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Runup Characteristics of Symmetrical Solitary Tsunami Waves of "Unknown" Shapes

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Abstract—The problem of tsunami wave runup on a beach is discussed in the framework of the rigorous solutions of the nonlinear shallow-water theory. We present an analysis of the runup characteristics for various shapes of the incoming symmetrical solitary tsunami waves. It will be demonstrated that the extreme (maximal) wave characteristics on a beach (runup and draw-down heights, runup and draw-down velocities and breaking parameter) are weakly dependent on the shape of incident wave if the definition of the "significant" wavelength determined on the 2/3 level of the maximum height is used. The universal analytical expressions for the extreme wave characteristics are derived for the runup of the solitary pulses. They can be directly applicable for tsunami warning because in many cases the shape of the incident tsunami wave is unknown.

Key words: Tsunamis, nonlinear shallow-water theory, long-wave runup.

1. Introduction

The reliable estimation of inundation extent is a key problem of coastal wave dynamics, and in particular for tsunami mitigation. Since the characteristic length of a tsunami wave in the coastal zone is several kilometers, the nonlinear shallow water theory is an appropriate theoretical model to describe the process of tsunami runup on a beach. The problem of the runup of long non-breaking waves on a plane beach is well described mathematically within the framework of a nonlinear shallow water theory. This approach leads to an analytical solution based on the Carrier–Greenspan transform (CARRIER and GREENSPAN, 1958). Various shapes of the periodic incident wavetrains such as the sine wave (KAISTRENKO *et al*., 1991; MADSEN and FUHRMAN, 2008), cnoidal wave (SYNOLAKIS, 1991) and nonlinear deformed periodic wave (DIDENKULOVA *et al*., 2006, 2007b) have been analyzed in the literature. The relevant analysis has also been performed for a variety of solitary waves and single pulses such as soliton (PEDERSEN and GJEVIK, 1983; SYNOLAKIS, 1987; KÂNOĞLU, 2004), sine pulse (MAZOVA et al., 1991), Lorentz pulse (PELINOVSKY and MAZOVA, 1992), Gaussian pulse (CARRIER *et al*., 2003;

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KÂNOĞLU and SYNOLAKIS, 2006), *N*-waves (TADEPALLI and SYNOLAKIS, 1994), "characterized tsunami waves" (Tinti and Tonini, 2005) and the random set of solitons (BROCHINI and GENTILE, 2001). As is often the case in nonlinear problems, reaching an analytical solution is seldom possible. Runup of solitary pulses is, however, often easily implemented experimentally in measuring flumes, and various experimental expressions are available (see MADSEN and FUHRMAN, 2008 for references).

The existing results for the water wavefield are based on various initial conditions (shapes of the incident waves) and are therefore not directly comparable with each other. Sometimes, the shape of the incident wave is unknown, and this situation is typical for tsunamis. To get universal expressions for runup characteristics several parameters can be used. MADSEN and FUHRMAN (2008) suggest expressing the formula for runup height in terms of a surf-similarity. These expressions are applicable for nonbreaking waves on a plane beach and for breaking waves as well. DIDENKULOVA *et al.* (2007a, 2008) parameterize runup expressions using various definitions of the wavelength of nonbreaking wave pulses. In this study we demonstrate that the definition of wavelength on the 2/3 level from a maximal value (as the "significant wavelength" in physical oceanography and ocean engineering) is optimal. In this case formulas for various extreme runup characteristics (runup and draw-down heights and velocities, breaking parameter) are universal and the influence of the initial wave form on extreme runup characteristics is weak. This result is obtained for incident symmetrical solitary waves.

The paper is organized as follows. The analytical theory of long wave runup on a beach in the framework of shallow-water theory is briefly described in section 2. Numerical computations of the tsunami waves far from the beach and on the shoreline, based on spectral Fourier series are presented in Section 3. The parameterization of wave shapes in formulas for the extreme (maximal) wave characteristics on a beach is discussed in section 4. The main results are summarized in section 5.

2. *Analytical Theory of the Long-Wave Runup on a Beach*

The runup of tsunami waves on a beach can be described in the framework of the nonlinear shallow-water equations. If the wave propagates perpendicularly to the isobaths, basic equations are,

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(h(x) + \eta)u] = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0, \tag{2}$$

where $\eta(x, t)$ is the vertical displacement of the water surface, $u(x, t)$ is the depth-averaged water flow, $h(x)$ – unperturbed water depth, and g is the gravitational acceleration. Analytical solutions of this system are obtained for a plane beach only,

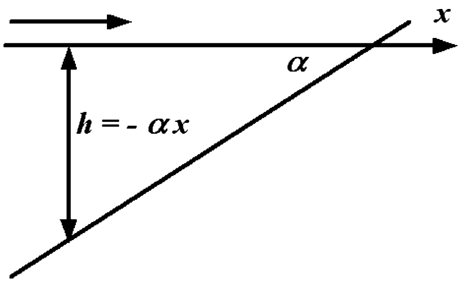

Figure 1
Sketch of the problem.

where the depth $h(x) = -\alpha x$ (Fig. 1). The procedure of the solution is based on the hodograph transformation initially described in the pioneering work of CARRIER-GREENSPAN (1958), and reproduced in different papers cited in section 1.

