Efficiency and probabilistic properties of bridge volatility estimator

S. Lapinova[a], A. Saichev[b,*], M. Tarakanova[b]

[a] *National Research University Higher School of Economics, Russia*
[b] *Nizhni Novgorod State University, Russia*

Abstract

We discuss efficiency of the quadratic bridge volatility estimator in comparison with Parkinson, Garman-Klass and Roger-Satchell estimators. It is shown in particular that point and interval estimations of volatility, resting on bridge estimator, are considerably more efficient than analogous estimations, resting on Parkinson, Garman-Klass and Roger-Satchell one.

Keywords: volatility estimators, probabilistic properties, efficiency, unbiasedness

1. Introduction

Volatility, defined as the variance of the increments of the log-price over a specific time interval, is an universally used risk indicator. With the growing availability of high-frequency tick-by-tick price time series, a number of new efficient volatility estimators have been developed (see, for instance, [1, 2, 3, 4, 5]).

We present here a comparative analysis of the efficiency of the quadratic bridge volatility estimator and the well-known Parkinson (PARK) [6], Garman-Klass (GK) [7] and Roger-Satchell (RS) [8, 9] volatility estimators, based on high and low values of the log-price increments within given time intervals. Some fruitful information and detailed analysis of stochastic prices models and efficiency of high-low-close volatility estimators one can find at the recent article [10]. Detailed and valuable discussion of stochastic volatility

*Bodenacherstrasse 67, 8121, Benglen, Switzerland. Phone 044577050541
Email address: saichev@hotmail.com (A. Saichev)

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models and volatility estimators, related to the topic of the present paper, are provided in G. Ramey and V. Ramey [11] and in Bonanno et al. [12, 13].

We find that the high-low quadratic bridge estimator, suggested in this work, is significantly more efficient than the above-mentioned PARK, GK and RS estimators, at least in the frame of the geometric Brownian motion with a drift model of the price stochastic process. Notice that some related results concerning statistical properties of volatility estimators was obtained in Saichev et al. [14], were have discussed constructions of most efficient volatility estimators. It was shown that efficiency of pointed out most efficient estimators are close to efficiency of suggested in this paper quadratic bridge estimator. The shortcoming of most efficient estimators, discussed in [14], is that they have much more complicated construction than quadratic bridge estimator.

For the Brownian motion model of log-price process, advantage of the quadratic bridge estimator can be intuitively understand as follows: It is well-known that the high and low values of a Brownian motion process are most probably found in the neighborhood of the edges of the observation interval. In contrast, by construction of the bridge, its high and low values are in general distant from the edges. As a result, the high and low of a bridge incorporate significantly more information about the variability of the original stochastic process than its own high and low values.

The paper is organized as follows. In section 2 a short description of high-low volatility estimators, including the quadratic bridge estimator, suggested in this work, is given. In section 3, the statistical description of high-low volatility estimators, in the frame of the Brownian motion model of the log-price stochastic process, is discussed in detail. In section 4, we compare the efficiency of PARK and quadratic bridge estimators. It is shown that, in contrast to the PARK estimator, the bridge one is unbiased for all drift values and has considerable smaller variance than the PARK estimator. In section 5, we give a comparative probabilistic analysis of the interval estimations, resting on bridge, PARK, GK and RS volatility estimators. In section 6 the results of statistical testing of the above-mentioned volatility estimators are described. In section 7 we draw the conclusions.

2. Examples of volatility estimators

Consider dependence on time t of the price $P(t)$ of some financial instrument. As a rule, at discussing of volatility, one consider its logarithm

$$X(t) := \ln P(t).$$

Let point out one of the conventional volatility $V(T)$ definition, which we are using in this work: It is the variance

$$V(T) := \mathbf{Var}\left[Y(t,T)\right] = \mathbf{E}\left[Y^2(t,T)\right] - \mathbf{E}^2\left[Y(t,T)\right]$$

of the log-price increment $Y(t,T) := X(t+T) - X(t)$ within given time interval duration T.

Recall, GK [7], PARK [6] and RS [8] volatility estimators are resting on the high and low values:

$$H := \sup_{t' \in (0,T)} Y(t,t'), \qquad L := \inf_{t' \in (0,T)} Y(t,t').$$

Accordingly, PARK estimator is equal to

$$\hat{V}_p := (H-L)^2 \big/ \ln 16, \tag{1}$$

while GK estimator given by expression

$$\begin{gathered} \hat{V}_g := k_1 (H-L)^2 - k_2 (C(H-L) - 2HL) - k_3 C^2, \\ k_1 = 0.511, \qquad k_2 = 0.0109, \qquad k_3 = 0.383. \end{gathered} \tag{2}$$

Here $C := Y(t,T)$ is the close value of the log-price increment. Recall else RS estimator, equal to

$$\hat{V}_r := H(H-C) + L(L-C).$$

Besides of mentioned well-known estimators, we discuss the quadratic bridge estimator. Below we call it shortly by *bridge estimator*. Before to define it, recall definition of the bridge $Z(t,t')$ of stochastic process $Y(t,t')$. It is equal to

$$Z(t,t') := Y(t,t') - \frac{t'}{T}\, Y(t,T), \qquad t' \in (0,T). \tag{3}$$

Let introduce high and low of the bridge:

$$\mathcal{H} := \max_{t'\in(0,T)} Z(t,t'), \qquad \mathcal{L} := \min_{t'\in(0,T)} Z(t,t').$$

Accordingly, mentioned above bridge volatility estimator given by

$$\hat{V}_b := \kappa \left(\mathcal{H} - \mathcal{L}\right)^2 . \tag{4}$$

The value of the factor κ will be calculated later on.

