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LINEAR BOUNDARY VALUE PROBLEMS AND CONTROL PROBLEMS FOR A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CONTINUOUS AND DISCRETE TIMES *

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Abstract. For a functional differential system with continuous and discrete times, the general linear boundary value problem and the problem of control with respect to an on-target vector-functional are considered. Conditions for the solvability of the problems are obtained. Questions of computer-aided techniques for studying these problems are discussed.

Key Words. Abstract functional differential equations, discrete-continuous systems, boundary value problems, control problems, hybrid controlled systems with aftereffect.

AMS(MOS) subject classification. 34K10, 34K30, 34K35, 91B74

1. Introduction.

We consider here a system of functional differential equations (FDE, FDS) that, formally speaking, is a concrete realization of the so-called abstract functional differential equation (AFDE). Theory of AFDE is thoroughly treated in [7, 9]. On the other hand, the system under consideration is a typical one met with in mathematical modeling economic dynamic processes and covers many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference, difference) and impulsive perturbations resulting in system's state jumps at prescribed time moments [13, 3, 4, 14, 19]. The equations of the system contain

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simultaneously terms depending on continuous time, $t \in [0,T]$, and discrete, $t \in \{0, t_1, ..., t_N, T\}$, this is why the term "hybrid" seems to be suitable. As this term is deeply embedded in the literature in different senses, we will follow the authors employing the more definite name "continuous-discrete systems" (CDS), see, for instance, [1, 2, 16, 17, 18] and references therein. Notice that in [1, 2] a detailed motivation for studying CDS and examples of applications can be found together with results on stabilization, observability and controllabality for a class of linear CDS with continuous-time dynamics described by ordinary differential equations. To finish with the terminology, it is pertinent to note that the name "concrete systems" could be used for short (as it was interpreted by V.I. Arnold, "concrete" means *con*(tinuous-dis)*crete*).

First we descript in detail a class of continuous-discrete functional differential equations (CDFDE) with linear Volterra operators and appropriate spaces where those are considered. We are concerned with the representation of general solution to the system and derive basic relationships for the Cauchy operator and the fundamental matrix. Next the setting of the general linear boundary value problem (BVP) for CDFDE is given, and conditions for the solvability of BVP are obtained. The control problem (CP) for CDFDE is set up and considered then. Here we give conditions for the solvability of CP and propose a technique of constructing the solutions to the problem.Questions of computer-aided techniques for studying these problems are discussed. Conclusively, we give a remark concerning CDFDE as an AFDE.

2. A class of continuous-discrete functional differential systems. First, let us introduce the Banach spaces where operators and equations are considered.

Fix a segment $[0,T] \subset R$. By $L^n = L^n[0,T]$ we donote the space of summable functions $v : [0,T] \to R^n$ under the norm $||v||_{L^n} = \int_0^T |v(s)|_n \, ds$, where $|\cdot|_n$ ($|\cdot|$ for short if the value of dimension is clear) stands for the norm of R^n.

Given set $\{\tau_1, ..., \tau_m\}, 0 < \tau_1 < ... < \tau_m < T$, the space $DS^n(m) = DS^n[0, \tau_1, ..., \tau_m, T]$ is defined (see [5, 8, 9]) as the space of piecewise absolutely continuous functions $y : [0,T] \to R^n$ representable in the form

$$y(t) \;=\; \int_0^t v(s)\, ds \;+\; y(0) \;+\; \sum_{k=1}^{m} \chi_{[\tau_k,T]}(t)\Delta y(\tau_k),$$

where $v \in L^n$, $\Delta y(\tau_k) = y(\tau_k) - y(\tau_k - 0)$, $\chi_{[\tau_k,T]}(t)$ is the characteristic function of the segment $[\tau_k, T]$: $\chi_{[\tau_k,T]}(t) = 1$ if $t \in [\tau_k, T]$ and $\chi_{[\tau_k,T]}(t) = 0, t \notin [\tau_k, T]$. Thus the elements of $DS^n(m)$ are the functions being absolutely continuous on each $[0,\tau_1), [\tau_1,\tau_2), ..., [\tau_m, T]$ and continuous from the right at the points $\tau_1, ..., \tau_m$. Under the norm

$$||y||_{DS^n(m)} = ||\dot{y}||_{L^n} + |y(0)|_n + \sum_{k=1}^{m} |\Delta y(\tau_k)|_n$$

the space $DS^n(m)$ is Banach.