According to this method all variables can be expressed through the "nonlinear" wave function $\Phi(\sigma, \lambda)$ by means of the hodograph transformation:

$$\eta = \frac{1}{2g}\left(\frac{\partial\Phi}{\partial\lambda} - u^2\right), \tag{3}$$

$$u = \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}, \tag{4}$$

$$t = \frac{1}{\alpha g}\left(\lambda - \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}\right), \tag{5}$$

$$x = \frac{1}{2\alpha g}\left(\frac{\partial\Phi}{\partial\lambda} - u^2 - \frac{\sigma^2}{2}\right), \tag{6}$$

and the wave function, $\Phi(\lambda, \sigma)$ satisfies the cylindrical linear wave equation

$$\frac{\partial^2\Phi}{\partial\lambda^2} - \frac{\partial^2\Phi}{\partial\sigma^2} - \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma} = 0. \tag{7}$$

The variables λ and σ denote generalized coordinates. Since

$$\sigma = 2\sqrt{g(-\alpha x + \eta)}, \tag{8}$$

the point $\sigma = 0$ corresponds to the instantaneous position of the shoreline (called moving shoreline in what follows).

It is interesting to note that if we analyze the linear system of shallow-water wave theory

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}[(-\alpha x)u] = 0, \tag{9}$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0, \tag{10}$$

the linear version of the hodograph transformation

$$\eta_l = \frac{1}{2g}\left(\frac{\partial \Phi_l}{\partial \lambda_l}\right), \quad u_l = \frac{1}{\sigma_l}\frac{\partial \Phi_l}{\partial \sigma_l}, \quad t = \frac{\lambda_l}{\alpha g}, \quad x = -\frac{\sigma_l^2}{4\alpha g}, \tag{11}$$

transforms (9)–(10) to the wave equation

$$\frac{\partial^2 \Phi_l}{\partial \lambda_l^2} - \frac{\partial^2 \Phi_l}{\partial \sigma_l^2} - \frac{1}{\sigma_l}\frac{\partial \Phi_l}{\partial \sigma_l} = 0, \tag{12}$$

which coincides with the wave equation (7) in the nonlinear problem. However, in this case the point $\sigma_l = 0$ corresponds to the unperturbed shoreline ($x = 0$).

Tsunami waves in the deep ocean have small amplitudes and can be described by the linear theory with very high accuracy. For such an incident wave the boundary conditions for the "nonlinear" (7) and "linear" (12) wave equations coincide, provided they are defined in a far and deep enough area. Consequently, the solutions of the nonlinear and linear problems also coincide in terms of solutions of the wave equation, as the functional forms of its "linear" and "nonlinear" solutions coincide: $\Phi(\sigma, \lambda) \equiv \Phi_l(\sigma_l, \lambda_l)$. Moreover, if the "linear" solution $\Phi_l(\sigma_l, \lambda_l)$ is known, the solution of the nonlinear problem (1)–(2) can be directly found from expressions (3)–(6). In particular, the description of properties of the moving shoreline $\sigma(x, t) = 0$ is straightforward. If the velocity of water particles on the unperturbed shoreline ($x = 0$) is calculated in frames of the linear theory, a real "nonlinear" velocity of the moving shoreline can be expressed in an implicit form (Pelinovsky and Mazova, 1992; Didenkulova *et al.*, 2007b):

$$u(t) = U(\tilde{t}), \quad \text{where } \tilde{t} = t + \frac{u(t)}{\alpha g}. \tag{13}$$

Mathematically the function $U(t)$ in the linear theory is defined as $U(t) = \lim_{\sigma_l \to 0} u_l$.

The described features allow use of a rigorous "two-step" method to calculate the runup characteristics under assumption that the linear theory adequately describes their motion far offshore. Firstly, the wave properties on the unperturbed shoreline ($x = 0$) such as the vertical displacement $R(t)$ or the velocity of wave propagation $U(t)$

$$U(t) = \frac{1}{\alpha}\frac{dR}{dt}, \tag{14}$$

are determined from the linear shallow-water theory. Secondly, the properties of the solution of the nonlinear problem (e.g., the real "nonlinear" speed of the moving shoreline) are found from Equation (13). Finally, the vertical displacement of the water level and position of the shoreline (equivalently, the horizontal extent of the inundation) at any instant of time is

$$r(t) = \eta(t, \sigma = 0) = R\left(t + \frac{u}{\alpha g}\right) - \frac{u^2}{2g}. \quad (15)$$

The important conclusion from Equations (13) and (15) is that the maxima of vertical displacements (equivalently, the runup height or the draw-down depth) and the velocity of the shoreline displacement in the linear and nonlinear theories coincide, as noted by CARRIER and GREENSPAN (1958) and SYNOLAKIS (1987, 1991) and as rigorously demonstrated by PELINOVSKY and MAZOVA (1992).

