ISSN 0005-1179, Automation and Remote Control, 2014, Vol. 72, No. 8, pp. 345–357. © Pleiades Publishing, Ltd., 2014.
Original Russian Text © D.A. Shvarts, 2012, published in Problemy Upravleniya, 2012, No. 1, pp. 33–41.

CONTROL SCIENCES

Axiomatics for Power Indices in the Weighted Games

D. A. Shvarts

State University, Higher School of Economics, Moscow, Russia
e-mail: dshvarts@mail.ru

Received November 10, 2011

Abstract—This paper demonstrates that most existing voting schemes represent or can be rewritten as weighted games. However, axiomatics for power indices defined on simple games are not directly applied to weighted games, since related operations become ill-posed. The author shows that the majority of axiomatics can be adapted to weighted games. Finally, a series of examples are provided.

1. INTRODUCTION

As a matter of fact, numerous publications study the axiomatization problem for power indices. In this context, we mention the papers [1] (the first axiomatics for the Shapley–Shubik index [2]) and [3] (the first axiomatics for the Banzhaf index [4]), as well as the research works [5–8] (the axiomatics for power indices that depend on the preferences of participants [9]).

On the other hand, most existing voting schemes represent or can be described as weighted games. And the following question arises naturally. How should one axiomatize power indices in this class of voting rules?

It appears impossible to apply directly any of the above axiomatics (and other axiomatics the author knows) to the case of weighted games. The reason lies in that, in contrast to simple games that define power indices, the set of weighted games is not closed with respect to many operations (joining, intersection, crossing out a coalition).

The book [10] suggested axiomatics for the Banzhaf index adapted to weighted games. Several new axioms were introduced. However, the author of the present paper believes that their statements are more complex against axiomatics for the Banzhaf index in simple games.

The proposed framework seems *per se* interesting. As it has turned out, many (more specifically, most of) axiomatics can be adapted to weighted games by adding the phrase "if the result of operation also represents a weighted games" where necessary.

This simple approach allows reformulating to the case of weighted games the axiomatics for power indices that depend on the coalition preferences of participants, see [9].

The scope of a paper makes it impossible (or even unreasonable) to provide reformulations and proofs for all possible axiomatics. An interested reader will do this independently. The present paper focuses on the Dubey–Shapley axiomatics [3] for the Banzhaf index [4] and one of the axiomatics for power indices that account for the preferences of participants [8, 9].

2. SIMPLE GAMES, WEIGHTED GAMES AND POWER INDICES

Definition 1. A simple game is a pair (N, v), where N designates a set and $v : 2^N \to \{0, 1\}$ specifies a function assigning to each subset of N either 0 or 1 under the monotonicity property: if S and T are subsets of N and $S \subseteq T$, then $v(S) \leq v(T)$. ♦

This definition is given according to [11]. A more traditional definition of a simple game also presupposes that $v(\varnothing) = 0, v(N) = 1$. The latter condition excludes just two trivial games, where the function $v(S)$ is identically equal to 0 or 1. Denote these games by **0** and **1**, respectively.

In the sequel, we believe that N represents a finite set of elements numbered from 1 to n, i.e., $N = \{1, \ldots, n\}$. Elements of the set N are called players, whereas subsets of N are called coalitions. Whenever no confusion occurs, a simple game (N, v) will be denoted by v, and the number of players in a coalition S will be denoted by s. The symbol SG_n corresponds to the set of all simple games of n players.

A coalition S is winning, if $v(S) = 1$, and losing, if $v(S) = 0$.

Player i is called the key player in a coalition S, if the coalition S is winning and the coalition $S\backslash\{i\}$ is losing (clearly, this requires that $i \in S$). A player is called a null player, if it is not key in any coalition. This term was pioneered in [2] by analogy to bridge (a popular card game); in simple games and bridge, a null player has no impact on events. The notation $W_i(v)$ indicates the set of all coalitions, where player i is key.

A winning coalition is called minimal, if all players in it are key. In other words, a minimal winning coalition contains no other winning coalition. The sets of winning coalitions and minimal winning coalitions will be designated by $W(v)$ and $M(v)$, respectively. A simple game is often defined by enumerating all (or just minimal) winning coalitions. Such approach seems well-grounded, since $M(v)$ uniquely defines $W(v)$, and $W(v)$ uniquely defines the function v.

Remark 1. Interestingly, a simple game (except the cases of $v = 0, 1$) has at least one winning coalition (N); therefore, it has a minimal winning coalition being nonempty as far as $\varnothing \notin W(v)$. Recall that all players are key in a minimal winning coalition. Hence, any simple game comprises a player being key in some coalition. ⧫

Consider an arbitrary coalition S. The game u^S, where S represents a unique minimal winning coalition, will be called the oligarchical game. If $i \notin S$, then player i is key in all coalitions containing S. If $i \notin S$, then player i makes a null player.

Let v be a simple game not coinciding with $u^N, S \notin M(v)$. Denote by v_{-S} the game resulting from v by moving a coalition S from winning coalitions to losing coalitions. According to formal considerations, $W(v_{-S}) = W(v)/\{S\}$. Transition from v to v_{-S} is called *crossing out a coalition* S. The game v_{-S} turns out simple, either (the coalition S is minimal and crossing out the latter does not violate the monotonicity property). Under crossing out the coalition S, all players in it forfeit a coalition where they are key players; at the same time, all other players (outside S) gain such a coalition. More specifically, the following statement holds true.