Let us give some remarks concerning the approach to the impulse systems based on the use of the space $DS^n(m)$. An approach to the study of differential equations with discontinuous solutions is associated with the so called "generalized ordinary differential equations" whose theory was initiated by J.Kurzweil [10]. Nowadays this theory is highly developed (see, for instance, [21, 6]). According to the accepted approaches impulsive equations are considered within the class of functions of bounded variation. In this case the solution is understood as a function of bounded variation satisfying an integral equation with the Lebesgue-Stiltjes integral or Perron-Stiltjes one. Integral equations in the space of functions of bounded variation became to be the subject of its own interest and are studied in detail in [22]. Recall that the function of bounded variation is representable in the form of the sum of an absolutely continuous function, a break function, and a singular component (a continuous function with the derivative being equal zero almost everywhere). The solutions of equations with impulse impact, which are considered below, do not contain the singular component and may have discontinuity only at finite number of prescribed points. We consider these equations on a finite-dimensional extension $DS^n(m)$ of the traditional space of absolutely continuous functions. This approach to the equations with impulsive impact was offered in [5]. It does not use the complicated theory of generalized functions, turned out to be rich in content and finds many applications in the cases where the question about the singular component does not arise.

Let us fix a set $J = \{t_0, t_1, ..., t_\mu\}, 0 = t_0 < t_1 < ... < t_\mu = T$.

$FD^\nu(\mu) = FD^\nu\{t_0, t_1, ..., t_\mu\}$ denotes the space of functions $z : J \to R^\nu$ under the norm

$$||z||_{FD^\nu(\mu)} = \sum_{i=0}^{\mu} |z(t_i)|_\nu.$$

We consider the system

$$\begin{aligned} \dot{y} &= \mathcal{T}_{11} y + \mathcal{T}_{12} z + f, \\ z &= \mathcal{T}_{21} y + \mathcal{T}_{22} z + g, \end{aligned} \qquad (1)$$

where the linear operators $\mathcal{T}_{ij}$, $i, j = 1, 2$, are defined as follows.

$$(\mathcal{T}_{11}) \qquad \mathcal{T}_{11} : DS^n(m) \to L^n;$$

$$(\mathcal{T}_{11}y)(t) = \int_0^t K^1(t,s)\dot{y}(s)\,ds \, + A_0^1(t)y(0) \, + \sum_{k=1}^m A_k^1(t)\Delta y(\tau_k)\,,\ t \in [0,T].$$

Here the elements $k_{ij}^1(t,s)$ of the kernel $K(t,s)$ are measurable on the set $0 \le s \le t \le T$ and such that $|k_{ij}^1(t,s)| \le \kappa(t)$, $i, j = 1, ..., n$, $\kappa(\cdot)$ is summable on $[0,T]$, $(n \times n)$-matrices $A_0^1, ..., A_m^1$ have elements summable on $[0,T]$. Recall [8, 9] that such a form of $\mathcal{T}_{11}$ covers many kinds of linear operators with concentrated and distributed delays including the so-called inner superposition operator.

$$(\mathcal{T}_{12}) \qquad \mathcal{T}_{12} : FD^\nu(\mu) \to L^n;\ \ (\mathcal{T}_{12}z)(t) = \sum_{\{j: t_j \le t-\Delta_1\}} B_j^1(t)z(t_j),\ t \in [0,T],$$

where elements of matrices B_j^1, $j = 0, , ..., \mu$, are summable on $[0,T]$, $\Delta_1 \ge 0$. As is it usually is, here and in the sequel $\sum_{i=k}^l F_i = 0$ for any F_i if $l < k$.

$$(\mathcal{T}_{21}) \qquad \mathcal{T}_{21} : DS^n(m) \to FD^\nu(\mu);$$

$$(\mathcal{T}_{21}y)(t_i) = \int_0^{t_i - \Delta_2} K_i^2(s)\dot{y}(s)ds \ + \ A_{i0}^2 y(0) \ + \sum_{k=1}^m A_{ik}^2 \Delta y(\tau_k),\ i = 0, 1, ..., \mu,$$

with measurable and essentially bounded on $[0,T]$ elements of matrices K_i^2 and constant $(\nu \times n)$-matrices A_{ik}^2 , $i = 0, 1, ..., \mu$, $k = 0, 1, ..., m$; $\Delta_2 \ge 0$. .

$$(\mathcal{T}_{22}) \qquad \mathcal{T}_{22} : FD^\nu(\mu) \to FD^\nu(\mu);\ \ (\mathcal{T}_{22}z)(t_i) = \sum_{j=0}^{i-1} B_{ij}^2 z(t_j),\ i = 1, ..., \mu,$$

with constant $(\nu \times \nu)$-matrices B_{ij}^2.