Another important outcome from Equations (13) and (15) is the simple definition of the conditions for the first breaking of waves on a beach. The temporal derivative of the velocity of the moving shoreline, found from Equation (13), approaches infinity (equivalently, wavebreaking occurs) when

$$Br = \frac{\max(dU/dt)}{\alpha g} = 1. \quad (16)$$

This condition has a simple physical interpretation: The wave breaks if the maximal acceleration of the shoreline $R''\alpha^{-1}$ along the sloping beach exceeds the along-beach gravity component αg. This interpretation is figurative, because formally R'' only presents the vertical acceleration of the shoreline in the linear theory and the "nonlinear" acceleration *du/dt* (that is not explicitly calculated here) is what actually approached infinity at the moment of breaking.

3. *Method for Computing Extreme Runup Characteristics*

Following the "two-step" method described above, the linear theory can be used for the computation of extreme characteristics of the tsunami wave runup. An effective method for solving linear partial differential equations is the Fourier method and its generalizations (for instance, the Hankel transformation for cylindrical wave equation). It is convenient to describe the wavefield in terms of its complex (amplitude-phase) spectrum $A(\omega)$ (equivalently, Fourier integral of the associated sea-level variations). The particular bounded solution of the cylindrical wave equation can be represented by the Fourier integral

$$\eta(x, t) = \int_{-\infty}^{+\infty} A(\omega) J_0\left(\frac{2\omega|x|}{\sqrt{gh(x)}}\right) \exp(i\omega t) d\omega, \quad (17)$$

where $J_0(y)$ is the zero-order Bessel function. The spectrum $A(\omega)$ can be found from the spectrum $H(\omega)$ of the incident wave with the use of the asymptotic representation of the wavefield (17) at $x \to -\infty$ as the superposition of the incident η_+ and reflected η_- waves

$$\eta(x \to -\infty, t) = \eta_+[x, t + \tau(x)] + \eta_-[x, t - \tau(x)]. \tag{18}$$

Here

$$\tau(x) = \frac{2|x|}{\sqrt{gh(x)}} = \int_{-|x|}^{0} \frac{dx}{\sqrt{gh(x)}} \tag{19}$$

is the travel time from a given location x to the unperturbed original shoreline. This measure can also be interpreted as the phase shift of the reflected wave in space.

In the same limit $x \to -\infty$, Equation (17) gives

$$\eta_\pm(x \to \infty, t) = \frac{1}{\sqrt{2\pi\tau(x)}} \int_{-\infty}^{+\infty} \frac{A(\omega)}{\sqrt{|\omega|}} \exp\left[i\left(\omega(t \pm \tau(x)) \mp \frac{\pi}{4}\mathrm{sign}(\omega)\right)\right] d\omega. \tag{20}$$

We assume that the incident wave at a fixed point $|x| = L$ (located distantly from the shoreline) is specified by the Fourier integral

$$\eta_+(t) = \int_{-\infty}^{+\infty} H(\omega)\exp(i\omega t)d\omega. \tag{21}$$

Its complex spectrum $H(\omega)$ is easily found in an explicit form in terms of the inverse Fourier transform:

$$H(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \eta_+(t)\exp(-i\omega t)dt. \tag{22}$$

We assume now that the point $|x| = L$ is located so remotely from the unperturbed shoreline that decomposition (20) (that formally is correct for $x \to -\infty$) can be used at this point. Comparison of Equations (20) and (22) then reveals that

$$A(\omega) = \sqrt{2\pi|\omega|\tau(L)} \exp\left[\frac{i\pi}{4}\mathrm{sign}(\omega)\right] H(\omega). \tag{23}$$

The solution in Equation (17) is thus completely determined by the incident wave.

The vertical displacement at the unperturbed shoreline $x = 0$ is a function of the location L

$$R(t) = \sqrt{2\pi\tau(L)} \int_{-\infty}^{+\infty} \sqrt{|\omega|} H(\omega) \exp\left\{i\left(\omega(t - \tau(L)) + \frac{\pi}{4}\mathrm{sign}(\omega)\right)\right\} d\omega. \tag{24}$$

The "linear" horizontal velocity of water particles at this point ($x = 0$) can be found from Equation (14). As was mentioned above, the extreme wave amplitudes (understood as the maximum displacement of water surface) and velocities at the unperturbed shoreline $x = 0$ in linear theory coincide with the maximum runup (draw-down) heights and velocities in the nonlinear theory. Therefore, we would like to emphasize that solving of nonlinear equations is not necessary if only extreme characteristics of tsunami waves are analyzed.

