arXiv:1512.00968v1 [math.DG] 3 Dec 2015

Compact pluricanonical manifolds are Vaisman

Liviu Ornea[1], and Misha Verbitsky[2]

Abstract

A locally conformally Kähler manifold is a Hermitian manifold (M, I, ω) satisfying $d\omega = \theta \wedge \omega$, where θ is a closed 1-form, called **the Lee form** of M. It is called **pluricanonical** if $\nabla \theta$ is of Hodge type $(2,0) + (0,2)$, where ∇ is the Levi-Civita connection, and **Vaisman** if $\nabla \theta = 0$. We show that a compact LCK manifold is pluricanonical if and only if the Lee form has constant length and the Kähler form of its covering admits an automorphic potential. Using a degenerate Monge-Ampère equation and the classification of surfaces of Kähler rank one, due to Brunella, Chiose and Toma, we show that any pluricanonical metric on a compact manifold is Vaisman. Several *errata* to our previous work are given in the last Section.

Contents

1 Introduction

1.1 LCK manifolds

Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M \geqslant 2$. It is called **locally conformally Kähler** (LCK) if it admits a Hermitian metric g whose fundamental 2-form $\omega(\cdot, \cdot) := g(\cdot, I\cdot)$ satisfies

$$d\omega = \theta \wedge \omega, \quad d\theta = 0, \tag{1.1}$$

for a certain closed 1-form θ called **the Lee form**.

Definition (1.1) is equivalent to the existence of a covering $\tilde{M}$ endowed with a Kähler metric Ω which is acted on by the deck group $\mathrm{Aut}_M(\tilde{M})$ by homotheties. Let

$$\chi : \mathrm{Aut}_M(\tilde{M}) \longrightarrow \mathbb{R}^{>0}, \quad \chi(\gamma) = \frac{\tau^*\Omega}{\Omega}, \tag{1.2}$$

be the group homomorphism which associates to a homothety its scale factor.

For definitions and examples, see [DO] and our more recent papers.

Three subclasses of LCK manifolds will be of interest to us.

An LCK manifold (M, ω, θ) is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection of g. Note that, unlike the LCK condition, which is conformally invariant (if g is LCK, then any $e^f \cdot g$ is still LCK), the Vaisman condition is not. The main example of Vaisman manifold is the diagonal Hopf manifold ([OV8]). The Vaisman compact complex surfaces are classified in [Be].

An LCK manifold is called **with potential** if it admits a Kähler covering on which the Kähler metric has global, positive and proper potential function which is acted on by homotheties by the deck group. Among the examples: Vaisman manifolds, but also non-Vaisman ones, such as the non-diagonal Hopf manifolds, [OV5].

The notion of pluricanonical LCK manifold, motivated by the theory of harmonic maps, was introduced by G. Kokarev in [Kok]. An LCK manifold (M, ω, θ) is called **pluricanonical** if the symmetric form $\nabla \theta$ is of type (2,0) plus (0,2), that is, $(\nabla \theta)^{1,1} = 0$. Obviously, all Vaisman manifolds are pluricanonical. In [OV6] we mentioned (without a proof) that compact pluricanonical manifolds are the same as LCK with potential. This claim is false: all compact pluricanonical manifolds are LCK with potential, but not all LCK manifolds with potential admit a pluricanonical metric.

This paper grew out from our effort to correct the above error (see Section 6) and to clarify the definitions and relations among these three subclasses of LCK manifolds.

In fact, as our main result here, we show that on compact complex manifolds, pluricanonical metrics are Vaisman. To this end, we prove (1) that compact pluricanonical manifolds admit LCK metrics with potential (generally different from the pluricanonical one), then we prove (2) that all compact LCK manifolds with potential do contain a finite quotient of a linear Hopf surface which, if the ambient manifold is not Vaisman, admits no Vaisman metric, and finally, we prove that (3) compact pluricanonical surfaces should be Vaisman, yielding a contradiction.

So, technically, what we proved is that on a *compact* Hermitian manifold, the equations $d\omega = \theta \wedge \omega$, $d\theta = 0$ and $(\nabla \theta)^{1,1} = 0$ imply $\nabla \theta = 0$. We don't know whether this could be proven directly, locally, using tensor computations. Our approach passed through LCK manifolds with potential and, *a posteriori*, clarified further their geometry.

1.2   LCK manifolds with potential

"LCK manifolds with potential" can be defined as LCK manifolds (M, ω, θ) equipped with a smooth function $\psi \in \mathscr{C}^{\infty}(M)$,

$$\omega = d_{\theta} d_{\theta}^c \psi, \tag{1.3}$$

where $d_{\theta}(x) = dx - \theta \wedge x$, $d_{\theta}^c = I d_{\theta} I^{-1}$, and the following properties are satisfied:

(i) $\psi > 0$;

(ii) the class $[\theta] \in H^1(M, \mathbb{R})$ is proportional to a rational one. $\tag{1.4}$

For more details and historical context of this definition, please see Subsection 2.1. The differential d_θ is identified with the de Rham differential with coefficients in a flat line bundle L called **the weight bundle**. In this context, ψ should be considered as a section of L. After passing to the smallest covering $\tilde{M} \xrightarrow{\pi} M$ where θ becomes exact, the pull-back bundle $\pi^* L$ can be trivialized by a parallel section. Then the equation (1.4) becomes $\tilde{\omega} = d\,d^c\tilde{\psi}$, where $\tilde{\omega}$ is a Kähler form on $\tilde{M}$, and $\tilde{\psi}$ the Kähler potential.

Since $\tilde{M} \xrightarrow{\pi} M$ is the smallest covering where θ becomes exact, its monodromy is equal to $\mathbb{Z}^k$, where k is the rank of the smallest rational subspace $V \subset H^1(M,\mathbb{R})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contain $[\theta]$. In particular, the condition (1.4) (ii) means precisely that $\tilde{M} \xrightarrow{\pi} M$ is a $\mathbb{Z}$-covering. This implies that the definition (1.3)-(1.4) is equivalent to the historical one (Definition 2.1).

In Theorem 2.2, we prove that the condition (1.4) (i) is in fact unnecessary: it automatically follows from (1.3).

However, the condition (1.4) (ii) is more complicated: there are examples of LCK manifolds satisfying (1.3) and not (1.4) (ii) (Subsection 2.4). Still, any complex manifold admitting an LCK metric (M,ω,θ) with potential ψ satisfying (1.3), admits an LCK metric satisfying (1.3)-(1.4) in any $\mathscr{C}^\infty$-neighbourhood of (ω,θ). Therefore the conditions (1.4) are not restrictive, and for most applications, unnecessary.

It makes sense to modify the notion of LCK manifold with potential to include the following notion (Subsection 2.4):

Definition 1.1: Let (M,ω,θ) be an LCK manifold, and $\psi \in \mathscr{C}^\infty(M)$ a function satisfying $d_\theta d^c_\theta \psi = \omega$. Denote by k the rank of the smallest rational subspace $V \subset H^1(M,\mathbb{Q})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contain $[\theta]$. Then ψ is called **proper potential** if $k = 1$ and **improper potential** if $k > 1$.

Remark 1.2: By Claim 2.3, the condition $k = 1$ is equal to the Kähler potential $\tilde{\psi} \in \mathscr{C}^\infty(\tilde{M})$ being proper in the usual sense (that is, having compact level sets). This explains the term.

1.3 Pluricanonical versus Vaisman: scheme of the proof

The main result of this paper is the following theorem.

Theorem 1.3: Let (M,ω,θ) be a compact pluricanonical LCK manifold. Then it is Vaisman.

    Proof: Theorem 5.13. ∎

Here we survey its proof and explain its key points.

We start from an observation which can be obtained by a straightforward tensorial calculation. Any pluricanonical manifold (M, ω, θ) satisfies

$$d\theta^c = \theta \wedge \theta^c - |\theta|^2 \omega,$$

and, moreover, $|\theta| = \mathrm{const}$, (see (3.3)). We then rescale the metric such that $|\theta| = 1$. Then the eigenvalues[1] of the (1,1)-form $d\theta^c$ are constant, all equal to 1 *but one which is* 0, and hence $\omega_0 := -d\theta^c$ is a semipositive $(1,1)$-form of constant rank $n - 1$.

From now on, we assume that M is a compact pluricanonical LCK manifold.

Let Σ be the zero eigenbundle of $d\theta^c$. Then Σ is independent from the choice of the pluricanonical metric on M. Indeed, suppose that two different pluricanonical metrics give exact semipositive forms ω_0 and ω_0'. Unless the corresponding zero eigenbundles coincide, the semi-Hermitian form $\omega_0 + \omega_0'$ would be strictly positive, which is impossible, because it is exact.

Now consider two pluricanonical LCK metrics (ω, θ) and (ω', θ') on M, with $\theta - \theta' = d\psi$. Then ψ is a solution of the degenerate Monge-Ampère equation

$$(\omega_0 + dd^c\psi)^n = 0, \quad \omega_0 + dd^c\psi \geqslant 0. \tag{1.5}$$

The standard argument from the theory of Monge-Ampère equations is used to show that the function ψ is constant on the leaves of Σ (Proposition 3.12).

If, in addition, (M, ω, θ) admits a Vaisman structure, one has a holomorphic vector field $X \in \mathscr{X}(M)$ tangent to Σ and acting by homotheties on its Kähler covering. Then Proposition 3.12 implies that X acts on (M, ω, θ) by isometries lifting to non-trivial homotheties on its Kähler covering. A theorem of Kamishima-Ornea ([KO]) then implies that (M, ω, θ) is Vaisman (see Proposition 5.10). This result can be stated as follows:

Proposition 1.4: (Proposition 5.10)
Let (M, ω, θ) be a compact pluricanonical LCK manifold admitting a Vaisman structure. Then (M, ω, θ) is also Vaisman.

This proposition is used to deduce our main result (Theorem 1.3) as follows. First, we prove Theorem 1.3 for surfaces. In this case, it follows from the classification theorem due to Chiose, Toma and Brunella.