3. Geometric Brownian motion

One of conventional models of price stochastic behavior is geometric Brownian motion (see [15, 16, 17]). In particular, it is used in theoretical justification of GK, PARK and RS estimators. Below we discuss statistics of mentioned volatility estimators in the frame of geometric Brownian motion model. Namely, we assume that increment of the log-price is of the form

$$Y(t,T) = \mu T + \sigma B(T).$$

Here μ is the drift of the price, while $B(t)$ is the standard Brownian motion $B(t) \sim \mathcal{N}(0,t)$. Factor σ^2 is the intensity of the Brownian motion.

Recall, Brownian motion posses by self-similar property

$$B(t) \sim \sqrt{T}\, B\left(t/T\right), \qquad \forall\; T > 0,$$

where and below sign $\sim$ means identity in law.

Using pointed out self-similar property, one can ensure that

$$\begin{gathered} Y(t,t') \sim \sigma\sqrt{T}\; x(\tau,\gamma), \\ x(\tau,\gamma) := \gamma\tau + B(\tau), \qquad \gamma := \mu\sqrt{T}/\sigma, \qquad \tau := t'/T \in (0,1). \end{gathered} \tag{5}$$

Henceforth we call process $x(\tau,\gamma)$ by *canonical Brownian motion*, while factor γ by *canonical drift*. Using relations (1), (2), (4) and (5), one find that

$$\begin{gathered} \hat{V}_p \sim V(T)\cdot \hat{v}_p(\gamma), \qquad \hat{V}_g \sim V(T)\cdot \hat{v}_g(\gamma), \qquad \hat{V}_b \sim V(T)\cdot \hat{v}_b, \\ \hat{V}_r \sim V(T)\cdot \hat{v}_r(\gamma), \qquad V(T) = \sigma^2 T. \end{gathered}$$

We have used above *canonical estimators*:

$$\begin{aligned} &\hat{v}_p(\gamma) := d^2\big/\ln 16, \qquad \hat{v}_b := \kappa s^2, \qquad d := h - l, \qquad s := \xi - \zeta, \\ &\hat{v}_g(\gamma) := k_1 d^2 - k_2(cd - 2hc) - k_3 c^2, \qquad \hat{v}_r = h(h-c) + l(l-c), \end{aligned} \tag{6}$$

containing high, low and close values

$$h := \sup_{\tau\in(0,1)} x(\tau,\gamma), \qquad l := \inf_{\tau\in(0,1)} x(\tau,\gamma), \qquad c := x(1,\gamma), \tag{7}$$

of canonical Brownian motion, and high and low values

$$\xi := \sup_{\tau\in(0,1)} z(\tau), \qquad \zeta := \inf_{\tau\in(0,1)} z(\tau), \tag{8}$$

of the canonical bridge

$$z(\tau) := x(\tau,\gamma) - \tau x(1,\gamma) = B(\tau) - \tau\cdot B(1), \qquad \tau\in(0,1). \tag{9}$$

Plots of the typical paths of the canonical Brownian motion $x(\tau,\gamma)$ (5) for $\gamma = 1$ and corresponding canonical bridge $z(\tau)$ (9) are given in figure 1.

It is worthwhile to note that the closer expected values of canonical estimators $\hat{v}_p(\gamma)$, $\hat{v}_g(\gamma)$, $\hat{v}_r$ and $\hat{v}_b$ to unity, the less biased corresponding original volatility estimators. Analogously, the smaller variances of canonical estimators the more efficient original volatility estimators $\hat{V}_p$, $\hat{V}_g$, $\hat{V}_r$ and $\hat{V}_b$.

Notice additionally that canonical drift γ of the canonical Brownian motion $x(\tau,\gamma)$ (5) is, as a rule, unknown. Nevertheless, to get some idea about dependence on drift μ of bias and efficiency of volatility estimators, we will discuss below in detail dependence of canonical estimators statistical properties on possible values of the factor γ.

4. Comparative efficiency of PARK and bridge estimators

Resting on, given at Appendix, analytical formulas for probability density functions (pdfs) of random variables (7) and (8), we explore in this section some atatistical properties of canonical PARK estimator $\hat{v}_p(\gamma)$ and bridge one $\hat{v}_b$ (6).