Integral properties of a wave runup dynamics also can be found from the linear theory. For instance, an integrated vertical displacement of the shoreline ("set-up") is

$$\hat{R} = \int_{-\infty}^{+\infty} R(t)dt = \sqrt{2\pi\tau(L)} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} \sqrt{|\omega|}H(\omega)\exp\left\{i\left(\omega(t-\tau(L)) + \frac{\pi}{4}\mathrm{sign}(\omega)\right)\right\}d\omega \tag{25}$$

After changing the order of integration and taking into account the properties of delta-function Equation (25) can be rewritten as

$$\hat{R} = 2\pi\sqrt{2\pi\tau(L)} \int_{-\infty}^{+\infty} \sqrt{|\omega|}H(\omega)\exp[i(\pi/4)\mathrm{sign}(\omega)]\delta(\omega)d\omega. \tag{26}$$

A physical sense of $H(0)$ is an integrated displacement of the incident wave which is bounded. Since the integrand is a continuous function, the integral in Equation (26) is equal to zero. Therefore, the tsunami runup is always presented as reversal oscillations of the shoreline, and a runup phase changes into a receding phase, the process of which does not depend on the shape of the incident wave.

We consider an incident tsunami wave having pulse shape (for example, a positively defined disturbance — wave of elevation or crest) with amplitude H_0 and duration T_0 at $|x| = L$ propagating onshore. It can be nondimensionalized as

$$\eta(t) = H_0 f(t/T_0), \quad f(\zeta) = \int_{-\infty}^{\infty} B(\Omega)\exp(i\Omega\zeta)d\Omega, \tag{27}$$

where

$$\zeta = t/T_0, \quad \Omega = \omega T_0, \quad B(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\zeta)\exp(-i\Omega\zeta)d\zeta. \tag{28}$$

In this case, formulas for velocity of the moving shoreline (14) and linear acceleration, connected with the breaking parameter of the wave (16), and for the maximal vertical displacement (24), can be presented as

$$R_{\max} = R_0 p_R, \quad p_R = \max\{I\}, \quad I = \int_{-\infty}^{\infty} \sqrt{|\Omega|}B(\Omega)\exp\left[i\left(\Omega\zeta + \frac{\pi}{4}\mathrm{sign}(\Omega)\right)\right]d\Omega, \tag{29}$$

$$U_{\max} = \frac{R_0}{\alpha T_0} p_U, \quad p_U = \max\left\{\frac{dI}{d\zeta}\right\}, \tag{30}$$

$$(dU/dt)_{\max} = \frac{R_0}{\alpha T_0^2} p_{Br}, \quad p_{Br} = \max\left\{\frac{d^2 I}{d\zeta^2}\right\}, \tag{31}$$

$$R_0 = \sqrt{\frac{4\pi L}{\lambda_0}} H_0, \quad \lambda_0 = \sqrt{g h_0} T_0, \tag{32}$$

where λ_0 is the wavelength and h_0 is the water depth at $|x| = L$.

In many cases the manner of determining the wavelength λ_0 and the duration T_0 of a solitary pulse is not clear. In particular, most of the wave shapes (represented by analytical functions that are continuous in all derivatives) are nonzero everywhere at $-\infty < t < \infty$. There is obvious ambiguity in the definition of their wavelength (or duration) that can be interpreted as their width at any level of elevation, or by the value of an appropriate integral (Didenkulova *et al.*, 2007a, 2008).

A convenient definition of the wavelength is the extension (spatial or temporal) of the wave profile elevation exceeding the 2/3 level of the maximum wave height. This choice is inspired by the definition of the significant wave height and length in physical oceanography and ocean engineering. For symmetric solitary waves, "significant" wave duration and "significant" wavelength are

$$T_s = 2T_0 f^{-1}\left(\frac{2}{3}\right), \quad \lambda_s = \sqrt{g h_0} T_s, \tag{33}$$

where f^{-1} is the inverse function of f. Thus the formulas for the maximal displacement, the velocity of the moving shoreline, and the breaking parameter can be expressed as

$$R_{\max} = \mu_R^+ H_0 \sqrt{\frac{L}{\lambda_s}}, \quad U_{\max} = \mu_U^+ \frac{H_0 L}{\lambda_s} \sqrt{\frac{g}{\alpha \lambda_s}}, \quad Br = \mu_{Br} \frac{H_0 L}{\alpha \lambda_s^2} \sqrt{\frac{L}{\lambda_s}}, \tag{34}$$

where coefficients (below called form factors) μ_R^+, μ_U^+ and μ_{Br} depend on the wave form:

$$\mu_R^+ = 2\sqrt{2\pi f^{-1}\left(\frac{2}{3}\right)} p_R, \quad \mu_U^+ = 4\sqrt{2\pi\left[f^{-1}\left(\frac{2}{3}\right)\right]^3} p_U, \quad \mu_{Br} = 8\sqrt{2\pi\left[f^{-1}\left(\frac{2}{3}\right)\right]^5} p_{Br}. \tag{35}$$

