NEWTON–OKOUNKOV POLYTOPES OF FLAG VARIETIES

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Abstract. We compute the Newton–Okounkov bodies of line bundles on the complete flag variety of GL_n for a geometric valuation coming from a flag of translated Schubert subvarieties. The Schubert subvarieties correspond to the terminal subwords in the decomposition $(s_1)(s_2s_1)(s_3s_2s_1)(\ldots)(s_{n-1}\ldots s_1)$ of the longest element in the Weyl group. The resulting Newton–Okounkov bodies coincide with the Feigin–Fourier–Littelmann–Vinberg polytopes in type A.

1. Introduction

Newton–Okounkov convex bodies generalize Newton polytopes from toric geometry to a more general algebro-geometric as well as representation-theoretic setting. In particular, Newton–Okounkov bodies of flag varieties and of Bott–Samelson resolutions for different valuations have recently attracted much interest due to connections with representation theory and Schubert calculus. The Newton–Okounkov body can be assigned to a line bundle on an algebraic variety X [KaKh, LM]. In contrast with Newton polytopes, Newton–Okounkov bodies depend heavily on a choice of a valuation on the field of rational functions $\mathbb{C}(X)$. In the case of flag varieties, it is especially interesting to consider various geometric valuations, namely, valuations coming from a complete flag of subvarieties $pt = Y_d \subset \ldots \subset Y_1 \subset Y_0 = X$, where $d := \dim X$, since the resulting Newton–Okounkov convex bodies can often be identified with polytopes that arise in representation theory.

The first explicit computation of Newton–Okounkov polytopes of flag varieties is due to Okounkov [O]. For a geometric valuation, he identified Newton–Okounkov polytopes of symplectic flag varieties with symplectic Gelfand–Zetlin polytopes. Since then several other computations were made for different valuations [An, Fu, FFL14, HY, Ka, Ki14], see also [An15, FK, SchS] for related results. In the present paper, we use a natural geometric valuation introduced by Anderson in [An, Section 6.4] who computed an example for GL_3. In this example, the Newton–Okounkov polytope was identified with the 3-dimensional Gelfand–Zetlin polytope.

Let X be the complete flag variety for $GL_n(\mathbb{C})$. We compute Newton–Okounkov convex bodies of semiample line bundles on X for the geometric valuation coming from the flag of translated Schubert subvarieties

$$w_0X_{\mathrm{id}} \subset w_0w_{d-1}^{-1}X_{w_{d-1}} \subset w_0w_{d-2}^{-1}X_{w_{d-2}} \subset \ldots \subset w_0w_1^{-1}X_{w_1} \subset X,$$

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where w_1, w_2,..., w_{d-1} are terminal subwords of the decomposition

$$(s_1)(s_2s_1)(s_3s_2s_1)(\ldots)(s_{n-1}\ldots s_1)$$

of the longest element in S_n (see Section 2.1 for a precise definition). The valuation can be alternatively described as the lowest term valuation associated with a natural coordinate system on the open Schubert cell in X (see Section 2.2). The computation is based on simple algebro-geometric and convex-geometric arguments. The only representation-theoretic input is the well-known fact that the number of integer points in the Gelfand–Zetlin polytope for a dominant weight λ is equal to the dimension of the irreducible representation of GL_n with the highest weight λ.

Surprisingly, the resulting polytopes for $n > 3$ are not, in general, combinatorially equivalent to the Gelfand–Zetlin polytopes and coincide instead with Feigin–Fourier–Littelmann–Vinberg polytopes in type A. The complete list of cases when Feigin–Fourier–Littelmann–Vinberg polytopes in type A are combinatorially equivalent to the Gelfand–Zetlin polytopes can be found in [Fo]. Though Feigin–Fourier–Littelmann–Vinberg polytopes can also be defined in type C an analogous result for Newton–Okounkov polytopes does not hold already for $Sp_4(\mathbb{C})$ (see Section 2.4 for more details). In both types A and C, Feigin–Fourier–Littelmann–Vinberg polytopes were earlier obtained as Newton–Okounkov bodies for a completely different valuation that does not come from any decomposition of the longest element (see [FFL14, Examples 8.1,8.2]). The fact that valuations considered in [FFL14] and in the present paper yield the same Newton–Okounkov polytopes served as the starting point for the recent preprint [FaFL15], which gives a conceptual explanation for this coincidence (see [FaFL15, Example 17]).

The paper is organized as follows. In Section 2, we define the valuation, formulate the main result and consider several examples. Section 3 contains the proof of the main theorem modulo the result on comparison between the Gelfand–Zetlin and Feigin–Fourier–Littelmann–Vinberg polytopes. The latter result is explained in Section 4 using purely convex-geometric arguments.

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2. Main result

In this section, we define the valuation on $\mathbb{C}(X)$, recall the inequalities defining Feigin–Fourier–Littelmann–Vinberg polytopes and formulate the main theorem. We also define a geometrically natural coordinate system on the open Schubert cell and use it do the simplest examples by hand. Finally, we discuss the case of symplectic flag varieties.

2.1. Valuation

Fix the decomposition $\underline{w_0} = (s_1)(s_2s_1)(s_3s_2s_1)\ldots(s_{n-1}\ldots s_1)$ of the longest element $w_0 \in S_n$. Here $s_i := (i\ i+1)$ is the i-th elementary transposition. Denote by $d := \binom{n}{2}$ the length of w_0.

Fix a complete flag of subspaces $F^\bullet := (F^1 \subset F^2 \subset \ldots \subset F^{n-1} \subset \mathbb{C}^n)$ (this amounts to fixing a Borel subgroup $B \subset GL_n$). In what follows, $\underline{w_k}$ for $k = 1,\ldots, d$ denotes the subword of $\underline{w_0}$ obtained by deleting the first k simple reflections in $\underline{w_0}$, and w_k denotes the corresponding element of S_n. Consider the flag of translated Schubert subvarieties:

$$w_0X_{\mathrm{id}} \subset w_0w_{d-1}^{-1}X_{w_{d-1}} \subset w_0w_{d-2}^{-1}X_{w_{d-2}} \subset \ldots \subset w_0w_1^{-1}X_{w_1} \subset GL_n/B, \qquad (*)$$

where Schubert subvarieties are taken with respect to the flag $F^\bullet$, i.e., $X_w = \overline{BwB/B}$ (cf. [An, Section 6.4] and [Ka, Remark 2.3]). Let y_1, ..., y_d be coordinates on the open Schubert cell C (with respect to $F^\bullet$) that are compatible with $(*)$, i.e., $w_0w_k^{-1}X_{w_k} \cap C = \{y_1 = \ldots = y_k = 0\}$. A possible choice of such coordinates is described in Section 2.2.