In what follows we will use some results from [8, 9] concerning the equation

$$(2) \qquad \dot{y} \ = \ \mathcal{T}_{11}y \ + \ f$$

and the results of [3] concerning the equation

$$(3) \qquad z \ = \ \mathcal{T}_{22}z \ + \ g.$$

Recall that the homogeneous equation (2) ($f(t) = 0$, $t \in [0,T]$) has the fundamental matrix $Y(t)$ of dimension $n \times (n+mn)$:

$$Y(t) \;=\; \Theta(t) \;+\; X(t), \tag{4}$$

where

$$\Theta(t) \;=\; (E_n, \chi_{[\tau_1,T]}E_n, ..., \chi_{[\tau_m,T]}E_n),$$

E_n is the identity $(n \times n)$-matrix, each column $x_i(t)$ of the $(n \times (n+mn))$-matrix $X(t)$ is a unique solution to the Cauchy problem

$$\dot{x}(t) = \int_0^t K^1(t,s)\dot{x}(s)\,ds \;+\; \tilde{a}_i^1(t),\; x(0) = 0\;, t \in [0,T]. \tag{5}$$

Here $\tilde{a}_i^1(t)$ is the i-th column of the matrix $\tilde{A}^1 = (A_0^1, A_1^1, ..., A_m^1)$.

The solution of (2) with the initial condition $y(0) = 0$ has the representation

$$y(t) \;=\; (C_1 f)(t) \;=\; \int_0^t C_1(t,s)f(s)\,ds, \tag{6}$$

where $C_1(t,s)$ is the Cauchy matrix [11, 12] of the operator $d/dt - \mathcal{T}_{11}$. This matrix can be defined (and constructed) as the solution to

$$\frac{\partial}{\partial t}C_1(t,s) \;=\; \int_s^t K^1(t,\tau)\frac{\partial}{\partial \tau}C_1(\tau,s)\,d\tau \;+\; K^1(t,s),\; 0 \le s \le t \le T, \tag{7}$$

under the condition $C_1(s,s) = E_n$.

The matrix $C_1(t,s)$ is expressed in terms of the resolvent kernel $R(t,s)$ of the kernel $K^1(t,s)$. Namely,

$$C_1(t,s) \;=\; E_n \;+\; \int_s^t R(\tau,s)\,d\tau.$$

The general solution of (2) has the form

$$y(t) \;=\; Y(t)\alpha \;+\; \int_0^t C_1(t,s)f(s)\,ds, \tag{8}$$

with arbitrary $\alpha \in R^{n+mn}$.

As for equation (3), it has the immediate analogs of the above terms. Namely, the fundamental matrix $Z(t_i)$, $i = 0, ..., \mu$, of the homogeneous equation (3):

$$z(t_i) \;=\; \sum_{j=0}^{i-1} B_{ij}^2 z(t_j),\;\; i = 1, 2, ..., \mu,$$

is defined as the solution of the initial problem

$$Z(t_i) = \sum_{j=0}^{i-1} B_{ij}^2 Z(t_j), \quad i = 1, 2, \dots, \mu, \; Z(t_0) = E_\nu. \tag{9}$$

The Cauchy matrix $C_2(i, j)$ is defined by the recurrent relationships

$$C_2(i, j) = E_\nu + \sum_{k=j}^{i-1} B_{ik}^2 C_2(k, j), \quad 1 \leqslant j \leqslant i \leqslant \mu, \tag{10}$$

and gives the solution to (3) under the condition $z(t_0) = 0$:

$$z(t_i) = (C_2 g)(t_i) = \sum_{j=1}^{i} C_2(i, j) g(t_j), \quad i = 0, 1, \dots, \mu.$$

Thus, the general solution of (3) has a representation

$$z(t_i) = Z(t_i)\beta + (C_2 g)(t_i), \quad i = 0, 1, \dots, \mu, \tag{11}$$

with arbitrary $\beta \in R^\nu$.