Let check, first of all, unbiasedness of canonical PARK estimator. To make it, let calculate, with help of pdf $q_x(\delta)$ (A.4), mean square of oscillation

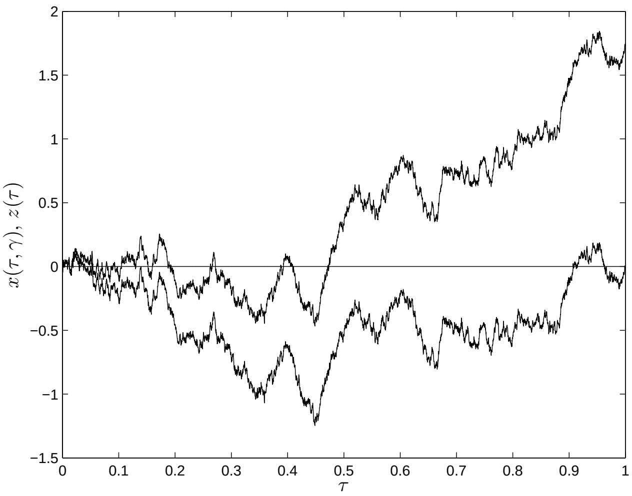

Figure 1: Typical paths of canonical Brownian motion $x(\tau, \gamma)$ (5) for $\gamma = 1$ and corresponding canonical bridge $z(\tau)$ (9)

$d = h - l$ of the canonical Brownian motion $x(\tau, \gamma)$ at the zero canonical drift ($\gamma = 0$). After simple calculations obtain

$$\mathbf{E}[d^2] = 2 + \sum_{m=1}^{\infty} \frac{2}{m(4m^2 - 1)} = \ln 16.$$

From here and from expression (6) of canonical PARK estimator $\hat{v}_p(\gamma)$ one can see that the following expression is true

$$\mathbf{E}[\hat{v}_p(\gamma = 0)] = 1.$$

Let find now the factor κ at expressions (4) and (6). To make it, calculate first of all the mean square of the bridge oscillation. Due to expression (A.5) for the bridge oscillation s (6) pdf, one have

$$\mathbf{E}[s^2] = \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

Accordingly, unbiased canonical bridge estimator has the form

$$\mathbf{E}[\hat{v}_b] = 1 \quad \Rightarrow \quad \kappa = 1\big/\mathbf{E}[s^2] \quad \Rightarrow \quad \hat{v}_b = 6\, s^2\big/\pi^2. \tag{10}$$

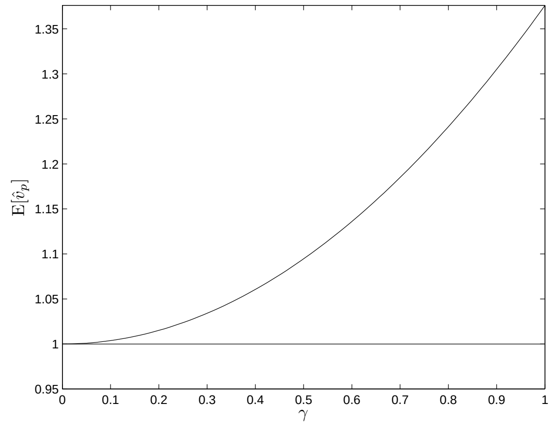

Figure 2: Plot of canonical PARK estimator $\hat{v}_p(\gamma)$ mean value, as function of canonical drift γ. It is seen that with growth of γ PARK estimator becomes more and more biased. Straight line is the plot of canonical bridge $\hat{v}_b$, mean value

The great advantage of the bridge estimator is its unbiasedness for any drift. This remarkable property of the pointed out estimator is the consequence of the fact that the bridge $Z(t, t')$ (3) and its canonical counterpart $z(\tau)$ don't depend on the drift μ (canonical drift γ) at all. On the contrary, PARK estimator becomes essentially biased at nonzero drift. In figure 2 depicted dependence on γ of canonical PARK estimator expected value, illustrating bias of PARK estimator at nonzero drift. Corresponding curve obtained with help of analytical expression (A.3) for canonical Brownian motion's oscillation d pdf.

Let calculate variances of canonical PARK and bridge estimators. After substitution into the rhs of expression

$$\mathbf{E}[\hat{v}_p^2(\gamma = 0)] := \frac{1}{\ln^2 16} \int_0^\infty \delta^4 q_x(\delta) d\delta$$

the sum (A.4) for the canonical Brownian motion oscillation pdf $q_x(\delta)$, and after summation obtain for $\gamma = 0$:

$$\mathbf{E}[\hat{v}_p^2(\gamma = 0)] = 9\,\zeta(3) \big/ \ln^2 16 \simeq 1.40733.$$

Accordingly, variance of canonical PARK estimator $\hat{v}_p$ is

$$\mathbf{Var}[\hat{v}_p(0)] = \frac{9\,\zeta(3)}{\ln^2 16} - 1 \simeq 0.407. \tag{11}$$

As the next step, we calculate variance of canonical bridge estimator $\hat{v}_b$ (10). Sought variance is equal to

$$\mathbf{Var}[\hat{v}_b] := \frac{36}{\pi^4}\,\mathbf{E}[s^4] - 1.$$

After substitution here, following from (A.5), relation

$$\mathbf{E}[s^4] := \int_0^2 \delta^4 q_b(\delta) d\delta = 3 \sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{30},$$

obtain

$$\mathbf{Var}[\hat{v}_b] = \frac{6}{5} - 1 = 0.2. \tag{12}$$

Comparing equalities (11) and (12), one can see that variance of bridge estimator is approximately twice smaller than variance of PARK estimator.