Fix the lexicographic ordering on monomials in coordinates $y_1, \ldots, y_d$, i.e., $y_1^{k_1} \cdots y^{k_d} \succ y^{l_1} \cdots y^{l_d}$ iff there exists $j \le d$ such that $k_i = l_i$ for $i < j$ and $k_j > l_j$. Let v denote the lowest order term valuation on $\mathbb{C}(X_{w_0}) = \mathbb{C}(GL_n/B)$ associated with these coordinates and ordering. Let L_λ be the line bundle on GL_n/B corresponding to a dominant weight $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of GL_n (*dominant* means that $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$). Recall that the bundle L_λ is semiample iff λ is dominant and very ample iff λ is strictly dominant, i.e., $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Denote by $\Delta_v(GL_n/B, L_\lambda) \subset \mathbb{R}^d$ the Newton–Okounkov convex body corresponding to GL_n/B, L_λ and v (see [KaKh, LM] for a definition of Newton–Okounkov convex bodies).

Theorem 2.1. *The Newton–Okounkov convex body $\Delta_v(GL_n/B, L_\lambda)$ coincides with the Feigin–Fourier–Littelmann–Vinberg polytope $FFLV(\lambda)$.*

We now recall the definition of $FFLV(\lambda)$. Label coordinates in $\mathbb{R}^d$ corresponding to $(y_1, \ldots, y_d)$ by $(u^1_{n-1}; u^2_{n-2}, u^1_{n-2}; \ldots; u_1^{n-1}, u_1^{n-2}, \ldots, u_1^1)$. Arrange the coordinates into the table

$$\begin{array}{ccccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & & & \ldots & & & & \lambda_n \\
& u_1^1 & & u_2^1 & & & \ldots & & & & u_{n-1}^1 & & \\
& & u_1^2 & & \ldots & & & & & u_{n-2}^2 & & & \\
& & & \ddots & & & \ddots & & & & & & \\
& & & & u_1^{n-2} & & & u_2^{n-2} & & & & & \\
& & & & & u_1^{n-1} & & & & & & &
\end{array} \qquad (FFLV)$$

The polytope $FFLV(\lambda)$ is defined by inequalities $u^l_m \ge 0$ and

$$\sum_{(l,m)\in D} u_m^l \le \lambda_i - \lambda_j$$

for all Dyck paths going from λ_i to λ_j in table $(FFLV)$ where $1 \le i < j \le n$ (see [FFL] for more details).

Example 2.2. (a) For $n = 3$, there are six inequalities

$$0 \le u_1^1 \le \lambda_1 - \lambda_2; \quad 0 \le u_2^1 \le \lambda_2 - \lambda_3; \quad 0 \le u_1^2; \quad u_1^1 + u_1^2 + u_2^1 \le \lambda_1 - \lambda_3.$$

In this case, there is a unimodular change of coordinates that maps $FFLV(\lambda)$ to the Gelfand–Zetlin polytope $GZ(\lambda)$ (see Section 4 for a definition of $GZ(\lambda)$).

(b) For $n = 4$, there are 13 inequalities

$$0 \le u_1^1 \le \lambda_1 - \lambda_2; \quad 0 \le u_2^1 \le \lambda_2 - \lambda_3; \quad 0 \le u_3^1 \le \lambda_3 - \lambda_4; \quad 0 \le u_1^2,\ u_2^2,\ u_1^3;$$

$$u_1^1 + u_1^2 + u_2^1 \le \lambda_1 - \lambda_3; \quad u_2^1 + u_2^2 + u_3^1 \le \lambda_2 - \lambda_4;$$
$$u_1^1 + u_1^2 + u_2^1 + u_2^2 + u_3^1 \le \lambda_1 - \lambda_4; \quad u_1^1 + u_1^2 + u_1^3 + u_2^2 + u_3^1 \le \lambda_1 - \lambda_4.$$

In this case, $FFLV(\lambda)$ and $GZ(\lambda)$ are combinatorially different whenever λ is strictly dominant because they have different number of facets (cf. [Fo, Proposition 2.1.1]).

2.2. Coordinates

We now introduce coordinates on the open Schubert cell in GL_n/B that are compatible with the flag $(*)$. These coordinates seem to be natural from a geometric viewpoint and will be used to compute by hand some examples in the end of this section. However, they are not needed for the proof of the main result.

To motivate the definition consider first the Bott–Samelson variety $X_{\underline{w_0}}$. Its points are collections of d subspaces $\{V_j^i \subset \mathbb{C}^n \mid i+j \le n,\ i,j>0\}$ such that $\dim V_j^i = i$, and V_j^i, $V_{j+1}^i \subset V_j^{i+1}$ where we put $V_{n-i}^{i+1} := F^{i+1}$. Incidence relations between subspaces V_j^i can be organized into the following table (similar to the Gelfand–Zetlin table).

$$\begin{array}{ccccccccccc}
V_1^1 & & V_2^1 & & \dots & & & V_{n-1}^1 & & F^1 \\
& V_1^2 & & \dots & & & V_{n-2}^2 & & F^2 & \\
& & \ddots & & \ddots & & \dots & & & \\
& & & V_1^{n-2} & & V_2^{n-2} & & F^{n-2} & & \\
& & & & V_1^{n-1} & & F^{n-1} & & &
\end{array}$$

where the notation

$$\begin{array}{ccc} U & & V \\ & W & \end{array}$$

means $U, V \subset W$.

Collections of spaces $(V_j^i \subset \mathbb{C}^n \mid i+j \le n,\ i,j \ge 1)$ appear naturally when we start from the fixed flag $F^\bullet$ and apply d one parameter deformations to get the moving flag $M^\bullet := (V_1^1 \subset V_1^2 \subset \ldots \subset V_1^{n-1} \subset \mathbb{C}^n)$. The deformations are encoded by the word $\underline{w_0}$ as follows. The elementary transposition s_i corresponds to $\mathbb{P}^1$-family of complete flags that differ only in the i-th subspace. To go from $F^\bullet$ to $M^\bullet$ we first move F^1 inside F^2 and get the flag $(V_{n-1}^1 \subset F^2 \subset \ldots \subset F^{n-1})$, second we move F^2 inside F^3 and get $(V_{n-1}^1 \subset V_{n-2}^2 \subset F^3 \subset \ldots \subset F^{n-1})$, third we move V_{n-1}^1 inside V_{n-2}^2 to get V_{n-2}^1 and so on.