Now we can apply (8) and (11) to the first equation and the second one of (1) respectively. Thus we obtain in operator form

$$\begin{aligned} y &= Y\alpha + C_1 \mathcal{T}_{12} z + C_1 f, \\ z &= Z\beta + C_2 \mathcal{T}_{21} y + C_2 g, \end{aligned} \tag{12}$$

or

$$\begin{aligned} &\begin{pmatrix} I & -C_1 \mathcal{T}_{12} \\ -C_2 \mathcal{T}_{21} & I \end{pmatrix} \cdot \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &\qquad + \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \cdot \begin{pmatrix} f \\ g \end{pmatrix}, \end{aligned} \tag{13}$$

where I is the identity operator in a proper space.

To obtain a representation of the general solution to (1) and derive the key relationships for the fundamental matrix and the Cauchy operator of CDS (1), we shall solve (13) with respect to $x = \operatorname{col}(y, z)$. This will be done making use of the following Lemma.

Lemma 1. *Let Δ_1 and Δ_2 in definition of $\mathcal{T}_{12}$ and $\mathcal{T}_{21}$ be such that the condition*

$$\Delta_1 + \Delta_2 \neq 0 \tag{14}$$

holds. Then the operator

$$P = \begin{pmatrix} I & -C_1\mathcal{T}_{12} \\ -C_2\mathcal{T}_{21} & I \end{pmatrix} : DS^n(m) \times FD^\nu(\mu) \to DS^n(m) \times FD^\nu(\mu)$$

is invertible.

Proof. It is easy to verify that a linear operator $M = \begin{pmatrix} I & A \\ B & I \end{pmatrix}$ with linear operators $A : Z \to Y$ and $B : Y \to Z$ (Y, Z are Banach spaces) is invertible if $(I - BA) : Z \to Z$ has the inverse $(I - BA)^{-1} : Z \to Z$. In such a situation, the inverse $(I - AB)^{-1}$ exists too and

$$M^{-1} = \begin{pmatrix} (I - AB)^{-1} & -(I - AB)^{-1}A \\ -B(I - AB)^{-1} & (I - BA)^{-1} \end{pmatrix}.$$

In the case under consideration, $BA = \mathcal{C}_2\mathcal{T}_{21}\mathcal{C}_1\mathcal{T}_{12} : FD^\nu(\mu) \to FD^\nu(\mu)$ is a τ-Volterra [12, p.106] operator with $\tau = \Delta_1 + \Delta_2$ and, therefore, is a nilpotent operator. In such a case, the spectral radius of BA equals zero. ∎

In the sequel we assume that (14) holds. Thus, it is follows from (13) that

$$\begin{aligned} \begin{pmatrix} y \\ z \end{pmatrix} &= \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \cdot \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &+ \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \cdot \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \cdot \begin{pmatrix} f \\ g \end{pmatrix}, \end{aligned} \tag{15}$$

where

$$H_{11} = (I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}\ ;\ H_{12} = -(I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}C_1\mathcal{T}_{12}\ ; \tag{16}$$
$$H_{21} = C_2\mathcal{T}_{21}(I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}\ ;\ H_{22} = (I - C_2\mathcal{T}_{21}C_1\mathcal{T}_{12})^{-1}.$$

Finally, the general solution $x = \begin{pmatrix} y \\ z \end{pmatrix} \in DS^n(m) \times FD^\nu(\mu)$ of (1) has the form

$$x = \mathcal{X}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \mathcal{C}\begin{pmatrix} f \\ g \end{pmatrix}, \tag{17}$$

where the fundamental matrix $\mathcal{X}$ is expressed in terms of the fundamental matrices Y and Z by the equality

$$\mathcal{X} = \begin{pmatrix} H_{11}Y & H_{12}Z \\ H_{21}Y & H_{22}Z \end{pmatrix} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{pmatrix} \tag{18}$$

and the Cauchy operator $\mathcal{C}$ is expressed in terms of the Cauchy operators C_1 and C_2:

$$\mathcal{C} = \begin{pmatrix} H_{11}C_1 & H_{12}C_2 \\ H_{21}C_1 & H_{22}C_2 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix}. \tag{19}$$