The analogous formulas for draw-down height and velocity can be obtained from (34, 35) by replacing $p_R \to \bar{p}_R = \min\{I\}, p_U \to \bar{p}_U = \min\{dI/d\zeta\}$ and $p_{Br} \to \bar{p}_{Br} = \min\left\{d^2 I/d\zeta^2\right\}$ in (29) and (30).

A remarkable property of this choice is that if the solitary wave duration is determined at the 2/3 level of the maximum height (33), the effect of the difference in the wave shapes will be fairly small. The analytical expressions for maximal runup characteristics (runup and draw-down heights, runup and draw-down velocities and breaking parameter) become universal and depend on the height and duration of the incoming onshore wave only.

4. Results of Calculations

We first consider the runup of incident symmetrical positive waves which have the shape of various "powers" of a sinusoidal pulse

$$f(\zeta) = \cos^n(\pi\zeta), \quad \text{where } n = 2, 3, 4, \ldots \tag{36}$$

which are defined on the segment [–1/2, 1/2]. Their shapes have a certain similarity, however, their wave characteristics, such as mean water displacement, energy and wave duration on various levels, differ considerably (Fig. 2). The functions representing such impulses have different smoothness: their n-th order derivatives are discontinuous at their ends. The case of $n = 1$ is not considered, as the relevant integrals in Equations (29) and (30) do not converge. The runup of sinusoidal pulse for the case of $n = 2$ is presented on Figure 3. Such oscillations are typical for runup of symmetric solitary waves on a beach of constant slope.

Form factors for runup μ_R^+ and draw-down μ_R^- height, runup μ_U^+ and draw-down μ_U^- velocity and breaking parameter $\mu_{Br,}$ calculated for all sine power pulses (36) with the use of the definition of the characteristic wave length λ_s (33) at the 2/3 level of the maximum height, are presented in Figure 6 at the end of section 4. Calculating means and root-mean-square deviations results in the values of form factors for the maximum wave runup $\mu_R^+ = 3.61 \cdot (1 \pm 0.02)$ and draw-down $\mu_R^- = 1.78 \cdot (1 \pm 0.28)$ that have a fairly limited variation (Table 1).

Initially, it is significant that the runup height is higher than the draw-down height. This feature is observed for all sets of positive impulses. The form factor for the

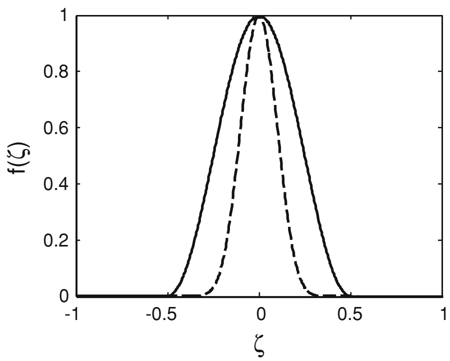

Figure 2
Family of sine power pulses (36): Solid line n = 2 and dashed line n = 10.

Table 1

Calculated form factors for different wave shapes

μ	Sine power	Soliton power	Lorentz pulse power
μ_R^+	$3.61 \cdot (1 \pm 0.02)$	$3.55 \cdot (1 \pm 0.05)$	$3.53 \cdot (1 \pm 0.08)$
μ_R^-	$1.78 \cdot (1 \pm 0.28)$	$1.56 \cdot (1 \pm 0.28)$	$1.51 \cdot (1 \pm 0.44)$
μ_U^+	$4.65 \cdot (1 \pm 0.30)$	$4.15 \cdot (1 \pm 0.22)$	$4.07 \cdot (1 \pm 0.26)$
μ_U^-	$6.98 \cdot (1 \pm 0.01)$	$6.98 \cdot (1 \pm 0.02)$	$6.99 \cdot (1 \pm 0.04)$
μ_{Br}	$13.37 \cdot (1 \pm 0.10)$	$12.90 \cdot (1 \pm 0.03)$	$12.99 \cdot (1 \pm 0.13)$

maximum wave runup in Equation (35) is almost independent on the power n, demonstrating that the influence of the initial wave shape on the extreme runup characteristics can be made fairly small by an appropriate choice of the characteristic wavelength. The above choice of the (significant) wavelength reduces the variation of the form factor for the sine power pulses to a remarkably small value, about 2%.

