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D. S. Malyshev & P. M. Pardalos

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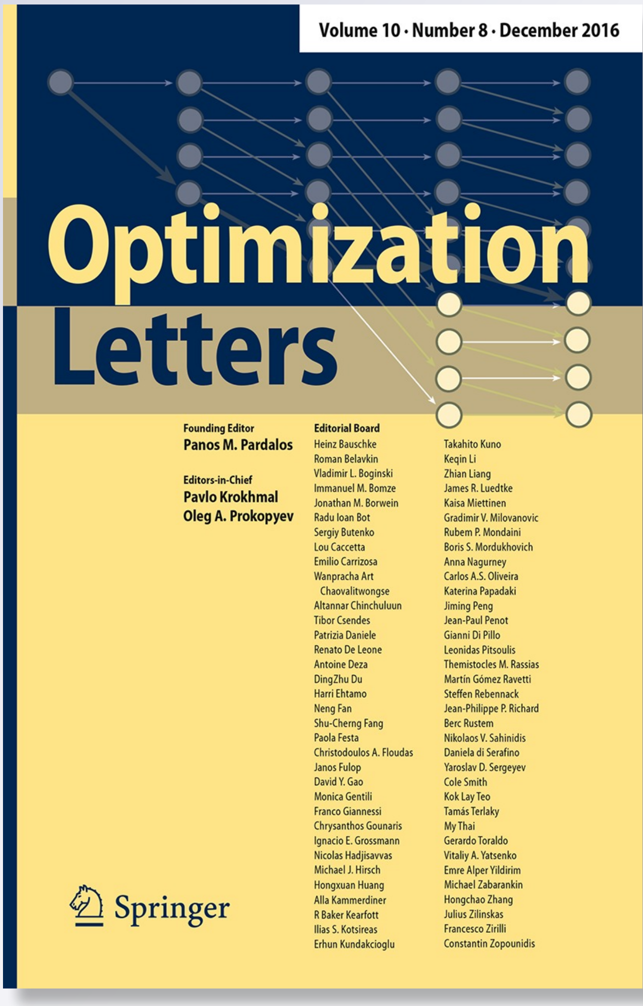

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REVIEW ARTICLE

Critical hereditary graph classes: a survey

D. S. Malyshev[1] · P. M. Pardalos[2]

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Abstract The task of complete complexity dichotomy is to clearly distinguish between easy and hard cases of a given problem on a family of subproblems. We consider this task for some optimization problems restricted to certain classes of graphs closed under deletion of vertices. A concept in the solution process is based on revealing the so-called "critical" graph classes, which play an important role in the complexity analysis for the family. Recent progress in studying such classes is presented in the article.

Keywords Computational complexity · Polynomial-time algorithm · Hereditary graph class · Independent set problem · Dominating set problem · Coloring problem · List edge-ranking problem

1 Introduction

A large number of results on polynomial-time solvability and NP-completeness has been accumulated for many graph problems under various restrictions of graph classes [45]. The existing extensive literature is constantly updated with new papers in this area. Despite the critical importance of studying the complexity of graph problems for individual classes, there is a noticeable absence of the generality in papers in the

✉ D. S. Malyshev
dsmalyshev@rambler.ru; dmalishev@hse.ru

P. M. Pardalos
pardalos@ufl.edu

1 National Research University Higher School of Economics, 25/12 Bolshaja Pecherskaja Ulitsa, Nizhny Novgorod 603155, Russia

2 University of Florida, 401 Weil Hall, P.O. Box 116595, Gainesville, FL 326116595, USA

field. Usually, the complexity status of an NP-complete graph problem is determined for “standard” classes, like the classes of bipartite graphs, planar graphs, bounded degree graphs, and etc. One could say that the standard approach in the area is to enumerate several “famous” classes and point out the complexity of a given problem for at least one of them. For example, the classical independent set problem is known to be polynomial-time solvable for bipartite graphs and graphs with degrees at most two [48], but it is NP-complete for planar graphs and graphs with degrees at most three [14]. However, the approach does not allow to clarify what is the reason of different complexity of the same graph problem for distinct restrictions of the class of all graphs. At the same time, it would be more natural to look at the issue more generally. A novel approach for a systematic study of the computational complexity is considered in this paper.

When considering representative families of graph classes, one could set more general problems than the complexity analysis of some concrete graph problem for a given class of graphs. One could ask the following two general questions. How to classify classes in a family with respect to the computational complexity of a considered graph problem? Is there a “boundary” separating “easy” and “hard” instances? To answer these questions, a suitable choice of the corresponding conceptual apparatus is necessary. Human intuition says that we should focus our attention on classes of the family critical with respect to some “complexity-topological” sense. For example, minimal “hard” and maximal “easy” classes are natural critical classes, as they are phase-transition elements. Possible absence of the “boundary points” above leads to the idea to consider the limits of monotonically decreasing sequences of “hard” classes. Intuitively, these limits may also be critical classes.

This paper is a survey about some types of critical classes (boundary and minimal hard) in the family of hereditary graph classes, i.e. sets of graphs closed under isomorphism and deletion of vertices.

2 Hereditary classes

All graphs in this paper are finite, unlabelled, undirected, without loops and multiple edges. A graph H is a *subgraph* of G if H is obtained from G by deletion of some edges and vertices with incident edges. A graph H is an *induced subgraph* of G if H is obtained from G by deletion of some vertices with incident edges. A *class of graphs* is a set of graphs closed under isomorphism. A graph H is called a *subgraph* of a graph G if H can be obtained from G by deletion of vertices and edges. A graph H is called an *induced subgraph* of a graph G if H can be obtained from G by deletion of only vertices. A class of graphs is called *hereditary* if it is closed under deletion of vertices. A class is hereditary if and only if it contains all induced subgraphs of each its graph. Any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$ (see Theorem 15 of [28]). We write $\mathcal{X} = Free(\mathcal{Y})$ in this case. In other words, $\mathcal{X}$ consists of those and only those graphs that deletion of their vertices does not produce any graph in $\mathcal{Y}$. There is a unique minimal set $\mathcal{Y}$ with this property denoted by $Forb(\mathcal{X})$. If $Forb(\mathcal{X})$ is finite, then $\mathcal{X}$ is called *finitely defined*. For example, if $\mathcal{X}_1$ is the set of all forests, then $Forb(\mathcal{X}_1)$

consists of all cycles. If $\mathcal{X}_2$ is the class of graphs, whose connected components are complete graphs, then a path with three vertices is a unique element of $Forb(\mathcal{X}_2)$. The class $\mathcal{X}_2$ is finitely defined, but $\mathcal{X}_1$ is not. Notice that if $\mathcal{X}$ and $\mathcal{Y}$ are hereditary classes such that $\mathcal{Y} \subset \mathcal{X}$, then there is a subset $\mathcal{Z} \subseteq \mathcal{X}$ such that $\mathcal{Y} = \mathcal{X} \cap Free(\mathcal{Z})$.

For some hereditary graph classes, determining the minimal set of forbidden induced subgraphs is a simple problem. However, in general, the problem of finding this set is far from being trivial, as the example of perfect graphs shows [10]. It has been open for almost 40 years.

The choice of the family is motivated by many reasons. Firstly, many known classes are hereditary. For example, the classes of bipartite and planar graphs, bounded degree graphs are hereditary. Secondly, the family is continuum and, hence, representative, which makes the questions raised in the introduction to be interesting for it. Indeed, taking any two different infinite subsets of the set of all simple cycles and forbidding their graphs as induced subgraphs produces different hereditary classes. The set of all simple cycles is countable, the power set of a countable set is continuum [47], the set of all finite subsets of a countable set is also countable [47]. The difference of a continuum set and a countable set is also continuum. Hence, the set of all those hereditary classes is continuum. Thirdly, for hereditary classes, the concept of critical graph classes really does its job, i.e. it helps to answer when a difficult problem becomes easy. More precisely, a graph problem is NP-complete for a finitely defined class if and only if the class contains a subclass critical for the problem. Hence, a known list of classes critical for a given problem enables to classify its complexity in the family of all finitely defined graph classes. Additional motivation to consider specifically hereditary graph classes will be presented through one section.

