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# Weak error for Continuous Time Markov Chains related to fractional in time P(I)DEs

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**Abstract**

We provide sharp error bounds for the difference between the transition densities of some multidimensional Continuous Time Markov Chains (CTMC) and the fundamental solutions of some fractional in time Partial (Integro) Differential Equations (P(I)DEs). Namely, we consider equations involving a time fractional derivative of Caputo type and a spatial operator corresponding to the generator of a non degenerate Brownian or stable driven Stochastic Differential Equation (SDE).


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## 1. Introduction

We are interested in the probabilistic approximation of P(I)DEs of the following type:

$$\begin{cases} \partial_t^\beta u(t,x) = Lu(t,x), & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

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where $\partial_t^\beta, \beta \in (0,1)$, stands for the Caputo–Dzherbashyan derivative and $L$ is the generator of a Brownian or stable driven non degenerate SDE (see Eqs. (2.9), (2.10) for the respective definitions of the Caputo–Dzherbashyan derivative and the spatial operators considered). Equations of the previous type appear in many applicative fields from natural sciences to finance, see, e.g., Meerschaert et al., [28,29] and references therein.

Under suitable assumptions, the solution of (1.1) can be represented as $u(t,x) := \mathbb{E}[f(X^{0,x}_{Z_t^\beta})]$ where $(X_s^{0,x})_{s\geq 0}$ solves the SDE with generator $L$ and $(Z_t^\beta)_{t\geq 0}$ is the inverse of a stable subordinator of index $\beta \in (0,1)$ independent from $X^{0,x}$. This therefore extends the "usual" Feynman–Kac representation, corresponding to $\beta = 1$, to the fractional case $\beta \in (0,1)$.

Many probabilistic numerical approximations of $u(t,x)$ have been considered when $\beta = 1$. We can for instance mention the works of Konakov et al. (see [19–21] in the non degenerate diffusive case or [22] for SDEs driven by symmetric stable processes) that investigate the Euler scheme or, more generally, the Markov Chain approximation of the spatial motions in terms of Edgeworth expansions or Local Limit Theorems (LLTs). We can also refer to the works of Bally and Talay for the Euler scheme of some hypoelliptic diffusions [2,3] or [23] for associated LLTs.

In the current (strictly) fractional framework two additional difficulties appear. Firstly, it is known that the fundamental solutions to (1.1) exhibit, additionally to the *usual* time singularity in short time, a diagonal spatial singularity, see e.g. Eĭdelman and Kochubeĭ [11] for the current case or Kochubeĭ [16] for extensions to higher order fractional derivatives $\beta \in (1,2)$. Secondly, the inverse of the subordinator leads to consider random integration times that might be either very small or very long. The analysis of the discretization error in the previously mentioned works was always performed for a fixed final time horizon $T$ and the constants controlling the error estimates depend explosively on $T$ for small and large times. We must therefore carefully control these explosions. In short time we must handle the spatial singularity of the fractional heat kernel whereas in long time, the explosion is compensated by the fast decay of the density of the inverse subordinator. Let us mention that those difficulties may deteriorate the "usual" convergence rate for the weak error even for the Euler approximation. We establish error bounds for the Euler scheme and the Markov Chain approximation of a diffusive SDE, Theorem 3.1, and for the Euler scheme of a stable driven SDE, Theorem 3.2. We emphasize that Kolokoltsov [18] also considered probabilistic schemes to approximate equations of type (1.1) in a more general setting, namely considering a fractional like time derivative that could depend on space as well. He establishes convergence of the schemes towards the expected solution but since the approach relies on semigroup techniques, no convergence rates are provided.

The article is organized as follows. We first recall in Section 2 which are the probabilistic tools needed to give representations and approximations of the solution to Eq. (1.1). We then state our main results in Section 3. The proofs are given in Section 4 which is the technical core of the work. Some perspectives are considered in Section 5. Technical results concerning the parametrix expansions, which are needed to establish the short and long time behavior of the involved spatial densities, are collected in Appendix A. As a by-product of our analysis, we obtain two-sided heat kernel estimates for the fundamental solution of (1.1) in Appendix B.