Example 2.3. Let $n=4$. Below is the sequence of intermediate flags between $F^\bullet$ and $M^\bullet$.

$$F^\bullet \xrightarrow{s_1} (V_3^1 \subset F^2 \subset F^3) \xrightarrow{s_2} (V_3^1 \subset V_2^2 \subset F^3) \xrightarrow{s_1} (V_2^1 \subset V_2^2 \subset F^3) \xrightarrow{s_3}$$

$$(V_2^1 \subset V_2^2 \subset V_1^3) \xrightarrow{s_2} (V_2^1 \subset V_1^2 \subset V_1^3) \xrightarrow{s_1} M^\bullet$$

Remark 2.4. The word $\underline{w_0}$ is the same (after switching s_i and s_{n-i}) as the word used in [V, 2.2] to encode the path from the fixed flag to the moving flag in order to establish a geometric Littlewood–Richardson rule for Grassmannians. According to [V, 3.12] not every reduced decomposition of w_0 can be used for this purpose which is another manifestation of the special properties of $\underline{w_0}$.

Note that if $F^\bullet$ and $M^\bullet$ are in general position (that is, $M^\bullet$ lies in the open Schubert cell C with respect to $F^\bullet$), then all subspaces V_j^i are uniquely defined by $M^\bullet$, namely, $V_j^i = F^{n-j+1} \cap M^{i+j-1}$. In particular, the natural projection

$$\pi_{\underline{w_0}} : X_{\underline{w_0}} \to GL_n/B; \quad \pi_{\underline{w_0}} : (V_j^i) \mapsto M^\bullet$$

is one to one over C. Fix a basis $e_1, \ldots, e_n$ in $\mathbb{C}^n$ compatible with $F^\bullet$, i.e., $F^i = \langle e_1, \ldots, e_i \rangle$ (fixing such a basis is equivalent to fixing a maximal torus $T \subset B$, and hence, an action of the Weyl group on flags). Using the word $\underline{w_0}$ we now introduce natural coordinates $(x_{n-1}^1; x_{n-2}^2, x_{n-2}^1; \ldots; x_1^{n-1}, x_1^{n-2}, \ldots, x_1^1)$ on $C \simeq \pi_{\underline{w_0}}^{-1}(C)$. The origin in this coordinate system is the flag $w_0F^\bullet := (w_0F^1 \subset w_0F^2 \subset \ldots \subset w_0F^{n-1})$. The coordinate x_j^i determines the position of V_j^i inside the $\mathbb{P}^1$-family of dimension i subspaces $V_j^i(x_j^i)$ such that $V_{j+1}^{i-1} \subset V_j^i(x_j^i) \subset V_j^{i+1}$. To define the coordinate x_j^i on $\mathbb{P}^1$ uniquely up to a constant factor it is enough to choose $V_j^i(0)$ and $V_j^i(\infty)$. The following choice seems to be the most natural.

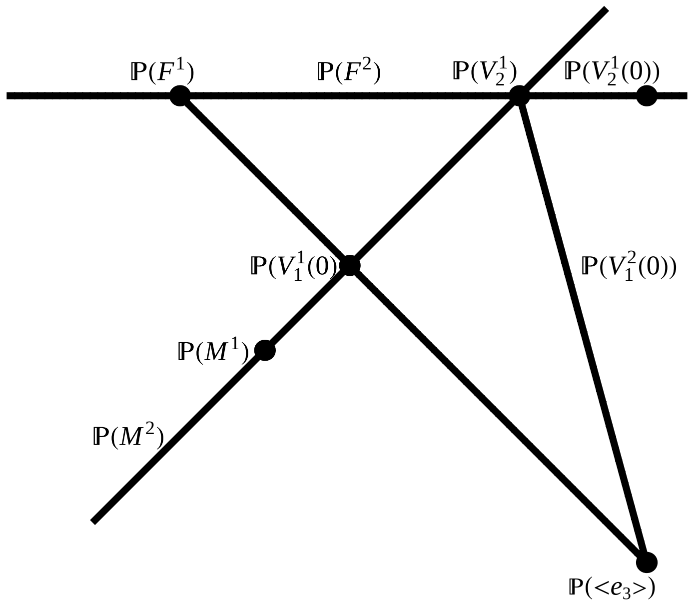

FIGURE 1. Coordinates on flags for $n = 3$.

Since $M^\bullet$ and $F^\bullet$ are in general position, that is, $\dim(F^{n-j} \cap M^{i+j}) = i$, we have inclusions of pairwise distinct subspaces:

$$V_j^i = F^{n-j+1} \cap M^{i+j-1} \quad \begin{matrix} V_{j+1}^{i-1} = F^{n-j} \cap M^{i+j-1} \\ \neq \\ V_j^{i+1} = F^{n-j+1} \cap M^{i+j} \end{matrix} \quad V_{j+1}^i = F^{n-j} \cap M^{i+j}$$

Put $V_j^i(\infty) := V_{j+1}^i$ and $V_j^i(0) := \langle F^{n-i-j}, e_{n-j+1}\rangle \cap M^{i+j} + V_{j+1}^{i-1}$. Note that $\langle F^{n-i-j}, e_{n-j+1}\rangle \cap M^{i+j}$ is the line spanned by a vector $e_{n-j+1} + v$ for some $v \in F^{n-i-j}$ since $F^{n-i-j} \cap M^{i+j} = \{0\}$. It follows that $\dim V_j^i(0) = i$, and $V_j^i(0) \neq V_j^i(\infty)$ because $e_{n-j+1} \notin F^{n-j}$. By construction, $V_{j+1}^{i-1} \subset V_j^i(0) \subset V_j^{i+1}$. Note also that V_j^i lies in $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{V_j^i(\infty)\}$ when $M^\bullet$ and $F^\bullet$ are in general position.

Remark 2.5. It is not hard to check that coordinates $(y_1, \ldots, y_d) := (x_{n-1}^1; x_{n-2}^2, x_{n-2}^1; \ldots; x_1^{n-1}, x_1^{n-2}, \ldots, x_1^1)$ are compatible with the flag $(*)$ of Schubert subvarieties.

Example 2.6. Let $n = 3$. Then

$$V_1^1 = \langle (x_1^1 x_2^1 - x_1^2)e_1 + x_1^1 e_2 + e_3\rangle; \quad V_2^1 = \langle x_2^1 e_1 + e_2\rangle;$$

$$V_1^2 = \langle x_2^1 e_1 + e_2, -x_1^2 e_1 + e_3\rangle.$$

Figure 1 depicts projectivizations in $\mathbb{P}^2$ of various subspaces involved in this example.

2.3. Examples

Theorem 2.1 will be proved in the next section. Here we verify it by hand in three simplest examples.