Table 1. Crossing out a coalition

Player	Coalitions						
	{1, 2}	{3, 4}	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}	{1, 2, 3, 4}
1	+/–	–/–	+/+	+/+	–/–	–/–	–/–
2	+/–	–/–	+/+	+/+	–/–	–/–	–/–
3	–/–	+/+	–/+	–/–	+/+	+/+	–/–
4	–/–	+/+	–/–	–/+	+/+	+/+	–/–

Lemma 1 [5]. *Assume that* $S \notin M(v)$. *Then*

$$W_i(v_{-S}) = \begin{cases} W_i(v) \setminus \{S\}, & \text{if} \quad i \in S \\ W_i(v) \cup \{S \cup \{i\}\}, & \text{if} \quad i \notin S. \end{cases}$$

Example 1. Select $N = \{1, 2, 3, 4\}$. The winning coalitions in the game v are all three- and four-element subsets, the coalitions $\{1, 2\}$ and $\{3, 4\}$, $S = \{1, 2\}$.

The game v_{-S} has the following winning coalitions: $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, and $\{1, 2, 3, 4\}$.

In the table below, the sign "+" highlights coalitions, where an appropriate participant is a key player (from the left side of slash—for the game v, and from the right side of slash—for the game v_{-S}).

Crossing out a coalition

After transition to the game v_{-S}, players 1 and 2 are no more key in the coalition $\{1, 2\}$ (it becomes a losing coalition). However, players 3 and 4 turn out key in the coalitions $\{1, 2, 3\}$ and $\{1, 2, 4\}$, respectively. The rest cells of the table remain the same.

2.1. Weighted Games

This term describes a special case of simple games covering most existing voting schemes.

Definition 2. Let $N = \{1, \ldots, n\}$ be the set of players. A weighted game is an ordered set of $n + 1$ nonnegative numbers, where the first number (q) is called the quota and the rest numbers ($w_1, \ldots, w_n$) are called the numbers of votes or weights of corresponding players. A weighted game has the short notation $(q; w_1, \ldots, w_n)$.

The number of votes (or weight) of a coalition is the sum of votes given by players belonging to this coalition: $w(S) = \sum\limits_{i \in S} w_i$. A coalition is winning, if the total number of votes of its players appears not smaller than the quota (and losing, otherwise). Therefore, a weighted game gets associated with a simple game. ♦

Example 2. In June 2011 (the period of preparation of this paper), the State Duma of the Russian Federation comprised 450 seats within 4 factions, namely, United Russia (315 seats), The Communist Party of the Russian Federation (CPRF) (57 seats), The Liberal Democratic Party (LDPR) (40 seats), and A Just Russia (38 seats). Decision-making requires simple majority, i.e., at least 226 votes. Thus, the decision-making rule is the weighted game (226; 315, 57, 40, 38). Here winning coalitions are any coalitions containing the first faction. ♦

The correspondence between weighted games and simple games is not univocal. For instance, the weighted games (51; 34, 33, 33) and (51; 49, 49, 2) define a same simple game—winning coalitions form two- and three-element sets exclusively.

Definition 3. We say that a simple game v can be reexpressed as a weighted game, if there exist nonnegative numbers q, $w_1, \ldots, w_n$ such that the weighted game $(q; w_1, \ldots, w_n)$ defines the game v. ♦

Whenever the difference is not important, we will identify a weighted game with a corresponding simple game.

Denote by WG_n the set of all simple games that can be reexpressed as weighted games.

Example 3 [12]. The United Nations Security Council consists of 15 members, *viz.*, five permanent members (the UK, China, Russia, the USA, and France) and 10 non-permanent members elected for two-year terms. A decision is made by the majority of nine votes, where five necessarily belong to permanent members (the latter possess veto power).

This decision-making rule admits the representation as the weighted game (39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1). The same winning coalitions appear if the permanent members of the Security Council receive 7 votes, each of the rest members obtain one vote and the quota makes up 39 votes. ♦

However, some simple games have no characterization in terms of weighted games. We provide a "minimal" example.

Example 4. Select $N = \{1, 2, 3, 4\}$ and define the game by the set of minimal winning coalitions: $M(v) = \{\{1, 2\}, \{3, 4\}$. Demonstrate that this game is not reexpressed as a weighted game.

By evidence to the contrary, suppose that there is a set $(q; w_1, w_2, w_3, w_4)$ defining the game v. The coalitions $\{1, 2\}$ and $\{3, 4\}$ are winning, therefore $w_1 + w_2 \geq q$, $w_3 + w_4 \geq q$ and, consequently, $w_1 + w_2 + w_3 + w_4 \geq 2q$. The coalitions $\{1, 3\}$ and $\{2, 4\}$ are losing, thus $w_1 + w_3 < q$, $w_2 + w_4 < q$ and, hence, $w_1 + w_2 + w_3 + w_4 < 2q$. This result makes a contradiction. ♦

Decision-making rules not reexpressible as weighted games arise in real elective bodies. An example (though, rather cumbersome) can be found in [11].

