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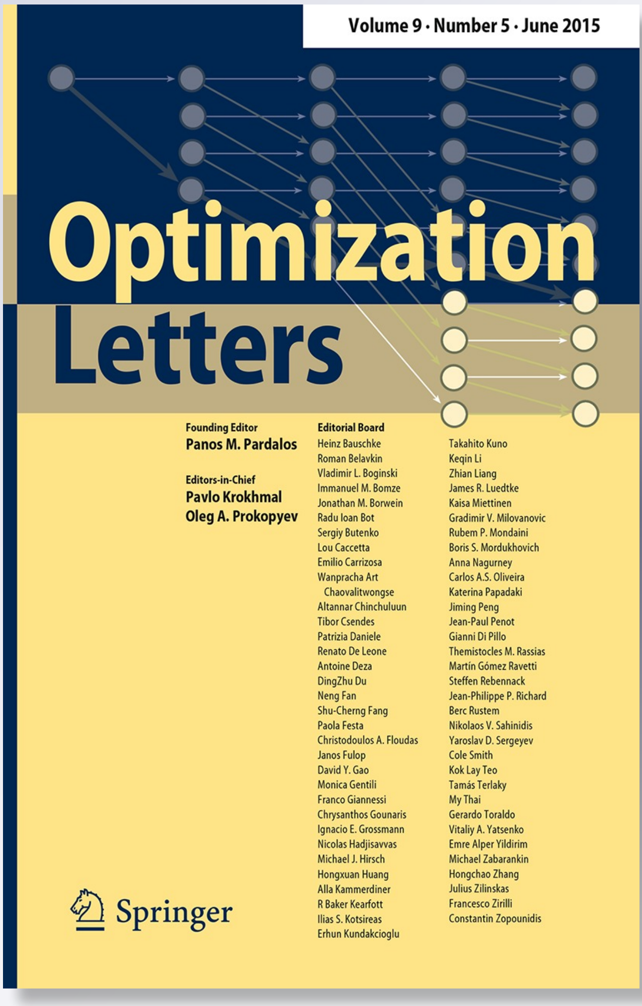
Optimization Letters

ISSN 1862-4472
Volume 9
Number 5

Optim Lett (2015) 9:839-843
DOI 10.1007/s11590-014-0805-z



Springer

Optim Lett (2015) 9:839–843
DOI 10.1007/s11590-014-0805-z

ORIGINAL PAPER

The clique problem for graphs with a few eigenvalues of the same sign

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Received: 26 July 2014 / Accepted: 17 September 2014 / Published online: 4 October 2014

Abstract The quadratic programming problem is known to be NP-hard for Hessian matrices with only one negative eigenvalue, but it is tractable for convex instances. These facts yield to consider the number of negative eigenvalues as a complexity measure of quadratic programs. We prove here that the clique problem is tractable for two variants of its Motzkin-Strauss quadratic formulation with a fixed number of negative eigenvalues (with multiplicities).

Keywords Quadratic programming · Computational complexity · Clique problem

1 Introduction

The general quadratic programming problem (QPP) is formulated as follows:

$$\begin{aligned}&\text{minimize } \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x}\\&\text{subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b},\end{aligned}$$

where $\mathbf{Q}$ is a given symmetric real $n \times n$ matrix, $\mathbf{c}$ is a given real $n \times n$ vector, $\mathbf{A}$ and $\mathbf{b}$ are given real $m \times n$ matrix and real $m \times 1$ vector, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector of variables to be determined. It is well known to be tractable for positive

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semi-definite cases (that is, when all eigenvalues of $\mathbf{Q}$ are nonnegative) by the ellipsoid method [5]. On the other hand, QPP is NP-hard even for the objective function $x_1 - x_2^2$ (i.e., when Hessian matrix has only one negative eigenvalue) [8]. So, existing negative eigenvalues yields a phase transition in the complexity of QPP.

Many classical optimization problems can be formulated as QPP. Intuitively, there are important classes of QPP, where the general phase transition does not hold. In other words, these QPP are polynomial-time solvable for matrices with negative eigenvalues (see, for example, [4]). The present paper is devoted to revealing cases of such type.