3 Graphs and classical graph problems

As usual, P_n, C_n, and K_n are a simple path, a simple cycle, and a complete graph with n vertices, respectively. A graph $K_{p,q}$ is complete bipartite with p vertices in the first part and q vertices in the second.

A graph $\overline{G}$ is the complement of a graph G. A graph $G_1 + G_2$ is the disjoint sum of graphs G_1 and G_2 with non-intersected sets of vertices. A graph kG is isomorphic to k disjoint copies of a graph G. For a graph G and its vertex x, $deg_G(x)$ is degree of x in the graph G.

In this paper, we will refer to the following classical graph problems.

An *independent set* of a graph is a subset of its pairwise non-adjacent vertices. The size of a maximum independent set of G is said to be the *independence number* of G and denoted by $\alpha(G)$. The *independent set problem*, for a graph G and a natural k, is to verify the inequality $\alpha(G) \geq k$.

A *vertex cover* of a graph G is a subset $V' \subseteq V(G)$ such that any edge in $E(G)$ is incident to an element of V'. It is easy to see that V' is a vertex cover of G if and only if $V(G) \backslash V'$ is independent. The size of a minimum vertex cover of G is denoted by $\beta(G)$. Clearly, $\alpha(G) + \beta(G) = |V(G)|$ for each graph G. The *vertex cover problem*, for a graph G and a natural k, is to verify the inequality $\beta(G) \leq k$.

A *clique* in a graph is a set of pairwise adjacent vertices. For a given graph G and a natural k, the *clique problem* is to determine whether G contains a clique with at least k vertices.

A *dominating set* of a graph G is a subset $V' \subseteq V(G)$ such that any element of $V(G)\setminus V'$ has a neighbor in V'. The size of a minimum dominating set of G is said to be the *domination number* of G and denoted by $\gamma(G)$. The *dominating set problem* is to check, for a given graph G and a natural k, whether $\gamma(G) \leq k$ or not.

A *proper coloring* (or simply a *coloring*) is an arbitrary mapping from the set of vertices or edges of a graph into a set of colors of the graph such that any adjacent vertices (or edges) are colored by different colors. The *chromatic number* of graph G denoted by $\chi(G)$ is a minimal number of colors needed to properly color G. The *vertex k-colorability problem* is to verify whether vertices of a given graph can be properly colored with at most k colors. The *edge k-colorability problem* is the edge analogue of the vertex k-colorability problem. The *chromatic number problem*, for a given graph G and a given natural k, is to check the validity of the inequality $\chi(G) \leq k$. Notice, the vertex k-colorability and the chromatic number problems are distinct problems, because we know k for the first problem in advance, i.e. before giving G. At the same time, k is a part of an input for the second problem.

A *Hamiltonian cycle* of a graph is a cycle that once visits all its vertices. For a given graph, the *Hamiltonian cycle problem* is to check whether a given graph contains a Hamiltonian cycle or not.

4 Boundary graph classes

We use the following natural formal definitions for "easy" and "hard" hereditary classes. For a given NP-complete graph problem Π, a hereditary class is said to be Π-*easy* if Π can be polynomially solved for its graphs. A hereditary class is Π-*hard* if Π is NP-complete for it. For instance, bipartite graphs constitute an easy case for the independent set problem [48], but the class of planar graphs is hard for it [14].

Maximal Π-easy and minimal Π-hard classes are natural boundary elements in the family of hereditary classes. It turns out that the boundary may be absent at all. First, there are no maximal Π-easy classes, as any Π-easy class $\mathcal{X}$ can be extended by adding a graph $G \notin \mathcal{X}$ and all proper induced subgraphs of G. Clearly, the resultant class is also Π-easy, as we added a finite set of graphs to $\mathcal{X}$. Second, minimal hard classes exist for some problems and do not exist for some others. For a given graph G' and a function $f : E(G') \longrightarrow \{1, 2\}$, the *travelling salesman problem with distances one and two* is to check whether the minimum length of its Hamiltonian cycles is at most a given number or not. It is NP-complete in the class of all complete graphs [44]. Forbidding any fixed complete graph in the class of all complete graphs restricts the number of vertices of graphs in the resultant class. Hence, each proper hereditary subclass of the class of all complete graphs contains a finite set of graphs. Hence, the problem can be solved in $O(1)$ time for the subclass. Hence, the class of all complete graphs is a minimal hard case for the problem. On the other hand, for the vertex and edge variants of the k-colorability problem, any hard class contains a proper hard subclass. Indeed, if $\mathcal{Y}$ is a hard case for the problem, then it must contain a graph that cannot be properly colored in k colors. Let $H \in \mathcal{Y}$ be a fixed graph of this type. Therefore, $\mathcal{Y} \backslash Free(\{H\})$ contains only graphs that also cannot be properly colored in k colors. There is a polynomial-time algorithm to test whether a given graph $G'' \in \mathcal{Y}$ belongs to $\mathcal{Y} \cap Free(\{H\})$ by enumerating all subsets of $V(G'')$ with $|V(H)|$ elements and verifying whether one of them induces H. If $G'' \in \mathcal{Y} \backslash Free(\{H\})$, then G'' is not k-colorable, as H is not k-colorable. Hence, $\mathcal{Y} \cap Free(\{H\})$ must be hard for the problem, and we have a contradiction. The phenomena of the absence of the boundary we just considered was noticed in [31].

So, to classify hereditary classes, we have to take into account that the sets of easy and hard classes can be open with respect to the inclusion relation. In other words, there may be infinite monotonically decreasing sequences of hard classes. Intuitively, the limits of such chains should play a special role in the analysis of the complexity. This observation leads to the notion of a boundary graph class. A class $\mathcal{X}$ is Π-*limit* if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \ldots$ of Π-hard classes such that $\mathcal{X} = \bigcap_{k=1}^{\infty} \mathcal{X}_k$. Clearly, any Π-limit class is hereditary. Moreover, any Π-hard class $\mathcal{X}'$ is Π-limit, as the stationary sequence $\{\mathcal{X}_i\}$, where $\mathcal{X}_i = \mathcal{X}'$ for each i, converges to $\mathcal{X}'$. A Π-limit class that is minimal under inclusion is said to be Π-*boundary*. The following theorem shows the significance of the boundary class notion (see [2,4]).

Theorem 1 *A finitely defined class is Π-hard if and only if it includes some Π-boundary class.*

The theorem shows that knowledge of the Π-*boundary system* (i.e. the set of all Π-boundary classes) gives a dichotomy with respect to NP-completeness and non-NP-completeness of Π for the family of finitely defined classes. Note, the theorem does not state that a finitely defined class is Π-easy if it does not contain any Π-boundary class. One more interesting fact is that there is a boundary class for each NP-complete graph problem (in contrast to minimal hard classes), as the set of all graphs is finitely defined. Unfortunately, Theorem 1 cannot be extended to the whole family of all hereditary classes, since it is wrong for it. The corresponding counterexample will be presented later.