Example 2.7. cf. [An, Section 6.4] Let $n = 3$, and $\lambda = (2, 1, 0)$. The flag variety GL_3/B can be regarded as a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^{2*}$ under the embedding $(V_1^1, V_1^2) \mapsto V_1^1 \times V_1^2$. The line bundle L_λ on GL_3/B is the pullback of the dual tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}^8$ under the embedding:

$$p_\lambda : GL_3/B \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^{2*} \stackrel{\text{Segre}}{\longrightarrow} \mathbb{P}^8.$$

Using Example 2.6 we get that in coordinates $(y_1, y_2, y_3) = (x_2^1, x_1^2, x_1^1)$ the map p_λ takes the form

$$p_\lambda : (y_1, y_2, y_3) \mapsto \begin{pmatrix} y_1y_3 - y_2 \\ y_3 \\ 1 \end{pmatrix} \times \begin{pmatrix} y_2 & y_1 & 1 \end{pmatrix}.$$

Hence, $H^0(GL_3/B, L_\lambda)$ has the basis 1, y_1, y_2, y_3, y_1y_3, y_2y_3, $y_1y_2y_3 - y_2^2$, $y_1^2y_3 - y_1y_2$. Applying the valuation v we get 8 integer points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$, $(0, 2, 0)$, $(1, 1, 0)$, whose convex hull in $\mathbb{R}^3$ is given exactly by inequalities of Example 2.2*(a)*.

Example 2.8. Let $n = 4$, and $\lambda = (1, 1, 0, 0)$. The line bundle L_λ on GL_4/B is the pullback of the dual tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}^5$ under the natural projection $GL_4/B \to G(2, 4)$ composed with the Plücker embedding $G(2, 4) \hookrightarrow \mathbb{P}^5$ of the Grassmannian. Using Example 2.3 we get that in coordinates $(y_1, \ldots, y_6)$ the plane V_1^2 is spanned by the vectors $(y_4y_6 + y_5, y_4, 1, 0)$ and $(y_2y_6 + y_3, y_2, 0, 1)$. Hence, the map p_λ has the form

$$p_\lambda : (y_1, \ldots, y_6) \mapsto (y_2y_5 - y_3y_4 : -(y_2y_6 + y_3) : y_4y_6 + y_5 : -y_2 : y_4 : 1).$$

The valuation v takes the sections of $H^0(GL_4/B, L_\lambda)$ to 6 integer points in the 4-space $\{u_1^1 = u_3^1 = 0\}$. In coordinates $(u_1^2, u_1^3, u_2^1, u_2^2)$, these points are $(0, 1, 1, 0)$, $(0, 1, 0, 0)$, $(0, 0, 0, 1)$, $(1, 0, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 0)$. Their convex hull in $\mathbb{R}^4$ is given exactly by inequalities of Example 2.2*(b)*.

Example 2.9. The previous example can be extended to $G(3, 6)$, that is, $n = 6$ and $\lambda = (1, 1, 1, 0, 0, 0)$. This is the minimal example when $FFLV(\lambda)$ and $GZ(\lambda)$ are not combinatorially equivalent (cf. [Fo, Proposition 2.1.1]). When computing V_1^3 in coordinates $(y_1, \ldots, y_{15})$ one can immediately ignore all monomials that contain y_{15}, y_{14}, y_{13} since they never appear as the lowest order terms. The same holds for y_3, y_2, y_1. If $y_{15} = y_{14} = y_{13} = 0$, then p_λ takes the following simple form:

$$p_\lambda : (y_4, \ldots, y_{12}) \mapsto 3 \times 3 \text{ minors of } \begin{pmatrix} y_{10} & y_{11} & y_{12} & 1 & 0 & 0 \\ y_7 & y_8 & y_9 & 0 & 1 & 0 \\ y_4 & y_5 & y_6 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, we have to compute the lowest order terms of all minors of the 3×3 matrix formed by the first three columns. After rotating this matrix as follows

$$\begin{array}{ccccc} & & y_{10} & & \\ & y_7 & & y_{11} & \\ y_4 & & y_8 & & y_{12} \\ & y_5 & & y_9 & \\ & & y_6 & & \end{array}$$

it is easy to see that the lowest order monomials in the minors are in bijective correspondence with those collections of u_j^i (where $3 \le i + j \le 6$, $j \le 3$) in table $(FFLV)$ that can not occur in the same Dyck path. By definition, $FFLV(\lambda)$ contains an integer point with $u_j^i = 1$ and $u_m^l = 1$ iff no Dyck path passes through both u_j^i and u_m^l. Hence, the valuation v maps bijectively the minors to the integer points in $FFLV(\lambda)$.

Remark 2.10. Arguments of Example 2.9 allow one to identify $\Delta_v(GL_n/B, L_{\omega_i})$ with $FFLV(\omega_i)$ for any fundamental weight ω_i of GL_n. This might lead to an alternative proof of Theorem 2.1 if one uses that $\Delta_v(GL_n/B, L_\lambda)$ for $\lambda = k_1\omega_1 + \ldots + k_{n-1}\omega_{n-1}$ contains the Minkowski sum $k_1\Delta_v(GL_n/B, L_{\omega_1}) + \ldots + k_{n-1}\Delta_v(GL_n/B, L_{\omega_{n-1}})$.

2.4. Symplectic case

A statement analogous to Theorem 2.1 does not hold in type C already in the case of Sp_4. We now discuss this case in more detail. For the rest of this section, X denotes the complete flag variety for Sp_4. The flag of translated Schubert subvarieties analogous to $(*)$ has the form

$$s_1s_2s_1s_2X_{\mathrm{id}} \subset s_1s_2s_1X_{s_2} \subset s_1s_2X_{s_1s_2} \subset s_1X_{s_2s_1s_2} \subset X,$$

where s_1, s_2 are simple reflections. The resulting Newton–Okounkov polytopes were computed in [Ki14, Proposition 4.1]. Regardless of whether s_1 corresponds to the shorter or the longer root, these polytopes have 11 vertices (for a strictly dominant weight) while Feigin–Fourier–Littelmann–Vinberg polytopes (as well as string polytopes) for Sp_4 have 12 vertices. In particular, the former are not combinatorially equivalent to the latter.

Note that the string polytopes for the decomposition

$$\underline{w_0} = (s_1)(s_2s_1s_2)(\ldots)(s_ns_{n-1}\ldots s_2s_1s_2\ldots s_{n-1}s_n), \qquad (Sp)$$

where s_1 corresponds to the longer root, coincides (after a unimodular change of coordinates) with the symplectic Gelfand–Zetlin polytopes by [L, Corollary 6.3]. The latter were exhibited in [O] as the Newton–Okounkov bodies of the symplectic flag variety Sp_{2n}/B for the lowest term valuation associated with the B-invariant flag of (not translated) Schubert subvarieties corresponding to the initial subwords of $\underline{w_0}$:

$$X_{\mathrm{id}} \subset X_{w_0w_1^{-1}} \subset \ldots \subset X_{w_0w_{d-1}^{-1}} \subset Sp_{2n}/B,$$

where $d = n^2 = \dim Sp_{2n}/B$.