## 2. Probabilistic objects associated with Cauchy problems with time fractional derivatives

We first recall how a probabilistic representation for the solution of (1.1) can be derived considering the special case of $L$ being associated with a $d$-dimensional symmetric stable process of index $\alpha \in (0,2]$, thus including the Brownian motion. We write for $\phi \in C_0^2(\mathbb{R}^d, \mathbb{R})$ (space of twice continuously differentiable functions with compact support), $L\phi(x) = \frac{1}{2}\Delta\phi(x)$ for $\alpha = 2$ whereas for $\alpha \in (0, 2)$:

$$\begin{aligned} L\phi(x) &= \int_{\mathbb{R}^d} \{\phi(x+y) - \phi(x) - \nabla\phi(x)\cdot y\mathbb{I}_{|y|\le 1}\}\mu(dy) \\ &= \int_{\mathbb{R}^+\times S^{d-1}} \{\phi(x+|y|\bar{y}) - \phi(x) - \nabla\phi(x)\cdot |y|\bar{y}\mathbb{I}_{|y|\le 1}\}|y|^{-(1+\alpha)}d|y|\nu(d\bar{y}), \end{aligned} \quad (2.1)$$

where $\nu$ is the spherical part of $\mu$ and is assumed to be non degenerate, i.e. there exists

$$\Lambda \ge 1,\ \forall \xi \in \mathbb{R}^d, \quad \Lambda^{-1}|\xi|^\alpha \le \int_{S^{d-1}} |\langle \xi, \eta\rangle|^\alpha \nu(d\eta) \le \Lambda|\xi|^\alpha. \quad (2.2)$$

Observe that if $(S^\alpha_u)_{u\ge 0}$ is a stable process with generator $L$ then

$$\forall t \ge 0,\ \forall \xi \in \mathbb{R}^d, \quad \varphi_{S^\alpha_t}(\xi) := \mathbb{E}[\exp(i\langle \xi, S^\alpha_t\rangle)] := \exp\left(-tC_\alpha \int_{S^{d-1}} |\langle \xi, \eta\rangle|^\alpha \nu(d\eta)\right),$$

where the explicit value of $C_\alpha$ can be found in Sato [36].

Let us now discuss heuristically how some suitable scaling limits of Continuous Time Random Walks (CTRWs) actually give a probabilistic interpretation of the solution to (1.1). A rigorous connection between CTRWs and the solution to (1.1) can be found in Section 5 of the work by Meerschaert and Scheffler [27], when the spatial motion is $\alpha$-stable. These objects thus provide a natural approximation scheme to (1.1). Basically, CTRWs are random walks that wait a certain amount of time between their jumps. In the i.i.d. case, the mechanism can be described as follows: let $(T_i, Y_i)_{i\in\mathbb{N}^*}$ be a sequence of $\mathbb{R}^+ \times \mathbb{R}^d$-valued i.i.d pairs of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For all $t \ge 0$ one defines:

$$\Gamma_t := \sum_{i=0}^{N_t} Y_i, \quad N_t := \max\left\{m \in \mathbb{N} : \sum_{i=1}^m T_i \le t\right\}.$$

If the $(T_i)_{i\in\mathbb{N}^*}$ are exponentially distributed and independent of the $(Y_i)_{i\in\mathbb{N}^*}$ the process $(\Gamma_t)_{t\ge 0}$ is easily shown to be Markovian. This property clearly fails in the general case. Of particular interest is the case when the $(T_i)_{i\in\mathbb{N}^*}$ and the $(Y_i)_{i\in\mathbb{N}^*}$ are independent and respectively belong to the domain of attraction of a $\beta$-stable and $\alpha$-stable law with $\beta \in (0, 1)$ and $\alpha \in (0, 2]$. This property indeed yields that:

$$\left(n^{-1/\beta}\sum_{i=1}^{\lfloor nu\rfloor} T_i\right)_{u\ge 0} \Rightarrow (S^{\beta,+}_u)_{u\ge 0}, \quad \text{and} \quad \left(n^{-1/\alpha}\sum_{i=1}^{\lfloor nu\rfloor} Y_i\right)_{u\ge 0} \Rightarrow (S^\alpha_u)_{u\ge 0}, \quad (2.3)$$