Apparently, still there is no simple technique to define whether an arbitrary simple game is a weighted game or not. For a detailed treatment of this issue, we refer to [11].

A power index, $\Phi : SG_n \to R^n$, assigns each simple game v to a vector $\Phi(v)$, where component i characterizes the power of player i. The power index of a weighted game is the power index of a corresponding simple game. Among most widespread power indices, we mention the ones suggested by Banzhaf and Shapley–Shubik. The present paper *par excellence* addresses the former index.

The Banzhaf index (BI) [4] is evaluated under the assumption that the power of a player appears directly proportional to the number of coalitions, where this player is key. The total Banzhaf index of player i constitutes $TBz_i = |W_i|$.

The Banzhaf index Bz_i results from normalization of the total index:

$$Bz_i = |W_i| \Big/ \sum_{j=1}^{n} |W_j|.$$

As a matter of fact, this power index was pioneered by Penrose—see the paper [13], where the number of coalitions with key player i is divided by the number of all coalitions containing player i:

$$P_i = \frac{1}{2^{n-1}} |W_i|.$$

Many publications (particularly, the work [15]) comprehend the Banzhaf index in the Penrose sense. To combine the historical justice and well-established traditions, the present paper reformulates the results of [3, 5] in terms of the total Banzhaf index. To pass to the Penrose index, one should simply divide the total Banzhaf index by 2^{n-1}.

An alternative form of the total Banzhaf index is

$$TBz_i = \sum_{S \subseteq N} (v(S) - v(S \setminus \{i\})).$$

Here we employ the key player property: $v(S) - v(S \setminus \{i\})$ equals 1, if player i is key in a coalition S, and 0 otherwise.

The Shapley–Shubik index (SSI) [2] stemmed from game theory as a particular case of the Shapley vector. In this index, the number added by a coalition to player's power depends on its size:

$$SS_i = \sum_{S \in W_i(v)} \frac{(n-s)!(s-1)!}{n!} = \sum_{S \subseteq N} (v(S) - v(S \setminus \{i\})) \frac{(n-s)!(s-1)!}{n!}.$$

2.2. Axiomatics for the Shapley–Shubik Index and the Banzhaf Index

For these (perhaps, most widespread) power indices, researchers have constructed many axiomatics. Consider the first and second ones in chronological sequence.

The Shapley–Shubik index is uniquely defined by the following four axioms.

Null Player (NP). In any simple game v, null player i possesses zero power.

Anonymity (An). For any game $v \in SG_n$, any permutation π of the set N and any $i \in N$:

$$\Phi_i(\pi v) = \Phi_{\pi(i)}(v),$$

where $(\pi v)(S) = v(\pi(S))$.

Transfer (T). For any games $v, w \in SG_n$ such that $v \vee w \in SG_n$:

$$\Phi(v) + \Phi(w) = \Phi(v \vee w) - \Phi(v \wedge v),$$

where $(v \vee w)(S) = \max(v(S), w(S))$, and $(v \wedge w)(S) = \min(v(S), w(S))$. ♦

This axiom admits an equivalent statement, see [5].

Transfer* (T*). For any games $v, w \in SG_n$, any coalition $S \in M(v) \cap M(w)$ and any $i \in N$:

$$\Phi_i(v) - \Phi_i(v_{-S}) = \Phi_i(w) - \Phi_i(w_{-S}).$$

Efficiency (E). If $v \neq \mathbf{0}, \mathbf{1}$, then

$$\sum_{i=1}^{n} \Phi_i(v) = 1,$$

i.e., the following result holds true.

Theorem 1 [2]. *Let $\Phi : SG_n \to R^n$. Then Φ satisfies axioms NP, An, $T(T^*)$ and E iff Φ represents the Shapley–Shubik index.* ♦

The Banzhaf index disagrees with the efficiency axiom. Therefore, it is replaced by a slightly more complex condition.

Banzhaf Total Power (BzTP).

$$\sum_{i=1}^{n} \Phi_i(v) = \sum_{i=1}^{n} \sum_{S \subset N} (v(S) - v(S \setminus \{i\})).$$

The rest three axioms coincide with those for the Shapley–Shubik index.

Theorem 2 [3, 5]. *Let $\Phi : SG_n \to R^n$. Then Φ satisfies axioms NP, An, $T(T^*)$ and $BzTP$ iff Φ is the Banzhaf index.*

3. GAMES AND POWER INDICES DEPENDENT ON THE PREFERENCES OF PARTICIPANTS

The structure below generalizes the definitions in [9] (see Example 3). The definition of a simple game is supplemented by the following information: each player i and each coalition S are assigned a quantity $f(i, S)$ interpreted as the measure of this player's desire for entering the coalition S.

Definition 4. A simple game with preferences is a triplet (N, v, f), where $N = \{1, \ldots, n\}$ denotes the set of players, the pair (N, v) forms a simple game, and f is a function which assigns a positive number $f(i, S)$ to each coalition S and each player i. ♦

A simple game can be viewed as a simple game with preferences, where all coalitions appear equally preferable: $(N, v){\equiv}(N, v, 1)$. Whenever no confusion occurs, a game (N, v, f) is designated simply by v. By assumption, two games mentioned in a proof have the same function f.