3. General linear boundary value problem. The general linear BVP is the system (1) supplemented by the linear boundary conditions

$$\ell x = \ell \begin{pmatrix} y \\ z \end{pmatrix} = \gamma, \quad \gamma \in R^N, \tag{20}$$

where $\ell : DS^n(m) \times FD^\nu(\mu) \to R^N$ is a linear bounded vector functional. Let us give the representation of ℓ:

$$\ell \begin{pmatrix} y \\ z \end{pmatrix} = \int_0^T \Phi(s)\dot{y}(s)\,ds + \Psi_0 y(0) + \sum_{k=1}^{m} \Psi_k \Delta y(\tau_k) + \sum_{j=0}^{\mu} \Gamma_j z(t_j). \tag{21}$$

Here Ψ_k, $k = 0, 1, \ldots, m$, are constant $(N \times n)$-matrices, Γ_j, $j = 0, 1, \ldots, \mu$ are constant $(N \times \nu)$-matrices, Φ is $(N \times n)$-matrix with measurable and essentially bounded on $[0, T]$ elements. We assume that the components $\ell_i : DS^n(m) \times FD^\nu(\mu) \to R$, $i = 1, \ldots, N$, of $\ell = \operatorname{col}(\ell_1, \ldots, \ell_N)$ are linearly independent.

BVP (1),(20) is well-defined if $N = n + mn + \nu$. In such a situation, BVP (1),(20) is uniquely solvable for any f, g if and only if the matrix

$$\ell\mathcal{X} = \left(\ell\mathcal{X}^1, \ldots, \ell\mathcal{X}^{n+mn+\nu}\right), \tag{22}$$

where $\mathcal{X}^j$ is the j-th column of $\mathcal{X}$ is nonsingular, i.e.

$$\det \ell\mathcal{X} \neq 0. \tag{23}$$

Hence the result may be summarized up as the following theorem.

THEOREM 1. *Suppose that $N = n + mn + \nu$. Then BVP (1),(20) is uniquely solvable for any f, g if and only if (23) holds, where $(N \times N)$-matrix $\ell\mathcal{X}$ is defined by (22),(21),(18),(16).*

4. Problem of control with respect to a system of linear on-target functionals. Let us write CDS (1) in the form

$$\delta x = \Theta x + \varphi, \tag{24}$$

where $x = \begin{pmatrix} y \\ z \end{pmatrix} \in DS^n(m) \times FD^\nu(\mu)$, $\varphi = \begin{pmatrix} f \\ g \end{pmatrix} \in L^n(m) \times FD^\nu(\mu)$,

$$\Theta = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix},$$

and consider the system under control

$$\delta x = \Theta x + Fu + \varphi. \tag{25}$$

Here $u \in H$ is a control, H is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, $F : H \to L^n \times FD^\nu(\mu)$ is a liner bounded operator responsible for realization of control actions. To formulate the control problem for (25), we introduce an on-target vector-functional $\ell : DS^n(m) \times FD^\nu(\mu) \to R^N$ of the general form (21). The control problem (CP) with respect to a given finite system of functionals ℓ_j, $\mathrm{col}\,(\ell_1, \ldots, \ell_N) = \ell$, for CDS (25) is the problem

$$\delta x = \Theta x + Fu + \varphi, \tag{26}$$
$$x(0) = \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in R^{n+\nu}; \quad \ell x = \gamma \in R^N$$

as the problem of the existense of a control $\bar{u} \in H$ such that BVP

$$\delta x = \Theta x + F\bar{u} + \varphi, \tag{27}$$
$$x(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \quad \ell x = \gamma$$

is solvable. If such a control exists for any $\varphi \in L^n \times FD^\nu(\mu)$, $\alpha \in R^n$, $\beta \in R^\nu$, $\gamma \in R^N$, then the CDS under control (25) is said to be controllable with respect to the vector-functional ℓ.