up to some possible additional multiplicative slowly varying function in the normalized constants $n^{-1/\beta}, n^{-1/\alpha}$, which we omit here for simplicity. Above, $\lfloor\cdot\rfloor$ stands for the integer part and we use the convention that $\sum_{i=1}^0 = 0$. Also, $S^{\beta,+}$ is a $\beta$-stable subordinator, i.e. a Lévy process with positive jumps and Laplace transform $\psi_{S^{\beta,+}_u}(\lambda) = \mathbb{E}[\exp(-\lambda S^{\beta,+}_u)] = \exp(-u\lambda^\beta),\ \lambda \ge 0$. On the other hand, we will assume that $S^\alpha$ is a symmetric $\mathbb{R}^d$-valued stable process with generator as in (2.1).

In the above equation, the symbol $\Rightarrow$ stands for the usual convergence in law for processes, in the respective Skorokhod spaces of càdlàg functions $D(\mathbb{R}^+, \mathbb{R}^+)$ and $D(\mathbb{R}^+, \mathbb{R}^d)$, for the $J_1$-topology. The second identity in (2.3) follows from Theorem 4.1 in Meerschaert and

Scheffler [27]. On the other hand, equation (2.5) in [27] gives the convergence in distribution over all finite dimensional marginals for the first identity. Convergence in $D(\mathbb{R}^+, \mathbb{R}^+)$ can then be established following the arguments of the proof of Corollary 3.4 of the same reference.

Now, the rescaled process associated with the number of jumps $(N_t)_{t\geq 0}$ also has a limit, namely Corollary 3.4 in [27] yields $(n^{-\beta} N_{nt})_{t\geq 0} \Rightarrow (Z_t^\beta)_{t\geq 0}$ in $D(\mathbb{R}^+, \mathbb{R}^+)$ for the $J_1$-topology, where $Z_t^\beta := \inf\{s \geq 0 : S_s^{\beta,+} > t\}$ which is the inverse process of a $\beta$-stable subordinator. Since $S_s^{\beta,+}$ is increasing in $s$, $Z_t^\beta$ also corresponds to the first passage time of $S^{\beta,+}$ above the level $t$. Thus, one formally has:

$$
\begin{aligned}
\left(n^{-\beta/\alpha}\Gamma_{nt}\right)_{t\geq 0} &= \left((n^\beta)^{-1/\alpha}\sum_{i=1}^{N_{nt}} Y_i\right)_{t\geq 0}\\
&= \left((n^\beta)^{-1/\alpha}\sum_{i=1}^{n^\beta(n^{-\beta}N_{nt})} Y_i\right)_{t\geq 0} \Rightarrow \left(S^\alpha_{Z_t^\beta}\right)_{t\geq 0}.
\end{aligned}
$$

The above convergence has been first rigorously established in $D(\mathbb{R}^+, \mathbb{R}^d)$ for the $M_1$-topology in Theorem 4.2 of [27] whereas the convergence in the $J_1$-topology follows from Theorem 3.6 of Straka and Henry [39]. From the independence of $S^\alpha$ and $Z^\beta$, the limit random variable $S^\alpha_{Z_t^\beta}$ has, for any $t > 0$, an *explicit* density that writes:

$$
q(t,x) = \int_0^{+\infty} p_{S^\alpha}(u,x)p_{Z^\beta}(t,u)du, \tag{2.4}
$$

where $p_{S^\alpha}(u,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp(-i\langle x,\lambda\rangle)\varphi_{S_u^\alpha}(\lambda)d\lambda$ and $p_{Z^\beta}(t,u)$ stands for the density of $Z_t^\beta$ at point $u$. We refer to Corollary 4.2 in [27] for a proof of (2.4).