The deepest draw-down is more affected by the wave shape: The relevant form factor varies up to 28%. This feature can be explained by the presence of a complex field of motions in the draw-down phase. A positive wave first executes runup and only later draw-down (see Fig. 3). Therefore the runup process is predominantly governed by the incident wave dynamics while the draw-down phenomena occurs under the influence of a set of distributed wave reflections and re-reflections from the slope and consequently it is more sensitive to the wave shape variations.

A similar analysis can be applied to maximum runup and draw-down velocities of the moving shoreline. Calculated form factors for maximum run-up μ_U^+ and draw-down μ_U^- velocities are presented on Figure 7 with triangles. The maximal values for the draw-down velocity are always greater than for the runup velocity for initial unidirectional impulses. The form factor for the draw-down velocity $\mu_U^- = 6.98 \cdot (1 \pm 0.01)$ is almost constant for all values of n (root-mean-square deviation is 1%) whereas the runup velocity $\mu_U^+ = 4.65 \cdot (1 \pm 0.30)$ changes in a wider range (±30%); see Table 1.

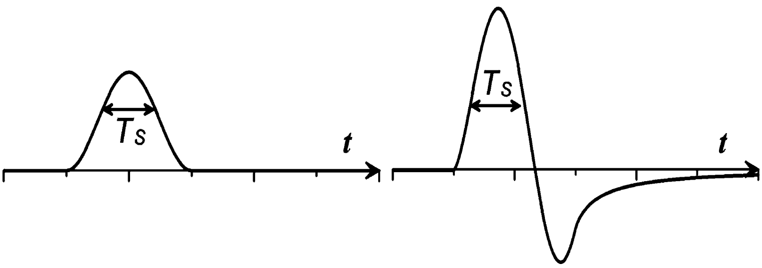

Figure 3

The runup of a symmetric solitary wave on a beach of constant slope. Left panel shows water surface elevation at $x = -L$, right panel shows wave runup on shore. Notice that the duration T_s of the incident wave is defined at $x = -L$ and is not necessarily conserved during the run-up and rundown process.

Variations of the form factor for the breaking parameter are also weak (see Fig. 8, triangles). The case of $n = 2$, corresponding to the discontinuity of the second-order derivative, is excluded since the integral in Equation (31) diverges. The relevant form factors $\mu_{Br} = 13.37 \cdot (1 \pm 0.10)$ can be considered a constant with a reasonable accuracy (Table 1).

Thus, form factors for the most important parameters such as runup height, draw-down velocity, and to some extent for breaking parameters, are universal and do not depend on the particular shape of a sine power impulse. The variations of form factors for draw-down height and runup velocity are more significant (about 30%), however, they also can be neglected for engineering estimates.

As the second example of the proposed approach we consider the family of solitary waves, described by the following expression,

$$f(\zeta) = \mathrm{sec\,h}^n(4\zeta), \quad n = 1, 2, 3, \ldots \tag{37}$$

These impulses are unlimited in space with exponential decay of the elevation at their ends (see Fig. 4). The case $n = 2$ corresponds to the well-known soliton solution of the Korteweg-de Vries (KdV) equation, which is frequently used as a generic example of shallow water solitary waves.

The runup of the KdV solitons on a constant beach was studied previously by Synolakis (1987) who presented both experimental and theoretical results. In our notation, the Synolakis formula (Synolakis, 1987) is

$$\frac{R_{\max}}{H_0} = 2.8312\sqrt{\frac{L}{h_0}}\left(\frac{H_0}{h_0}\right)^{1/4}. \tag{38}$$

The "significant" wavelength of the soliton is easily calculated from the well-know analytical expression for a soliton in a constant-depth basin

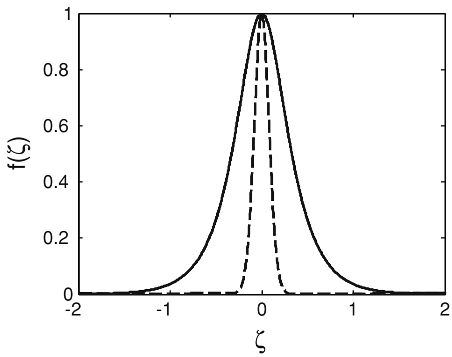

Figure 4
Family of soliton-like impulses (37): Solid line $n = 1$ and dashed line $n = 10$.

$$\eta(x) = H_0 \mathrm{sech}^2\left(\sqrt{\frac{3H_0}{4h_0}}\frac{x}{h_0}\right) \tag{39}$$

and has the explicit form:

$$\lambda_s = 4\mathrm{sech}^-1\left(\sqrt{\frac{2}{3}}\right) h_0\sqrt{\frac{h_0}{3H_0}}, \tag{40}$$

where $\mathrm{sech}^{-1}(z)$ is an inverse function of sech(z). Substituting the expression for H_0/h_0 from (40) into the right-hand side of (38), we obtain

$$\frac{R_{\max}}{H_0} = 3.4913\sqrt{\frac{L}{\lambda_s}}. \tag{41}$$

Our numerical calculations lead to the same value of the form factor, $\mu_R^+ = 3.4913$ at $n = 2$. This example indicates that the theory of soliton runup on a beach, which leads to a nonlinear relation between the runup height and the soliton amplitude, is consistent with a general theory of the runup of solitary waves on a beach and represents a special case.

