This implies that $[B^{1,0}, B^{1,0}] \subset T_{\text{hor}}X \otimes \mathbb{C}$. Conversely, if $[B^{1,0}, B^{1,0}] \subset T_{\text{hor}}X \otimes \mathbb{C}$, this means that R_z is of type $(1, 1)$ on every 6-dimensional subspace $W \subset T_mM$, and Proposition 3.2 implies that $R \in \Lambda_{14}^2 \otimes \text{End}(TM)$.

This already proves that integrability of the CR structure $B^{1,0} \subset TX \otimes \mathbb{C}$ implies that (M, ρ) is a holonomy G_2 manifold. The converse implication takes more work.

Step 2: To study the bundle $B = \theta^\perp \cap T_{\text{hor}}X$, we identify $X = S^6TM$ with the total space of the unit bundle $S^6T^*M \subset \text{Tot}(T^*M)$. This is done using the Riemannian metric on TM . On $\text{Tot}(T^*M)$ there is a standard 1-form Θ , dual to θ (see Section 2). The zero-space of this form is identified with θ^\perp , and the corresponding Frobenius bracket $[\theta^\perp, \theta^\perp] \longrightarrow TX/\theta^\perp$ is a 2-form which is proportional to the restriction $d\Theta \Big|_{\theta^\perp}$. However, $d\Theta = \Xi$ is the standard symplectic form on T^*M , and it vanishes on the horizontal sub-bundle, as follows from Lemma 2.1. This implies that the bracket $[\theta^\perp, \theta^\perp] \longrightarrow TX/\theta^\perp$ vanishes on $B \subset \theta^\perp$. In Step 1, we have shown that $[B^{1,0}, B^{1,0}] \subset T_{\text{hor}}X \otimes \mathbb{C}$; the above argument implies that

$$[B^{1,0}, B^{1,0}] \subset (T_{\text{hor}}X \cap \theta^\perp) \otimes \mathbb{C} = B \otimes \mathbb{C}.$$

Step 3: The 6-dimensional bundle $B \subset T_{\text{hor}}M$ by construction comes equipped with a natural $SU(3)$ -structure. The corresponding $(3, 0)$ -form can

be written down as follows (Remark 1.8):

$$\left(\pi^* \rho + \sqrt{-1} (\pi^* \rho^*) \lrcorner \theta \right) \Big|_B \quad (4.2)$$

Here, $\pi : X \rightarrow M$ is the standard projection, ρ and $\rho^* := *\rho$ are the fundamental 3-form and 4-form on M , and (4.2) is the standard equation expressing the standard $(3,0)$ -form on a codimension 1 subspace $W \subset V = \mathbb{R}^7$ in terms of the fundamental 3-form and 4-form on V . Let $\Omega := \pi^* \rho + \sqrt{-1} (\pi^* \rho^*) \lrcorner \theta$. In Step 3, we are going to prove that $(d\Omega) \Big|_{T_{\text{hor}} X} = 0$.

Since ρ is closed, this would follow from $d((\pi^* \rho^*) \lrcorner \theta) \Big|_{T_{\text{hor}} X} = 0$. Let $Z := \text{Tot}(\Lambda^3 M)$, and $\varphi : X \rightarrow Z$ be an embedding mapping $(v, m) \in X = S^6 M$ to $(\rho^* \lrcorner v, m)$. Clearly, $\pi^*(\rho^*) = \varphi^*(\Theta)$, where Θ is a 3-form on $\text{Tot}(\Lambda^3 M)$ defined in Section 2. Then $d((\pi^* \rho^*) \lrcorner \theta) = \varphi^*(\Xi)$. However, the connection on $\Lambda^3 M$ is compatible with the embedding $\Lambda^1(M) \rightarrow \Lambda^3 M$, hence the image $\varphi(T_{\text{hor}} X)$ lies in the horizontal subspace $T_{\text{hor}} Z \subset TZ$. Therefore, to prove that

$$d((\pi^* \rho^*) \lrcorner \theta) \Big|_{T_{\text{hor}} X} = \varphi^*(\Xi) \Big|_{T_{\text{hor}} X} = 0$$

it would suffice to show that $\Xi \Big|_{T_{\text{hor}} Z} = 0$. This follows from Lemma 2.1.

Step 4: Let $Z, T \in B^{0,1}$. Since Ω is of type $(3,0)$ on B , one has $\Omega \lrcorner Z = \Omega \lrcorner T = 0$. Therefore, Cartan's formula gives

$$d\Omega(X, Y, Z, T) = \Omega(X, Y, [Z, T]). \quad (4.3)$$

Taking a complex conjugate of the equation $[B^{1,0}, B^{1,0}] \subset B \otimes \mathbb{C}$ (Step 2), we obtain that $[Z, T] \in B \otimes \mathbb{C}$. Since $d\Omega \Big|_B = 0$, (4.3) implies that the 2-form $\Omega([Z, T], \cdot, \cdot)$ is identically zero on B . However, Ω is a non-degenerate $(3,0)$ -form on B , hence for any $\xi \in B \otimes \mathbb{C}$, the vanishing $\Omega(\xi, \cdot, \cdot) = 0$ implies that $\xi \in B^{0,1}$. We proved that $B^{0,1}$ is involutive. The same is true for $B^{1,0}$ by complex conjugation. ■

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