Observe also that, since $\mathbb{P}[Z_t^\beta \leq u] = \mathbb{P}[S_u^{\beta,+} > t]$, we have the following relation between the density of the inverse $Z^\beta$ and the density of the subordinator $S^{\beta,+}$:

$$
p_{Z^\beta}(t,u) = \partial_u \mathbb{P}[Z_t^\beta \leq u] = \partial_u(1 - \mathbb{P}[S_u^{\beta,+} \leq t]) = -\partial_u \int_0^t p_{S^{\beta,+}}(u,y)dy. \tag{2.5}
$$

We also have the following important property for this density.

**Proposition 2.1** (*PDE Associated with $p_{Z^\beta}$*). *For fixed $t > 0$, the density $p_{Z^\beta}(t,u)$ satisfies in the distributional sense the equation:*

$$
(\mathbb{D}_t^\beta + \partial_u)p_{Z^\beta}(t,u) = \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(u), \tag{2.6}
$$

*where $\delta$ stands for the Dirac mass at 0 and $\mathbb{D}_t^\beta$ denotes the Riemann–Liouville derivative. Namely, for a function $h$ and $\beta \in (0,1)$:*

$$
\mathbb{D}_t^\beta h(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\left\{\int_0^t h(t-u)u^{-\beta}du\right\}.
$$

Proposition 2.1 is a special case of Theorem 3.1 in Baeumer and Meerschaert [1]. Additional details can be found in Section 5.1 of [29].

From (2.4) and (2.6) we therefore formally derive that for all $t > 0$:

$$\begin{aligned}
\mathbb{D}_t^\beta q(t,x) &= \int_0^{+\infty} p_{S^\alpha}(u,x) \left\{ -\partial_u p_{Z^\beta}(t,u) + \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(u) \right\} du \\
&= \int_0^{+\infty} \partial_u p_{S^\alpha}(u,x) p_{Z^\beta}(t,u) du + \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(x) \\
&= Lq(t,x) + \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(x),
\end{aligned} \tag{2.7}$$

where we used that $\partial_u p_{S^\alpha}(u,x) = Lp_{S^\alpha}(u,x)$ (the spatial density satisfies the Kolmogorov backward equation) and the fact that $S_u^\alpha \underset{u\to 0}{\to} 0$. We have also denoted by $\delta$, with a slight abuse of notation, the Dirac mass at 0 in $\mathbb{R}^d$. A rigorous proof for Eq. (2.7) again follows from Theorem 3.1 in [1].

In other words, the transition p.d.f. $q$ of $(S^\alpha_{Z_t^\beta})_{t\geq 0}$ can be viewed as the fundamental solution of (2.7) (see also Theorem 5.1 in [27]). Hence, for a given continuous initial data $f$, the associated Cauchy problem solved by $u(t,x) = \mathbb{E}[f(x + S^\alpha_{Z_t^\beta})]$ writes:

$$\begin{cases}
\mathbb{D}_t^\beta u(t,x) = Lu(t,x) + \dfrac{t^{-\beta}}{\Gamma(1-\beta)} f(x), \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d, \\
u(0,x) = f(x), \quad x \in \mathbb{R}^d.
\end{cases} \tag{2.8}$$

Alternatively, Eq. (2.8) can be rewritten in terms of the Caputo–Dzherbashyan fractional derivative, denoted $\partial_t^\beta$ hereafter, recalling that for a smooth function $h$ and $\beta \in (0,1)$:

$$\begin{aligned}
&\partial_t^\beta h(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t h'(t-u) u^{-\beta} du, \\
&\mathbb{D}_t^\beta h(t) - \frac{t^{-\beta}}{\Gamma(1-\beta)} h(0) = \partial_t^\beta h(t), \quad t > 0.
\end{aligned} \tag{2.9}$$

Applying identity (2.9) to the function $h(t) = u(t,x)$ yields that (2.8) rewrites:

$$\begin{cases}
\partial_t^\beta u(t,x) = Lu(t,x), \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d, \\
u(0,x) = f(x),
\end{cases}$$

which corresponds to the initial equation (1.1). The probabilistic representation of the solution to Eq. (1.1) thus writes $u(t,x) = \mathbb{E}[f(x + S^\alpha_{Z_t^\beta})]$.

Let us mention that, in the special case $L = \lambda^2 \Delta$, some explicit expansions of the density of $\sqrt{2}\lambda S^2_{Z_t^\beta}$ have been obtained by Beghin and Orsingher [6].