Recall, variance of bridge estimator does not depend on drift. On the contrary, variance of PARK estimator essentially depends on the drift. One can see it in figure 3, where depicted plot of dependence, on canonical drift γ, of canonical PARK estimator variance.

Notice else that bias of some estimator is insignificant only if it is much smaller than rms of corresponding estimator, i.e. is small the relative bias:

$$\varrho := \frac{\mathbf{E}[\hat{v}(\gamma)] - 1}{\sqrt{\mathbf{Var}[\hat{v}(\gamma)]}}. \tag{13}$$

Plot of canonical PARK estimator relative bias, as function of canonical drift γ depicted in figure 4.

5. Interval estimations on the basis of PARK and bridge estimators

Given at Appendix analytical expressions (A.3), (A.4) and (A.5) for canonical Brownian motion and canonical bridge random oscillations pdfs allow us to explore in detail probabilistic properties of PARK and bridge canonical estimators. Let find, at first, pdfs of mentioned canonical estimators random values. It is well-known from Probabilistic Theory that pdf

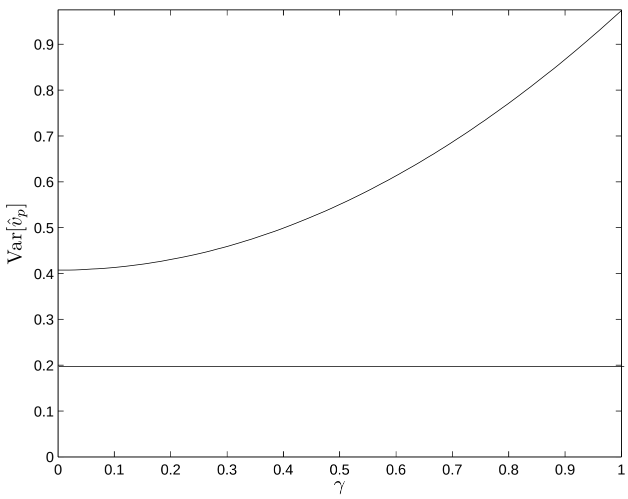

Figure 3: Plots of dependence on γ of canonical PARK estimator variance. Straight line is the variance of canonical bridge estimator

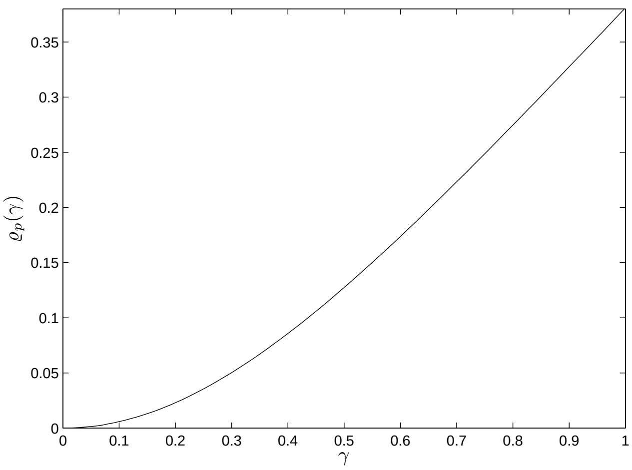

Figure 4: Plot of relative bias (13) of canonical PARK estimator as function of canonical drift γ

$W_p(x;\gamma)$ of canonical PARK estimator is expressed through pdf $q_x(\delta;\gamma)$ (A.3) of canonical Brownian motion oscillation by the relation

$$W_p(x;\gamma) = \sqrt{\frac{\alpha}{4x}}\, q_x\left(\sqrt{\alpha x};\gamma\right), \qquad \alpha = \ln 16. \tag{14}$$

Similarly, pdf of canonical bridge estimator is equal to

$$W_b(x) = \sqrt{\frac{\alpha}{4x}}\, q_b\left(\sqrt{\alpha x}\right), \qquad \alpha = \frac{\pi^2}{6}. \tag{15}$$

Here $q_b(\delta)$ (A.5) is the pdf of canonical bridge oscillation. Plots of canonical PARK estimator pdf, for $\gamma = 0$, and pdf of canonical bridge estimator are depicted in figure 5. In figure 6 are comparing pdfs of canonical PARK estimator, for $\gamma = 1$, and pdf of canonical bridge estimator. It is seen in both figures that pdf of canonical bridge estimator is better concentrated around its expected value $\mathbf{E}[\hat{v}_b] = 1$ than canonical PARK estimator pdf.