The notions of winning, losing and minimal winning coalitions, a key player, crossing out a coalition and a weighted game are *in extenso* adopted from simple games. The presence of an additional function f introduces no fundamental modifications so far. Under crossing out a coalition, changes affect only v, the function f remains the same.

Example 5 [9]. The preferences of players are specified by a matrix P of dimensions $n \times n$. Speaking informally, its element $p_{ij} \in [0,\,1]$ defines the desire of player i for entering a coalition with player j. The matrix P is not necessarily symmetrical, i.e., generally $p_{ij} \neq p_{ji}$. For the sake of computational convenience, we believe that $p_{ii} = 0$.

The paper [9] suggested several techniques for preference matrix evaluation in real elective bodies and over 10 versions of the index based on the preference matrix. Recall four of them in the current notation:

$$f^{+}(j, S, P) = \sum_{i \in S} \frac{p_{ji}}{s-1};$$

$$f^{-}(j, S, P) = \sum_{i \in S} \frac{p_{ij}}{s-1};$$

$$f(j, S, P) = [f^{+}(j,\ S,\ P) + f^{-}(j, S, P)]/2;$$

$$f(S, P) = \sum_{j \in S} \frac{f^{+}(j, S, P)}{s} = \sum_{j \in S} \frac{f^{-}(j, S, P)}{s} = \frac{1}{s(s-1)} \sum_{i,j \in S} p_{ij}.$$

If a coalition S comprises one element, suppose that all functions equal unity. The function $f^{+}(j, S, P)$ can be comprehended as the average desire of player j for entering a coalition with the rest players S. Similarly, the function $f^{-}(j, S, P)$ characterizes the average desire of the rest players from the coalition S for making a coalition with player j. And finally, the function $f(S, P)$ reflects the average desire of all players for entering a coalition with their colleagues from the coalition S.

If all players in a coalition S possess good relations, i.e., $p_{ij} = 1$ for all $i, j \in S$, then $f^{+}(j, S, P) = f^{-}(j, S, P) = f(S, P) = 1$. If the relations among all players from a coalition S are bad, i.e., $p_{ij} = 0$ for all $i, j \in S$, then $f^{+}(j, S, P) = f^{-}(j, S, P) = f(S, P) = 0$.

Similarly to the case of simple games, for each game v with symmetrical or asymmetrical preferences a power index $\Phi : SGP_n \to R^n$ assigns a vector $\Phi(v)$, whose component i is interpreted as the power of player i. ♦

Definition 5. The α-index of power is defined by the formula

$$\alpha_i(v) = \sum_{S \in W_i(v)} f(i, S).$$

Suppose that $f(i, S) > 0$ for all players and coalitions, and v is not identically equal to 0 or 1. Introduce the normalized α-index [9] in the form

$$N\alpha_i(v) = \frac{\alpha_i(v)}{\sum\limits_{j \in N} \alpha_j(v)}. \quad ♦$$

The conditions of Definition 5 guarantee a nonzero denominator in the faction above.

The proof of the following result (though, in a slightly modified statement) can be found in the paper [8].

Lemma 2. *Let $S \in M(v)$. In this case,*

$$\alpha_i(v) - \alpha_i(v_{-S}) = \begin{cases} f(i,S), & \text{if} \quad i \in S \\ -f(i, S \cup \{i\}), & \text{if} \quad i \notin S. \end{cases}$$

Example 6. Assume that the function $f(S)$ depends only on the number of players in a coalition S. If $f(S) = 1$, then the α-index coincides with the total Banzhaf index and the normalized α-index coincides with the Banzhaf index:

$$St_i(v) = \sum_{S \in W_i(v)} 1 = |W_i(v)| = Bz_i(v).$$

If $f(S) = 1/2^{n-1}$, the α-index coincides with the Penrose index.

And finally, if $f(S) = \frac{(n-s)!(s-1)!}{n!}$, the $\alpha(v)$-index coincides with the Shapley–Shubik index:

$$\alpha(v) = \sum_{S \in W_i(v)} \frac{(n-s)!(s-1)!}{n!} = SS_i(v). \quad \blacklozenge$$

Many other indices (e.g., the ones proposed by Johnston [14], Deegan–Packel [16], Holler–Packel [17]) are also expressed via the α-index [15]. Therefore, the α-index can be considered as their generalization.

4. AXIOMATICS FOR THE α-INDEX

The α-index allows an axiomatization by analogy to the indices above [8]. For variety's sake, we provide another axiomatics from the same paper [8]. Interestingly, two axioms are sufficient.

Null Player (NP). The payoff of a null player appears independent from the preference intensities and equals zero.

Strong Transfer (ST). For any game $v \in SGP_n$, any coalition $S \in M(v)$ and any $i \in S$

$$\Phi_i(v) - \Phi_i(v_{-S}) = f(i,S).$$

If $i \in S$, then axiom ST strengthens axiom T* (the latter indicates that the difference $\Phi_i(v) - \Phi_i(v_{-S})$ is constant with respect to v, and the former evaluates this difference).

However, in the case of $i \notin S$, axiom ST declares nothing, in contrast to axiom T*.