---

[1]The eigenvalues of a Hermitian form η are the eigenvalues of the symmetric operator L_η defined by the equation $\eta(x, Iy) = g(Lx, y)$.

Theorem 1.5: Let M be a compact complex surface admitting an exact semipositive form of rank 1. Then M admits a Vaisman metric.

Proof: Section 4. ∎.

Comparing Theorem 1.5 and Proposition 1.4, we obtain Theorem 1.3 for surfaces.

To prove it in general situation, a version of a theorem of Ma. Kato is used. In [Kat], Kato studied subvarieties in a general Hopf manifold H, which is defined as a quotient of $\mathbb{C}^n \backslash 0$ by a holomorphic contraction A. Note that elsewhere in this paper, we consider only **linear Hopf manifolds**, for which A is a linear endomorphism. The (more general) manifolds considered by Kato are also embeddable into linear Hopf manifolds ([OV5]), hence admit an LCK metric with potential ([OV8]).

Kato proves that H admits a sequence of Hopf submanifolds $H \supset H_1 \supset H_2 \supset \cdots \supset H_n = \varnothing$ with $\dim H_i = \dim H - i$.

For LCK manifolds with potential, a similar flag exists. Moreover, the following useful result can be proven.

Theorem 1.6: Let M be an LCK manifold with potential. Then M contains a finite quotient of a linear Hopf surface H. Moreover, if M admits no Vaisman metric, then H can be chosen to admit no Vaisman metric.

Proof: Theorem 4.9. ∎

Theorem 1.6 and Theorem 1.5 imply our main theorem easily. Indeed, any complex submanifold of a pluricanonical manifold is again pluricanonical (Proposition 5.12). Pluricanonical surfaces are Vaisman. Theorem 1.6 implies that any non-Vaisman pluricanonical manifold contains a surface which is not Vaisman, hence not pluricanonical: contradiction!

1.4   Some errors found

This paper is much influenced by Paul Gauduchon, who discovered an error in our result mentioned as obvious in [OV6]. In [OV6], we claimed erroneously that an LCK metric is pluricanonical if and only if it admits an LCK potential. This was obvious because (as we claimed) the equations for LCK with potential and for pluricanonical metric are the same. Unfortunately, a scalar multiplier was missing in our equation for the pluricanonical (see Subsection 3.2).

From an attempt to understand what is brought by the missing multiplier, this paper grew, and we found an even stronger result: any compact pluricanonical manifold is Vaisman.

However, during our work trying to plug a seemingly harmless mistake, we discovered a much more offensive error, which has proliferated in a number of our papers.

In [OV1], we claimed that any Vaisman manifold admits a $\mathbb{Z}$-covering which is Kähler. This is true for locally conformally hyperkähler manifolds, as shown in [Ve1]. However, this result is false for more general Vaisman manifolds, such as a Kodaira surface (Theorem 6.1).

It is easiest to state this problem and its solution using the notion of "LCK rank" (Definition 2.4), defined in [GOPP] and studied in [PV]. Briefly, LCK rank is the smallest r such that there exists a $\mathbb{Z}^r$-covering $\tilde{M}$ of M such that the pullback of the LCK metric is conformally equivalent to a Kähler metric on $\tilde{M}$.

It turns on that the Kähler rank of a Vaisman manifold can be any number between 1 and $b_1(M)$ (Theorem 6.1). Moreover, for each r, the set of all Vaisman metrics of Kähler rank r is dense in the space of all Vaisman metrics (say, with $\mathscr{C}^\infty$-topology).

It is disappointing to us (and even somewhat alarming) that nobody has discovered this important error earlier.

However, not much is lost, because the metrics which satisfy the Structure Theorem of [OV1] are dense in the space of all LCK metrics, hence all results of complex analytic nature remain true. To make the remaining ones correct, we need to add "Vaisman manifold of LCK rank 1" or "Vaisman manifold with proper potential" (Subsection 2.4) to the set of assumptions whenever [OV1] is used.

Still, we want to offer our apologies to the mathematical community for managing to mislead our colleagues for such a long time.

For more details about our error and an explanation where the arguments of [OV1] failed please see Subsection 6.2.

2 LCK manifolds: properness and positivity of the potential

2.1 LCK manifolds with potential: historical definition

When the notion of LCK manifold with potential was introduced in [OV5], we assumed properness of the potential. Later, it was "proven" that the potential is always proper ([OV7]). Unfortunately, the proof was false (see the *Errata* to this paper, Section 6). In view of this error and other results in Section 6, it

makes sense to generalize the notion of LCK manifold with potential to include the manifolds with LCK rank > 1. For the old notion of LCK with potential we should attach "proper" to signify that the potential is a proper function on the minimal Kähler covering.

Definition 2.1: ([OV5]) An **LCK manifold with proper potential** is a manifold which admits a Kähler covering $(\tilde{M}, \tilde{\omega})$ and a smooth function $\varphi : \tilde{M} \to \mathbb{R}^{>0}$ (the **LCK potential**) satisfying the following conditions:

(i) φ is proper, *i.e.* its level sets are compact;

(ii) The deck transform group acts on φ by multiplication with the character χ (see (1.2)): $\tau^*(\varphi) = \chi(\tau)\varphi$, where $\tau \in \mathrm{Aut}_M(\tilde{M})$ is any deck transform map[1].

(iii) φ is a Kähler potential, *i.e.* $dd^c\varphi = \tilde{\omega}$.

We are now able to show that an automorphic global potential is always positive, and that once an automorphic global potential exists on a Kähler covering, then another one, which is proper, exists too. The precise statement is the following:

Theorem 2.2: Let M be an LCK manifold admitting a Kähler covering $(\tilde{M}, \tilde{\omega})$ and an automorphic function $\varphi : \tilde{M} \to \mathbb{R}$ satisfying $dd^c\varphi = \tilde{\omega}$. Then φ is strictly positive.

Moreover, M admits a covering, possibly different from $\tilde{M}$, and an automorphic potential on it which is positive and proper, hence satisfies all conditions of Definition 2.1.

We prove Theorem 2.2 in the next two subsections.

2.2 Properness of the LCK potential

In [OV5], it was also shown that the properness condition is equivalent to the following condition on the deck transform group of $\tilde{M}$. Recall that a group is **virtually cyclic** if it contains $\mathbb{Z}$ as a finite index subgroup. The following claim is clear.

Claim 2.3: Let M be a compact manifold, $\tilde{M}$ a covering, and $\varphi : \tilde{M} \to \mathbb{R}^{>0}$ an automorphic function, that is, a function which satisfies $\gamma^*\varphi = c_\gamma\varphi$ for any deck

---

[1] In general, differential forms $\eta \in \Lambda^\bullet \tilde{M}$ which satisfy $\tau^*\eta = \chi(\tau)\eta$ are called **automorphic**. In particular, so is the Kähler form on $\tilde{M}$.

transform map γ, where c_γ is constant. Then φ is proper if and only if the deck transform group of $\tilde{M}$ is virtually cyclic. ∎

Definition 2.4: Let (M, ω, θ) be an LCK manifold. Define the **LCK rank** as the dimension of the smallest rational subspace $V \subset H^1(M, \mathbb{Q})$ such that the Lee class $[\theta]$ lies in $V \otimes_{\mathbb{Q}} \mathbb{R}$.

Remark 2.5: The character $\chi : \mathrm{Aut}_M(\tilde{M}) \longrightarrow \mathbb{R}^{>0}$ can be defined on any LCK manifold, because the Kähler form $\tilde{\omega}$ is automorphic by definition: $\tau^*(\tilde{\omega}) = \chi(\tau)\tilde{\omega}$. Then one can see that the LCK rank as defined above coincides with the rank of the image of $\chi : \mathrm{Aut}_M(\tilde{M}) \longrightarrow \mathbb{R}^{>0}$ which is also called **the weight monodromy group** of the LCK manifold. See also [GOPP] for another interpretation of the LCK rank and see [PV] for examples on non-Vaisman compact LCK manifolds with Kähler rank greater than 1. Clearly, LCK rank 0 corresponds to globally conformally Kähler structures.

From Claim 2.3 below it follows that condition (i) is equivalent to M being of LCK rank 1.

In [OV4], we managed to get rid of the need to take the covering in Definition 2.1, by using the Morse-Novikov (twisted) differential $d_\theta := d - \theta \wedge \cdot$, where $\theta \wedge \cdot(x) = \theta \wedge x$, and θ is the Lee form. In [OV4] the definition of LCK manifold with potential was restated equivalently as follows.

Definition 2.6: Let (M, ω, θ) be an LCK manifold of LCK rank 1. Then M is called **LCK manifold with potential** if there exists a positive function $\varphi_0 \in \mathscr{C}^\infty(M)$ satisfying $d_\theta d_\theta^c(\varphi_0) = \omega$, where $d_\theta^c = I d_\theta I^{-1}$.

Claim 2.7: Definition 2.1 is equivalent to Definition 2.6.

Proof: To see that Definition 2.1 and Definition 2.6 are equivalent, consider the smallest covering $\pi : \tilde{M} \longrightarrow M$ such that $\pi^*\theta$ is exact, and take a function ψ satisfying $d\psi = \pi^*\theta$. Since $\pi^*\theta$ is invariant under the deck transform group Γ, for each $\gamma \in \Gamma$ one has $\gamma^*\psi = \psi + c(\gamma)$, where $c(\gamma)$ is a constant. Consider the multiplicative character $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ given by $\chi(\gamma) = e^{c(\gamma)}$. Let $\Lambda_\chi^*(M)$ denote the space of automorphic forms on $\tilde{M}$ which satisfy $\gamma^*\eta = \chi(\gamma)\eta$. The map $\Lambda^*(M) \xrightarrow{\Psi} \Lambda_\chi^*(M)$ mapping η to $e^{-\psi}\pi^*\eta$ makes the following diagram commu-

tative:

$$
\begin{array}{ccc}
\Lambda^*(M) & \xrightarrow{\ \Psi\ } & \Lambda^*_\chi(M) \\[2pt]
d_\theta \downarrow & & d \downarrow \\[2pt]
\Lambda^*(M) & \xrightarrow{\ \Psi\ } & \Lambda^*_\chi(M)
\end{array}
$$

Then Ψ maps a "potential" φ_0 in the sense of Definition 2.6 to a potential ψ in the sense of Definition 2.1 and vice versa. Properness of $\Psi(\varphi_0)$ is equivalent to Γ being virtually cyclic, as Claim 2.3 implies. The existence of a Kähler covering with virtually cyclic deck transform group is clearly equivalent to M having LCK rank 1. $\blacksquare$

We now show that automorphic potentials can be approximated by proper ones. The following argument is taken from [OV4].