Finally, note that string polytopes for any connected reductive group G and any reduced decomposition $\underline{w_0}$ were obtained in [Ka] as the Newton–Okounkov bodies of the complete flag variety G/B for the **highest** term valuation associated with the B-invariant flag of Schubert subvarieties:

$$X_{\mathrm{id}} \subset X_{w_{d-1}} \subset \ldots \subset X_{w_1} \subset G/B.$$

Here d denotes the dimension of G/B (and the length of $\underline{w_0}$). Note that for $G = GL_n$ and $\underline{w_0}$ as in Section 2.1, the string polytope coincides with the Gelfand–Zetlin polytope in type A by [L, Corollary 5.2]. While the highest term valuation comes naturally when dealing with crystal bases and string polytopes the lowest term valuation is more natural from a geometric viewpoint since it can be interpreted using the order of the pole of a rational function along a hypersurface.

3. Proof of Theorem 2.1

We first formulate and prove simple general results about Newton–Okounkov bodies and recall classical facts about divisors on Schubert varieties. Then we prove Theorem 2.1.

3.1. Preliminaries

We will need the following two simple lemmas on Newton–Okounkov convex bodies.

Lemma 3.1. *Let X be a variety, L a line bundle on X, and v a valuation on $\mathbb{C}(X)$. If D is an effective divisor on X, then*

$$\Delta_v(X, L) \subset \Delta_v(X, L \otimes \mathcal{O}(D)).$$

Proof. Since D is effective, $1 \in H^0(X, \mathcal{O}(D))$. The lemma follows directly from the definition of Newton–Okounkov bodies since for any $l \in \mathbb{N}$ we have the inclusion $i : H^0(X, L^{\otimes l}) \subset H^0(X, (L \otimes \mathcal{O}(D))^{\otimes l})$ given by $i(s) = s \otimes 1$.

The lemma below is a partial case of [LM, Theorem 4.24]. We provide a short proof for the reader's convenience.

Lemma 3.2. *Let $X \subset \mathbb{P}^N$ be a projective variety of dimension d, and $Y_\bullet = (\{x_0\} = Y_d \subset \ldots \subset Y_1 \subset Y_0 = X)$ a complete flag of subvarieties at a smooth point $x_0 \in X$. Consider a valuation v on $\mathbb{C}(X)$ associated with the flag $Y_\bullet$, and the corresponding coordinates $a_1, \ldots, a_d$ on $\mathbb{R}^d$. Let v_1 be the restriction of the valuation v to $\mathbb{C}(Y_1)$. Denote by L the restriction of the dual tautological bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ to X. Then we have*

$$\Delta_{v_1}(Y_1, L|_{Y_1}) = \Delta_v(X, L) \cap \{a_1 = 0\}.$$

Proof. It is well-known that the natural restriction map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l)) \to H^0(X, L^{\otimes l})$ is surjective for sufficiently large l. Similarly, the map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l)) \to H^0(Y_1, L^{\otimes l}|_{Y_1})$ is surjective. Hence, the map $H^0(X, L^{\otimes l}) \to H^0(Y_1, L^{\otimes l}|_{Y_1})$ is surjective, and $\Delta_{v_1}(Y_1, L|_{Y_1}) \subset \Delta_v(X, L)$. For a section $s \in H^0(X, L^{\otimes l})$, denote by $\bar{s}$ its restriction to Y_1. Then $\bar{s} \neq 0$ iff $v(s) \in \{a_1 = 0\}$. Hence, $\Delta_{v_1}(Y_1, L|_{Y_1}) = \Delta_v(X, L) \cap \{a_1 = 0\}$ as desired.

We will also use the classical Chevalley formula [B, Proposition 1.4.3] and the description of Cartier divisors on Schubert varieties [B, Proposition 2.2.8]. When applied to X_w from $(*)$ and L_λ these propositions immediately yield the following

Lemma 3.3. *Let $w = (s_i \ldots s_1)(s_{n-j+1} \ldots s_1) \ldots (s_{n-1} \ldots s_1)$ where $i + j \leq n$. Then the Picard group of X_w is spanned by the classes of X_{ws} where s runs through transpositions s_1, $s_2 \ldots$, s_{j-1}; $(j\ j+1)$, $(j\ j+2)$,..., $(j\ i+j)$ and $(j-1\ i+j+1)$, $(j-1\ i+j+2)$,..., $(j-1\ n)$. In particular,*

$$L_\lambda|_{X_w} = \bigotimes_{l=1}^{j-1} \mathcal{O}(X_{ws_l})^{\lambda_l - \lambda_{l+1}} \otimes \bigotimes_{l=1}^{i} \mathcal{O}(X_{w(j\ l+j)})^{\lambda_j - \lambda_{l+j}} \otimes$$

$$\otimes \bigotimes_{l=i+j+1}^{n} \mathcal{O}(X_{w(j-1\ l)})^{\lambda_{j-1} - \lambda_l}.$$

Remark 3.4. Lemma 3.3 implies the following important property of the decomposition $\underline{w_0}$. For every $k \leq d$, the Schubert subvariety X_{w_k} is a Cartier divisor on $X_{w_{k-1}}$. This property is used in the proof below. It would be interesting to find decompositions with this property for other reductive groups (decomposition (Sp) for Sp_n does not have this property).

Moreover, it is easy to check that all X_{w_k} are smooth by [M, Theorem 3.7.5] but this is not used in the proof.

3.2. Proof of Theorem 2.1

We will prove by induction the following more general statement. Put $Y_k := w_0 w_k^{-1} X_{w_k}$, and let v_k be the restriction of the valuation v to $\mathbb{C}(Y_k) \simeq \mathbb{C}(y_{k+1}, \ldots, y_d)$ (see Remark 2.5). We will also use an alternative labeling of coordinates in $\mathbb{R}^d$, namely, $(a_1, a_2, \ldots, a_d) = (u^1_{n-1}; u^2_{n-2}, u^1_{n-2}; \ldots; u^{n-1}_1, u^{n-2}_1, \ldots, u^1_1)$. Let $F_k(\lambda)$ be the face of $FFLV(\lambda)$ given by equations $u^l_m = 0$ for all pairs (l, m) such that either $m > j$, or $m = j$ and $l \geq i$. Here k and (i, j) are related via the above identification of coordinates a_k and u^i_j, i.e., $a_k = u^i_j$.

Theorem 3.5. *The Newton–Okounkov convex body $\Delta_{v_k}(Y_k, L_\lambda|_{Y_k})$ coincides with the face $F_k(\lambda)$.*

In particular, this theorem reduces to Theorem 2.1 when $k = 0$ (we put $F_0(\lambda) = FFLV(\lambda)$). The main idea of the proof is to identify the slices of $\Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}})$ by hyperplanes $\{a_k = \mathrm{const}\}$ with $F_k(\mu)$ for suitable μ. We will need a convex-geometric lemma for slices of $F_{k-1}(\lambda)$ and a similar algebro-geometric lemma for $\Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}})$.