Theorem 3. *The power index $\Phi(v)$ satisfies axioms NP and ST iff $\Phi(v) = \alpha(v)$.*$\blacklozenge$

This axiomatics is analogous to stating that a linear function is defined by two properties: (1) vanishing in the origin and (2) having a constant derivative at each point.

5. AXIOMATICS FOR POWER INDICES IN THE CASE OF WEIGHTED GAMES

Any of the mentioned axiomatics are not directly applied to weighted games, since the results of many operations over weighted games (joining, intersection, crossing out a coalition) do not represent weighted games.

Example 7. Let $N = \{1,2,3,4\}$. Denote by v a simple game with two minimal winning coalitions $\{1,2\}$ and $\{3,4\}$. According to Example 2, v could not be rewritten as a weighted game. However,

- $v = u^{\{1,2\}} \cup u^{\{3,4\}}$, i.e., the joining of two weighted games;
- consider four weighted games:

$$w_1 = (3; 2, 1, 2, 1), \quad w_2 = (3; 1, 2, 2, 1),$$
$$w_3 = (3; 2, 1, 1, 2), \quad w_4 = (3; 1, 2, 1, 2);$$

here winning coalitions are all three- and four-element sets of players and all two-element sets except the coalition $\{2, 4\}$ for the vote w_1, except $\{1, 4\}$ for w_2, except $\{2, 3\}$ for w_3 and except $\{1, 3\}$ for w_4. Therefore, the intersection of these weighted games yields the game v;

- in the vote w_1, five minimal winning coalitions are all two-element sets except the coalition $\{2, 4\}$. Cross out the coalition $\{1, 3\}$. The resulting simple game could not be reexpressed as a weighted game. Otherwise, since the coalitions $\{1, 3\}$ and $\{2, 4\}$ are winning, the sum of their votes appears not smaller than the two quotas; the coalitions $\{1, 3\}$ and $\{2, 4\}$ are losing, and the sum of their votes is smaller than the two quotas. But in both cases the matter concerns the sum of votes of all players. We have arrived at a contradiction. ♦

On the other hand, a series of "basic" games admit weighted game representations. Although it is impossible to cross out an arbitrary minimal winning coalition from a game $v \in WG_n$ still staying in the set WG_n, one can cross out a certain minimal winning coalition. Thus and so, some proofs remain in force if all axioms get supplemented by the phrase "if the result of operation also represents a weighted game" where necessary. Formalization of the aforesaid begins with the following lemma.

Lemma 3. *(a) Simple games* $\mathbf{0}$, $\mathbf{1} \in WG_n$*; (b) for any* S*:* $u^S \in WG_n$*; (c) for any* $S \neq N$ *:* $u^S_{-S} \in WG_n$ *; (d) let* $v \in WG_n$ *and player* i *not be a null player in a game* v*. Then there exists a minimal winning coalition* $S \ni i$ *such that* $v_{-S} \in WG_n$*.*

Proof. (a) Suppose that for all $i \in$: $Nw_i = 1$. In this case, if $q = 0$, all coalitions appear winning and, if $q = n + 1$, winning coalitions are absent.

(b) Let $w_i = n + 1$, if $i \in S$, $v_i = 1$, if $i \notin S$, $q = |S|(n + 1)$. Then a coalition is winning iff it contains S, *quod erat demonstrandum.*

(c) Define the weights of players just as in (b) and decrease the quota by unity: $q|S|(n+1) - 1$. A coalition is winning iff it contains S with the only exception: S makes a losing coalition. Such a construction becomes ill-defined if $|S| = 0$. But then we obtain $S = \varnothing$ and $v = 1$, and this case has been explored in (a).

(d) Decompose this statement into two ones.

(d1) If a game can be reexpressed as a weighted game, this is possible to do such that the payoffs of all coalitions become pairwise different.

(d2) If the payoffs of all coalitions are pairwise different, then an appropriate reduction in the weight of player i guarantees that the same coalitions remain winning except the coalition containing player i.

First, prove (d1). Let ε be the difference between the quota and the weight of the most powerful losing coalition. Choose positive numbers $\varepsilon_1, \ldots, \varepsilon_n$ that are smaller ε/n and consider the weighted game $(q; v_1 + \varepsilon_1, \ldots, v_n + \varepsilon_n)$.

Demonstrate that the new weighted game defines the same simple game as the previous one. The quota remains unchanged, whereas the weight of each player increases. Therefore, winning coalitions are still winning. But the weight of each coalition increases at most by the sum of all ε_i; each of these quantities is smaller than ε/n, i.e., their sum increases at most by ε. Consequently, all losing coalitions are again losing.

Now, show that it is possible to choose ε_i such that the weights of all coalitions differ.

The set of all admissible quantities ε_i forms an open hypercube in R^n defined by the inequalities $0 < \varepsilon_i < \varepsilon_i/n$. Its measure constitutes $(\varepsilon/n)^n > 0$. Each equality condition for the weights of two coalitions specifies a linear equation in ε_i. In other words, inappropriate quantities ($\varepsilon_1, \ldots,$ ε_{n}) lie in a finite set of hyperplanes in R^n, i.e., have zero measure in R^n. Therefore, the set of appropriate quantities possesses the same (positive) measure as the set of all admissible quantities ε_i, guaranteeing that the set of all appropriate quantities appears nonempty.

d2) Gradually decrease the weight of player i under fixed quota and weights of the rest players. When the weight of player i vanishes, all coalitions containing player i as key become losing. Player i is not a null player, and such coalitions do exist.

