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Systems of resultants

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Abstract

Writing down convenient explicit formulas for systems of resultants is an important but essentially open problem. In this paper I'll give such a formula derived from the ordinary multivariate resultant.

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1 Intro

Fix some algebraically closed field $\Bbbk$.

Problem 1. *Given a system of polynomial equations*

$$\begin{cases} f_0(x_0,\ldots,x_n)=0 \\ \ldots \\ f_m(x_0,\ldots,x_n)=0 \end{cases} \tag{1}$$

$\deg f_i = N_i$,

$$f_j(x_0,\ldots,x_n)=\sum_{\sum\limits_i s_i=N_j} a_{j,s_0,\ldots,s_n} x_0^{s_0}\cdot\ldots\cdot x_n^{s_n}.$$

How to determine if there exists a non-zero solution of (1)?

It is well-known after [WdW] that there exists a finite set of polynonials on $a_{j,s_0,\ldots,s_n}$ with integer coefficients $R_l(a)\in\mathbb{Z}[a_{j,s_0,\ldots,s_n}]_{j,s_0,\ldots,s_n}$, such that

$$(\text{there exists a non-zero solution of (1)}) \Longleftrightarrow \forall l\ R_l(a)=0$$

Such a set of polynomials ($R_l(a)$) is called a **system of resultants**.

Example. Let $\deg f_j = 1$, $j=0,\ldots,m$, $f_j(x)=\sum_i a_{ji}x_i$. Then the system of resultants is the set of maximal minors of matrix

$$\begin{pmatrix} a_{00} & \ldots & a_{0n} \\ \ldots & \ldots & \ldots \\ a_{m0} & \ldots & a_{mn} \end{pmatrix}$$

Problem 2. *Given a system of polynomial equations*

$$\begin{cases} f_0(x_0,\ldots,x_n)=0 \\ \ldots \\ f_n(x_0,\ldots,x_n)=0 \end{cases} \tag{2}$$

$$\deg f_i = N_i,$$

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$$f_j(x_0,\dots,x_n)=\sum_{\sum_i s_i=N_j} a_{j,s_0,\dots,s_n} x_0^{s_0}\cdot\dots\cdot x_n^{s_n}.$$

How to determine if there exists a non-zero solution of (2)?

It is also well-known (see [GZK] for a modern explanation) that there exist an irreducible polynonial on $a_{j,s_0,\dots,s_n}$ with integer coefficients

$$R(a)\in\mathbb{Z}[a_{j,s_0,\dots,s_n}]_{j,s_0,\dots,s_n},$$

such that

$$\text{(there exists a non-zero solution of (2))} \Longleftrightarrow R(a)=0$$

Such a polynomial ($R(a)$) is called a **resultant** and also denoted as $R(f_0,\dots,f_n)$

Example. Let $f_i(x)=\sum_j a_{ij}x_j$ then $R(f_0,\dots,f_n)=\det(a_{ij})$.

Example. Let $f(x,y)=\sum_i a_i x^i y^{n-i}$, $g(x,y)=\sum_i b_i x^i y^{m-i}$, $a_0\neq 0$, $b_0\neq 0$. Then

$$R(f(x,y),g(x,y))=Res(f(z,1),g(z,1))$$

where Res is a famous Sylvester determinant.

$$\det\underbrace{\begin{pmatrix} a_0 & a_1 & \dots & a_n & & & \\ & a_0 & a_1 & \dots & a_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_m & & & \\ & b_0 & b_1 & \dots & b_m & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \dots & b_m \end{pmatrix}}_{m+n}\begin{matrix}\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n\end{matrix}$$

2 Results on resultants

Consider the system

$$\begin{cases} f_0(x_0,\dots,x_n)=0 \\ \dots \\ f_m(x_0,\dots,x_n)=0 \end{cases} \tag{3}$$

$$\deg f_j=n_j,$$

$$f_j(x_0,\dots,x_n)=\sum_{\sum_i s_i=n_j} a_{j,s_0,\dots,s_n}.$$

Fix some positive integer numbers m_i, $i=0,\dots,n$, and k_{ij}, $i=0,\dots,n; j=0,\dots,m$, such that $m_i=k_{ij}+n_j$. Consider polynomials

$$A_{ij}(x_0,\dots,x_n)=\sum_{\sum_l s_l=k_{ij}} b_{i,j,s_0,\dots,s_n} x_0^{s_0}\cdot\dots\cdot x_n^{s_n}$$

with indeterminate coefficients $b_{i,j,s_0,\dots,s_n}$.

I will consider

$$R(\sum_{j=0}^m A_{0j}f_j,\sum_{j=0}^m A_{1j}f_j,\dots,\sum_{j=0}^m A_{nj}f_j)$$

as a polynomial in $b_{i,j,s_0,\dots,s_n}$ for various $i,j,s_0,\dots,s_n$.