We shall obtain conditions of the solvability to (26) on the base of the representation (17) which gives the description of all solutions to (25) under the initial conditions $y(0) = \alpha \in R^n$, $z(0) = \beta \in R^\nu$.

$$x = \mathcal{X} \begin{pmatrix} \alpha \\ \sigma \\ \beta \end{pmatrix} + \mathcal{C}\varphi + \mathcal{C}Fu. \tag{28}$$

Here $\sigma = \mathrm{col}\,(\Delta y(\tau_1), \ldots, \Delta y(\tau_m)) \in R^{mn}$ is arbitrary. Applying the vector-functional ℓ to both sides of (28) and taking into account the goal of controlling as reaching the given value $\gamma \in R^N$ for ℓx along the trajectories of (26), we arrived at the equation

$$\ell\mathcal{X} \begin{pmatrix} \alpha \\ \sigma \\ \beta \end{pmatrix} + \ell\mathcal{C}\varphi + \ell\mathcal{C}Fu = \gamma \tag{29}$$

with respect to $\sigma \in R^{mn}$ and $u \in H$.

We shall reduce (29) to a linear algebraic system. Note that $\lambda_j = \ell_j \mathcal{C} F$ is a linear bounded functional defined on the Hilbert space H, this is why there exists $v_j \in H$ such that $\lambda_j = \langle v_j, u \rangle$ ($v_j = (\mathcal{C}F)^* \ell_j$, $\cdot^*$ stands for notation of adjoint operator).

Let us seek the control $\bar{u}$ in the form of the linear span

$$\bar{u} = \sum_{i=1}^{N} d_i v_i$$

(recall that the space H can be represented as the direct sum $\operatorname{span}(v_1, \ldots, v_N) \oplus [\operatorname{span}(v_1, \ldots, v_N)]^{\perp}$).

Thus, we have

$$\ell \mathcal{C} F \bar{u} = V d, \tag{30}$$

where $V = \{\langle v_i,\, v_j\rangle\}_{i,j=1,\ldots,N}$ is the Gram $(N \times N)$-matrix for the system $v_1, \ldots, v_N \in H$.

Let us write the matrix $\ell\mathcal{X}$ in the form

$$\ell\mathcal{X} = (\Xi_y|,\, \Xi_\Delta|,\, \Xi_z)\,, \tag{31}$$

where the matrices Ξ_y, Ξ_Δ, Ξ_z have dimensions $N \times n$, $N \times (mn)$, $N \times \nu$, respectively.

Now we arrive at the system

$$\Xi_\Delta \sigma + V d = \gamma - \ell\mathcal{C}\varphi - \Xi_y \alpha - \Xi_z \beta \tag{32}$$

and formulate the result as the following theorem.

Theorem 2. *(cf. Theorem 2 [13]) The control problem (26) for CDS (25) is solvable if and only if the linear algebraic system (32) is solvable in $(mn+N)$-vector $col(\sigma, d)$. Each solution $col(\sigma_0, d_0)$, $\sigma_0 = col(\sigma_0^1, \ldots, \sigma_0^m)$, of the system (32) defines the control that solves CP (26) including the impulses $\sigma_0^k = \Delta y(\tau_k)$, $k = 1, \ldots, m$, and the control $\bar{u} \in H$, $\bar{u} = \sum_{j=1}^{N} d_{0j} v_j$.*