Therefore, under gradually decreasing weight of player i, we encounter a certain moment when the first of these coalitions becomes losing. Since the weights of all coalitions differ, it is possible to choose a moment when just one of these coalitions is losing. ♦

Note an important aspect. If the weight of players represent integer numbers (a common situation in practice), it is possible to make the modified weights of players integer, either. There is nothing to prevent from selecting ε_i rational; then the modified weights of players turn out rational. (The point is that the set of appropriate quantities $(\varepsilon_1, \ldots, \varepsilon_n)$ is not just nonempty, but also open (as the difference between an open set and a finite number of closed sets). Any nonempty open subset in R^n contains a point with rational coordinates.)

Interestingly, for any positive a, the weighted games $(q; w_1, \ldots, w_n)$ and $(aq; aw_1, \ldots, aw_n)$ define the same simple game. And so, by multiplying the quota and the weights of all players by the common denominator ε_i, one obtains a weighted game having integer coefficients.

Generally, analogous reasoning allows to demonstrate that any game representable in the form of a weighted game can be reexpressed as a weighted game having integer quotas and weights of players. It seems curious to obtain the same result without an "intermediate" weighted game.

Example 8. Consider the weighted game $(2; 1, 1, 1)$. Set $\varepsilon_1 = 1/2$, $\varepsilon_2 = 1/3$, $\varepsilon_3 = 1/6$. Add them to the weights of players to get the weighted game $(2, 3/2, 4/3, 7/6)$ or $(12; 9, 8, 7)$.

According to Table 2, the weights of all coalitions differ, and winning coalitions are two- and three-elements sets, as previously.

Table 2. An illustration to Lemma 3

X	Ø	{*C*}	{*B*}	{*A*}	*q*	{*B*,*C*}	{*A*,*C*}	{*A*,*B*}	{*A*,*B*,*C*}
w(*X*)	0	7	8	9	12	15	16	17	24

Remark 2. Points (a), (b) and (c) of the lemma are obvious. They serve for a complete statement. The work [10] established a result analogous to point (2): if a game v admits representation in the form of a weighted game, then there exists a minimal winning coalition S such that the game v_{-S} is also reexpressed as a weighted game. However, the lemma has a more accurate formulation and the author believes that the proof above is simpler and better fits the essence of the problem.

5.1. Adapted Axioms and Characterization

Axioms NP, An, E and BzTP remain unchanged. But the domain of an index is reduced from all simple games to weighted games.

Furthermore, axioms T and T* get even weakened.

Transfer (T). For any $v, w \in WG_n$ such that $v \vee w \in WG_n$ and $v \wedge w \in WG_n$:

$$\Phi(v) + \Phi(w) = \Phi(v \vee w) - \Phi(v \wedge v),$$

where $i \in N(v \vee w)(S) = \max(v(S), w(S))$, and $(v \wedge w)(S) = \min(v(S), w(S))$.

Transfer* (**T***). For any games $v, w \in WG_n$, any coalition $S \in M(v) \cap M(w)$ such that $v_{-S}, w_{-S} \in WG_n$ and any player i:

$$\Phi_i(v) - \Phi_i(v_{-S}) = \Phi_i(w) - \Phi_i(w_{-S}).$$

Proof employs the following lemma.

Lemma 4. *Suppose that a power index* Φ *meets axiom T. Then it satisfies axiom T*.*

Proof. Let v and w be reexpressed as weighted games, $S \in M(v) \cap M(w)$, v_{-S} and w_{-S} be representable as weighted games. If $S = N$, then $v = w = u^N$ and this lemma becomes trivial. In the sequel, we believe that $S \neq N$.

By virtue of Lemma 5, the games u^S and u^S_{-S} are reexpressed as weighted games; moreover, $v_{-S} \cup u^S = v$, $v_{-S} \cap u^S = u^S_{-S}$, $w_{-S} \cup u^S = w$, and $w_{-S} \cap u^S = u^S_{-S}$. Hence, due to axiom T,

$$\Phi(v) = \Phi(v_{-S} \cup u^S) = \Phi(v_{-S}) + \Phi(u^S) - \Phi(u^S_{-S}),$$
$$\Phi(w) = \Phi(w_{-S} \cup u^S) = \Phi(w_{-S}) + \Phi(u^S) - \Phi(u^S_{-S}),$$

i.e.,

$$\Phi(v) - \Phi(v_{-S}) = \Phi(w) - \Phi(w_{-S}) = \Phi(u^S) - \Phi(u^S_{-S}).$$

This concludes the proof of Lemma 4. ♦

We demonstrate the correctness of the above axiomatics for the Banzhaf index by analogy to similar assertions in [5, 8]. The introduced amendments allow evading the specifics of weighted games.