The definition of a boundary graph class also shows the importance of the family of hereditary graph classes, as critical graph classes may be absent at all for some other families. For example, the method does not work for the family of all graph classes. Indeed, if $\mathcal{X}$ is an arbitrary class of graphs, then it is a finite or a countable set, as the class of all graphs is countable. If $\mathcal{X}$ is Π-hard, then $\mathcal{X}$ must be countable. Hence, $\mathcal{X} = \{G_1, G_2, \ldots\}$. Therefore, for each fixed i, the problem Π is also NP-complete for $\mathcal{X}_i = \mathcal{X} \backslash \bigcup_{j=1}^{i}\{G_j\}$. The sequence $\{\mathcal{X}_i\}$ converges to the empty set, as $\bigcap_{i=1}^{\infty} \mathcal{X}_i = \emptyset$. Hence, an infinite monotonically decreasing sequence of classes with NP-complete problem Π converging to the empty set can be stretched from any class with NP-complete problem Π. So, applied to the family of all graph classes, the empty set is an analogue of a boundary class. In other words, critical classes do not exist for the family. At the same time, removing a graph G from a hereditary class $\mathcal{X}$ forces to remove all supergraphs of G from $\mathcal{X}$, i.e. we come to the class $\mathcal{X} \cap Free(\{G\})$. The computational status of a graph problem for $\mathcal{X}$ and $\mathcal{X} \cap Free(\{G\})$ may be distinct. So, to form an infinite monotone sequence $\{\mathcal{X}_i\}$ of Π-hard classes, for each i, one can forbid only particular graphs in $\mathcal{X}_i$ to obtain $\mathcal{X}_{i+1}$. This restriction imposes "lower bounds" on Π-boundary classes in the family of hereditary classes.

The notion of a boundary graph class was originally introduced by V. E. Alekseev for the independent set problem [2]. It was later applied for the dominating set problem [5]. Nowadays, boundary classes are known for several algorithmic graph problems [2,4,5,19,30,34,36,39].

The aim of the following three sections is to present some known boundary classes for the independent set, dominating set, and edge 3-colorability problems. The boundary of some of them will be equipped by a complete or a partial proof to demonstrate key ideas in this area.

5 Boundary classes for some classical graph problems

5.1 The independent set problem

The notion of a boundary class was introduced in [2] applied to the independent set problem, where the first boundary class was also found for the problem. This class is $\mathcal{S}$, which consists of all forests with at most three leaves in every connected component. In other words, any connected component of every graph in $\mathcal{S}$ is a graph of the form $S_{i,j,k}$ for some non-negative numbers i, j, and k (see Fig. 1).

Now, we are ready to give an example showing that Theorem 1 is not true for general hereditary classes. Indeed, the class of all forests is easy for the problem [48] and it contains $\mathcal{S}$.

Any proof that a class is boundary for some graph problem can be split into two parts. First, the fact that it is a limit class should be proved. Next, its minimality should be shown. There are two tools to discover limit classes: graph transformations and reducibility between NP-complete problems. We demonstrate the first tool in this and the third subsections and the second tool in the following subsection.

A graph is *subcubic* if degrees of all its vertices are at most three. Let $\mathcal{D}eg(3)$ be the set of all subcubic graphs. The *hereditary closure* $[\mathcal{X}]$ of a graph class $\mathcal{X}$ is the set of all induced subgraphs (not necessary proper) of all members of $\mathcal{X}$.

Lemma 1 *The class $\mathcal{S}$ is limit for the independent set problem.*

Proof The independent set problem is NP-complete in the class $\mathcal{D}eg(3)$ [14]. Denote this class by $\mathcal{X}_0$. A *k-subdivision of an edge* (a, b) of a graph is to delete it from the graph, add new vertices $c_1, c_2, \ldots, c_k$ and the edges $(a, c_1), (c_1, c_2), \ldots, (c_{k-1}, c_k)$,

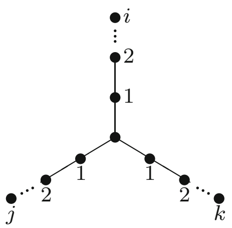


Fig. 1 A graph $S_{i,j,k}$

(c_k, b). It is known that a 2-subdivision of any edge of any graph increases its independence number by exactly one [43]. Let us apply a $2i$-subdivision to every edge of each graph in $\mathcal{X}_0$. Let $\mathcal{X}_i$ be the hereditary closure of the set of all resultant graphs. Clearly, the problem is NP-complete for $\mathcal{X}_i$ for each i. Let $\mathcal{Y}_i$ be equal to $\bigcup_{j=i}^{\infty} \mathcal{X}_i$. Clearly, $\mathcal{Y}_1 \supseteq \mathcal{Y}_2 \supseteq \ldots$ and $\mathcal{Y}_i$ is a hard case for the problem for each i. In addition, $\bigcap_{i=1}^{\infty} \mathcal{Y}_i = \mathcal{S}$. Hence, $\mathcal{S}$ is a limit class for the independent set problem by the definition. □

The dominating set problem is NP-complete for the class $\mathcal{D}eg(3)$ [14]. It is also known that a 3-subdivision of any edge of any graph increases its domination number by exactly one [18]. Hence, similar to the proof of Lemma 1, it is easy to show that $\mathcal{S}$ is limit for the dominating set problem.

There are no common ideas in proving the minimality of limit classes. That is, the most of known proofs are individual and based on a structure of a limit class. Perhaps, a proof for $\mathcal{S}$ and reduced to it are the only exceptions due to some interesting property of monotone graph classes not including $\mathcal{S}$.

A hereditary graph class is *monotone* if it is additionally closed under deletion of edges. For example, the classes of bipartite and planar graphs are monotone, but the class of all complete graphs is not. Any monotone class (and only monotone) can be defined by its forbidden subgraphs [28].

Clique-width is an important parameter of graphs. This is explained by the fact that many graph problems can be solved in polynomial time for graphs of bounded clique-width (see [11] for more information). More precisely, for each fixed number C, many problems that are NP-complete for the set of all graphs become polynomial-time solvable for the class of all graphs having clique-width at most C. In particular, this category includes the independent and dominating set problems, the vertex 3-colorability problem [11].

Lemma 2 [8] *If $\mathcal{X}$ is a monotone class and $\mathcal{S} \not\subseteq \mathcal{X}$, then there is a constant $C(\mathcal{X})$ such that any graph in $\mathcal{X}$ has clique-width at most $C(\mathcal{X})$.*

Theorem 2 *If $P \neq NP$, then the class $\mathcal{S}$ is boundary for the independent set problem.*

Proof Assume that there is a class $\mathcal{X}$, boundary for the problem, such that $\mathcal{X} \subset \mathcal{S}$. As $\mathcal{X}$ is hereditary and $\mathcal{X} \subset \mathcal{S}$, there is a number k such that $kS_{k,k,k} \notin \mathcal{X}$. Then $\mathcal{X} \subseteq \mathcal{S} \cap Free(\{kS_{k,k,k}\})$. Let $\mathcal{Y}$ be the set of all possible graphs obtained from $kS_{k,k,k}$ by addition of one or more edges. Any graph in this set is not a forest with at most three leaves in every connected component. Hence, $\mathcal{Y} \cap \mathcal{S} = \emptyset$ and $\mathcal{X} \subseteq Free(\{kS_{k,k,k}\} \cup \mathcal{Y})$. The class $Free(\{kS_{k,k,k}\} \cup \mathcal{Y})$ is monotone, as it coincides with the set of all graphs that do not contain $kS_{k,k,k}$ as a subgraph. In addition, it is finitely defined and does not include $\mathcal{S}$. Hence, it must easy for the problem by the previous lemma. But, by Theorem 1, it must be hard, as it includes a boundary class $\mathcal{X}$. We have a contradiction with $P \neq NP$. □

By Theorem 2, if $P \neq NP$, then $\mathcal{S}$ is boundary for the vertex cover problem. Similarly, the class $co(\mathcal{S}) = \{G | \overline{G} \in \mathcal{S}\}$ is boundary for the clique problem if $P \neq NP$. A proof that $\mathcal{S}$ is boundary for the dominating set problem (assuming that $P \neq NP$) is similar to the proof of Theorem 2.