5. Reliable computing experiment. The effective study of the original problem, (BVP (1),(20) or CP (26)) is based on the use of the corresponding linear algebraic system (LAS), $\ell\mathcal{X} \cdot c = \gamma$ for BVP and (32) for CP. In doing so we have to understand that all parameters of such a system can be only approximately calculated. Thus the study of LAS for solvability requests a special technique with use of the so-called reliable computing experiment (RCE) [9, 20]. Both the theoretical background and practical implementation of RCE need the elaboration of some specific constructive methods of investigation based on the fundamental statements of the general theory with making use of contemporary software. It is relevant to notice that the main destination of such methods is reliable establishing the fact of the solvability of the problem. If it is done, the next task is to construct an approximate solution in common with an error bound of quite high quality. RCE as a tool for the study of differential and integral models is very actively developing during last 20 years. There are some main directions in this field: the study of the Cauchy problem for ordinary differential equations (ODE) as well as for certain classes of partial DE (PDE) (H.Bauch, M.Berz, G.Corliss, B.Dobronetz, E.Kaucher and W.Miranker); the study of boundary value problems (BVP) for ODE and PDE (S.Godunov, M.Plum, N.Ronto and A.Samoilenko); the study of integral equations (E.Kaucher and W.Miranker, C.Kennedy, R.Wang); the study of nonlinear operator equations (S.Kalmykov, R.Moor, Yu.Shockin, Z.Yuldashev). A common idea in this studies is the interval calculus in finite-dimensional and functional spaces and, as a consequence, the special techniques of rounding off when calculations are produced by real computer. Our approach allows us to consider essentially more wide class of problems that are complicated by such properties as the property of not being a local operator, the presence of discontinuous solutions, the presence of the inner superposition operator, as well as the general form of boundary conditions. In addition we do not use interval calculations, which are characterized by high speed of the accumulation of rounding errors, but make use of the rational numbers arithmetics with a specific technique of definitely oriented rounding. The key idea of the constructive study is as follows: by the original problem there is being constructed an auxiliary problem with reliably computable parameters, which allows one to produce the efficient computer-assisted testing for the solvability. If such the problem is solvable, the final result depends on the closeness of the original problem and the auxiliary one (recall that the inequality $||\ell\mathcal{X} - \tilde{\ell}\tilde{\mathcal{X}}|| < 1/||[\tilde{\ell}\tilde{\mathcal{X}}]^{-1}||$ for approximations $\tilde{\ell}$, $\tilde{\mathcal{X}}$ to ℓ, $\mathcal{X}$, implies that $\ell\mathcal{X}$ is nonsingular). The theorems, which stand for a background of RCE, give efficiently testable (by means of computer) conditions of the solvability for the original problem. In the case these conditions are failed one has to construct a new (and more close to the original problem) auxiliary problem and then to test the conditions again. The implementation of the constructive methods in the form of a computer program (of course, it must be oriented to quite definite class of problems) allows one to study a concrete problem by a many-times repeated RCE. A theoretical background and some details of the practical implementation of RCE for the study of functional differential systems are presented in [20]. It is clear that RCE includes the construction and the successive refinement of approximation to the key parameters of LAS with reliable error bounds. An efficient computer-aided technique of such the construction for certain classes of FDE under some natural conditions is proposed in [15] (see also [9]).

6. Conclusive remark. CDFDS as an AFDE. First, recall the definition of AFDE. Let D and B be Banach spaces such that D is isomorphic to the direct product $B \times R^p$ ($D = B \times R^p$ for short).

The equation

$$\mathcal{L}x = \varphi \tag{33}$$

with a linear bounded operator $\mathcal{L} : D \to B$ is called the linear abstract functional differential equation (AFDE). The theory of the equation (33) was thoroughly treated in [7, 9]. Let us fix an isomorphism $J = \{\Lambda, Y\} : B \times R^p \to D$ and denote the inverse $J^{-1} = [\delta, r]$. Here $\Lambda : B \to D$, $Y : R^p \to D$ and $\delta : D \to B$, $r : D \to R^p$ are the corresponding components of J and J^{-1}:

$$J\{z, \alpha\} = \Lambda z + Y\alpha \in D, \; z \in B, \; \alpha \in R^p,$$

$$J^{-1}x = \{\delta x, rx\} \in B \times R^p, \; x \in D.$$

The system

$$\delta x = z, \; rx = \alpha \tag{34}$$

is called the principal boundary value problem (PBVP). Thus, for any $\{z, \alpha\} \in B \times R^p$,

$$x = \Lambda z + Y\alpha \tag{35}$$

is the solution of (34). The representation (35) gives the representation of $\mathcal{L}$: $\mathcal{L}x = \mathcal{L}(\Lambda z + Y\alpha) = \mathcal{L}\Lambda z + \mathcal{L}Y\alpha = Qz + A\alpha$, where the so-called principal part of $\mathcal{L}$, $Q : B \to B$, and the finite-dimensional operator $A : R^p \to D$ are defined by $Q = \mathcal{L}\Lambda$ and $A = \mathcal{L}Y$. The general theory of (33) assumes Q to be a Fredholm operator (i.e. a Noether one with the zero index).

The system (1) written in the form (24) is an AFDE with $\mathcal{L}x = \delta x - \Theta x$ considered as a linear bounded operator from the space D to the space B, where $B = L^n \times FD^\nu(\mu)$, $D = [L^n \times FD^\nu(\mu)] \times [R^{n+mn} \times R^\nu]$. In the case under consideration the principal part, Q, is invertible.

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