Knowing estimators pdfs, one can produce interval estimations of possible volatility values. Consider typical interval estimation: Let $\hat{V}$ is some volatility estimator, equal to

$$\hat{V} = V(T)\cdot\hat{v}. \tag{16}$$

Here $\hat{v}$ is corresponding canonical estimator, while $V(T)$ is the measured volatility. One needs to find probability

$$F(N) := \mathbf{Pr}\left\{V(T) < N\cdot\hat{V}\right\}$$

that unknown (random) volatility $V(T)$ is not more than N times exceeds known (measured) volatility estimated value $\hat{V}$. It follows from (16) that following inequalities are equivalent:

$$V(T) < N\cdot\hat{V} \qquad \Leftrightarrow \qquad \hat{v} > 1/N.$$

Last means in turn that sought probability $F(N)$ is expressed through pdf of canonical estimator $\hat{v}$ by the following way:

$$F(N) = \mathbf{Pr}\left\{\hat{v} > 1/N\right\} = \int_{1/N}^{\infty} W(x)dx. \tag{17}$$

Here $W(x)$ is the pdf of canonical estimator $\hat{v}$.

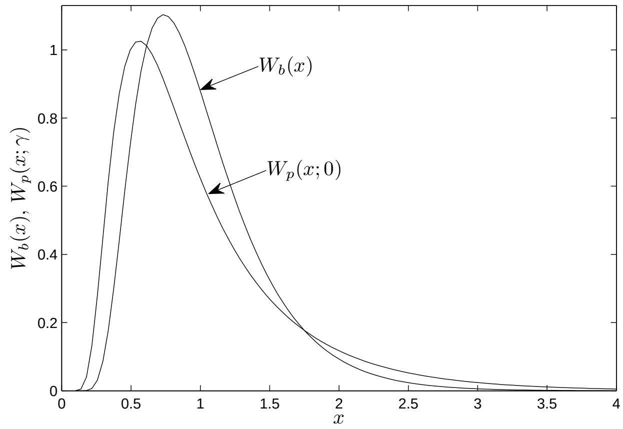

Figure 5: Plots of canonical PARK and bridge estimators pdfs, clearly demonstrating "probabilistic preference" of bridge estimator in compare with PARK one

Calculations, based on the relations (14), (15) and (17) give the result that the probability that the true volatility is less than twice of given bridge volatility estimator value $\hat{V}_b$ is equal to $F_b(2) = 0.918$. This is substantially larger than the analogous probability in the case of the PARK estimator: $F_p(2, \gamma = 0) \simeq 0.813$. The plots of the probabilities $F(N)$ (Eq. (17)) as a function of the level N, for the PARK estimator (in the case of zero drift $\mu = 0$) and for the bridge volatility estimator, are shown in figure 7.

6. Comparative statistics of canonical estimators

Above, we explored in detail statistical properties of two, PARK and bridge estimators. Here we compare their statistics and statistics of another well-known volatility estimators: GK and RS one. Despite to previous chapters, where we have used known analytical expressions for pdfs of canonical PARK and the bridge estimators, below we use predominantly results of numerical simulations.

Namely, we produce $M \gg 1$ numerical simulations of random sequences

$$x_n(\gamma) := \gamma \frac{n}{N} + \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \epsilon_n, \qquad n = 0, 1, \ldots, N, \qquad x_0(\gamma) = 0,$$

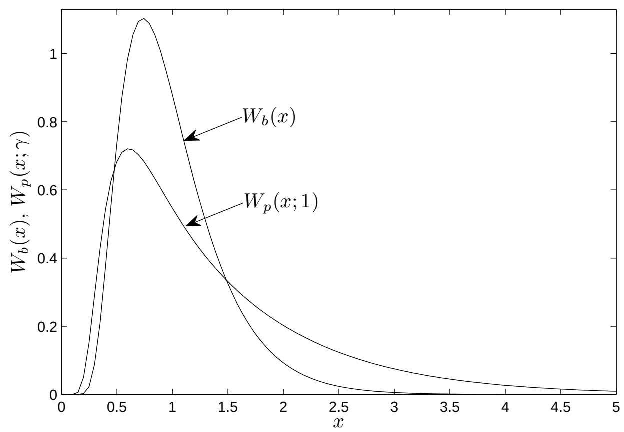

Fig. 6

Figure 6: Plots of PARK and bridge canonical estimators pdfs for $\gamma = 1$

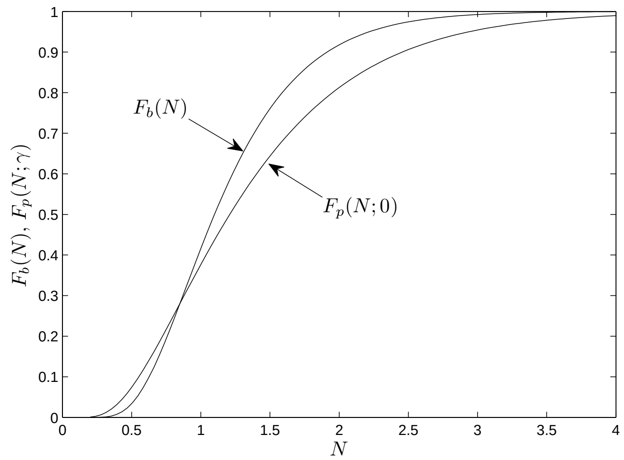

Figure 7: Plots of probabilities $F_p(N)$ and $F_b(N)$ that true volatility is less than N times exceeds values of PARK and bridge estimators

where $\{\epsilon_n\}$ are iid Gaussian variables $\sim \mathcal{N}(0,1)$. Notice that stochastic process $x_n(\gamma)$ of discrete argument n rather accurately approximates, for large $N \gg 1$, paths of canonical Brownian motion $x(\tau,\gamma)$ (5).