Theorems 1 and 2 not only shed some new light on known results on the complexity of the independent set problem, but also give numerous new facts of this type. The class of all graphs without triangles is finitely defined. For each fixed k, the class of all graphs with degrees of all vertices at most k is also finitely defined. Indeed, any its minimal forbidden induced subgraph is obtained from a graph H with $k+1$ vertices by adding a new vertex adjacent to all vertices of $V(H)$. The first class and the second class for $k > 2$ are classical cases with NP-complete independent set problem [14]. These facts known more than 35 years completely correspond to those recent theorems, as each of the two classes includes $\mathcal{S}$. Moreover, for arbitrary graphs $G_1, \ldots, G_s$ not belonging to $\mathcal{S}$, the independent set problem is NP-complete for $Free(\{G_1, \ldots, G_s\})$ by Theorems 1 and 2. So, the new approach generalizes some previously known intractability results and discovers a lot of new hard cases for the problem.

Assuming $P \neq NP$, V. E. Alekseev conjectured that $\mathcal{S}$ is a unique boundary class for the independent set problem [2]. This conjecture is true if and only if $Free(\{G\})$ is easy for the problem for each $G \in \mathcal{S}$ [2]. Progress on the way to prove or disprove this conjecture is modest. At the moment, polynomial-time solvability of the independent set problem for $Free(\{G\})$ was proved for all graphs $G \in \mathcal{S}$ having at most five vertices [3,21,24]. On the other hand, the complexity of the problem is already unknown for $Free(\{S_{1,1,3}\})$ and $Free(\{P_6\})$, i.e. for classes defined by forbidding some six-vertex graphs in $\mathcal{S}$. Nevertheless, there are several indirect evidences that the Alekseev's conjecture is likely true [6,7,23,35].

5.2 The dominating set problem

The class $\mathcal{S}$ is boundary for the dominating set problem [5]. Three more boundary graph classes are known for it. For a graph G, its *line graph* $L(G)$ has vertex set $E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. Let $\mathcal{T}$ be the set of all line graphs of graphs in $\mathcal{S}$, i.e. the set $\{L(G) | \, G \in \mathcal{S}\}$. In other words, any connected component of any graph in $\mathcal{T}$ is a path or of the form shown in the figure below.

If $P \neq NP$, $\mathcal{T}$ is boundary for the dominating set problem [5]. A proof of this fact is somewhat similar to the proof presented for $\mathcal{S}$ and the independent set problem (Fig. 2).

For a graph G, a graph $Q(G)$ has vertex set $V(G) \cup E(G)$ and edge set $\{(v_i, v_j) | \, v_i, v_j \in V(G)\} \cup \{(v, e) | \, v \in V(G), e \in E(G), v \text{ is incident to } e\}$ (see Fig. 3).

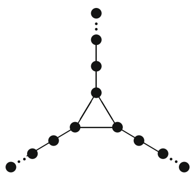
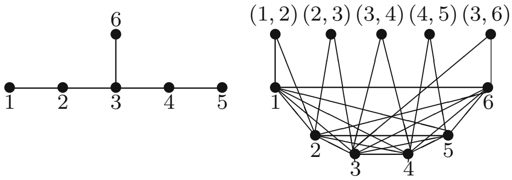


Fig. 2 A representative of the class $\mathcal{T}$

Fig. 3 The graphs $S_{1,2,2}$ and $Q(S_{1,2,2})$



The class $\mathcal{Q}$ is the set $[\{G|\ \exists H \in \mathcal{S}, G = Q(H)\}]$. It is the third boundary class for the problem. Partially proving this fact, we also demonstrate the idea of polynomial-time reducibility between two graph problems.

Lemma 3 *For every connected graph G, we have* $\gamma(Q(G)) = \beta(G)$.

Proof The set $V(Q(G))$ can be split into a clique A and an independent set B, where $A = V(G)$ and $B = E(G)$. Clearly, any vertex cover of G corresponds to a subset of A that is a dominating set of $Q(G)$. Hence, $\gamma(Q(G)) \leq \beta(G)$. It is easy to see that if D is a dominating set of $Q(G)$ and $x \in D \cap B$, then $D \backslash \{x\} \cup \{y\}$ is also a dominating set of G, where $y \in A$ is an arbitrary neighbor of x. Therefore, there is a minimum dominating set of $Q(G)$ included in A. It corresponds to some vertex cover of G. Hence, $\gamma(Q(G)) \geq \beta(G)$. □

Theorem 3 *If* $P \neq NP$, *then the class* $\mathcal{Q}$ *is boundary for the dominating set problem.*

Proof For a hereditary class $\mathcal{X}$, by $Q(\mathcal{X})$ we denote the hereditary closure of $\{G|\ \exists H \in \mathcal{X}, G = Q(H)\}$. The independent set problem for $\mathcal{X}$ is polynomially equivalent to the dominating set problem for $Q(\mathcal{X})$ [5]. Lemma 3 is the most important result to prove this fact. Hence, $\mathcal{Q} = Q(\mathcal{S})$ is a limit class for the dominating set problem, as it is so for $\mathcal{S}$ and the independent set problem. In addition, the class $Q(\mathcal{G})$ is finitely defined, where $\mathcal{G}$ is the class of all graphs [5]. Hence, any monotone sequence $\{\mathcal{X}_i\}$ of hard classes for the dominating set problem converging to a proper subset of $\mathcal{Q}$ must contain an element $\mathcal{X}_j$ such that $\mathcal{X}_j \subseteq Q(\mathcal{G})$. Moreover, there is a graph $G' \in \mathcal{S}$ such that $\mathcal{X}_j \subseteq Q(\mathcal{G}) \cap Free(\{Q(G')\})$ for some j. The dominating set problem for $\mathcal{X}_j$ can be polynomially reduced to the independent set problem for the class of all graphs that do not contain G' as a subgraph. The last class is monotone, and it does not include $\mathcal{S}$. Hence, the independent set problem is easy for the class. Hence, $\mathcal{X}_j$ is easy for the dominating set problem. We have a contradiction with $P \neq NP$. □

The fourth boundary class is defined similar to $\mathcal{Q}$. Let G be a subcubic graph. Let $V'(G)$ be the set of all degree three vertices of G and $V''(G) = V(G) \backslash V'(G)$. We define a graph $Q^*(G)$ as follows. The set $V(Q^*(G))$ coincides with $V''(G) \cup E(G)$. A vertex $x \in V'(G)$ is incident to edges $e_1(x), e_2(x), e_3(x)$ in the graph G. The set $E(Q^*(G))$ coincides with $\{(v_i, v_j)|\ v_i, v_j \in V''(G)\} \cup \{(v, e)|\ v \in V''(G), e \in E(G), v \text{ is incident to } e\} \cup \bigcup_{x \in V'} \{(e_1(x), e_2(x)), (e_1(x), e_3(x)), (e_2(x), e_3(x))\}$. The class $\mathcal{Q}^*$ is the set $[\{G|\ \exists H \in \mathcal{S}, G = Q^*(H)\}]$.

A proof of the following result is similar to the presented proof of Theorem 3 (Fig. 4).

Theorem 4 [39] *If* $P \neq NP$, *then the class* $\mathcal{Q}^*$ *is boundary for the dominating set problem.*

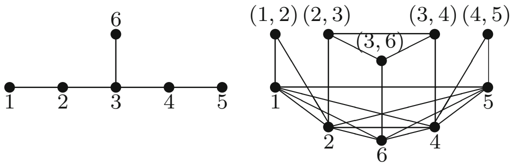


Fig. 4 The graphs $S_{1,2,2}$ and $Q^*(S_{1,2,2})$

Unfortunately, a complete description of all boundary classes for the dominating set problem is unknown.

5.3 The edge 3-colorability problem

In this subsection, we present boundary classes for the edge 3-colorability problem. Graph stretching is the main idea to reveal limit classes for it. The proof for their minimality presented below is surprising. Namely, we show that boundary classes included in the revealed limit classes must contain graphs of a special form. Next, we prove an "extendability property"—if a boundary class $\mathcal{X}$ contains a graph G having a vertex x of a special type, then there is a graph $H \in \mathcal{X}$ such that G is an induced subgraph of H and x is not a special-type vertex in G. This property and those mandatory graphs impose some structural "lower bounds" on boundary classes for the problem, which mean that those limit classes must be boundary.