Theorem 4. *Let* $\Phi : WG_n \rightarrow R^n$. *Then* Φ *satisfies axioms NP, An, T and BzTP iff* Φ *forms the Banzhaf index.*

Proof. Note that the reformulated axioms turn out weaker in comparison with their counterparts (they state the same but under essential constraints). According to Theorem 2, the Banzhaf index defined on SG_n meets axioms NP, An, T and BzTP. Hence, the same index defined on WG_n necessarily satisfies these axioms.

The converse assertion will be argued by mathematical induction for the number of winning coalitions; argumentation utilizes the results from part 1 of the proof.

The basis. Suppose that $|W(v)| = 0$, i.e., $v = \mathbf{0}$. Owing to Lemma 5, $v \in WG_n$. There is no key player in any coalition. Axiom NP implies that $\Phi_i(v) = 0$ for all i. Since $Bz(v)$ also meets axiom NP, then $Bz_i(v) = 0$. And so, $\Phi_i(v) = Bz_i(v)$.

The inductive step. Two cases are possible, as follows.

1. The game v has a unique minimal winning coalition S, i.e., $v = u^S$. By virtue of Lemma 5, $u^S \in WG_n$. In this case, a coalition T is winning iff it contains S, *viz.*, incorporates all players from S. Therefore, if player $j \notin S$, its belonging to the coalition T affects nothing: the coalitions T and $T \backslash \{j\}$ are winning or losing simultaneously. Thus, all players outside S represent null players in the game v. And so, if $i \notin S$, then $\Phi_i(v) = Bz_i(v) = 0$.

Now, consider players entering S. The anonymity axiom claims that these players have an identical power, i.e., for any $i, j \in S$: $\Phi_i(v) = \Phi_j(v)$ and $Bz_i(v) = Bz_j(v)$. According to axiom

BzTP, the total powers of the players evaluated using the indices Bz and Φ do coincide, i.e.,

$$\sum_{i\in N} \Phi_i(v) = \sum_{i\in N} Bz_i(v),$$
$$\sum_{i\in S} \Phi_i(v) = \sum_{i\in S} Bz_i(v),$$
$$|S|\Phi_j(v) = |S|Bz_j(v),\ \forall j \in S,$$
$$\Phi_j(v) = Bz_j(v),\ \forall j \in S,$$

quod erat demonstrandum.

2. The game v has two minimal winning coalitions (S and S'), i.e., $M(v) > 1$. In addition, it is possible to believe that $v_{-S} \in WG_n$. Other winning coalitions are all coalitions containing S. Therefore, the number of winning coalitions in v appears not smaller than in u^S. However, $S \not\subset S'$ (otherwise, the coalition S' fails to be minimal winning). Hence, v includes more winning coalitions than u^S, and the inductive hypothesis can be applied to u^S. Cross out S from v and u^S; $v_{-S} \in WG_n$ by the inductive hypothesis, $u_{-S} \in WG_n$ owing to Lemma 5. Due to axiom T* for the index Φ, the inductive hypothesis for u^S and u^S_{-S} and axiom T* for the index Bz, we obtain

$$\Phi(v) - \Phi(v_{-S}) = \Phi(u^S) - \Phi(u^S_{-S}) = Bz(u^S) - Bz(u^S_{-S}) = Bz(v) - Bz(v_{-S}).$$

At the same time, the inductive hypothesis for v_{-S} dictates that $\Phi(v_{-S}) = Bz(v_{-S})$. Hence, $\Phi(v) = Bz(v)$. ♦

By analogy, one can formulate and prove a similar theorem for the Shapley–Shubik index.

Theorem 5. *Let* $\Phi\colon WG_n \to R^n$. *Then* Φ *meets axioms NP, An, T and iff* Φ *represents the Shapley–Shubik index.* ♦

The line of reasoning word-by-word repeats the proof of the previous theorem; we merely replace axiom BzTP by the efficiency axiom. Formally speaking, the axiom states nothing if $v = \mathbf{0}$ or $\mathbf{1}$, but this follows from axiom NP.

If $v = \mathbf{0}$ or $\mathbf{1}$, there exist no key players in any coalition in the game v (in the former case, winning coalitions disappear, whereas in the latter case losing coalitions are absent). Consequently, all players are null players and, if the power index Φ agrees with axiom NP, then $\Phi_i(v) = 0$ for all players i (similarly to the proof of Theorem 4).

6. AXIOMATICS FOR THE α-INDEX IN THE CASE OF WEIGHTED GAMES

Lemma 5 allows adapting the axiomatics for the α-index to the case of weighted games. Let us reformulate the axioms.

Null Player (NP). The payoff of a null player is independent from preference intensities and equals 0.

Strong Transfer (ST). For any weighted game v and any coalition $S \in M(v)$ such that v_{-S} is a weighted game and any $i \in S$:

$$\Phi_i(v) - \Phi_i(v_{-S}) = f(i, S).$$

Theorem 6. *In the case of weighted games, the α-index is uniquely defined by axioms NP and ST restated for weighted games.*

Proof. By analogy to the proof of Theorem 4, note that the reformulated axioms turn out weaker than their counterparts. Since the α-index meets the original versions of axioms NP and ST (without restatement for weighted games), it naturally does so for the reformulated ones.

The converse assertion will be argued by mathematical induction for the number of winning coalitions; argumentation utilizes the results from part 1 of the proof.