Knowing M iid sequences $\{x_n(\gamma)\}$ one can find corresponding iid samples of pointed out above canonical estimators. Everywhere below we take number of iid samples M and discretization number N equal to

$$N = 5 \cdot 10^3, \qquad M = 5 \cdot 10^5.$$

Plots in figure 8 demonstrate rather convincingly accuracy of numerical simulations. In figure 9 are given two hundred samples of canonical GK, RS, bridge and PARK estimators, ensuring "by naked eye" that canonical bridge estimator is more efficient than GK one.

In figures 10 and 11 are given, obtained by numerical simulations, plots of canonical GK, PARK, RS and bridge estimators mean values and variances. These plots clearly demonstrate unbiasedness and high efficiency of the bridge estimator in comparison with PARK, GK and RS estimators.

Finally, in figure 12 are shown the plots of probabilities that the true volatility $V(T)$ is larger than half of corresponding estimator value and less than twice of it:

$$P_\Delta := \mathbf{Pr}\left\{\hat{V}/2 < V(T) < 2\hat{V}\right\} = \int_{1/2}^{2} W(x)dx. \tag{18}$$

It is seen that for any γ mentioned probability is essentially larger for bridge estimator, than for GK, RS and PARK estimators.

7. Conclusions

In this work we have analyzed statistical properties of the quadratic bridge volatility estimator, which is significantly more efficient than most, known before, high-low-close volatility estimators.

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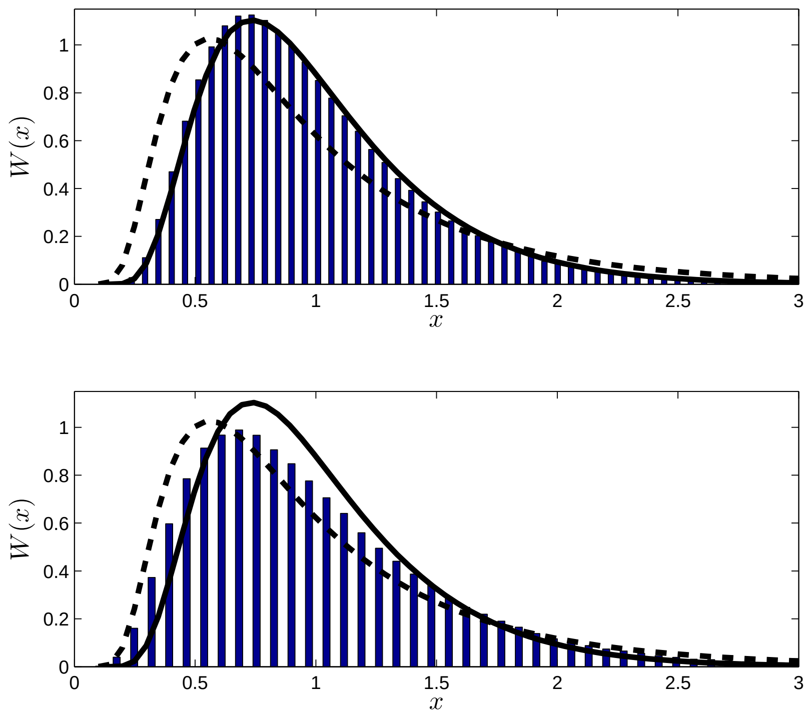

Figure 8: **Upper panel:** Histogram of M samples of canonical bridge estimator $\hat{v}_b$. Solid line is the plot of canonical bridge estimator's pdf, given by analytical expression (15), (A.5). Dashed line is the pdf of canonical PARK estimator for $\gamma = 0$. **Lower panel:** Histogram of M samples of canonical GK estimator $\hat{v}_g$ for $\gamma = 0$. Solid line is the plot of the canonical bridge estimator pdf. Dashed line is the canonical PARK estimator pdf for $\gamma = 0$