Let G be a graph with two chosen vertices such that there is an automorphism of G mapping these vertices to each other. *Replacement of an edge* $e = (a, b)$ *by the graph* G is to delete e from a graph, identify a with one of the chosen vertices of G and b with the other chosen vertex of G. Clearly, the resultant graph does not depend on the choice of a vertex identified with a.

For a finite binary sequence π of length l, a π*-garland* is a graph obtained from a path with $2l+2$ vertices by replacements of its edges. For each $i \in \{1, 2, \ldots, l\}$, $2i$-th edge of this path is replaced by a $diamond$ (if $\pi_i = 0$) or by a *bug* (if $\pi_i = 1$), where the degree two vertices of the $diamond$ and the *bug* are chosen (see Figs. 5, 6).

Lemma 4 *For every graph and every finite binary sequence* π*, replacement of any its edge by a* π*-garland preserves edge 3-colorability.*

Proof Follows from the fact that replacement of any edge of any graph by the (1)-garland or by the (0)-garland preserves edge 3-colorability. This fact can be checked

Fig. 5 The graphs $diamond$ and *bug*

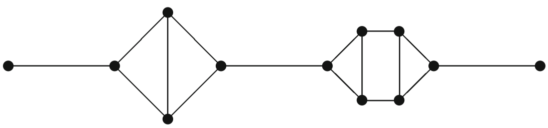


Fig. 6 The (0, 1)-garland

as follows. Let G be an arbitrary graph and $e = (a, b)$ be its edge. Let G' be obtained by replacement of e by the (0)-garland. That is, we delete e from G, add vertices v_1, v_2, v_3, v_4 and the edges $(a, v_1), (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_4, b)$. Suppose that G has a proper edge 3-coloring, in which e has the first color. For the graph G', keeping the colors of $E(G) \cap E(G')$, we color the edges $(a, v_1), (v_2, v_3), (v_4, b)$ in the first color, the edges (v_1, v_2) and (v_3, v_4) in the second one, the edges (v_1, v_3) and (v_2, v_4) in the third color. Hence, G' is edge 3-colorable. Suppose that G' is edge 3-colorable. The colors of $(a, v_1), (v_1, v_2), (v_1, v_3)$ are pairwise distinct. The same is true for $(b, v_4), (v_2, v_4), (v_3, v_4)$; $(v_1, v_2), (v_2, v_3), (v_2, v_4)$; $(v_1, v_3), (v_2, v_3), (v_3, v_4)$. Hence, the edges $(a, v_1), (v_2, v_3), (v_4, b)$ have the same color c^* in the coloring of G'. For the graph G, keeping the colors of $E(G) \cap E(G')$, we color the edge e in c^* to obtain a proper edge 3-coloring of G. The case of replacement of e by the (1)-garland is considered in a similar way. □

By S_π we denote a graph obtained by replacements of all edges of an $S_{1,1,1}$ by π-garlands.

Let π be an infinite binary sequence now and $\pi^{(l)}$ be its subsequence that consists of the first l members of π. The class $\mathcal{S}_\pi$ is the set $[\bigcup_{l=1}^{\infty}\{lS_{\pi^{(l)}}\}]$.

Lemma 5 *For every infinite binary sequence π, the class $\mathcal{S}_\pi$ is limit for the edge 3-colorability problem.*

Proof The edge 3-colorability problem is NP-complete in the class $\mathcal{X}_0$ of all graphs with degrees of all vertices equal to three [17]. For a graph $G \in \mathcal{X}_0$, let $G_{\pi^{(i)}}$ be a graph obtained from G by replacements of all its edges by $\pi^{(i)}$-garlands, where $\pi^{(i)}$ is the sequence that consists of the first i members of π. Let $\mathcal{X}_i$ be the hereditary closure of $\bigcup_{G\in\mathcal{X}_0}\{G_{\pi^{(i)}}\}$. Clearly, for each i, $\mathcal{X}_i$ is hard for the edge 3-colorability problem, as the problem for $\mathcal{X}_0$ can be polynomially reduced to the same problem for $\mathcal{X}_i$ by Lemma 4. Let $\mathcal{Y}_j = \bigcup_{i=j}^{\infty}\mathcal{X}_i$. Clearly, $\mathcal{Y}_1 \supseteq \mathcal{Y}_2 \supseteq \ldots$ and $\bigcap_{i=1}^{\infty}\mathcal{Y}_i = \mathcal{S}_\pi$. Hence, $\mathcal{S}_\pi$ is a limit class for the edge 3-colorability problem. □

A vertex x of a graph $G \in \mathcal{D}eg(3)$ is called *specific* if one of the following conditions holds:

(a) $deg_G(x) \leq 1$
(b) $deg_G(x) = 2$, and there exists a neighbor y of x such that $deg_G(y) \leq 2$
(c) $deg_G(x) = 2$, and x belongs to an induced *diamond* of G
(d) $deg_G(x) = 2$, and x belongs to an induced *bug* of G

Lemma 6 *Let $\mathcal{X}$ be an arbitrary boundary class for the edge 3-colorability problem, $G_1 \in \mathcal{X}$, and x be a specific vertex of G_1. Then $\mathcal{X} \subseteq \mathcal{D}eg(3)$ and it contains a graph G_2 such that G_1 is an induced subgraph of G_2 and x is not specific in G_2.*

Proof For every subcubic graph H and every its specific vertex x, the graph H is edge 3-colorable if and only if it so for $H\backslash\{x\}$. To make sure the correctness of this fact, one should verify that H is edge 3-colorable whenever $H\backslash\{x\}$ is edge 3-colorable. It is clear if $deg_H(x) \leq 1$. If $deg_H(x) = 2$, y is a neighbor of x in H having degree at most two, $H\backslash\{x\}$ is edge 3-colorable, then $H\backslash\{(x, y)\}$ is also edge 3-colorable, as x and y are degree one vertices in $H\backslash\{(x, y)\}$. As x and y are degree one vertices in $H\backslash\{(x, y)\}$, then H is also edge 3-colorable. If $deg_H(x) = 2$, x belongs to a *diamond* of H induced by vertices x, a, b, c, where $(x, c) \notin E(G)$, $H\backslash\{x\}$ is edge 3-colorable, then coloring of (x, a) in the color of (b, c) and (x, b) in the color of (a, c) produces a proper edge 3-coloring of H. If $deg_H(x) = 2$, x belongs to a *bug* of H induced by vertices x, a_1, b_1, a_2, b_2, y, where (x, a_1, b_1) and (y, a_2, b_2) are triangles of H, $(a_1, a_2) \in E(H)$, $(b_1, b_2) \in E(H)$, $H\backslash\{x\}$ is edge 3-colorable, then coloring of (x, a_1) in the color of (y, a_2) and (x, b_1) in the color of (y, b_2) produces a proper edge 3-coloring of H.

A necessary condition for a graph to be edge 3-colorable is to be subcubic. Let $\mathcal{Y}$ be a hard case for the edge 3-colorability problem. We remove all non-subcubic graphs from it. Next, we consider all induced subgraphs having no specific vertices of all graphs in the remaining part of $\mathcal{Y}$. The hereditary closure of the class of all subgraphs of this type is denoted by $\mathcal{Y}'$. It is a hard case for the problem, as the edge 3-colorability problem for $\mathcal{Y}$ can be polynomially reduced to the same problem for $\mathcal{Y}'$ by the first sentence of the previous paragraph. Notice that every graph in $\mathcal{Y}'$ is subcubic. In addition, the class $\mathcal{Y}'$ has the following extendability property. For any graph $G_1 \in \mathcal{Y}'$ and its specific vertex x, there is a graph $G_2 \in \mathcal{Y}'$ such that G_1 is an induced subgraph of G_2 and x is not specific in G_2. This follows from the fact that $\mathcal{Y}'$ is the hereditary closure of a set of subcubic graphs, whose every vertex is not specific. As $\mathcal{X}$ is boundary for the edge 3-colorability problem, there is a monotonically decreasing sequence $\{\mathcal{X}_i\}$ of hard classes for the problem converging to $\mathcal{X}$, each member of which is included in $\mathcal{D}eg(3)$ and has the extendability property. Therefore, $\mathcal{X} \subseteq \mathcal{D}eg(3)$. We will show that $\mathcal{X}$ also has the extendability property.