Claim 2.8: Let (M,ω,θ) be an LCK manifold, and $\varphi \in \mathscr{C}^\infty(M)$ a function satisfying $d_\theta d_\theta^c(\varphi) = \omega$. Then M admits an LCK structure (ω',θ') of LCK rank 1, approximating (ω,θ) in $\mathscr{C}^\infty$-topology.

Proof: Replace θ by a form θ' with rational cohomology class $[\theta']$ in a sufficiently small $\mathscr{C}^\infty$-neighbourhood of θ, and let $\omega' := d_{\theta'} d_{\theta'}^c(\varphi)$. Then ω' approximates ω in $\mathscr{C}^\infty$-topology, hence for θ' sufficiently close to θ, the form ω' is positive. It is $d_{\theta'}$-closed, because $d_{\theta'}^2 = 0$, hence $0 = d_{\theta'}\omega' = d\omega' - \theta' \wedge \omega'$. This implies that (ω',θ') is an LCK structure. The Kähler rank of an LCK manifold is the dimension of the smallest rational subspace $W \subset H^1(M,\mathbb{Q})$ such that $W \otimes_\mathbb{Q} \mathbb{R}$ contains the cohomology class of the Lee form. Since $[\theta']$ is rational, (M,ω',θ') has LCK rank 1. $\blacksquare$

2.3   Positivity of automorphic potentials

Finally, we prove that automorphic potentials are necessarily positive:

Proposition 2.9: Let (M,ω,θ) be a compact LCK manifold, $\dim_\mathbb{C} M > 2$, and $\varphi \in \mathscr{C}^\infty(M)$ a function satisfying $d_\theta d_\theta^c(\varphi) = \omega$. Then φ is strictly positive.

Proof: From Claim 2.8 it is clear that it suffices to prove Proposition 2.9 assuming that (M,ω,θ) has LCK rank 1.

In this situation, the deck transform group of the smallest Kähler covering $(\tilde{M},\tilde{\omega})$ is $\mathbb{Z}$ and, therefore, the fundamental domains of the covering are compact. Denote by ψ the automorphic Kähler potential of $(\tilde{M},\tilde{\omega})$ (Claim 2.7).

Assume that the generator of Γ acts on ψ by multiplication with a constant $c > 1$. Then the covering $\tilde{M}$ can be written as

$$\tilde{M} = \bigcup_{x>0} \psi^{-1}([x, cx]) \cup \psi^{-1}(0) \cup \bigcup_{x<0} \psi^{-1}([cx, x]).$$

Since M has LCK rank 1, ψ descends to a continuous map from M to a circle, and therefore it is proper. Therefore, $\psi^{-1}(0)$ is a compact set on which $\mathbb{Z}$ cannot act freely. We conclude that $\psi^{-1}(0) = \emptyset$. But then $\tilde{M}$ would be disconnected unless $\bigcup_{x>0} \psi^{-1}([x, cx])$ or $\bigcup_{x<0} \psi^{-1}([cx, x])$ is empty, and hence ψ is strictly positive or strictly negative.

To show that the potential ψ cannot be strictly negative, we argue by contradiction.

Step 1. Consider the level set $S_t := \psi^{-1}(t)$, $t < 0$. Since ψ is plurisubharmonic, S_t is strictly pseudoconvex. Applying the Rossi-Andreotti-Siu theorem (cf. [R, Theorem 3, p. 245] and [AS, Proposition 3.2]) we find that S_t is the boundary of a compact Stein domain V_t with boundary, uniquely determined by the CR-structure on S_t.

Step 2. Let $\tilde{M}_{[t,t']}$ be V_t glued together with $\psi^{-1}([t, t'])$ over S_t. Since the boundary of a compact complex manifold $\tilde{M}_{[t,t']}$ is strictly pseudoconvex, this space is holomorphically convex. On the other hand, it contains no non-trivial compact complex subvarieties without boundary. Indeed, for such a subvariety Z, the restriction of ψ to $Z \cap \psi^{-1}([t, t'])$ must achieve a maximum, unless $Z \cap \psi^{-1}([t, t'])$ is empty. However, a plurisubharmonic function cannot achieve a maximum on a complex subvariety. Therefore, $Z \cap \psi^{-1}([t, t']) = \emptyset$, and Z is contained in V_t. This is impossible, because V_t is Stein. By Remmert's theorem, a holomorphically convex manifold without compact complex subvarieties is Stein. Then the union $\tilde{M}_t := \bigcup_{t'} \tilde{M}_{[t,t']}$ is Stein too, as a union of an increasing family of Stein varieties.

Since S_t is contained in $\psi^{-1}([t, 0[)$ for all $t' \in [t, 0[$, it is the boundary of a Stein subvariety within $\tilde{M}_t$. Therefore, $\tilde{M}_t$ can be obtained as $V_{t'}$ glued together with $\psi^{-1}([t', 0[)$. We have identified $\tilde{M}_t$ and $\tilde{M}_{t'}$. This implies, in particular, that $\tilde{M}_t$ contains $\bigcup_t \psi^{-1}([t, 0[) = \tilde{M}$, and the mapping class group of $\tilde{M}$ acts on $\tilde{M}_t$ holomorphically.

We extend ψ to $\tilde{M}_t$ by setting it to $-\infty$ on $\tilde{M}_t \backslash \tilde{M}$. This gives an automorphic plurisubharmonic function ψ on $\tilde{M}_t$.

We have included $\tilde{M}$ into a Stein manifold $\tilde{M}_t$, with the same monodromy action, and extended ψ to an automorphic plurisubharmonic function on $\tilde{M}_t$.

Step 3. Since $\tilde{M}_t$ is Stein, there exists a positive, smooth, plurisubharmonic function φ on $\tilde{M}_t$ Since $\tilde{M}_t$ is properly embedded to a bigger Stein domain, this function can be assumed to be bounded. We apply the standard technique for constructing regularized maxima of plurisubharmonic functions ([D2]; see also [OV2, Proposition 4.2]): for a very large $C \gg 0$, define

$$\xi := \max_{\varepsilon}(\psi, \tilde{\varphi} - C - \sup \tilde{\varphi})$$

thus obtaining a smooth plurisubharmonic function on $\tilde{M}_t$. Moreover, $\xi = \psi$ on an arbitrary big neighbourhood of the boundary including the level sets $S_t = \psi^{-1}(t)$ and $S_{ct} = \psi^{-1}(ct)$.

Now, V_t can be written as $V_t = \psi^{-1}(]-\infty, t[)$. Then V_t is a Stein subset of $\tilde{M}_t$ with boundary S_t. By Stokes theorem, we obtain:

$$\mathrm{Vol}_\xi(V_t) := \int_{V_t} (dd^c \xi)^n = \int_{\partial V_t} d^c \xi \wedge (dd^c \xi)^{n-1} = \int_{\partial V_t} d^c \psi \wedge (dd^c \psi)^{n-1} \quad (2.1)$$

Since the monodromy map τ maps V_t to V_{ct} multiplying ψ by λ, (2.1) gives $\mathrm{Vol}_\xi(V_t) = \lambda^n \mathrm{Vol}_\xi(V_{ct})$, with $\lambda, c > 1$. This is impossible, because $t < 0$ and V_{ct} is strictly included in V_t. ∎

2.4   LCK manifolds with proper and improper potential

It seems now that the equation $d_\theta d_\theta^c \psi = \omega$ is more fundamental than the notion of LCK manifold with (proper) potential. For most applications, this (more general) condition is already sufficient.

The relation between manifolds with $d_\theta d_\theta^c \psi = \omega$ and LCK with potential is similar to the relation between general Vaisman manifolds and quasiregular ones[2]. One could always deform an irregular Vaisman manifold to a quasiregular one, and quasiregular Vaisman manifolds are dense in the space of all Vaisman manifolds.

The notion of "LCK manifold with improper potential" is similar, in this regard, to the notion of irregular Vaisman or irregular Sasakian manifold[3], [BG].

Definition 2.10: Let (M, θ, ω) be an LCK manifold, and ψ a function (positive by Proposition 2.9) which satisfies $d_\theta d_\theta^c \psi = \omega$. Then (M, θ, ω) is called **a manifold with improper LCK potential** if its LCK rank is ≥ 2, and **a manifold with proper LCK potential** if it has LCK rank 1.

---

[2](Quasi-)Regularity and irregularity of a Vaisman manifold refers to the (quasi-)regularity and irregularity of the 2-dimensional canonical foliation generated by $\theta^\sharp$ and $I\theta^\sharp$.

[3]Here (ir)regularity refers to the 1-dimensional foliation generated by the Reeb field.

Remark 2.11: The expressions "proper potential" and "improper potential", when used for solutions of the equation $d_\theta d_\theta^c \psi = \omega$, as in the above Definition, do not refer to the properness of $\psi : M \longrightarrow \mathbb{R}$, which is always proper if M is compact.

Note that "LCK with potential" was previously used instead of "manifold with proper LCK potential"; now (in light of the discovery of Vaisman manifolds having improper potential, see Section 6) it makes sense to change the terminology by including improper potentials in the definition of LCK with potential.

Claim 2.8 can be rephrased as follows.