Lemma 3.6. *There exists a path of dominant weights $\mu(t)$ such that*

$$(t - \lambda_{i+j})e_k + F_k(\mu(t)) = F_{k-1}(\lambda) \cap \{a_k = t - \lambda_{i+j}\}.$$

for all $t \in [\lambda_{i+j}, \lambda_j]$. Here e_k denotes the k-th basis vector in $\mathbb{R}^d$. In particular,

$$F_{k-1}(\lambda) = \mathrm{conv}\{(t - \lambda_{i+j})e_k + F_k(\mu(t)) \mid \lambda_{i+j} \le t \le \lambda_j\}.$$

Proof. Define $\mu(t) = (\mu_1(t), \dots, \mu_n(t))$ as follows

$$\mu_l(t) = \begin{cases} \max\{\lambda_l, t\} & \text{if } j < l \le i + j \\ \lambda_l & \text{otherwise} \end{cases}$$

In particular, $\lambda = \mu(\lambda_{i+j})$, and every $\mu_l(t)$ is a piecewise linear concave function of t. The lemma now follows immediately from the definitions of $F_k(\lambda)$ and $FFLV(\lambda)$.

In particular, $F_{k-1}(\lambda)$ fibers over the segment $[0, \lambda_j - \lambda_{i+j}]$, and the fiber polytope is analogous to $F_k(\lambda)$ for strictly dominant λ.

Lemma 3.7. *Take $\mu(t)$ as in the proof of Lemma 3.6. Then*

$$(t - \lambda_{i+j})e_k + \Delta_{v_k}(Y_k, L_{\mu(t)}|_{Y_k}) \subset \Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}}) \cap \{a_k = t - \lambda_{i+j}\}$$

for all integer $t \in [\lambda_{i+j}, \lambda_j]$. In particular,

$$\mathrm{conv}\{(t - \lambda_{i+j})e_k + \Delta_{v_k}(Y_k, L_{\mu(t)}|_{Y_k}) \mid \lambda_{i+j} \le t \le \lambda_j,\ t \in \mathbb{Z}\} \subset \Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}}).$$

Proof. By definition, Y_k and Y_{k-1} are translates of the Schubert varieties X_{w_k} and $X_{w_{k-1}}$, respectively, where $w_k = (s_{i-1} \dots s_1)(s_{n-j+1} \dots s_1) \dots (s_{n-1} \dots s_1)$ and $w_{k-1} = s_i w_k$. Put $\tau = t - \lambda_{i+j}$. It is easy to check using Lemma 3.3 that

$$L_\lambda|_{Y_{k-1}} \otimes \mathcal{O}(-\tau Y_k) = L_{\mu(t)}|_{Y_{k-1}} \otimes \mathcal{O}(\tau(s_i Y_k - Y_k)) \otimes E(\tau)$$

for an effective Cartier divisor $E(\tau)$ on Y_{k-1}. Indeed, $E(\tau) = L_{(\lambda-\mu(t))}|_{Y_{k-1}} \otimes \mathcal{O}(-\tau s_i Y_k)$ is a translate of the following divisor on $X_{w_{k-1}}$:

$$\bigotimes_{l=1}^{i-1} \mathcal{O}(X_{w(j\ l+j)})^{\max\{0, t-\lambda_{l+j}\}}.$$

Note that $\Delta_{v_{k-1}}(Y_{k-1}, L_{\mu(t)}|_{Y_{k-1}} \otimes \mathcal{O}(\tau(s_i Y_k - Y_k))) = \tau e_k + \Delta_{v_{k-1}}(Y_{k-1}, L_{\mu(t)}|_{Y_{k-1}})$ since $s_i Y_k - Y_k$ is the divisor of the rational function y_k. Applying Lemma 3.1 to Y_{k-1}, $L_{\mu(t)}|_{Y_{k-1}} \otimes \mathcal{O}(\tau(s_i Y_k - Y_k))$ and $E(\tau)$ we get

$$\tau e_k + \Delta_{v_{k-1}}(Y_{k-1}, L_{\mu(t)}|_{Y_{k-1}}) \subset \Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}} \otimes \mathcal{O}(-\tau Y_k)).$$

Intersecting both sides with the hyperplane $\{a_k = \tau\}$ yields

$$\tau e_k + \Delta_{v_{k-1}}(Y_{k-1}, L_{\mu(t)}|_{Y_{k-1}}) \cap \{a_k = 0\} \subset \Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}} \otimes \mathcal{O}(-\tau Y_k)) \cap \{a_k = \tau\}.$$

Since $L_{\mu(t)}$ is semiample we can apply Lemma 3.2 and get that

$$\Delta_{v_k}(Y_k, L_{\mu(t)}|_{Y_k}) = \Delta_{v_{k-1}}(Y_{k-1}, L_{\mu(t)}|_{Y_{k-1}}) \cap \{a_k = 0\}.$$

It follows that

$$\tau e_k + \Delta_{v_k}(Y_k, L_{\mu(t)}|_{Y_k}) \subset \Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}} \otimes \mathcal{O}(-\tau Y_k)) \cap \{a_k = \tau\}.$$

It remains to note that $\Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}} \otimes \mathcal{O}(-\tau Y_k)) \subset \Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}})$ by Lemma 3.1.

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. Let us first prove that $F_k(\lambda) \subset \Delta_{v_k}(Y_k, L_\lambda|_{Y_k})$ for all dominant λ by backward induction on k. For $k = d$, we have that both convex bodies coincide with the origin in $\mathbb{R}^d$. Suppose the inclusion holds for k. We now prove it for $k-1$. By Lemma 3.6

$$F_{k-1}(\lambda) = \mathrm{conv}\{(t - \lambda_{i+j})e_k + F_k(\mu(t)) \mid \lambda_{i+j} \le t \le \lambda_j\}.$$

Moreover, when taking the convex hull it is enough to consider only integer values of t, since $\mu(t)$ is linear at all non-integer points. Using the induction hypothesis $F_k(\mu(t)) \subset \Delta_{v_k}(Y_k, L_{\mu(t)}|_{Y_k})$ we get that

$$F_{k-1}(\lambda) \subset \mathrm{conv}\{(t - \lambda_{i+j})e_k + \Delta_{v_k}(Y_k, L_{\mu(t)}|_{Y_k}) \mid \lambda_{i+j} \le t \le \lambda_j,\ t \in \mathbb{Z}\}.$$

Hence, $F_{k-1}(\lambda) \subset \Delta_{v_{k-1}}(Y_{k-1}, L_\lambda|_{Y_{k-1}})$ by Lemma 3.7.