We will now consider the problem of approximating solutions to the fractional Cauchy problem (2.8) (or in its Caputo–Dzherbashyan formulation (1.1)) for generators of the following form:

$$\begin{aligned}
L_1\phi(x) &= \langle b(x), \nabla\phi(x)\rangle + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(x) D_x^2\phi(x)) \quad \text{or} \\
L_2\phi(x) &= \langle b(x), \nabla\phi(x)\rangle \\
&\quad + \int_{\mathbb{R}^d} \{\phi(x+\sigma(x)y) - \phi(x) - \nabla\phi(x)\cdot\sigma(x)y\mathbb{I}_{|\sigma(x)y|\leq 1}\}\mu(dy),
\end{aligned} \tag{2.10}$$

for $\mu(dy) := |y|^{-(1+\alpha)}d|y|\nu(d\bar{y})$ as in (2.1), in Eqs. (2.8), (1.1). We consider homogeneous operators in time for notational simplicity. Let $(X_s)_{s\geq 0}$ solve the SDE with generator given in (2.10), i.e.

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_{s^-})dY_s \tag{2.11}$$

where $Y$ is a Brownian motion in the diffusive case and a symmetric $\alpha$-stable with $\alpha \in (0, 2)$ process with Lévy measure $\mu$ otherwise. Under suitable assumptions, typically smoothness and boundedness of $b, \sigma$ and non degeneracy of $\sigma$, the transition probability function of $X$ is absolutely continuous and the density is itself smooth and must satisfy the Fokker–Planck equation, see e.g. Nualart [33] in the diffusive case, Bichteler et al. [7] in the jump case under some integrability constraints on the jumps, and Chakraborty [8] in the stable driven case.

In all those cases, for $t > 0$, the density $p(t, x, \cdot)$ of $X_t$ in (2.11) is the fundamental solution of (1.1) with $\beta = 1$, i.e. $(\partial_t + L)p(t, x, \cdot) = 0$ for $L = L_1$ or $L = L_2$ and $p(t, x, \cdot) \to_{t\downarrow 0} \delta_x(\cdot)$. From this property, and provided some *good* integrability properties are available for $p(t, x, \cdot)$ (see Lemma 4.10), the arguments of Theorem 3.1 in [1] and Corollary 3.1 in [27] can be reproduced to derive that the density $p_{X_{Z_t^\beta}}$ of $X_{Z_t^\beta}$ at time $t > 0$, which writes through a conditioning argument for all $(x, y) \in (\mathbb{R}^d)^2$:

$$p_{X_{Z_t^\beta}}(x, y) = \int_0^{+\infty} p(u, x, y)p_{Z^\beta}(t, u)du, \tag{2.12}$$

is the fundamental solution to (1.1) for $\beta \in (0, 1)$, i.e. for a continuous $f$ in (1.1):

$$\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \quad u(t, x) = \mathbb{E}[f(X^{0,x}_{Z_t^\beta})]. \tag{2.13}$$

Let us also mention that modifying the laws of the waiting times of the initial CTRW including a possible dependence on the previous jump time and spatial position, leads to consider at the limit a *modified* fractional derivative in time. We refer to Kolokoltsov [18] for details and results concerning the extension of the formula (2.13) and convergence of the associated approximations. Let us emphasize that the semi-group approach (convergence of the generators) used in [18] gives convergence of the one dimensional marginals for these models. The process convergence in the Skorokhod space for an appropriate topology still seems to be an open problem.

## 3. Assumptions and main results

### 3.1. Assumptions on the coefficients and approximation scheme

We assume that the coefficients in Eq. (2.11) satisfy the following conditions:

**(S)** The coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are assumed to be bounded as well as their derivatives up to order 6.