Let G be a graph in $\mathcal{X}$ such that some vertex x of G is specific. Clearly, for any i, $G \in \mathcal{X}_i$. Let $G_i \in \mathcal{X}_i$ be a graph such that G is an induced subgraph of G_i and x is not specific in G_i. It obligatory exists. We construct a graph G_i' as follows. If $deg_{G_i}(x) = 3$, then we delete all elements of $V(G_i)\backslash V(G)$ non-adjacent to x. If $deg_{G_i}(x) = 2$, then we delete all elements of $V(G_i)\backslash V(G)$ lying at the distance at least four from x. Taking into account that x is not specific in G_i and the "locality" of the specific vertex notion, it is easy to see that x is also not specific in G_i'. Moreover, $|V(G_i')|-|V(G)| \leq 14$ for any i, as G_i' is subcubic. Hence, the sequence $(G_1', G_2', \ldots)$ contains finitely many distinct graphs. Therefore, for some i^*, a graph G_{i^*}' belongs to infinitely many of the classes $\mathcal{X}_1, \mathcal{X}_2, \ldots$. As $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \ldots$, then G_{i^*}' belongs to each of these classes. Therefore, $G_{i^*}' \in \mathcal{X}$. We have a contradiction with the assumption. □

Theorem 5 *If $P \neq NP$, then, for every infinite binary sequence π, the class $\mathcal{S}_\pi$ is boundary for the edge 3-colorability problem.*

Proof Assume that there exists a class $\mathcal{X}$, boundary for the problem, such that $\mathcal{X} \subset \mathcal{S}_\pi$. Clearly, $\mathcal{X} \subseteq Free(\{L(S_{2,2,2})\})$, as $\mathcal{S}_\pi \subseteq Free(\{L(S_{2,2,2})\})$ by the definition of $\mathcal{S}_\pi$. First, we will show that a graph $iS_{1,1,1}$ belongs to $\mathcal{X}$ for each i. It is known that for every graphs $G_1 \in \mathcal{S}$ and $G_2 \in \mathcal{T}$ clique-width of any graph in $\mathcal{D}eg(3) \cap Free(\{G_1, G_2\})$ is bounded by some constant $C(G_1, G_2)$ [26]. Hence, $\mathcal{D}eg(3) \cap Free(\{G_1, G_2\})$ is an easy case for the edge 3-colorability problem [11]. The class $\mathcal{D}eg(3)$ is a finitely defined superclass of $\mathcal{X}$ by the previous lemma. Hence, if $i^*S_{1,1,1} \notin \mathcal{X}$ for some i^*, then $\mathcal{D}eg(3) \cap Free(\{i^*S_{1,1,1}, L(S_{2,2,2})\})$ is a finitely defined superclass of $\mathcal{X}$. Therefore, any monotonically decreasing sequence of hard classes converging to $\mathcal{X}$

must contain a class that is included in $\mathcal{D}eg(3) \cap Free(\{i^*S_{1,1,1}, L(S_{2,2,2})\})$. This class is easy for the edge 3-colorability problem by [11,26]. We have a contradiction with $P \neq NP$.

As $\mathcal{X} \subset \mathcal{S}_\pi$, then a graph $lS_\pi^{(l)}$ does not belong to $\mathcal{X}$ for some l. Let $G \in \mathcal{X}$ be a maximal proper induced subgraph of $lS_\pi^{(l)}$ that contains an $lS_{1,1,1}$ as an induced subgraph. Then the graph G obligatory has a vertex x such that x is a specific vertex of G. Let H be a minimal graph in $\mathcal{X}$ such that G is an induced subgraph of H and x is not a specific vertex of H. This graph exists by Lemma 6. We may consider that $deg_G(x) \leq 2$, otherwise deleting all elements of $V(H)\setminus V(G)$ except any neighbor of x produces an supergraph of G, which is a proper induced subgraph of $lS_\pi^{(l)}$. Hence, G is not maximal in the case $deg_G(x) \leq 2$. Suppose $deg_G(x) = 2$. By the minimality of H and by the structure of $lS_\pi^{(l)}$, H is obtained from G by adding exactly one vertex adjacent to x or several vertices, each of which is adjacent to a neighbor of x. By the structure of $\mathcal{S}_\pi$, H is also an induced subgraph of $lS_\pi^{(l)}$. We have a contradiction with the maximality of G. Hence, the strict inclusion $\mathcal{X} \subset \mathcal{S}_\pi$ is impossible. So, the class $\mathcal{S}_\pi$ is boundary for the problem for every infinite binary sequence π. □

Clearly, $\mathcal{S}_{\pi_1} \neq \mathcal{S}_{\pi_2}$ for every distinct infinite binary sequences π_1 and π_2. Hence, as the set of all binary infinite sequences has the continuum cardinality, the boundary system for the edge 3-colorability problem is also continuum. This result was initially proved in [30,32]. The boundary systems for the vertex k-colorability and edge k-colorability problems for every $k \geq 3$, the chromatic number problem also have the continuum cardinality [19,33,34].

Advances in complete descriptions of the boundary systems for the independent set and dominating set problems are minor. A natural idea arises that for some graph problems structure of boundary systems is too complex that is impossible to describe them completely. By Theorem 1, the cardinality of a boundary system can be interpreted as a complexity measure of the corresponding graph problem. It has been conjectured in [4] that there is a graph problem with an infinite boundary system, i.e. with a large value of the measure. One could consider this conjecture as a Gödel argument in the sense that a boundary system may be quite complicated and attempts to get its exhaustive description may look hopeless. Theorem 5 shows that the conjecture is a true statement.

5.4 The chromatic number and Hamiltonian cycle problems

A *subcubic tree* is a tree with degrees of all vertices at most three. A vertex is said to be *cubic* if it has three neighbors. A *caterpillar with hairs of an arbitrary length* is a subcubic tree, in which all cubic vertices belong to a single path. An example of such a graph is shown in Fig. 7.

The class $\mathfrak{S}$ constitutes all graphs, in which every connected component is a caterpillar with hairs of an arbitrary length. The class $R(\mathfrak{S})$ is the hereditary closure of the result of inscribing a triangle in each cubic vertex of every graph in $\mathfrak{S}$ (see Fig. 8).

The following result was obtained in [19]. We do not present a proof, since its ideas are graph transformations and reducibility between graph problems and we have already met with them.

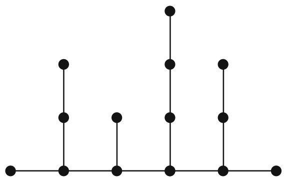


Fig. 7 A caterpillar with hairs of an arbitrary length

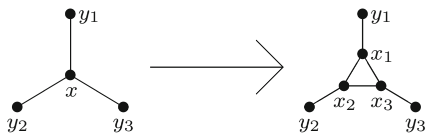


Fig. 8 Inscribing a *triangle* in x

Theorem 6 *If $P \neq NP$, then $co(\mathcal{T}) = \{G | \overline{G} \in \mathcal{T}\}$ is boundary for the chromatic number problem, $\mathfrak{S}$ and $R(\mathfrak{S})$ are boundary for the Hamiltonian cycle problem.*

The notion of a boundary graph class can be used for graph problems of a diverse nature, not only to algorithmic ones. An interested reader is referred to [20,25,27].