Proposition 2.12: Let $(M, \omega, \theta, \psi)$ be a compact LCK manifold with improper LCK potential. Then (ω, θ, ψ) can be approximated in the $\mathscr{C}^\infty$-topology by an LCK structure with proper LCK potential. ∎

Remark 2.13: We have just proven that existence of an LCK metric with improper LCK potential implies existence of a metric with proper LCK potential on the same manifold. The converse is clearly false: when $H^1(M, \mathbb{R})$ is 1-dimensional, any Lee class is proportional to an integral cohomology class, and any LCK structure has LCK rank 1, hence M admits no metrics with improper LCK potentials.

However, in all other situations improper potentials do exist.

Proposition 2.14: Let $(M, \omega, \theta, \psi)$ be an LCK manifold with potential, and suppose $b_1(M) > 1$. Then M admits an LCK metric $(M, \omega', \theta', \psi)$ with improper potential and arbitrary LCK rank between 2 and $b_1(M)$. Moreover, (ω', θ') can be chosen in arbitrary $\mathscr{C}^\infty$-neighbourhood of (ω, θ).

Proof: Choose a closed θ' in a sufficiently small neighbourhood of θ, and let V_θ be the smallest rational subspace of $H^1(M, \mathbb{R})$ such that $V_\theta \otimes_\mathbb{Q} \mathbb{R}$ contains θ. Since the choice of the cohomology class $[\theta']$ is arbitrary in a neighbourhood of $[\theta]$, the dimension of V_θ can be chosen in arbitrary way. Choosing θ' sufficiently close to θ, we can assume that the (1,1)-form $\omega' := d_{\theta'} d_{\theta'}^c(\psi)$ is positive definite. Then $(M, \omega', \theta', \psi)$ is an LCK manifold with improper potential and arbitrary LCK rank. ∎

3   Pluricanonical condition *versus* LCK with potential

3.1   Exterior derivative of the Lee form: preliminary computations

For a one-form η, we set $\eta^c := I\eta$ and $I\eta(\cdot) = \eta(I\cdot)$. On an LCK manifold, we then have $\theta^\sharp \lrcorner \omega = \theta^c$, where $\theta^\sharp$ is the dual vector field of θ.

Remark 3.1: Note the difference of sign with respect to other authors' conventions who put $\omega(\cdot,\cdot) = g(I\cdot,\cdot)$ and not $\omega(\cdot,\cdot) = g(\cdot,I\cdot)$, and define $I\eta(\cdot) = -\eta(I\cdot)$.

As ω is nondegenerate, similarly to the Kähler case, one has:

Lemma 3.2: On an LCK manifold (M, I, g) with $\dim_{\mathbb{C}} M \geqslant 2$, exterior multiplication with ω is injective.

It is known that the Levi-Civita connections of the local Kähler metrics glue to a global connection, here denoted D, which is almost complex ($DI = 0$) and satisfies $Dg = \theta \otimes g$ – and hence it is the Weyl connection of the couple (g, θ).

Using the well-known relation between the Levi-Civita connections of two conformal metrics, the LCK condition is equivalent ([DO]) with:

$$(\nabla_X I)Y = \frac{1}{2}\left(\theta(IY)X - \theta(Y)IX + g(X,Y)I\theta^\sharp - \omega(X,Y)\theta^\sharp\right) \tag{3.1}$$

where $\sharp$ refers to the g-raising of indices.

From this we can derive:

Lemma 3.3: On an LCK manifold, the exterior derivative of θ^c is:

$$d\theta^c(X,Y) = \left(-|\theta|^2\omega + \theta \wedge \theta^c\right)(X,Y) + \frac{1}{2}\left((\nabla_Y)(IX) - (\nabla_X\theta(IY))\right) \tag{3.2}$$

Proof: Using (3.1) we have

$$
\begin{aligned}
(\nabla_X\theta^c)Y &= \nabla_X\theta(IY) - \theta(I\nabla_X Y) = (\nabla_X\theta)(IY) + \theta(\nabla_X(IY)) - \theta(I\nabla_X Y) \\
&= (\nabla_X\theta)(IY) + \theta((\nabla_X I)Y) \\
&= (\nabla_X\theta)(IY) + \frac{1}{2}\left(\theta(IY)\theta(X) - \theta(Y)\theta(IX) - \omega(X,Y)|\theta|^2\right)
\end{aligned}
$$

Now (3.2) follows from $d\theta^c$ being the antisymmetrization of $\nabla\theta^c$. ∎

3.2   Pluricanonical manifolds and $(\nabla\theta)^{1,1} = 0$

Recall that a pluricanonical LCK manifold, as defined in [Kok], [KK], is an LCK manifold (M, ω, θ) satisfying $(\nabla\theta)^{1,1} = 0$, where ∇ is the Levi-Civita connection. This is equivalent with

$$(\nabla_{IX}\theta)(IY) + (\nabla_X\theta)(Y) = 0$$

which, changing Y into IY and using the symmetry of $\nabla\theta$ (recall that $d\theta = 0$) gives:

$$(\nabla_Y\theta)(IX) - (\nabla_X\theta)(IY) = 0$$

Together with (3.2) this gives (see [OV6] for a different proof) that the pluricanonical condition is equivalent to

$$d\theta^c = \theta \wedge \theta^c - |\theta|^2\omega \tag{3.3}$$

We now take the exterior derivative of the above (modulo itself and using $d\omega = \theta \wedge \omega$):

$$0 = -\theta \wedge d\theta^c - d|\theta|^2 \wedge \omega - |\theta|^2\theta \wedge \omega + = -d|\theta|^2 \wedge \omega$$

which, by Lemma 3.2, implies $|\theta| = \mathrm{const}$.

We may now rescale the pluricanonical metric such that, *from now on we assume* $|\theta| = 1$.

Corollary 3.4: Let (M, ω, θ) be a LCK manifold. Then it is pluricanonical if and only if

$$d\theta^c = \theta \wedge \theta^c - \omega. \tag{3.4}$$

and $|\theta| = 1$ ∎

Remark 3.5: Condition (3.4) is equivalent to (M, ω, θ) being "LCK with proper or improper potential" in the sense of Definition 1.1.

Remark 3.6: One can prove that the pluricanonical condition is also equivalent with $\mathrm{Lie}_{\theta^\sharp}\omega = 0$ ([G2]). See also similar computations in the recent [AD, Section 3].

In general, on an LCK manifold with potential ψ, the norm of the Lee form $d\psi$ is not constant. This, however, holds if the LCK metric is Gauduchon (see Proposition 3.9 below).

Definition 3.7: On a complex manifold of complex dimension n, a Hermitian metric whose Hermitian 2-form ω satisfies the equation $\partial\overline{\partial}\omega^{n-1} = 0$ is called **Gauduchon**.

Remark 3.8: On a compact Hermitian manifold, a Gauduchon metric exists in each conformal class and it is unique up to homothety. Moreover, it is characterized by the coclosedness of its Lee form. A Vaisman metric *is* a Gauduchon metric in its conformal class, [G1].

On the other hand, it was shown in [Kok] that a pluricanonical metric has coclosed Lee form and hence, if the manifold is compact, it is a Gauduchon metric in the given conformal class.

Proposition 3.9: Let (M,ω,θ) be a compact LCK manifold with a potential ψ (proper or improper; see Definition 1.1). Then the LCK form $\omega = \psi^{-1}dd^c\psi$ is Gauduchon if and only if $|\theta| = \text{const}$.

Proof: The Hermitian form ω is Gauduchon if and only if $dd^c\omega^{n-1} = 0$.

On the other hand, we compute $dd^c\omega^{n-1}$ using the equation (3.4) which is satisfied on an LCK manifold with automorphic potential. We obtain

$$dd^c\omega^{n-1} = (n-1)^2\omega^{n-1}\wedge\theta\wedge\theta^c + (n-1)\omega^{n-1}\wedge d\theta^c.$$

On the other hand,

$$\omega^{n-1}\wedge\theta\wedge\theta^c = \frac{1}{n}|\theta|^2\omega^n$$

and

$$d\theta^c\wedge\omega^{n-1} = -\omega\wedge\omega^{n-1} + \theta\wedge\theta^c\wedge\omega^{n-1} = \left(\frac{1}{n}|\theta|^2 - 1\right)\omega^n.$$

All in all we get:

$$dd^c\omega^{n-1} = \frac{(n-1)^2}{n}|\theta|^2\omega^n + (n-1)\left(\frac{1}{n}|\theta|^2 - 1\right)\omega^n = (n-1)\bigl(|\theta|^2 - 1\bigr)\omega^n.$$

Then $dd^c\omega^{n-1} = 0$ if and only if $|\theta| = 1$. This finishes the proof. ∎

We obtained the following corollary.

Corollary 3.10: Let M be a compact LCK manifold. Then the following are equivalent.

(i) M is pluricanonical.

(ii) M is LCK manifold with potential, and its LCK metric is Gauduchon.

Claim 3.11: Let (M, ω, θ) be a compact LCK manifold with potential. Consider the 2-form $\omega_0 := d\theta^c = \theta \wedge \theta^c - \omega$. Then ω_0 is semipositive if and only if (M, ω, θ) is pluricanonical.

Proof: On a pluricanonical manifold, the form ω_0 has one 0 eigenvalue and all other strictly positive. This can be seen by writing it in a diagonal basis which includes $\theta^\sharp$ and $I\theta^\sharp$ and taking into account that $|\theta| = 1$. In particular, $\omega_0^{\dim_{\mathbb{C}} M} = 0$. Note that ω_0 has the same eigenvalues on Vaisman manifolds, see [Ve1]. Conversely, if $\omega_0^n = 0$, then ω_0 has at least one 0 eigenvalue. $\blacksquare$

This simple observation has strong consequences and leads to a degenerate Monge-Ampère equation that we now discuss (compare also with [OV3], where a similar equation is considered).