A *clique* is a set of pairwise adjacent vertices of a graph. The size of a largest clique of a graph G is called the *clique number* of G denoted by $\omega(G)$. The *clique problem* (CP) for a given graph G and a number k is to verify the validity of the inequality $\omega(G) \leq k$. The quadratic establishment of CP for G was given by Motzkin and Strauss in [7] as follows:

$$\begin{aligned} &\text{maximize } \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A_G}\mathbf{x} \\ &\text{subject to } \sum_{i=1}^{n} x_i = 1 \text{ and } x_i \geq 0 \quad \text{for any} \quad i, \end{aligned}$$

where $\mathbf{A_G}$ is the adjacency matrix of G and $n = |V(G)|$. It will be denoted by $MS_1(G)$. The Motzkin-Strauss theorem says that the optimal value of $MS_1(G)$ is $\frac{1}{2}(1 - \frac{1}{\omega(G)})$ [7]. Formally, $MS_1(G)$ is not QPP, as $MS_1(G)$ is a maximization problem. So, we also consider the following reformulation of $MS_1(G)$ (denoted by $MS_2(G)$):

$$\begin{aligned} &\text{minimize } \frac{1}{2}\mathbf{x}^{\mathrm{T}}(-\mathbf{A_G})\mathbf{x} \\ &\text{subject to } \sum_{i=1}^{n} x_i = 1 \text{ and } x_i \geq 0 \quad \text{for any} \quad i \end{aligned}$$

The complexity of $MS_1(G)$ and $MS_2(G)$ is studied in the present paper by means of the quantities of negative eigenvalues of $\mathbf{A_G}$ and $-\mathbf{A_G}$. The class $\mathcal{NE}_k$ is a set of graphs whose adjacency matrices have at most k negative eigenvalues (with multiplicities). The set $\{G : -\mathbf{A_G}$ has at most k negative eigenvalues counting multiplicities$\}$ $=$ $\{G : \mathbf{A_G}$ has at most k positive eigenvalues (with multiplicities)$\}$ will be denoted by $\mathcal{PE}_k$. The main result of this paper is proving polynomial-time solvability of CP for $\mathcal{NE}_k, \mathcal{PE}_k$ for any fixed k.

2 Some notations

As usual, K_n is the complete graph with n vertices. The *complement graph* of G (denoted by $\overline{G}$) is a graph on the same set of vertices and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The disjoint union of k copies of a graph G is denoted by kG.

We refer to textbooks in graph theory and matrix theory for any notions not defined in the paper.

3 The hereditary closeness of $\mathcal{NE}_k$ and $\mathcal{PE}_k$

The following result is known as the Cauchy inequalities or the interlacing lemma (see, for example, p. 119 of [6]).

Lemma 1 *Let* $\mathbf{A}$ *be a real symmetric* $n \times n$ *matrix with eigenvalues* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$*, and let* $\mathbf{B}$ *be its* $m \times m$ *principal submatrix with eigenvalues* $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$*. Then* $\lambda_j \leq \mu_j \leq \lambda_{j+n-m}$ *for any* $1 \leq j \leq m$*.*

A graph H is called an *induced subgraph* of a graph G if H is obtained from G by deleting its vertices. A set of graphs is called a *hereditary graph class* if it is closed under deletion of vertices. In other words, a class of graphs is hereditary if it contains all induced subgraphs of every its graph. It is well known that any hereditary (and only hereditary) class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$. We write $\mathcal{X} = Free(\mathcal{Y})$ in this case.

Lemma 2 *For each fixed k, the classes* $\mathcal{NE}_k$ *and* $\mathcal{PE}_k$ *are hereditary.*

Proof Let $G \in \mathcal{NE}_k$ and H be its induced subgraph with $|V(H)| = |V(G)|-1 = n-1$. Clearly, $\mathbf{A_H}$ is a principal submatrix of $\mathbf{A_G}$, as deleting a vertex from a graph yields simultaneous deleting the corresponding row and column in its adjacency matrix. The matrices $\mathbf{A_G}, \mathbf{A_H}$ have the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ correspondingly. Assume, $\mu_{k+1} < 0$. By the interlacing lemma, the inequality $\lambda_{k+1} \leq \mu_{k+1}$ is true. Hence, $\mathbf{A_G}$ has at least $k+1$ negative eigenvalues i.e., $G \notin \mathcal{NE}_k$. We have a contradiction. Hence, $\mu_{k+1} \geq 0$ and $H \in \mathcal{NE}_k$.