Finally, for $k = 0$ we get $F_0(\lambda) \subset \Delta_v(GL_n/B, L_\lambda)$. Since both convex bodies have the same volume they must coincide. Here we use that by Theorem 4.3 the volume of $F_0(\lambda) = FFLV(\lambda)$ coincides with the volume of the Gelfand–Zetlin polytope $GZ(\lambda)$. Hence, inclusions $F_k(\lambda) \subset \Delta_{v_k}(Y_k, L_\lambda|_{Y_k})$ are equalities for all k.

Remark 3.8. Results of Section 4 (see Theorem 4.3 and Remark 4.1) imply that the number of integer points in $F_k(\lambda)$ (and hence, in the Newton–Okounkov polytope $\Delta_{v_k}(Y_k, L_\lambda|_{Y_k})$) is equal to the dimension of the Demazure module $H^0(Y_k, L_\lambda|_{Y_k})$ for all $k = 0, \dots, d$ and dominant λ.
To illustrate the proof of Theorem 3.5 consider the simplest meaningful example.

Example 3.9. Let $k = d-1$, i.e., $w_k = s_1$ and $w_{k-1} = s_2 s_1$. Then $Y_{k-1} = \hat{\mathbb{P}}^2$ is the blow up of $\mathbb{P}^2$ at one point, and $Y_k = \mathbb{P}^1$ is embedded into Y_{k-1} as one of the fibers of the $\mathbb{P}^1$-bundle $\hat{\mathbb{P}}^2 \to \mathbb{P}^1$. The Picard group of $\hat{\mathbb{P}}^2$ is spanned by $\mathcal{O}(Y_k)$ and $\mathcal{O}(E)$ where $E \subset \hat{\mathbb{P}}^2$ is the exceptional divisor. Note that $\mathcal{O}(E)^a \otimes \mathcal{O}(Y_k)^b$ is semiample iff $0 \le a \le b$. We have

$$L_\lambda|_{Y_{k-1}} = \mathcal{O}(E)^{\lambda_1 - \lambda_2} \otimes \mathcal{O}(Y_k)^{\lambda_1 - \lambda_3}.$$

Hence, the line bundle $L_\lambda|_{Y_{k-1}} \otimes \mathcal{O}(-(t - \lambda_3)Y_k))$ is no longer semiample if $\lambda_2 < t \le \lambda_1$. However, it has the same global sections (modulo multiplication by $y_k^{t-\lambda_3}$) as the semiample bundle $L_{\mu(t)} = \mathcal{O}(E)^{\lambda_1 - t} \otimes \mathcal{O}(Y_k)^{\lambda_1 - t}$. Hence, $L_{\mu(t)}$ can be used instead of $L_\lambda|_{Y_{k-1}} \otimes \mathcal{O}(-(t - \lambda_3)Y_k))$ when computing $\Delta_{v_{k-1}}(L_\lambda|_{Y_{k-1}}, Y_{k-1})$. Figure 2 shows the Newton–Okounkov polygons of $L_\lambda|_{Y_{k-1}}$ (trapezoid) and $L_{\mu(t)}|_{Y_{k-1}}$ (triangle), which are just Newton polygons since Y_{k-1} is toric.

4. Comparison of Gelfand–Zetlin polytopes and Feigin–Fourier–Littelmann–Vinberg polytopes

We start with an elementary construction of polytopes fibered over a segment. Then we apply this construction to get the Gelfand–Zetlin and Feigin–Fourier–Littelmann–Vinberg polytopes in a uniform way.

4.1. Construction with fiber polytope

Let $P \subset \mathbb{R}^l$ be a convex polytope. The set of linear functionals, whose restrictions to P attain their maximal values at a face $F \subset P$, form a cone C_F; the *normal fan* of P is defined as the set of cones C_F corresponding to all faces $F \subseteq Q$. We say that a polytope $Q \subset \mathbb{R}^l$ is *subordinate* to P if the normal fan of P is a subdivision of the normal fan of Q. Note that the set of all polytopes subordinate to P forms a semigroup under the Minkowski sum. Denote this semigroup by S_P.

Let $\mu(t)$ be a piecewise-linear continuous function from a segment $I \subset \mathbb{R}$ to S_P. We say that $\mu(t)$ is *convex* if

$$\frac{\mu(t_1) + \mu(t_2)}{2} \subset \mu\left(\frac{t_1 + t_2}{2}\right)$$

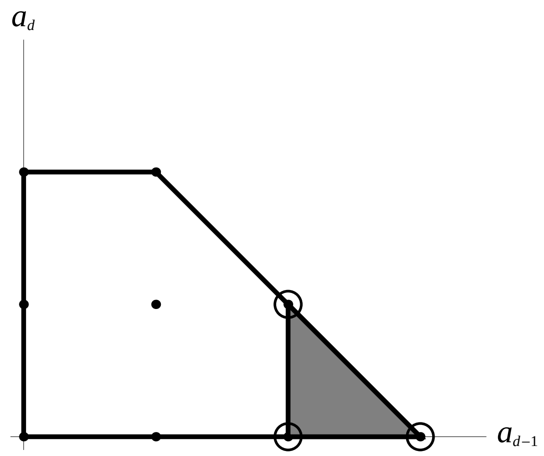

FIGURE 2. Newton polygons of $L_\lambda|_{Y_{d-2}}$ and $L_{\mu(t)}|_{Y_{d-2}}$ for $d = 3$, $\lambda = (3, 1, 0)$ and $t = 2$.

for all $t_1, t_2 \in I$. In other words, the set

$$P_\mu := \bigcup_{t\in I} \mu(t) \times \{t\} \subset \mathbb{R}^l \times \mathbb{R} = \mathbb{R}^{l+1}$$

is a convex polytope. In this case, P_μ fibers over I and the fiber polytope is subordinate to P.

Suppose now that $\mu'(t)$ is a convex function from I to S_Q for a convex polytope $Q \subset \mathbb{R}^l$. If the polytopes $\mu(t)$ and $\mu'(t)$ have the same Ehrhart polynomials for all $t \in I$ then obviously so do P_μ and $P_{\mu'}$. The simplest example is when $P = Q$ and $\mu'(t)$ is a parallel translate of $\mu(t)$. In this case, P_μ and $P_{\mu'}$ also have the same fiber polytope but might be combinatorially different even for quite simple $\mu(t)$ and $\mu'(t)$ (see Example 4.4).