Let θ be a closed 1-form such that $\omega_0 := d^c\theta$ is a semipositive (1,1)-form, satisfying $\omega_0^n = 0$. Assume that $\theta' = \theta + d\psi$ is another 1-form such that $d^c\theta'$ is semipositive. Then $(d^c\theta - dd^c\psi)^n = 0$ is a degenerate Monge-Ampère equation which can be studied in the usual way.

We write
$$(d\theta^c)^n - (d\theta'^c)^n = (d\theta^c - d\theta'^c) \wedge P,$$

where $P := \sum_k d^c\theta^k \wedge (d^c\theta')^{n-k-1}$ is a semipositive $(n-1, n-1)$-form. This equation gives
$$dd^c\psi \wedge P = 0, \text{ and hence } (dd^c\psi)\psi \wedge P = 0.$$

Using Stokes' theorem and integrating by parts, we obtain

$$0 = \int_M (dd^c\psi)\psi \wedge P = -\int_M d\psi \wedge d^c\psi \wedge P. \tag{3.5}$$

Now recall that the exterior product of semipositive forms is semipositive. By assumption, $d^c\theta$ and $d^c\theta'$ (and hence their powers) are semipositive. Then P is semipositive, as a sum of semipositive forms. As $d\psi \wedge d^c\psi$ is semipositive too, $\int_M d\psi \wedge d^c\psi \wedge P = 0$ implies $d\psi \wedge d^c\psi \wedge P = 0$.

On the other hand, $d^c\theta$ and $d^c\theta'$ have the same kernel Σ of (complex) dimension 1: otherwise, their sum would be strictly positive. But $d^c\theta + d^c\theta'$ is exact and hence, by Stokes theorem, its top power must be zero, contradiction. As P is a transversal volume form on TM/Σ, it follows that P has the same kernel Σ too. This implies:

Proposition 3.12: Let (M, ω, θ) and (M, ω', θ') be two pluricanonical LCK structures on M, and ω_0 and ω'_0 the corresponding semipositive forms (see Claim 3.11). Then $\ker \omega_0 = \ker \omega'_o$. Moreover, if θ is cohomologuous to θ', one has $\theta - \theta' = d\psi$, where ψ is a function which is constant on the leaves of $\Sigma = \ker \omega_0 = \ker \omega'_0$.

Proof: The forms ω_0 and ω'_0 are semipositive, with one-dimensional kernel, hence unless $\ker \omega_0 = \ker \omega'_o$, their sum is strictly positive. In the latter case, one has $\int_M (\omega_0 + \omega'_0)^n > 0$, which is impossible by Stokes' formula, since ω_0 and ω'_0 are exact.

Finally, (3.5) implies that $d\psi$ vanishes on Σ, otherwise the semipositive, exact form $d\psi \wedge d^c\psi \wedge P$ would have been strictly positive somewhere on M. ∎

4 Compact pluricanonical surfaces are Vaisman

4.1 Complex surfaces of Kähler rank 1

Recall that a compact complex surface is **of Kähler rank 1** if and only if it is not Kähler but it admits a closed semipositive (1,1)-form whose zero locus is contained in a curve ([HL]).

Lemma 4.1: A compact pluricanonical LCK surface M has Kähler rank 1.

Proof: The manifold M is non-Kähler, because it admits a positive, exact form. This form, multiplied by the Kähler one, would have given us an exact volume form, which is impossible by Stokes' theorem. On the other hand, M admits a semipositive form by Claim 3.11. ∎

Recall that a **Hopf surface** is a finite quotient of H, where H is a quotient of $\mathbb{C}^2 \backslash 0$ by a polynomial contraction. A Hopf surface is **diagonal** if this polynomial contraction is expressed by a diagonal matrix.

Compact surfaces of Kähler rank 1 have been classified in [CT] and [Br]. They can be:

1. Non-Kähler elliptic fibrations,

2. Diagonal Hopf surfaces and their blow-ups,

3. Inoue surfaces and their blow-ups.

Inoue surfaces have $b_1 = 1$ and hence, if they admit an automorphic potential, this has to be proper. But all compact LCK manifolds with proper potential can be deformed to Vaisman ones, and Inoue surfaces are not diffeomorphic to Vaisman manifolds (this result follows from a classification of Vaisman surfaces by F. Belgun, [Be]). Thus Inoue surfaces cannot have automorphic potential.

A cover of a blow-up of any complex manifold cannot admit plurisubharmonic functions because, by the lifting criterion, the projective spaces contained in the blow-up lift to the cover. Thus blow-ups cannot have global potential.

We are left with non-Kähler elliptic fibrations and diagonal Hopf surfaces which are known to admit Vaisman metrics, see *e.g.* [Be]. And hence:

Proposition 4.2: All compact pluricanonical LCK surfaces admit Vaisman metrics. ∎

For further use it is convenient to list all criteria used to distinguish Vaisman Hopf surfaces from non-Vaisman ones.

Theorem 4.3: Let M be a Hopf surface. Then the following are equivalent.

(i) M is Vaisman.

(ii) M is diagonalizable.

(iii) M has Kähler rank 1.

(iv) M contains at least two distinct elliptic curves.

Proof: Equivalence of the first three conditions is proven above. The equivalence of (iv) and (ii) is shown by Iku Nakamura and Masahide Kato ([N, Theorem 5.2]). Note that the cited result refers to primary Hopf surfaces, but we can always pass to a finite covering and the number of elliptic curves will not change because the eigenvectors for rationally independent eigenvalues cannot be mutually exchanged, and if they were dependent, they would produce infinitely many elliptic curves. ∎

4.2   Algebraic groups and the Jordan-Chevalley decomposition

In this section we let $V := \mathbb{C}^n$.

Lemma 4.4: Let $A \in GL(V)$ be a linear operator, and $\langle A \rangle$ the group generated by A. Denote by G the Zariski closure of $\langle A \rangle$ in $GL(V)$. Then, for any $v \in V$, the Zariski closure Z_v of the orbit $\langle A \rangle \cdot v$ is equal to the usual closure of $G \cdot v$.

Proof: Clearly, Z_v is G-invariant. Indeed, its normalizer $N(Z_v)$ in $GL(V)$ is an algebraic group containing $\langle A \rangle$, hence $N(Z_v)$ contains G. The converse is also true: since $\langle A \rangle$ normalizes $\langle A \rangle \cdot v$, its Zariski closure G normalizes the Zariski closure Z_v of the orbit. Therefore, the orbit $G \cdot v$ is contained in Z_v. Since $G \cdot v$ is a constructible set, its Zariski closure coincides with its usual closure, [H], [Kol]. This gives $\overline{G \cdot v} \subset Z_v$. As $\overline{G \cdot v}$ is algebraic and contains $\langle A \rangle \cdot v$, the inclusion $Z_v \subset \overline{G \cdot v}$ is also true. ∎

Let $G \subset GL(V)$ be an algebraic group over $\mathbb{C}$. Recall that an element $g \in G$ is called **semisimple** if it is diagonalizable, and **unipotent** if $g = e^n$, where n is a nilpotent element of its Lie algebra.

Theorem 4.5: (Jordan-Chevalley decomposition, [H, Section 15]) For any algebraic group $G \subset GL(V)$, any $g \in G$ can be represented as a product of two commuting elements $g = g_s g_u$, where g_s is semisimple, and g_u unipotent. Moreover, this decomposition is unique and functorial under homomorphisms of algebraic groups.

Corollary 4.6: Let M be a submanifold of a linear Hopf manifold $H = (V \backslash 0)/A$, $\tilde{M} \subset V \backslash 0$ its $\mathbb{Z}$-covering, and G the Zariski closure of $\langle A \rangle$ in $GL(V)$. Then $\tilde{M}$ contains the G-orbit of each point $v \in \tilde{M}$. Moreover, G is a product of $G_s := (\mathbb{C}^*)^k$ and a unipotent group G_u commuting with G_s, and bothe of these groups preserve $\tilde{M}$.

Proof: Let X be the closure of $\tilde{M}$ in $\mathbb{C}^N$. The ideal I_X of X is generated by polynomials, as shown in [OV5, Proof of Theorem 3.3]. As the polynomial ring is Noetherian, I_X is finitely generated, [AM]. Therefore, X is a cone of a projective variety.

This allows us to consider the smallest algebraic group G containing A. Then G acts naturally on X and preserves it. The last assertion of Corollary 4.6 is implied by the Jordan-Chevalley decomposition. ∎

4.3   Finding surfaces in LCK manifolds with potential

Lemma 4.7: Let M be a submanifold of a linear Hopf manifold $H = (V \backslash 0)/A$, $\dim M \geqslant 3$, and $G = G_s G_u$ the Zariski closure of $\langle A \rangle$ with its Jordan-Chevalley decomposition. Then M contains a surface M_0, with G_u acting non-trivially on its $\mathbb{Z}$-covering $\tilde{M}_0 \subset V$.

Proof: Another form of this statement is proven by Masahide Kato ([Kat]).

We shall use induction on dimension of M. To prove Lemma 4.7 it would suffice to find a subvariety $M_1 \subset M$ of codimension 1 such that G_u acts non-trivially on its $\mathbb{Z}$-covering $\tilde{M}_1 \subset \mathbb{C}^n \backslash 0$. Replacing V by the smallest A-invariant subspace containing $\tilde{M}$, we may assume that the intersection $\tilde{M} \cap V_1 \neq V_1$ for each proper subspace $V_1 \subset V$. Now take a codimension 1 subspace $V_1 \subsetneq V$ which is A-invariant and such that G_u acts on V_1 non-trivially (equivalently, such that A acts on V_1 non-diagonally). Using the Jordan decomposition of A, such V_1 is easy to construct. Then $\tilde{M}'_1 := V_1 \cap \tilde{M}$ gives a subvariety of M of codimension 1 and with non-trivial action of G_u. $\blacksquare$

Lemma 4.8: Let $M \subset H = (V \backslash 0)/\langle A \rangle$ be a surface in a Hopf manifold, and $G = G_s G_u$ the Zariski closure of $\langle A \rangle$ with its Jordan-Chevalley decomposition. Assume that G_u acts on the $\mathbb{Z}$-covering $\tilde{M}$ non-trivially. Then M is non-diagonal.