The form factors for the maximum height of the wave runup and draw-down for different values of n (Fig. 6) again virtually do not depend on the exponent n. This feature suggests that the proposed approach is not sensitive with respect to the shape of the impulses. The form factors, averaged over the range $n = 1$ - 20 are $\mu_R^+ = 3.55 \cdot (1 \pm 0.05)$ for the runup and $\mu_R^- = 1.56 \cdot (1 \pm 0.28)$ for the draw-down height (Table 1). Notice that these values are close to analogous coefficients for sine power pulses.

Form factors for runup and draw-down velocities and breaking parameters (Figs. 7 and 8) are $\mu_U^+ = 4.15 \cdot (1 \pm 0.22)$, $\mu_U^- = 6.98 \cdot (1 \pm 0.02)$, and $\mu_{Br} = 12.90 \cdot (1 \pm 0.03)$ (Table 1). The variation of these parameters for different values of n for soliton-like impulses is to some extent similar to the analogous dependence for sine power pulses. The largest difference is that the runup velocity form factor for sine-pulses increases with a decrease of the exponent, while the runup velocity form factor for soliton-like impulses decreases with a decrease of the exponent.

Similar results are obtained for solitary ridges of a Lorentz-like shape with algebraic decay (Fig. 5)

$$f(\zeta) = \frac{1}{\left[1 + (4\zeta)^2\right]^n}, \quad n = 1, 2, 3, \ldots \tag{42}$$

The calculated runup and draw-down height form factors for this class of solitary waves (Fig. 6) show some variability in the range of $n = 1 \div 20$. The average values are $\mu_R^+ = 3.53 \cdot (1 \pm 0.08)$ for the runup and $\mu_R^- = 1.51 \cdot (1 \pm 0.44)$ for the draw-down height (Table 1). The variation of μ_R^+ is still very reasonable. The form factors for runup and draw-down velocities and breaking parameter (Figs. 7 and 8) show even smaller

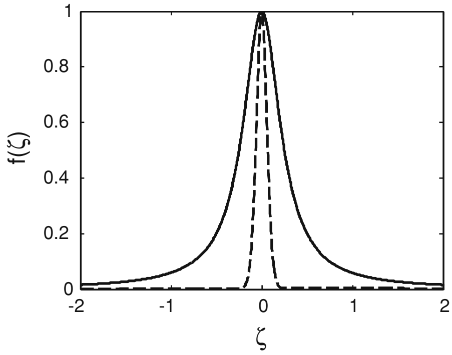

Figure 5
Family of Lorentz-like impulses (42): Solid line $n = 1$ and dashed line $n = 10$.

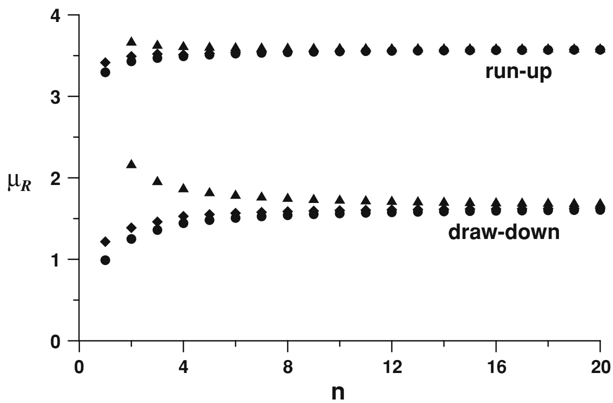

Figure 6
Calculated form factors for the maximum runup μ_R^+ and draw-down μ_R^- height for sine power pulses (triangles), soliton-like (diamonds) and Lorentz-like (circles) impulses.

variation: $\mu_U^+ = 4.07 \cdot (1 \pm 0.26)$, $\mu_U^- = 6.99 \cdot (1 \pm 0.04)$, and $\mu_{Br} = 12.99 \cdot (1 \pm 0.13)$. Their dependence on the exponent n is similar for other families of impulses.