Theorem 1. *System (3) has a non-zero solution iff*

$$R = R(\sum_{j=0}^{m} A_{0j}f_j, \sum_{j=0}^{m} A_{1j}f_j, \dots, \sum_{j=0}^{m} A_{nj}f_j) \equiv 0$$

as a polynomial in the coefficients $b_{i,j,s_0,\dots,s_n}$ *of* A_{ij} *. Thus, coefficients of* R *form the system of resultants of* $f_0, \dots, f_m$ *.*

Example. Let $f_j(x) = \sum_i a_{ij}x_i$ and $\deg A_{ij} = 0$ then

$$R(\sum_{j=0}^{m} A_{ij}f_j)_{i=0}^{n} = \sum_{J\subset\{0,\dots m\},|J|=n+1} \det(a_{ij})_{i=0,\dots,n,j\in J} \prod_{j\in J} b_j.$$

Proof. Assume the contrary. For $x \in \mathbb{P}^n$ I put

$$H_x = \{(A_{ij})_{ij} \mid \sum_{j=0}^{m} A_{ij}(x)f_j(x) = 0, i = 0, \dots, n\}.$$

The condition $R \equiv 0$ is equivalent to

$$\bigoplus_{i=0}^{n}\bigoplus_{j=0}^{m} S^{k_{ij}}(\mathbb{k}^{n+1}) = \bigcup_{x\in\mathbb{P}^n} H_x$$

If x is not a solution of (3) then H_x is a codimension $(n+1)$ linear subspace in

$$V = \bigoplus_{i=0}^{n}\bigoplus_{j=0}^{m} S^{k_{ij}}(\mathbb{k}^{n+1}).$$

If there are no non-zero solutions of (3) then V is a union of n-parametric family of codimension $(n+1)$ subspaces. We get the contradiction.

Remark. *In [GZK] there is a definition of mixed resultant for sections of very ample linear bundles* $L_0, \dots, L_n$ *on a dimension* n *projective variety. Theorem 1 can be generalised to the case of sections of very ample linear bundles*

$$f_j \in H^0(X, L_j), j = 0 \dots, m$$

on a dimension n *projective variety* X *. Consider a system of very ample line bundles* C_{ij} *,* $0 \le i \le n$ *,* $0 \le j \le n$ *, s.t.* $B_i = C_{ij} \otimes L_j$ *for all* i, j *. Then the system of resultants is just the collection of coefficients of*

$$R(\sum_{j=0}^{m} A_{ij} \otimes f_j)_{i=0}^{n}$$

considered as a polynomial in indeterminate

$$A_{ij} \in H^0(X, C_{ij}).$$

Remark. *We get only the set-theoretical (not the scheme-theoretical) system of resultants.*

There are also some related results (which may be used for simplification of calculations and which can be proved by almost exactly the same prooftext):

Theorem 2. *Let* $\deg f_0 \ge \deg f_1 \ge \dots \ge \deg f_m$ *and* $k_{ij} = \deg f_i - \deg f_j$ *. Then system (3) has a non-zero solution iff*

$$R(f_0 + \sum_{j=n+1}^{m} A_{0j}f_j, f_1 + \sum_{j=n+1}^{m} A_{1j}f_j, \dots, f_n + \sum_{j=n+1}^{m} A_{nj}f_j) \equiv 0$$

as a polynomial on coefficients of A_{ij} *.*

Consider vector subspaces V_{ij} of $S^{k_{ij}}(\mathbb{k}^{n+1})$, such that

$$\{A_{ij}(x) \mid A_{ij} \in V_{ij}\} = \mathbb{k}$$

for all $x \in \mathbb{k}^{n+1}$.

Example. $V_{ij} = S^{k_{ij}}(\mathbb{k}^{n+1})$

Example. $V_{ij} = \{\sum\limits_{l=0}^{n} a_{lij} x_l^{k_{ij}}\}$

Theorem 3. *System (3) has a non-zero solution iff*

$$R = R(\sum_{j=0}^{m} A_{0j} f_j, \sum_{j=0}^{m} A_{1j} f_j, \ldots, \sum_{j=0}^{m} A_{nj} f_j) \equiv 0,$$

(where $A_{ij} \in V_{ij}$) as a polynomial on $\bigoplus_{i=0}^{n} V_i$. Thus, coefficients of R form the system of resultants of $f_0, \ldots, f_m$.

Remark. *Theorem 3 is a generalisation of Theorem 1.*

Consider vector subspaces V_i of $\bigoplus_{j=0}^{m} S^{k_{ij}}(\mathbb{k}^{n+1})$, such that

$$\{(A_{i0}(x), A_{i1}(x), \ldots, A_{im}(x) \mid (A_{0m}, A_{1m}, \ldots, A_{im}) \in V_i\} = \mathbb{k}^{m+1}$$

for all $x \in \mathbb{k}^{n+1}$

Example. $V_i = \bigoplus_{j=0}^{m} S^{k_{ij}}(\mathbb{k}^{n+1})$

Example. $V_i = \bigoplus_{j=0}^{m} V_{ij}$

Example. $V_i = \{(\sum\limits_{l \neq 0} a_{li0} x_l^{k_{i0}} + b x_0^{k_{i0}}, \ldots \sum\limits_{l \neq n} a_{lin} x_l^{k_{in}} + b x_n^{k_{in}}, \sum\limits_{l=0}^{n} a_{li(n+1)} x_l^{k_{i(n+1)}}, \ldots, \sum\limits_{l=0}^{n} a_{lim} x_l^{k_{im}})\}$

Theorem 4. *System (3) has a non-zero solution iff*

$$R = R(\sum_{j=0}^{m} A_{0j} f_j, \sum_{j=0}^{m} A_{1j} f_j, \ldots, \sum_{j=0}^{m} A_{nj} f_j) \equiv 0$$

(where $(A_{i0}, A_{i1}, \ldots, A_{im}) \in V_i$) as a polynomial on $\oplus_{i=0}^{n} V_i$. Thus, coefficients of R form the system of resultants of $f_0, \ldots, f_m$.

Remark. *Theorem 4 is a generalisation of Theorem 3.*

References

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