Proof: Replacing G by its quotient by the subgroup acting trivially on $\tilde{M}$ if necessary, we may assume that G acts properly on a general orbit in $\tilde{M}$. Then G is at most 2-dimensional. However, it cannot be 1-dimensional because G_s contains contractions (hence cannot be 0-dimensional) and G_u acts non-trivially. Therefore, $G_s \simeq \mathbb{C}^*$ and $G_u \simeq \mathbb{C}$.

Since G_s acts by contractions, the quotient $S := \tilde{M}/G_s$ is a compact curve, equipped with G_u-action which has a dense orbit. The group G_u can act non-trivially only on a genus 0 curve, and there is a unique open orbit O of G_u, with $S \backslash O$ being one point. All complex subvarieties of M are by construction G-invariant, and the complement of an open orbit is an elliptic curve, hence M has only one elliptic curve, and is non-diagonalizable by Theorem 4.3. $\blacksquare$

Theorem 4.9: A compact LCK manifold M with potential which is not Vaisman contains an embedded non-diagonal Hopf surface.

Proof: Let M be a compact LCK manifold with potential, $\dim_{\mathbb{C}} M \geq 3$. Then M is holomorphically embedded into a Hopf manifold $\mathbb{C}^n \backslash 0/\langle A \rangle$, where $A \in \mathrm{GL}(N, \mathbb{C})$ is a linear operator, see [OV5, Theorem 3.4]. Applying Lemma 4.7 and Lemma 4.8, we find a non-diagonal Hopf surface in M. $\blacksquare$

5   Pluricanonical condition *versus* Vaisman

5.1   Bott-Chern cohomology for Vaisman manifolds

Definition 5.1: The **Bott-Chern cohomology** groups $H^*_{BC}(M)$ of a complex

manifold M are $\frac{\ker d \cap \ker d^c}{\operatorname{im} dd^c}$. Since these groups are manifestly invariant with respect to the $U(1)$-action inducing the Hodge decomposition, one has $H^*_{BC}(M) = \bigoplus_{p,q} H^{p,q}_{BC}(M)$, where

$$H^{p,q}_{BC}(M) = \frac{\ker d \cap \ker d^c \cap \Lambda^{p,q}(M)}{dd^c(\Lambda^{p-1,q-1}(M))}.$$

The Bott-Chern cohomology is relevant for complex manifolds which do not satisfy a global dd^c lemma, in particular for LCK manifolds (see, *e.g.*, the recent [AU]).

Proposition 5.2: Let (M, I, ω, θ) be a compact Vaisman manifold, and let $\psi : H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ be the tautological map. Then $\ker \psi$ is 1-dimensional and it is generated by $d^c \theta$.

The proof of Proposition 5.2 will occupy the remaining part of this subsection. Let $n = \dim_{\mathbb{C}} M$. Since $d^* \theta = 0$ (that is θ is coclosed), one has $d(*\theta) = d(\omega^{n-1} \wedge I(\theta)) = 0$ (where $*$ is the Hodge operator). Therefore, $dd^c(\omega^{n-1}) = 0$: the Vaisman metric is Gauduchon. This is well known.

Then for any differentiable function f, integration by parts and Stokes' formula give

$$\int_M dd^c f \wedge \omega^{n-1} = 0. \tag{5.1}$$

Define the **degree map**

$$\deg \colon H^{1,1}_{BC}(M) \longrightarrow \mathbb{C} \text{ by } [\alpha] \mapsto \int_M \alpha \wedge \omega^{n-1}.$$

From (5.1) it follows that the degree map is well defined.

We now define a second order elliptic operator on functions by the formula

$$D(f) := \frac{dd^c f \wedge \omega^{n-1}}{\omega^n}.$$

Its index is zero, because its symbol is the same as for the Laplacian on functions, and the index of the Laplacian is zero, because it is self-dual.

Note that $\ker D$ only contains constants by the Hopf maximum principle. Therefore, $\operatorname{coker} D$ is 1-dimensional. By (5.1), $\operatorname{im} D$ is the space of functions g such that $\int_M g \omega^n = 0$. This gives the following useful lemma:

Lemma 5.3: Let $[\alpha] \in H^{1,1}_{BC}(M)$ be a degree 0 Bott-Chern cohomology class. Then $[\alpha]$ can be represented by a closed (1,1)-form α such that $\alpha \wedge \omega^{n-1} = 0$ (such (1,1)-forms are called **primitive**).

Proof: Let α_1 be a (1,1)-form representing $[\alpha]$. Then $\int_M \alpha_1 \wedge \omega^{n-1} = 0$, and hence $\alpha_1 \wedge \omega^{n-1} = dd^c f \wedge \omega^{n-1}$ for some $f \in \mathscr{C}^\infty(M)$. Then $\alpha := \alpha_1 - dd^c f$ is primitive. ∎

Definition 5.4: Let M be a Vaisman manifold, $\theta^\sharp$ its Lee field, and $\Sigma \subset TM$ the subbundle generated by $\theta^\sharp$ and $I(\theta^\sharp)$. The subbundle $\Sigma \subset TM$ is a holomorphic foliation, called **the canonical, or fundamental foliation of the Vaisman manifold** M (see [Va2], [DO]).

A form η on M is called **transversal** or **basic** (with respect to Σ) if $v \lrcorner \eta = v \lrcorner (d\eta) = 0$ for any vector field $v \in \Sigma$. Locally in a neighbourhood where the leaf space of Σ exists, transversal forms are forms on this leaf space. See [To] and [Va2] for the definitions of basic forms and basic cohomology with respect to Σ.

In [Va2], transversal geometry of a Vaisman manifold is explored in depth. It is shown that the form $\omega_0 := -d^c\theta$ is transversal and defines a Kähler structure on the leaf space in any neighbourhood where the leaf space of Σ exists. Such a form is called **transversally Kähler**. It is straightforward to define the transversal de Rham cohomology, the Hodge decomposition, Lefschetz $SL(2)$-action and so forth on the space of transversal forms. It turns out that the same properties of the Hodge decomposition (including the Lefschetz $SL(2)$-action on cohomology) are true for the transversal cohomology of the Vaisman manifold.

For the relation between de Rham cohomology and basic cohomology of Vaisman manifolds, we recall the following result of T. Kashiwada ([Kas], [Va2, Theorem 4.1]):

Theorem 5.5: On a compact Vaisman manifold of complex dimension n, any (real) harmonic p-form η, $0 \leqslant p \leqslant n-1$, has a unique decomposition

$$\eta = \alpha + \theta \wedge \beta, \tag{5.2}$$

with α and β basic, transversally primitive, harmonic and transversally harmonic forms. ∎

This easily implies the following:

Lemma 5.6: The kernel K of the natural map $H_\Sigma^*(M) \longrightarrow H^*(M)$ from the basic cohomology to the de Rham cohomology is generated (multiplicatively) by ω_0.

Proof: Any class in K can be represented by a form α which is transversally harmonic and exact. Then the primitive part of α vanishes by Theorem 5.5. ∎

Lemma 5.7: Let $\eta \in \Lambda^{1,1}(M)$ be a primitive, closed (1,1)-form on a compact Vaisman manifold. Then η is transversal.

Proof: The lemma is essentially [Ve2, Proposition 4.2], and its proof is entirely similar to [Ve2, Proposition 4.2]. Choose an orthonormal basis z_i in $\Lambda^{1,0}(M)$ in such a way that:

$$\omega = -\sqrt{-1} \sum_{i=0}^{n-1} z_i \wedge \overline{z}_i, \quad \text{and} \quad \omega_0 = -\sqrt{-1} \sum_{i=1}^{n-1} z_i \wedge \overline{z}_i.$$

Clearly $z_0 = \theta$. It suffices to prove Lemma 5.7 for real (1,1)-forms, hence we may write

$$\sqrt{-1}\,\eta = \sum_i a_i z_i \wedge \overline{z}_i + \sum_{i \neq j} b_{ij} z_i \wedge \overline{z}_j, \quad \text{where } a_i \in \mathbb{R}, \ b_{ij} = \overline{b}_{ji}.$$

Since η is exact, one has $\int_M \eta \wedge \eta \wedge \omega_0^{n-2} = 0$. However, at each point

$$\frac{\eta \wedge \eta \wedge \omega_0^{n-2}}{\omega^n} = a_0 \sum_{i=1}^{n-1} a_i - \sum b_{0i} b_{i0} = a_0 \sum_{i=1}^{n-1} a_i - \sum |b_{0i}|^2 \qquad (5.3)$$

Since η is primitive, $\sum_{i=0}^{n-1} a_i = 0$, and hence $a_0 \sum_{i=1}^{n-1} a_i = -a_0^2$. Then (5.3) becomes

$$\frac{\eta \wedge \eta \wedge \omega_0^{n-2}}{\omega^n} = -a_0^2 - \sum |b_{0i}|^2 \leqslant 0,$$

with equality reached only when $a_0 = 0$ and $b_{0i} = 0$ for all i. However, the form $\eta \wedge \eta \wedge \omega_0^{n-2}$ is exact, hence the integral $\eta \wedge \eta \wedge \omega_0^{n-2}$ cannot be positive, giving $\eta \wedge \eta \wedge \omega_0^{n-2} = 0$ and $a_0 = b_{0i} = 0$. The latter equality is precisely the transversality of η. ∎

Comparing Lemma 5.3 and Lemma 5.7, we obtain the following:

Corollary 5.8: Let $[\alpha] \in H_{BC}^{1,1}(M)$ be a Bott-Chern class on a compact Vaisman manifold. Then $[\alpha]$ can be represented by a transversal form.