The basis. Suppose that winning coalitions are absent. This game is reexpressed as the weighted game $(n+1; 1, \ldots, 1)$. There are no key players in any coalition. Hence, by axiom NP, $\Phi_i(v) = 0$ for all i. As far as $\alpha(v)$ satisfies axiom NP, $\alpha_i(v) = 0$. Therefore, $\Phi_i(v) = \alpha_i(v)$.

The inductive step. Let $v \in WGP_n$. If i is a null player in the game v, then $\Phi_i(v) = \alpha_i(v) = 0$. Otherwise, Lemma 5 implies that there exists a coalition $S \in M(v)$ such that $v_{-S} \in WGP_n$. Apply the inductive hypothesis to the game v_{-S} to get

$$\Phi_i(v_{-S}) = \alpha(v_{-S}) = \sum_{T \in W_i(v_{-S})} f(i, S).$$

By virtue of axiom ST for $\Phi(v)$, the inductive hypothesis and axiom ST for $\alpha(v)$, we obtain:

$$\Phi_i(v) = \Phi_i(v_{-S}) + f(i, S) = \alpha_i(v_{-S}) + f(i, S),$$
$$\alpha_i(v) = \alpha_i(v_{-S}) + f(i, S).$$

Therefore, $\Phi_i(v) = \alpha_i(v)$. ♦

7. CONCLUSION

According to the author's viewpoint, the axiomatization problem for power indices restricted to weighted games does not seem appealing. The whole essence is that all well-known power indices are uniquely defined, e.g., by the set of winning coalitions and involve no specifics of weighted games.

On the other hand, this paper demonstrates that axiomatics for power indices in the case of weighted games can be constructed by simply reformulating the original axioms.

Concerning mathematical aspects, the major statement of the present paper lies in the following. Although simple games corresponding to weighted games do not form a lattice, it is possible to move from the maximal element to the minimal one by visiting any given node. Perhaps, this consideration will assist in exact description of the set of games reexpressed as weighted games.

REFERENCES

1. Dubey, P., On the Uniqueness of the Shapley Value, *Int. J. Game Theory*, 1975, vol. 4, pp. 131–139.
2. Shapley, L.S. and Shubik, M., A Method for Evaluating the Distribution of Power in a Committee System, *Am. Polit. Sci. Rev.*, 1954, vol. 48(3), pp. 787–792.
3. Dubey, P. and Shapley, L.S., Mathemaical Properties of the Banzhaf Power Index, *Math. Oper. Res.*, 1979, vol. 4, pp. 99–131.
4. Banzhaf, J.F., Weighted Voting Doesn't Work: A Mathematical Analysis, *Rutgers Law Rev.*, 1965, vol. 19, pp. 317–343.
5. Laruelle A. and Valenciano, F., Shapley–Shubik and Banzhaf Indices Revisited, *Math. Oper. Res.*, 2000, vol. 26, no. 1, pp. 89–104.
6. Lehrer, E., An Axiomatizaton of the Banzhaf Value, *Int. J. Game Theory*, 1988, vol. 17, pp. 88–99.
7. Nowak, A.S., An Axiomatizaton of the Banzhaf Value without the Additivity Axiom, *Int. J. Game Theory*, 1997, vol. 26, pp. 137–141.
8. Shvarts, D.A., Axiomatics for Power Indices that Account for Player Preferences, *Autom. Remote Control*, 2010, no. 1, pp. 128–141.

9. Aleskerov, F.T., Power Indices That Account for Coalition Preferences of the Participants, *Dokl. Akad. Nauk*, 2007, vol. 414, no. 5, pp. 594–597.

10. Batsyn, M.V. and Kalyagin, V.A., *Ob aksiomaticheskom opredelenii obshchikh indeksov vliyaniya v zadache golosovaniya s kvotoi* (On the Axiomatization of Total Power Indices in the Weighted Games), Moscow: Vysshaya Shkola Ekonomiki, 2009.

11. Taylor, A.D. and Zwicker, W.S., *Simple Games*, Princeton: Princeton Univ. Press, 1999.

12. Roberts, F.S., *Discrete Mathematical Models with Application to Social, Biological and Environmental Problems*, Englewood Cliffs: Prentice-Hall, 1976. Translated under the title *Diskretnye matematicheskie modeli s prilozheniyami k sotsial'nym, biologicheskim i ekologicheskim zadacham*, Moscow: Nauka, 1986.

13. Penrose, L.S., Elementary Statistics of Majority Voting, *J. Royal Statict. Soc.*, 1946, vol. 109, pp. 53–57.

14. Johnston, R.J., On the Measurement of Power: Some Reactions to Laver, *Environ. Plan.*, 1978, vol. 10, pp. 907–914.

15. Shvarts, D.A., On Calculation of the Power Indices with Allowance for the Agent Preferences, *Autom. Remote Control*, 2009, no. 3, pp. 484–490.

16. Deegan, J. and Packel, E.W., A New Index of Power for Simple n-Person Games, *Int. J. Game Theory*, 1978, vol. 7(2), pp. 113–123.

17. Holler, M.J. and Packel, E.W., Power, Luck and the Right Index, *J. Econom.*, 1983, vol. 43, pp. 21–29.