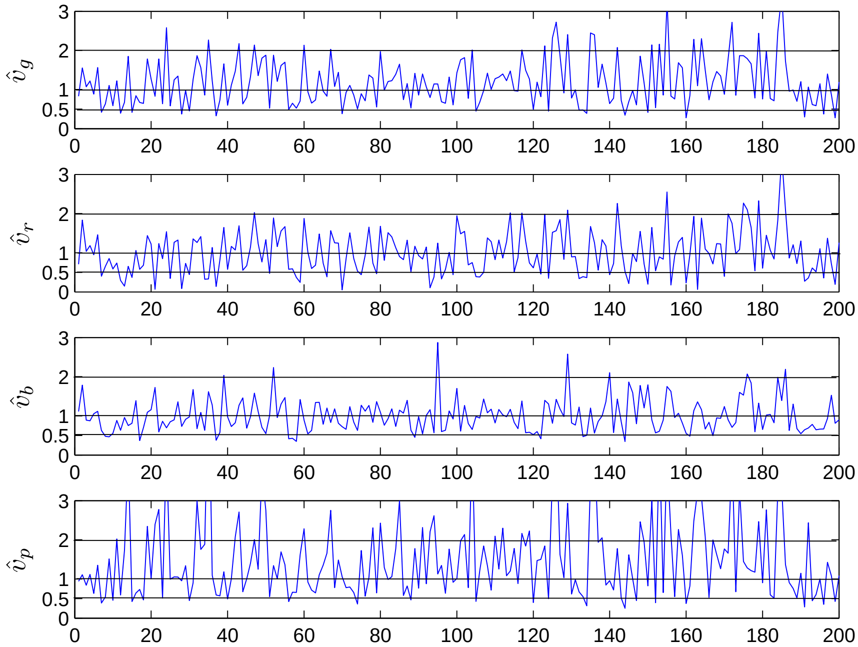

Figure 9: Plots of two hundreds samples of canonical estimators. Up to down are samples of GK, RS, bridge and PARK estimators. It is seen even by "naked eye" that bridge estimator estimates volatility more accurately than another mentioned estimators

Appendix A. Probabilistic properties of high, low and close values

Here are given pdfs of random variables (h, l, c) (7) and variables (ξ, ζ) (8), which one need for canonical estimators (6) statistical analysis. Let begin with random variable $c = x(1, \gamma)$. Obviously, its pdf is

$$f(\chi; \gamma) := \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\chi - \gamma)^2}{2}\right), \qquad \chi \in (-\infty, \infty).$$

It is easy to show, additionally, that joint pdf $q_x(\eta, \chi; \gamma)$ of high value h (7) of canonical Brownian motion $x(\tau, \gamma)$ and the close value $c = x(1, \gamma)$ is equal

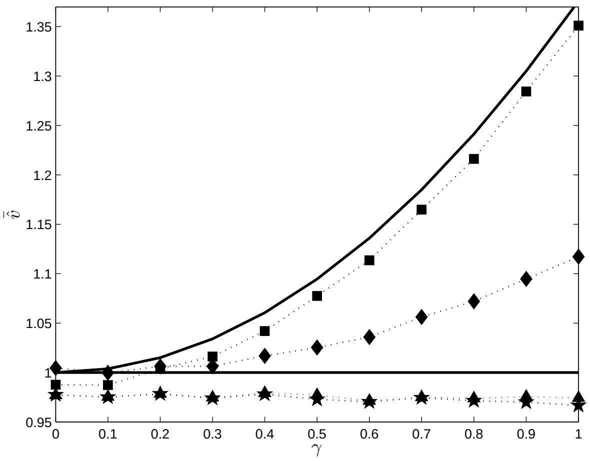

Figure 10: Mean values $\bar{\tilde{v}}$ of canonical PARK (■), GK (♦), RS (★) and bridge (▲) estimators. Solid lines are theoretical expectations, borrowing from figure 4

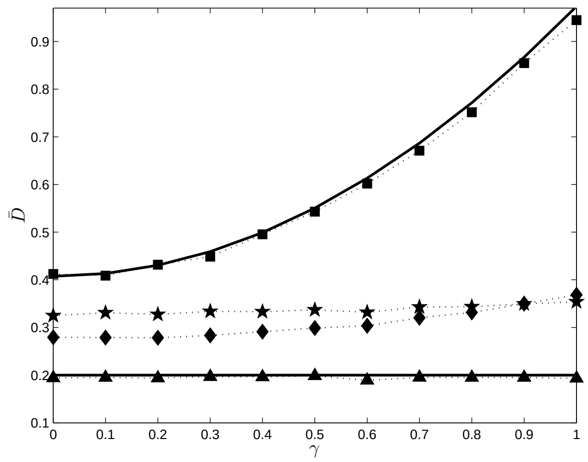

Figure 11: Estimations $\bar{D}$ of variance of PARK (■), RS (★), GK (♦) and bridge (▲) canonical estimators. Solid lines are plots of theoretical variances, borrowed from the figure 4. It is seen that for any γ bridge estimator's variance significantly smaller than variances of another mentioned estimators

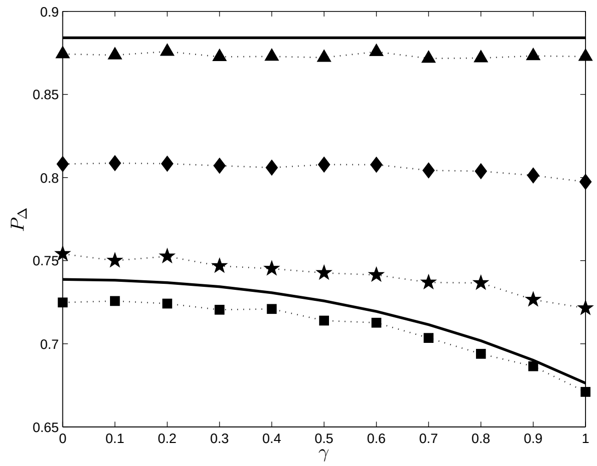

Fig. 12

Figure 12: Estimations of probability P_Δ (18) at different γ values, for PARK (■), RS (★), GK (⧫) and bridge (▲) estimators. Solid lines are results of theoretical calculations, resting on formula (18)