Proof: Since the form ω_0 is positive, it has positive degree. Then $[\alpha_1] := [\alpha] - c[\omega_0]$ has degree 0, for appropriate c. Lemma 5.3 and Lemma 5.7 then imply that any degree 0 Bott-Chern (1,1)-class $[\alpha_1]$ can be represented by a primitive,

transversal form α_1. As ω_0 is closed and $\theta^\sharp \lrcorner \omega_0 = I(\theta^\sharp) \lrcorner \omega_0 = 0$, the form ω_0 is also transversal. Then $\alpha_1 + c\omega_0$ is a transversal form representing $[\alpha]$. ∎

Remark 5.9: For another proof of this result, see [Ts1, Theorem 2.1].

Finally we can give:

Proof of Proposition 5.2: Let $\eta \in H^{1,1}_{BC}(M)$ be a Bott-Chern cohomology class vanishing in de Rham cohomology. By Corollary 5.8, we can represent η by a transversal, closed (1,1)-form. By Lemma 5.6, the kernel of the tautological map $H^{1,1}_\Sigma(M) \longrightarrow H^{1,1}_{BC}(M)$ is generated by $\omega_0 = d^c\theta$. ∎

5.2 Pluricanonical submanifolds

To prove the main result, we still need several preliminary facts.

It is known ([Va1]), that if a compact complex manifold admits Kähler metrics, then any LCK metric on it is globally conformally Kähler. The next result is an analogue for pluricanonical *versus* Vaisman metrics:

Proposition 5.10: Let M be a compact complex manifold which admits a Vaisman metric. Then any pluricanonical metric on it is Vaisman.

Proof: Let $\omega = \theta \wedge \theta^c - d\theta^c$ be a pluricanonical form on M and let ω' be a Vaisman one, with θ' its Lee form. Denote by ω_0, ω'_0 the corresponding exact semipositive forms, $\omega_0 = -d^c\theta$, $\omega'_0 = -d^c\theta'$. Since both ω_0 and ω'_0 are exact, their Bott-Chern classes are proportional (Proposition 5.2). Rescaling one of these forms if necessary, we may assume that $\omega_0 - \omega'_0 = dd^c\varphi$, where φ is a function which is transversal, that is, constant on the leaves of Σ. Then $\alpha := \theta - \theta' + d\varphi$ is d- and d^c-closed, hence holomorphic, and therefore transversal by [Ts2, Theorem 3.3]).

According to [KO, Theorem A], an LCK metric on M is Vaisman if and only it admits a holomorphic flow which leaves it invariant, but acts non-isometrically on the Kähler covering. We then show that the holomorphic Lee flow F generated by $\theta'^\sharp - \sqrt{-1}\, I(\theta'^\sharp)$ (which is tangent to the leaves of Σ) preserves the pluricanonical metric. As F is holomorphic, it is enough to show that ω is F-invariant.

Indeed, the form $\theta = \alpha + \theta' - d\varphi$ is the sum of the transversal (and hence, F-invariant) form $\alpha - d\varphi$ and the F-invariant form θ'. Therefore, it is also F-invariant. As F is holomorphic, $I(\theta)$ is F-invariant too and also $\omega_0 = -d^c\theta$, and hence ω is F-invariant and Vaisman. ∎

Remark 5.11: The case of compact pluricanonical surfaces was treated in Proposition 4.2. From the classification of surfaces of Kähler rank 1 it follows that any pluricanonical surface admits a Vaisman metric.

Obviously, complex submanifolds of LCK manifolds are LCK. As we already recalled, the Vaisman condition is inherited on compact complex submanifolds, [Ve1]. A similar result occurs for pluricanonical manifolds:

Proposition 5.12: A compact complex submanifold of a pluricanonical LCK manifold is pluricanonical.

Proof: Let M be a pluricanonical LCK manifold and let $i : N \hookrightarrow M$ be a compact submanifold. By Claim 3.11 the two-form $\omega_0 := d\theta^c$ is degenerate on M. Then its restriction $i^*\omega_0$ to N is degenerate too, otherwise $\int_N \left(i^*\omega_0\right)^{\dim_{\mathbb{C}} N} \neq 0$, contradiction with $i^*\omega_0$ being exact. This means that the induced LCK structure on N satisfies (3.3) and hence is pluricanonical. ∎

5.3 The main result: all compact pluricanonical manifolds are Vaisman

Theorem 5.13: Let (M, I) be a compact complex manifold and let g be an LCK pluricanonical metric on it. Then g is Vaisman.

Proof: Let M be a compact pluricanonical locally conformally Kähler manifold. As (3.4) is satisfied, the universal covering of M carries an automorphic potential which by Theorem 2.2 is strictly positive. Moreover, by the same result, M admits a locally conformally Kähler metric with potential (possibly different from the pluricanonical metric), call it g'. This LCK metric has Lee form of length 1, by Corollary 3.4.

Now we argue by contradiction. Suppose the metric g' is not Vaisman. Then, by Theorem 4.9, M contains an embedded non-diagonal Hopf surface H^2 which, by Proposition 5.12 (applied for the initial metric on M), is pluricanonical and, by Proposition 4.2, admits Vaisman metrics. But non-diagonal Hopf surfaces cannot admit Vaisman metrics, [Be], contradiction.

As M admits a Vaisman metric, g', then by Proposition 5.10, the pluricanonical metric g itself is Vaisman. ∎

In view of this result, Claim 3.11 now gives:

Corollary 5.14: Let (M, ω, θ) be a compact LCK manifold with potential. Consider the 2-form $\omega_0 := d\theta^c = \theta \wedge \theta^c - \omega$. Then ω_0 is semipositive if and only if (M, ω, θ) is Vaisman. $\blacksquare$

6   *Errata*

6.1   **Pluricanonical condition revisited**

In Section 3 of [OV6] the following erroneous claim was made: "We now prove that the pluricanonical condition is equivalent with the existence of an automorphic potential on a Kähler covering."

Then we proceeded to make calculations purporting to show that pluricanonical condition is equivalent to the LCK with potential condition $d(I\theta) = \omega - \theta \wedge I\theta$. Here, the scalar term is lost: the correct equation (in the notation of [OV6]) is $d(I\theta) = |\theta|^2\omega - \theta \wedge I\theta$.

In Corollary 3.4, we prove that this equation, indeed, implies $d(I\theta) = \omega - \theta \wedge I\theta$.

However, the converse statement is false: as shown in Theorem 5.13, not all LCK manifolds with potential admit a pluricanonical LCK structure, but only Vaisman ones.

6.2   **LCK rank of Vaisman manifolds**

Recall that the **LCK rank** of an LCK manifold (M, ω, θ) is the rank of the smallest rational subspace V in $H^1(M, \mathbb{R})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contains the cohomology class $[\theta]$. When the LCK rank is 1, the manifold admits a $\mathbb{Z}$-covering which is Kähler (Section 2).

In several papers published previously ([OV1], [OV4], [OV7]) we claimed that a Vaisman manifold and an LCK manifold with potential always have LCK rank 1. This is in fact false. In this section we produce a counterexample to these claims, and explain the error.

Notice, however, that, as we prove below, any complex manifold which admits a structure of a Vaisman manifold (or LCK manifold with potential) *also* admits a structure of a Vaisman manifold (or LCK manifold with potential) with LCK rank one, so the problems arising because of this error are all differential-geometrical in nature; all results of complex geometry remain valid. This is probably the reason why the error was not noticed for so many years. Moreover, *the set of Vaisman (or LCK with potential) structures with LCK rank 1 on a given manifold is dense in the set of all Vaisman (or LCK with potential) structures.*

To construct a Vaisman (or LCK) manifold with an LCK rank bigger than 1 we use the same construction as used in Proposition 5.10. Consider a Vaisman manifold (M, ω, θ), with $\omega = d_\theta d_\theta^c(1)$ (see Definition 2.6 and the following Claim) which admits a transversal holomorphic 1-form $\alpha^{1,0}$. Examples of such Vaisman manifolds include the Kodaira surface, which is an isotrivial elliptic fibration over an elliptic curve. In this case, $\alpha^{1,0}$ is the pullback of a holomorphic differential of the elliptic curve.

Now, let $\alpha := \operatorname{Re} \alpha^{1,0}$, let $\theta' := \theta + \alpha$, and consider the 2-form $\omega' := d_{\theta'} d_{\theta'}^c(1)$. By construction, this is a (1,1)-form, which is $d_{\theta'}$-closed and (for small values of α in the $\mathscr{C}^\infty$ norm) positive, hence it is an LCK form. This form is in fact conformally equivalent to a Vaisman one by Ornea-Kamishima criterion (see the proof of Theorem 1.3). By [Va2], $H^1(M, \mathbb{R})$ is generated by the cohomology class of $[\theta]$ and cohomology classes of the real parts of holomorphic 1-forms. This implies the following unexpected result (already used in Proposition 5.10).

Theorem 6.1: Let M be a Vaisman manifold or LCK manifold with potential and let $L \subset H^1(M, \mathbb{R})$ be the set of cohomology classes of all Lee forms for the Vaisman (LCK with potential) structures on M. Then L is open in $H^1(M, \mathbb{R})$.

Proof: The Vaisman case was considered above. The general LCK with potential case is elementary: given an LCK with potential structure (M, ω, θ), such that $\omega = d_\theta d_\theta^c(1)$, we can always replace θ by a closed form $\theta' = \theta + \alpha$, with α sufficiently small. Then $\omega' = d_{\theta'} d_{\theta'}^c(1)$ is a positive $(1,1)$-form, giving an LCK structure with potential. ∎

Now, for a general Vaisman structure (M, ω, θ), its LCK rank is equal to $b_1(M)$, and any number between 1 and $b_1(M)$ can be obtained as an LCK rank for an appropriate choice of θ.

Theorem 6.1 has the following consequences.

Corollary 6.2: Let M be a complex manifold which admits a structure of a Vaisman manifold (or LCK manifold with potential) (M, ω, θ). Then M admits a structure of a Vaisman manifold (or LCK manifold with potential) (M, ω', θ') with proper potential, that is, of LCK rank one. Moreover, such ω' and θ' can be chosen in any neighbourhood of (M, ω, θ). ∎

Now, let us explain where the proof of [OV1] (later refined in [OV7]) failed.