5. Conclusions

The central outcome from the presented study is that the influence of the initial wave form on maximal runup characteristics can be almost removed, or made fairly weak, by means of a proper choice of incident wave characteristics. The properties of the features with the largest variation such as the runup heights, draw-down velocities, and the

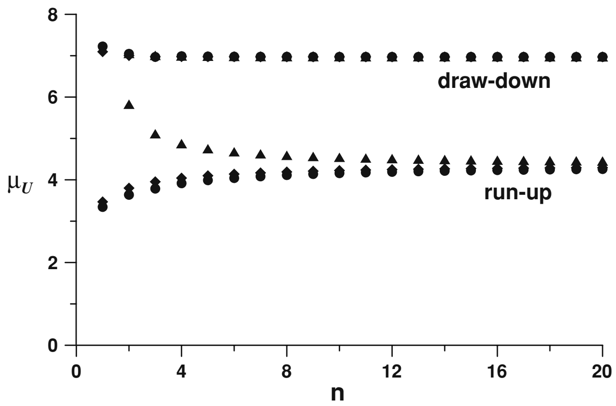

Figure 7
Calculated form factors for the maximum runup μ_U^+ and draw-down μ_U^- velocity for sine power pulses (triangles), soliton-like (diamonds) and Lorentz-like (circles) impulses.

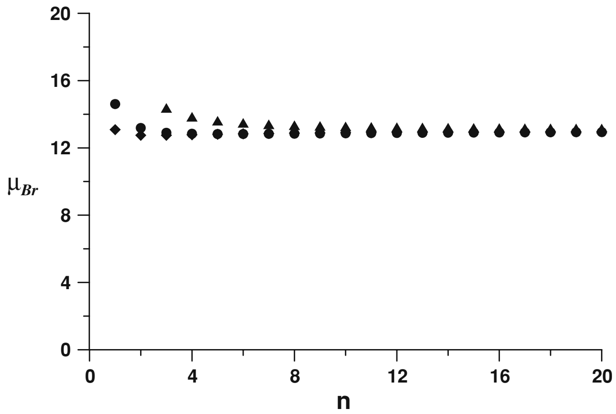

Figure 8
Calculated form factors for the breaking parameter μ_{Br} for sine power pulses (triangles), soliton-like (diamonds) and Lorentz-like (circles) impulses.

breaking parameter are at best described with this approach while the other key properties such as the draw-down depths of runup velocity are reasonably reproduced.

Another key result of the study is that the average values of calculated form factors for all concerned classes of symmetrical positive solitary waves (Table 1, Figs. 6–8) and formulas (34) for the maximum runup and draw-down characteristics of solitary waves virtually do not depend on the form of the incident wave if the wave duration is appropriately defined. This is especially evident with the form factors for the runup height and draw-down velocity, where the variations for all the wave classes in question do not exceed 8%.

This property suggests that the definition of the "significant" wavelength for solitary waves at the 2/3 level of their maximum height is optimal. In this case the following approximate analogues of formulas (34) for the runup and draw-down characteristics of the long waves on a beach are universal:

$$R_{\mathrm{run-up}} = 3.5H_0\sqrt{\frac{L}{\lambda_s}}, \quad R_{\mathrm{draw-down}} = 1.5H_0\sqrt{\frac{L}{\lambda_s}}, \tag{43}$$

$$U_{\mathrm{run-up}} = 4.5\frac{H_0L}{\lambda_s}\sqrt{\frac{g}{\alpha\lambda_s}}, \quad U_{\mathrm{draw-down}} = 7\frac{H_0L}{\lambda_s}\sqrt{\frac{g}{\alpha\lambda_s}}, \quad Br = 13\frac{H_0L}{\alpha\lambda_s^2}\sqrt{\frac{L}{\lambda_s}}. \tag{44}$$

Expressions (43–44) can be used for estimates of the runup and draw-down characteristics of approaching tsunamis as soon as rough estimates for their heights, significant lengths and periods in the open ocean become available.

Finally, we note that the obtained results hold only for symmetrical solitary waves. If the incident wave is shaped as an *N*-wave, also typical for the tsunami problem (Tadepalli and Synolakis, 1994, 1996; Tinti and Tonini, 2005), the magnitude of coefficients in the "runup" formulas will differ from that given in (43)–(44); see for comparison the difference between run-up heights of a soliton and its derivative (Synolakis, 1987; Tadepalli and Synolakis, 1994). If the incident wave is an asymmetrical wave, with different steepness of the front and back slopes, the runup characteristics depend on the front-slope steepness (Didenkulova *et al.*, 2006, 2007b). Therefore, the universal character of runup characteristics can be achieved "inside" each class of incident wave shapes. As a result, estimates of run-up characteristics require knowledge of only a few "robust" parameters of the incident tsunami wave (positive crest or negative trough, *N*-wave or asymmetrical wave) but not a detailed description of the wave shape. This conclusion is important in practice, as it allows prediction of runup characteristics of tsunami waves with "unknown" shapes.

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