6 Complete descriptions of boundary systems

Perhaps, the most important issue in the theory of boundary classes is obtaining a comprehensive description of boundary systems. This question appears to be difficult to answer for many graph problems. The first and unique known result about complete descriptions of boundary systems has recently been obtained by one of the authors in [36], where a generalization of the edge k-colorability problem has been considered. This problem is called the list edge-ranking problem, which can be stated as follows.

We are given a graph G and a set $\mathfrak{L} = \{L(e) : e \in E(G)\}$, where $L(e)$ is a finite set of naturals that are feasible colors to color e. The *list edge-ranking problem* is to recognize whether G admits a mapping $c : E(G) \longrightarrow \bigcup_{e \in E(G)} L(e)$ such that: a) $c(e) \in L(e)$ for each $e \in E(G)$ b) if $c(e_1) = c(e_2)$, $e_1 \neq e_2$, then any path connecting e_1 and e_2 contains an edge $e_3 \in E(G)$ with $c(e_3) > c(e_2)$. Clearly, the last requirement generalizes the definition of a proper edge coloring, as it forbids to color any adjacent edges in the same color. The problem was firstly introduced in [13], and it has applications in parallel query processing [29] and in parallel assembly of modular products [12].

To define the boundary classes, we need to define some graphs. Graphs $Comb_i$, $Star_i$, Cam_i, $Comet_i$ are drawn in Fig. 9.

The class $\mathcal{C}liques$ is the set of all complete graphs, $\mathcal{B}at$ is the set of all complete bipartite graphs with at most two vertices in one of the parts, $\mathcal{C}omb$, $\mathcal{S}tar$, $\mathcal{C}am$, $\mathcal{C}omet$ are the hereditary closures of $\bigcup_{i=1}^{\infty}\{Comb_i\}$, $\bigcup_{i=1}^{\infty}\{Star_i\}$, $\bigcup_{i=1}^{\infty}\{Cam_i\}$, $\bigcup_{i=1}^{\infty}\{Comet_i\}$, respectively.

Graphs $\tilde{S}_i$ and $\tilde{T}_i$ are isomorphic to $S_{1,i,i}$ and $L(S_{1,i+1,i+1})$, respectively, graphs $\hat{S}_i$ and $\hat{T}_i$ are drawn in Fig. 10.

The classes $\tilde{\mathcal{S}}$, $\hat{\mathcal{S}}$, $\tilde{\mathcal{T}}$, $\hat{\mathcal{T}}$ are the hereditary closures of $\bigcup_{i=1}^{\infty}\{i\tilde{S}_i\}$, $\bigcup_{i=1}^{\infty}\{i\hat{S}_i\}$, $\bigcup_{i=1}^{\infty}\{i\tilde{T}_i\}$, $\bigcup_{i=1}^{\infty}\{i\hat{T}_i\}$, respectively.

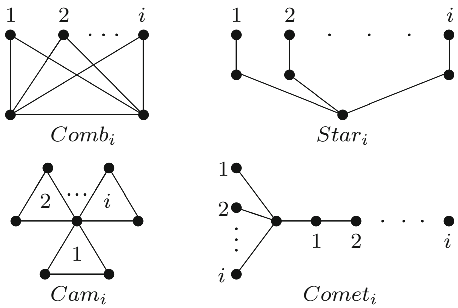


Fig. 9 Graphs $Comb_i, Star_i, Cam_i, Comet_i$

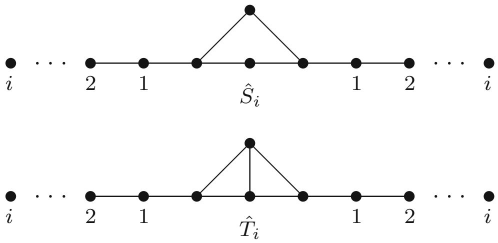


Fig. 10 Graphs $\hat{S}_i$ and $\hat{T}_i$

In [36], the following result was proved. We do not give its proof, since it is too long and difficult.

Theorem 7 *If $P \neq NP$, then the boundary system for the list edge-ranking problem consists of the classes $\mathcal{C}liques, \mathcal{B}at, \mathcal{C}omb, \mathcal{S}tar, \mathcal{C}am, \mathcal{C}omet, \tilde{\mathcal{S}}, \hat{\mathcal{S}}, \tilde{\mathcal{T}}, \hat{\mathcal{T}}$.*

Theorem 1, a general result, does not claim that a finitely defined class is Π-easy if it contains no Π-boundary classes. Applied to the list edge-ranking problem, we really have a complete complexity dichotomy (a "zero-one law") in the sense that any finitely defined class is easy or hard for the problem.

Theorem 8 [36] *If a finitely defined class contains at least one of the classes $\mathcal{C}liques, \mathcal{B}at, \mathcal{C}omb, \mathcal{S}tar, \mathcal{C}am, \mathcal{C}omet, \tilde{\mathcal{S}}, \hat{\mathcal{S}}, \tilde{\mathcal{T}}, \hat{\mathcal{T}}$, then it is hard for the list edge-ranking problem. Otherwise, it is easy for the problem.*

By Theorem 8, we have a complete description of all finitely defined easy cases for the edge list-ranking problem. This rises the following natural question. How to apply Theorem 8 for a given finitely defined class? How to decide whether it contains at least one of the ten classes? To this end, one could use a more simple, graphic form of Theorem 8. Let us demonstrate it on the example of the classes $Free(\{P_6, K_3, C_4\})$ and $Free(\{P_5, S_{1,1,1}, K_4\})$. We fill two criterion tables by pluses and minuses as follows. We put "+" if and only if a graph in the lists of forbidden induced subgraphs belongs to one of the ten classes (Tables 1, 2).

Theorem 8 can be reformulated as follows. A class is hard for the list edge-ranking problem if there is a column having only minuses. Otherwise, it is easy. Hence, $Free(\{P_6, K_3, C_4\})$ is hard, but $Free(\{P_5, S_{1,1,1}, K_4)\})$ is easy.

When the set of all Π-boundary classes is completely known, a table reformulation of Theorem 1 could be more useful than the original. Indeed, for a given class

Table 1 A criterion table for $Free(\{P_6, K_3, C_4\})$

Graph	$\mathcal{C}liques$	$\mathcal{B}at$	$\mathcal{C}omb$	$\mathcal{S}tar$	$\mathcal{C}am$	$\mathcal{C}omet$	$\tilde{\mathcal{S}}$	$\hat{\mathcal{S}}$	$\tilde{\mathcal{T}}$	$\hat{\mathcal{T}}$
P_6	–	–	–	–	–	+	+	+	+	+
K_3	+	–	+	–	+	–	–	–	+	+
C_4	–	+	–	–	–	–	–	+	–	–

Table 2 A criterion table for $Free(\{P_5, S_{1,1,1}, K_4)\})$

Graph	$\mathcal{C}liques$	$\mathcal{B}at$	$\mathcal{C}omb$	$\mathcal{S}tar$	$\mathcal{C}am$	$\mathcal{C}omet$	$\tilde{\mathcal{S}}$	$\hat{\mathcal{S}}$	$\tilde{\mathcal{T}}$	$\hat{\mathcal{T}}$
P_5	–	–	–	+	–	+	+	+	+	+
$S_{1,1,1}$	–	+	+	+	+	+	–	–	+	+
K_4	+	–	–	–	–	–	–	–	–	–

$\mathcal{X} = Free(\{G_1, \ldots, G_s\})$, one may construct a table, whose rows correspond to $G_1, \ldots, G_s$ and columns correspond to Π-boundary classes. If a graph G_i belongs to jth Π-boundary class, the we put "+" into the ij-cell and "−" otherwise. By Theorem 1, $\mathcal{X}$ is Π-hard if and only if there is a column containing only minuses.