Let M be a compact Vaisman manifold, and $\theta^\sharp$ its Lee field. Then $\theta^\sharp$ acts on M by holomorphic isometries, and on its smallest Kähler covering $(\tilde{M}, \tilde{\omega})$ by holomorphic homotheties. Denote by G the closure of the group generated by

$e^{t\theta^\sharp}$. This group is a compact Lie group, because isometries form a compact Lie group on a compact Riemannian manifold, and a closed subgroup of a Lie group is a Lie group by Cartan's theorem. Moreover, it is commutative, because $\langle e^{t\theta^\sharp}\rangle$ is commutative, and this gives $G = (S^1)^k$.

Let $\tilde{G}$ be the group of pairs $(\tilde{f} \in \mathrm{Aut}(\tilde{M}),\ f \in G)$, making the following diagram commutative:

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\ \tilde{f}\ } & \tilde{M} \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
M & \xrightarrow{\ f\ } & M
\end{array}
$$

Then $\tilde{G}$ is a covering of G, and the kernel of this projection is $\tilde{G}\cap\mathrm{Aut}_M(\tilde{M})$, where $\mathrm{Aut}_{\tilde{M}}(M)$ is the deck transform group of the covering $\tilde{M} \longrightarrow M$.

Consider the homomorphism $\chi : \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ mapping an element of $\pi_1(M)$ considered as an automorphism of $\tilde{M}$, to the Kähler homothety constant, $\gamma \mapsto \frac{\gamma^*\tilde{\omega}}{\tilde{\omega}}$. Since $\tilde{M}$ is the smallest Kähler covering, we identify $\mathrm{Aut}_M(\tilde{M})$ with $\chi(\pi_1(M)) \subset \mathbb{R}^{>0}$.

Now, let $\tilde{G}_0 \subset \tilde{G}$ be the subgroup acting on $\tilde{M}$ by isometries. Since the group $\tilde{G} \cap \mathrm{Aut}_M(\tilde{M})$ is a subgroup of $\mathrm{Aut}_{\tilde{M}}(M)$, $\tilde{G}_0$ maps to its image in G bijectively.

We assumed that G_0 (being the subgroup of elements of $\tilde{G}$ acting by isometries on both $\tilde{M}$ and M) is closed in G. Then, if $\tilde{G}_0 \cong S^{k-1}$, this would imply that $\tilde{G} \cong (S^1)^{k-1} \times \mathbb{R}$, proving that M is a quotient of $\tilde{M}$ by $\mathbb{Z}$-action.

However, this is false, because G_0 is closed in $\tilde{G}$, but not closed in G. This is where the argument fails.

References

[A]   K. Abe, *On a class of Hermitian manifolds*, Invent. Math. **51** (1979), 103–121.

[AU]   D. Angella, L. Ugarte, *Locally conformal Hermitian metrics on complex non-Kähler manifolds*, arXiv:1505.08007.

[AD]   V. Apostolov, G. Dloussky, *Locally conformally symplectic structures on compact non-Kähler complex surfaces*, arXiv:1501.02687.

[AS]   A. Andreotti, Y.T. Siu, *Projective embeddings of pseudoconcave spaces*, Ann. Scuola Norm. Sup. Pisa **24**, 231–278 (1970).

[AM]   M.F. Atyiah, I.G. Macdonald, Introduction to commutative algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.

[Be]   F.A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), no. 1, 1–40.

[BG]   C. Boyer, K. Galicki, Sasakian geometry, Oxford Univ. Press, 2008.

[Br]   M. Brunella, *A characterization of Inoue surfaces* Comment. Math. Helv. **88** (2013), no. 4, 859–874.

[CT]   I. Chiose, M. Toma, *On compact complex surfaces of Kähler rank one*, Amer. J. Math. **135** (2013), no. 3, 851–860.

[D1]   J.-P. Demailly, Complex analytic and differential geometry, `www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`.

[D2]   J.-P. Demailly, *Analytic methods in algebraic geometry,* Lecture Notes, École d'été de Mathématiques de Grenoble "Géométrie des variétés projectives complexes : programme du modèle minimal" (June–July 2007).

[DO]   S. Dragomir, L. Ornea, Locally conformally Kähler geometry, Progress in Math. **155**, Birkhäuser, 1998.

[G1]   P. Gauduchon, *La 1-forme de torsion d'une variété Hermitienne compacte*, Math. Ann. **267** (1984), 495–518.

[G2]   P. Gauduchon, *Private communication.*

[GOPP]   R. Gini, L. Ornea, M. Parton, P. Piccinni, *Reduction of Vaisman structures in complex and quaternionic geometry*, J. Geom. Physics, **56** (2006), 2501-2522.

[HL]   R. Harvey and H. B. Lawson Jr., *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74** (1983), no. 2, 169–198.

[H]   J.E. Humphreys, Linear algebraic groups, GTM **21**, 4th ed., Springer, 1998.

[KO]   Y. Kamishima, L. Ornea, *Geometric flow on compact locally conformally Kähler manifolds*, Tohoku Math. J., **57** (2) (2005), 201–221.

[Kas]   T. Kashiwada, *On V-harmonic forms in compact locally conformal Kähler manifolds with the parallel Lee form*, Kodai Math. J. **3** (1980), 70–82.

[Kat]   Ma. Kato, *Some Remarks on Subvarieties of Hopf Manifolds,* Tokyo J. Math. **2**, Nr. 1 (1979), 47-61.

[Kok]   G. Kokarev, *On pseudo-harmonic maps in conformal geometry*, Proc. London Math. Soc., **99** (2009), 168–94.

[KK]   G. Kokarev, D. Kotschick, *Fibrations and fundamental groups of Kähler-Weyl manifolds*, Proc. Amer. Math. Soc. **138** (2010), no. 3, 997–1010.

[Kol]   J. Kollár, Exercises in the birational geometry of algebraic varieties. Analytic and algebraic geometry, 495–524, IAS/Park City Math. Ser., **17**, Amer. Math. Soc., Providence, RI, 2010.

[N]   I. Nakamura, *On Surfaces of Class VII$_0$ with Curves*, Invent. Math. 78, 393-443 (1984).

[OV1]   L. Ornea and M. Verbitsky, *Structure theorem for compact Vaisman manifolds*, Math. Res. Lett. **10** (2003), 799–805. arxiv:math/0305259

[OV2]   L. Ornea, M. Verbitsky, *Embedding of compact Sasakian manifolds*, Math. Res. Lett. **14** (2007) no. 4, 703–710.

[OV3]   L. Ornea, M. Verbitsky, *Einstein-Weyl structures on complex manifolds and conformal version of Monge-Ampère equation*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **51(99)** (2008), no. 4, 339–353.

[OV4]   L. Ornea, M. Verbitsky, *Morse-Novikov cohomology of locally conformally Kähler manifolds*, J. Geom. Phys. **59**, No. 3 (2009), 295–305.

[OV5]   L. Ornea, M. Verbitsky, *Locally conformal Kähler manifolds with potential.* Math. Ann. **348** (2010), 25–33.

[OV6]   L. Ornea, M. Verbitsky, *Topology of Locally Conformally Kähler Manifolds with Potential*, IMRN, Vol. 2010, pp. 717–726.

[OV7]   L. Ornea, M. Verbitsky, *Automorphisms of locally conformally Kähler manifolds with potential*, Int. Math. Res. Not. 2012, no. 4, 894-903, arXiv:0906.2836.

[OV8]   L. Ornea and M. Verbitsky, *Locally conformally Kahler metrics obtained from pseudoconvex shells*, Proc. Amer. Math. Soc. **144** (2016), 325–335, arXiv:1210.2080.

[PV]   M. Parton, V. Vuletescu, *Examples of non–trivial rank in locally conformal Kähler geometry*, Math. Z. **270** (2012), no. 1–2, 179–187.

[R]   H. Rossi, *Attaching analytic spaces to an analytic space along a pseudo–convex boundary*, Proceedings of the Conference Complex Manifolds (Minneapolis), pp. 242–256. Springer, Berlin (1965).

[To]   P. Tondeur, Geometry of foliations, Monographs in Mathematics, **90**, Birkhäuser Verlag, Basel, 1997.

[Ts1]   K. Tsukada, *The canonical foliation of a compact generalized Hopf manifold*, Differential Geom. Appl. **11** (1999), no. 1, 13–28.

[Ts2]   K. Tsukada, *Holomorphic forms and holomorphic vector fields on compact generalized Hopf manifolds*, Compositio Math. **93** (1994), no. 1, 1–22.

[Va1]   I. Vaisman, On locally and globally conformal Kähler manifolds, Trans. Amer. Math. Soc., **262** (1980), 439–447.

[Va2]   I. Vaisman, Generalized Hopf manifolds, Geom. Dedicata **13** (1982), no. 3, 231–255.

[Ve1]   M. Verbitsky, *Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds*, Proc. Steklov Inst. Math. **246** no. 3 (2004), 54–78.

[Ve2]   M. Verbitsky, *Stable bundles on positive principal elliptic fibrations*, Math. Res. Lett. 12 (2005), no. 2-3, 251–264.

LIVIU ORNEA

UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS,

14 ACADEMIEI STR., 70109 BUCHAREST, ROMANIA, *and*

INSTITUTE OF MATHEMATICS "SIMION STOILOW" OF THE ROMANIAN ACADEMY,

21, CALEA GRIVITEI STREET 010702-BUCHAREST, ROMANIA

liviu.Ornea@imar.ro,   lornea@fmi.unibuc.ro

MISHA VERBITSKY

LABORATORY OF ALGEBRAIC GEOMETRY, FACULTY OF MATHEMATICS,

NRU HSE, 7 Vavilova Str. Moscow, Russia, *and*
Université Libre de Bruxelles, Département de Mathématique
Campus de la Plaine, C.P. 218/01, Boulevard du Triomphe
B-1050 Brussels, Belgium
verbit@mccme.ru