Let $G \in \mathcal{PE}_k$ and H be its induced subgraph with $|V(H)| = |V(G)| - 1 = n - 1$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of $\mathbf{A_G}$ and $\mathbf{A_H}$ respectively. Assume that $H \notin \mathcal{PE}_k$ i.e., $\mu_{n-k-1} > 0$. Hence, the eigenvalues $\lambda_{n-k}, \lambda_{n-k+1}, \ldots, \lambda_n$ are also positive (by the interlacing lemma). Thus, $G \notin \mathcal{PE}_k$. Therefore, our assumption was false. □

4 The spectrum of the complement of a regular graph

A graph is called *d-regular* if all its vertices have degrees equal to d. It is well known that any d-regular graph has the largest eigenvalue d (Theorem 2.6 of [3]). Moreover, the spectra of a d-regular graph and its complement are connected by the following relation (see Theorem 2.6 in [3]).

Lemma 3 *Let G be a d-regular graph of order n, and let the eigenvalues of G be* $\lambda_1 = d \geq \lambda_2 \geq \cdots \geq \lambda_n$*. Then the eigenvalues of* $\overline{G}$ *are* $n-1-d, -1-\lambda_2, -1-\lambda_3, \ldots, -1-\lambda_n$*.*

The graph pK_2 has p eigenvalues +1 and p eigenvalues -1. It is 1-regular. Hence, by the lemma, $\overline{pK_2}$ has the eigenvalues $\underbrace{-2, \ldots, -2}_{p-1 \ times}, \underbrace{0, \ldots, 0}_{p \ times}, 2p-2$.

5 Tractability of the clique problem for $\mathcal{NE}_k$ and $\mathcal{PE}_k$

The main claim of this paper is the following result.

Theorem 1 *For each fixed k, CP is polynomial-time solvable for $\mathcal{NE}_k$ and $\mathcal{PE}_k$.*

Proof By Lemma 2, the class $\mathcal{NE}_k$ is hereditary and $\overline{(k+2)K_2} \notin \mathcal{NE}_k$. Hence, $\mathcal{NE}_k \subseteq Free(\{\overline{(k+2)K_2}\})$. For each fixed s, a polynomial-time algorithm for $Free(\{\overline{sK_2}\})$ is known for the clique problem [2]. Its time complexity is $O(n^{2s+1})$. Thus, the clique problem is solved for $\mathcal{NE}_k$ in $O(n^{2k+1})$ time. The clique K_k has one eigenvalue $1-k$ and $k-1$ eigenvalues equal to 1. As $\mathcal{PE}_k$ is hereditary, then $\mathcal{PE}_k \subseteq Free(\{K_{k+2}\})$. Hence, CP can be solved for graphs in $\mathcal{PE}_k$ by any exhaustive search algorithm in $O\left(\sum_{i=0}^{k+1}\binom{n}{i}\right) = O\left(\sum_{i=0}^{k+1} n^i\right) = O(n^{k+1})$ time. □

6 Corollaries of the main result

The clique problem can be established by several quadratic formulations. For example, consider the following problem for an n-vertex graph G:

$$\text{maximize } \sum_{i=1}^{n} x_i - \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}_{\overline{\mathbf{G}}}\mathbf{x}$$
$$\text{subject to } \mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0, 1]^n$$

Its optimal value is equal to $\omega(G)$ (see [1]). Hence, this quadratic problem can be solved in polynomial time for graphs in $\mathcal{NE}_k \cup \mathcal{PE}_k$. The same is true for the continuous optimization problems below, as their optimal values are equal to $\omega(G)$ (see [1]).

$$\text{maximize } \sum_{i=1}^{n}(1-x_i)\prod_{(i,j)\in E(\overline{G})} x_j \qquad \text{maximize } \sum_{i=1}^{n} x_i - \sum_{(i,j)\in E(\overline{G})} x_i x_j$$
$$\text{subject to } (x_1, x_2, \ldots, x_n) \in [0, 1]^n \qquad \text{subject to } (x_1, x_2, \ldots, x_n) \in [0, 1]^n$$

An *independent set* of a graph is a set of its pairwise adjacent vertices. The size of a largest independent set of G is denoted by $\alpha(G)$. A *vertex cover* of a graph is a set of vertices, such that every edge is incident to at least one vertex in the set. The size of a minimum vertex cover of G is denoted by $\beta(G)$. It is well known that for any graph G the relations $\alpha(G) + \beta(G) = |V(G)|$ and $\alpha(G) = \omega(\overline{G})$ hold. Hence, computing $\alpha(G)$ or $\beta(G)$ (the so-called *independent set* and *vertex cover* problems) can be done in polynomial time for graphs in $\{G : \overline{G} \in \mathcal{PE}_k\}$ and for graphs in $\{G : \overline{G} \in \mathcal{NE}_k\}$.

Acknowledgments Research is partially supported by LATNA laboratory, National Research University Higher School of Economics, RF government grant, ag. 11.G34.31.00357.

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