7 Applications of the boundary class notion in the analysis of the computational complexity

Theorem 8 is a unique known example of a complete complexity dichotomy in the family of all finitely defined graph classes. This fact certifies the opinion that obtaining a complete dichotomy in the family is a difficult task for many graph problems. A natural idea comes to mind is to consider a subfamily of the hereditary classes family and try to solve the problem specifically for it. One of the best examples in this field is connected to monotone classes. Indeed, a finitely defined monotone graph class including $\mathcal{S}$ is hard for the independent set problem by Theorems 1 and 2. On the other hand, any monotone graph class not including $\mathcal{S}$ is easy for it by Lemma 2 and [11]. Hence, we have the following result.

Theorem 9 *A finitely defined monotone class is hard for the independent set problem if it contains $\mathcal{S}$. Otherwise, it is easy for it.*

The last theorem also holds for the dominating set problem.

Another example of a "good" subfamily is a set of all classes defined by small forbidden induced subgraphs. Combining results of [2,3,21,24], we obtain the following result.

Theorem 10 *Let $\mathcal{X}$ be a set of graphs with at most five vertices. Then the independent set problem is hard for $Free(\mathcal{X})$ if it contains $\mathcal{S}$. Otherwise, it is easy.*

Korobitsyn has considered in [18] the so-called *monogenic graph classes*, i.e. classes defined by a single forbidden induced structure. He also proved there that the dominating set problem is polynomial for $Free(\{G\})$ if G is isomorphic to $P_i + pK_1$, where $i \leq 4$ and p is arbitrary. Moreover, the problem is NP-complete for all other choices of G [18]. This result can be rewritten as follows.

Theorem 11 *A monogenic graph class $\mathcal{X}$ is hard for the dominating set problem if $\mathcal{S} \subseteq \mathcal{X}$ or $\mathcal{T} \subseteq \mathcal{X}$ or $\mathcal{Q} \subseteq \mathcal{X}$. It is easy in all other cases.*

The complexity of the dominating set problem was considered in [39,40] for classes defined by small forbidden induced structures. Namely, the following result has been proved.

Theorem 12 *Let $\mathcal{X}$ be a set of graphs with at most five vertices. The class $Free(\mathcal{X})$ is hard for the dominating set problem if $\mathcal{S} \subseteq \mathcal{X}$ or $\mathcal{T} \subseteq \mathcal{X}$ or $\mathcal{Q} \subseteq \mathcal{X}$. It is easy in all other cases.*

Of course, the boundary class notion helps to prove only half of each of Theorems 9, 11, 12, as, by Theorem 1, it can certify only NP-completeness of a graph problem for a finitely defined class. To prove the second half, the corresponding polynomial-time algorithms should be invented for all classes in the families not including the boundary classes. So, Theorems 9, 11, 12 are concrete examples of a successful application of the general method for obtaining complexity dichotomies based on boundary classes: prove NP-completeness for some classes in a family by applying Theorem 1 and design polynomial-time algorithms for all of the remaining classes.

Clearly, any result on a complexity dichotomy in a subfamily of hereditary classes defined by small forbidden induced subgraphs can be formulated in terms of an explicit description of "easy" prohibitions not in terms of boundary classes. It was done in [1,9,15,16,22,37,38,41,42,46] and many other papers. At the same time, the size of an answer can quickly grow with the size of the prohibitions. The notion of a boundary class helps to represent the answer more compactly.

8 Conclusions and open problems

In this paper, we considered the notion of a boundary graph class, which is a helpful tool for analyzing the computational complexity of graph problems in the family of finitely defined classes. This notion is interesting in that a graph problem is NP-complete for a finitely defined graph class $\mathcal{X}$ if and only if $\mathcal{X}$ includes a boundary class for the problem. Therefore, discovering boundary classes for various graph problems is of interest. We described all known boundary classes for some classical graph problems: the independent set and the dominating set problems, the Hamiltonian cycle problem. For the edge 3-colorability problem, we constructively showed that the boundary system has the continuum cardinality. We gave a complete description of all boundary classes for the so-called list edge-ranking problem. At length, we presented several examples on how boundary classes present a complete complexity dichotomy in a "simple" subfamily of the hereditary classes family.

Despite some achievements, the theory of boundary graph classes is still full of open questions. Perhaps, the oldest open question here is the Alekseev's conjecture.

Open problem 1 *Is $\mathcal{S}$ a unique boundary class for the independent set problem?*

Similar questions can be asked for the dominating set and Hamiltonian cycle problems.

Open problem 2 *Is there is a boundary graph class for the dominating set problem distinct to $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$, $\mathcal{Q}^*$, simultaneously?*

Open problem 3 *Is there is a boundary graph class for the Hamiltonian cycle problem distinct to $\mathfrak{S}$ and $R(\mathfrak{S})$, simultaneously?*

As we mentioned before, the cardinality of a boundary system can be considered as a complexity measure of a graph problem. It would be interesting to know what values can take this measure.

Open problem 4 *What are possible cardinal numbers of boundary systems of graph problems?*

The set of all graphs is countable. The set of all finite subsets of a countable set is also countable [47]. Every finitely defined class can be described by a finite set of its forbidden induced subgraphs. Hence, the set of all finitely defined graph classes is also countable. Therefore, the boundary system for the edge 3-colorability problem is redundant for complexity classifying in the family of all finitely defined classes, as the system is continuum. This observation leads to the notion of a criterial system. For a graph problem Π, a Π-*criterial system* is any countable subset of the Π-boundary system that is enough to classify the complexity of Π in the family of all finitely defined graph classes. Such a system obligatory exists, as we can take the union $\bigcup_{\mathcal{X}}\{\mathcal{Y}_{\mathcal{X}}\}$ over all Π-hard finitely defined classes $\mathcal{X}$, where $\mathcal{Y}_{\mathcal{X}}$ is any Π-boundary class included in $\mathcal{X}$.

Theorem 13 *If the Π-boundary system is finite, then there is a unique Π-criterial system coinciding with the Π-boundary system.*

Proof Let some Π-criterial system consists of classes $\mathcal{X}_1, \ldots, \mathcal{X}_k$ and do not contain a Π-boundary class $\mathcal{X}$. Let $Forb(\mathcal{X}) = \{G_1, G_2, \ldots, G_s, \ldots\}$. This set must be infinite, otherwise $\mathcal{X}$ is finitely defined and it includes a Π-boundary class $\mathcal{X}$. Hence, it must be a Π-hard class. As $\{\mathcal{X}_1, \ldots, \mathcal{X}_k\}$ is a Π-criterial system, $\mathcal{X}$ must contain one of its elements. We obtain that one of Π-boundary classes contains another Π-boundary class. It is impossible. We have a contradiction.

For each i, let $\mathcal{Y}_i = Free(\{G_1, \ldots, G_i\})$. It is a Π-hard class, as it includes $\mathcal{X}$. As $\{\mathcal{X}_1, \ldots, \mathcal{X}_k\}$ is a Π-criterial system, then, for each i, there is a number j_i such that $\mathcal{Y}_i$ includes $\mathcal{X}_{j_i}$. The infinite sequence $j_1, j_2, \ldots$ has finitely many distinct elements. Hence, some number $1 \leq i^* \leq k$ appears in the sequence infinitely many times. Therefore, we can find an infinite subsequence in the sequence $\{\mathcal{Y}_i\}$ such that each its member includes $\mathcal{X}_{i^*}$. This subsequence also converges to $\mathcal{X}$. Hence, $\mathcal{X} \supseteq \mathcal{X}_{i^*}$. In other words, one of Π-boundary classes includes other Π-boundary class. We have a contradiction. □

For the edge 3-colorability problem, any criterial system is distinct to the boundary system, as the boundary system has the continuum cardinality. Perhaps, the problem of finding out its criterial system is much simpler than the boundary system. This raises the following open problem.

Open problem 5 *What is a criterial system for the edge 3-colorability problem?*

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