to

$$q_x(\eta, \chi; \gamma) = \sqrt{\frac{2}{\pi}}\,(2\eta - \chi)\, e^{2\gamma\eta} \exp\left(-\frac{1}{2}(2\eta - x + \gamma)^2\right),$$

$$\chi < \eta, \qquad \eta > 0.$$

In turn, pdf of high value h (7)

$$q_x(\eta; \gamma) := \int_{-\infty}^{h} q_x(\eta, \chi; \gamma) d\chi$$

given by expression

$$q_x(\eta; \gamma) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{(\eta - \gamma)^2}{2}\right) - \gamma e^{2\gamma\eta}\, \text{erfc}\left(\frac{\eta + \gamma}{2}\right), \qquad \eta > 0.$$

Let write here explicit expression for joint pdf $q_x(\eta, \ell, \chi; \gamma)$ of random variables (h, l, c) (7). Using formulas, given at the monograph [18] and in the article [19], one might show that pointed out joint pdf given by:

$$\begin{gathered} q_x(\eta, \ell, \chi; \gamma) = f(\chi; \gamma)\, \mathcal{S}(\eta, \ell|\chi), \\ \chi \in (\ell, \eta), \qquad h > \chi \mathbf{1}(\chi), \qquad \ell < \chi \mathbf{1}(-\chi). \end{gathered} \tag{A.1}$$

Here $\mathbf{1}(\chi)$ is the unit step function, equal to unity for $\chi > 0$ and zero otherwise. Besides, above there is function

$$\mathcal{S}(\eta, \ell|\chi) :=$$

$$\sum_{m=-\infty}^{\infty} m\left[m\mathcal{F}(m(\eta-\ell),\chi) + (1-m)\mathcal{F}(m(\eta-\ell)+\ell,\chi)\right], \tag{A.2}$$

$$\mathcal{F}(\eta,\chi) := \left[(\chi-2\eta)^2 - 1\right] e^{2\eta(\chi-\eta)}.$$

To explore statistical properties of canonical GK estimator, we need in joint pdf $q_x(\delta, \chi; \gamma)$ of canonical Brownian motion $x(\tau, \gamma)$ (5) oscillation $d = h - l$ and the close value $c = x(1, \gamma)$. As it follows from (A.1), (A.2), mentioned pdf is equal to

$$q_x(\delta,\chi;\gamma) = 4f(\chi;\gamma) \sum_{m=-\infty}^{\infty} m\times$$

$$\left[m(\delta-|\chi|)[(|\chi|+2m\delta)^2 - 1] - (m+1)(|\chi|+2m\delta)\right] e^{-2m\delta(|\chi|+m\delta)},$$

$$\delta > |\chi|, \qquad \chi \in (-\delta, \delta).$$

After integration above joint pdf over all χ values obtain pdf $q_x(\delta;\gamma)$ of oscillation d:

$$q_x(\delta;\gamma) = 2\sum_{m=-\infty}^{\infty} m\left(\sqrt{\frac{8}{\pi}} e^{-\frac{\gamma^2}{2}-2m^2\delta^2}\times\right.$$

$$\left[e^{-\frac{\delta^2}{2}(1+4m)}\cosh(\delta\gamma)(1+2m+\gamma^2 m) - m(2+\gamma^2)\right] + \tag{A.3}$$

$$\left.\gamma\left[a(\delta,\gamma,m) + a(-\delta,\gamma,m)\right]\right), \qquad \delta > 0.$$

Here have used auxiliary function

$$a(\delta,\gamma,m) := e^{2m\delta\gamma}\left[1 + m(3+\gamma(2m\delta+\gamma+\delta))\right]\times$$

$$\left[\operatorname{erf}\left(\frac{2m\delta+\gamma+\delta}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{2m\delta+\gamma}{\sqrt{2}}\right)\right].$$

In particular case of zero drift ($\gamma = 0$), one get from (A.3) following expression

$$q_x(\delta) = \sqrt{\frac{32}{\pi}} \sum_{m=-\infty}^{\infty} m\left[(1+2m)e^{-\frac{(1+2m)^2\delta^2}{2}} - 2me^{-2m^2\delta^2}\right]. \tag{A.4}$$

All statistical properties of high and low values (8) of canonical bridge (9) are defined by their two-fold joint pdf $q_b(\eta, \ell)$, given by relation

$$q_b(\eta, \ell) = \sum_{m=-\infty}^{\infty} m \left[m\mathcal{F}(m(\eta - \ell)) + (1 - m)\mathcal{F}(m(\eta - \ell) + \ell) \right],$$
$$\mathcal{F}(\eta) := 4(4\eta^2 - 1)e^{-2\eta^2}.$$

Following from here pdf $q_b(\delta)$ of canonical bridge oscillation $s = \xi - \zeta$ given by equality

$$q_b(\delta) = 8\delta \sum_{m=1}^{\infty} m^2 (4m^2\delta^2 - 3)e^{-2m^2\delta^2}, \qquad \delta > 0. \tag{A.5}$$

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