4.2. $GZ(\lambda)$ **vs** $FFLV(\lambda)$

We now show that both $GZ(\lambda)$ and $FFLV(\lambda)$ can be obtained inductively from a point using the above construction. Recall that the Gelfand–Zetlin polytope $GZ(\lambda) \subset \mathbb{R}^d$ is defined by the following inequalities

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \ldots & & & & \lambda_n \\
& z_1^1 & & z_2^1 & & \ldots & & & & z_{n-1}^1 & \\
& & z_1^2 & & \ldots & & & & z_{n-2}^2 & & \\
& & & \ddots & & \ddots & & & & & \\
& & & & z_1^{n-2} & & z_2^{n-2} & & & & \\
& & & & & z_1^{n-1} & & & & &
\end{array}$$

where the notation

$$\begin{array}{ccc} a & & b \\ & c & \end{array}$$

means $a \geq c \geq b$. Let $G_k(\lambda)$ be the face of the Gelfand–Zetlin polytope $GZ(\lambda)$ given by the equations $z_m^l = z_{m+1}^{l-1}$ for all pairs (l, m) such that either $m > j$, or $m = j$ and $l \geq i$ (we put $z_m^0 = \lambda_m$).

Remark 4.1. In [Ki, Theorem 3.4], there is an inductive construction of the Gelfand–Zetlin polytope via convex geometric Demazure operators. The flag of faces

$$G_d(\lambda) \subset G_{d-1}(\lambda) \subset G_{d-2}(\lambda) \subset \ldots \subset G_1(\lambda) \subset GZ(\lambda) =: G_0(\lambda).$$

is exactly the flag used in this construction. In particular, by [Ki, Corollary 4.5] the number of integer points in G_k is equal to the dimension of the Demazure module $H^0(Y_k, L_\lambda|_{Y_k})$ for all $k = 0, \ldots, d$ and dominant λ.

Lemma 4.2. *Take $\mu(t)$ as in the proof of Lemma 3.7. There exists a path $z(t) \in \mathbb{R}^d$ such that*

$$G_{k-1}(\lambda) \cap \{z^i_j = t\} = z(t) + G_k(\mu(t))$$

for all integer $t \in [\lambda_{i+j}, \lambda_j]$. In particular,

$$G_{k-1}(\lambda) = \mathrm{conv}\{z(t) + G_k(\mu(t)) \mid \lambda_{i+j} \le t \le \lambda_j\}.$$

Proof. Define the coordinates $z^l_m(t)$ of $z(t) \in \mathbb{R}^d$ as follows:

$$z^l_m(t) = \begin{cases} (t - \lambda_{i+j}) & \text{if } m > j, l + m = i + j,\ \lambda_{i+j} \le t \\ (t - \lambda_{i+j-1}) & \text{if } m > j, l + m = i + j - 1,\ \lambda_{i+j-1} \le t \\ \vdots & \vdots \\ (t - \lambda_{j+2}) & \text{if } m > j, l + m = j + 2,\ \lambda_{j+2} \le t \\ 0 & \text{otherwise} \end{cases}.$$

In particular, $z(t) = 0$ if $i = 1$. The statement of the lemma now follows by direct calculation from the definition of $GZ(\lambda)$ and $G_k(\lambda)$.

Lemmas 3.6 and 4.2 together with the backward induction on k immediately yield an elementary proof of the following theorem.

Theorem 4.3. *Polytopes $F_k(\lambda)$ and $G_k(\lambda)$ have the same Ehrhart polynomial for all $k = 0,\ldots,$ d. In particular, Gelfand–Zetlin polytope $GZ(\lambda)$ and Feigin–Fourier–Littelmann–Vinberg polytope $FFLV(\lambda)$ have the same Ehrhart polynomial.*

The last statement of the theorem also follows from [FFL]. The first elementary proof of this statement was given in [ABS] using a different approach.

Lemmas 3.6 and 4.2 imply that both $FFLV(\lambda)$ and $GZ(\lambda)$ can be obtained inductively from a point by iterating the construction of Section 4.1. Note that both $F_{k-1}(\lambda)$ and $G_{k-1}(\lambda)$ fiber over a segment of length $\lambda_j - \lambda_{i+j}$, and fibers are equal (up to a parallel translation) to $F_k(\mu(t))$ and $G_k(\mu(t))$, respectively, for the same piecewise linear function $\mu(t)$ on the segment. The only difference between these two cases is the presence of the shift vector $z(t)$ in the second case.

Example 4.4. cf. [Fo] For $n = 3$, $k = 0, \ldots, 3$, and $n = 4$, $k = 2, \ldots, 6$, there exists a unimodular change of coordinates that maps F_k to G_k. Let $n = 4$, and $k = 1$. Then F_k provides the minimal example when F_k is not combinatorially equivalent to G_k.

We now illustrate how to obtain the inequalities defining F_1 from those of F_2 using Lemma 3.6 or equivalently the construction of Section 4.1 (and not the definition of F_1). For $k = 2$, we have $i = j = 2$, and

$$\mu(t) = \begin{cases} (\lambda_1, \lambda_2, \lambda_3, t) & \text{if } \lambda_4 \le t \le \lambda_3 \\ (\lambda_1, \lambda_2, \ \ t, t) & \text{if } \lambda_3 \le t \le \lambda_2 \end{cases}.$$

By Example 2.2 the inequalities defining F_2 are

$$0 \le u_1^1 \le \lambda_1 - \lambda_2; \quad 0 \le u_2^1 \le \lambda_2 - \lambda_3; \quad 0 \le u_1^2,\ u_1^3;$$

$$u_1^1 + u_1^2 + u_2^1 \le \lambda_1 - \lambda_3; \quad u_1^1 + u_1^2 + u_1^3 \le \lambda_1 - \lambda_4.$$

Put $u_2^2 := t - \lambda_4$. Using the last statement of Lemma 3.6 as a definition of F_1, we get that F_1 is defined by inequalities:

$$0 \le u_1^1 \le \lambda_1 - \lambda_2; \quad 0 \le u_2^1 \le \lambda_2 - \mu_3(u_2^2 + \lambda_4); \quad 0 \le u_1^2,\ u_1^3;$$

$$u_1^1 + u_1^2 + u_2^1 \le \lambda_1 - \mu_3(u_2^2 + \lambda_4); \quad u_1^1 + u_1^2 + u_1^3 \le \lambda_1 - (u_2^2 + \lambda_4);$$
$$0 \le u_2^2 \le \lambda_2 - \lambda_4.$$

Using that $\mu_3(t) = \max\{\lambda_3, t\}$ and eliminating redundant inequalities we get

$$0 \le u_1^1 \le \lambda_1 - \lambda_2; \quad 0 \le u_2^1 \le \lambda_2 - \lambda_3; \quad u_2^1 + u_2^2 \le \lambda_2 - \lambda_4; \quad 0 \le u_1^2,\ u_1^3,\ u_2^2;$$

$$u_1^1 + u_1^2 + u_2^1 \le \lambda_1 - \lambda_3; \quad u_1^1 + u_1^2 + u_2^1 + u_2^2 \le \lambda_1 - \lambda_4;$$
$$u_1^1 + u_1^2 + u_1^3 + u_2^2 \le \lambda_1 - \lambda_4.$$

Similarly, one can restore G_1 from G_2 and check that there are only 10 inequalities for G_1.

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