**(UE)** There exists $\Lambda \geq 1$ s.t. for all $(x, \xi) \in (\mathbb{R}^d)^2$:

$$\Lambda^{-1}|\xi|^2 \leq \langle \sigma\sigma^*(x)\xi, \xi\rangle \leq \Lambda|\xi|^2.$$

Now, we consider for a given time step $h > 0$, setting for $i \in \mathbb{N}$, $t_i := ih$, the following Markov Chain approximation of Eq. (2.11):

$$\forall i \in \mathbb{N}, \quad X^h_{t_{i+1}} = X^h_{t_i} + b(X^h_{t_i})h + \sigma(X^h_{t_i})h^{1/\alpha}\eta_{i+1}, \ X^h_0 = x, \tag{3.1}$$

for i.i.d. $\mathbb{R}^d$-valued random variables $(\eta_j)_{j\geq 1}$ and $\alpha \in (0, 2]$.

Also, since we are considering the fractional derivative, we can first explicitly and exactly simulate the $\beta$-stable subordinator on the infinite time grid, i.e. $(S^{\beta,+}_{t_i})_{i\in\mathbb{N}}$, using the scaling properties and the independence of increments, from the procedure described in Chambers et al. [9]. Precisely, we recall that if $\Theta$ is uniformly distributed on $[0, \pi]$ and $W$ is an exponential variable with mean 1 and independent from $\Theta$ then for all $\beta \in (0, 1)$:

$$S^{\beta,+}_1 \overset{(\text{law})}{=} (a_\beta(\Theta)/W)^{(1-\beta)/\beta}, \quad \forall \theta \in (0, \pi), \ a_\beta(\theta) = \frac{\sin((1-\beta)\theta)\sin(\beta\theta)^{\beta/(1-\beta)}}{\sin(\theta)^{1/(1-\beta)}}. \tag{3.2}$$

From the exact simulation on the time grid, a natural choice to approximate the inverse process $Z^\beta_t$ is to consider the *discrete* inverse process:

$$\forall t \geq 0, \quad Z^{\beta,h}_t := \inf\{s_i := ih : S^{\beta,+}_{s_i} > t\}. \tag{3.3}$$

Thus, for a given $T > 0$, we finally approximate $X_{Z^\beta_T}$ appearing in (2.13) by $X^h_{Z^{\beta,h}_T}$.

Let us mention that the smoothness assumption (**S**) could be weakened in the case of the Euler scheme. These smoothness conditions are actually those required to apply the results of [21] for the Markov Chain approximation in the diffusive setting, i.e. for $\alpha = 2$.

We also indicate that schemes of the type (3.1)–(3.3) had already been proposed by Magdziarz et al. [25], without convergence analysis, to approximate the solution density of $X_{Z^\beta_T}$. Let us also mention that "particle methods" had already been considered in Zhang et al. [40] to approximate the underlying PIDE(s) for the case of a stable like spatial motion for the usual time derivative. Again, no convergence rates are proved.

**Remark 3.1** (*About the Innovations in the Spatial Approximation Scheme* (3.1)).

- In the *diffusive case*, corresponding to generators of the form $L_1$ in (2.10), or equivalently to $\alpha = 2$, we will typically consider for the $(\eta_j)_{j\in\mathbb{N}^*}$ in (3.1) standard i.i.d Gaussian variables or i.i.d. random variables with polynomial tails whose moments match those of the standard Gaussian law up to a given order.
- In the *strictly stable case*, corresponding to generators of the form $L_2$ in (2.10) (or to $\alpha \in (0, 2)$), the situation is more complicated. Considering the Euler approximation in (3.1) amounts to say that $\eta_j \overset{(\text{law})}{=} Y_1, \ j \in \mathbb{N}^*$ and would require to be able to exactly simulate the driving process $(Y_s)_{s\geq 0}$ in (2.11). Now, few symmetric stable processes can be exactly simulated in $\mathbb{R}^d, d \geq 2$. This is for instance the case if the spectral measure $\nu$ in (2.1) writes as a sum of $n$ Dirac masses, $n \in \mathbb{N}$, on $S^{d-1}$. Then $Y_1$ can be realized as a sum of $n$ independent one-dimensional symmetric stable random variables that can be simulated using directly identities (2.3) and (2.4) in [9] or from (3.2) and Gaussian subordination. This last technique can be extended to the multidimensional case. Indeed, if $N$ is a standard Gaussian vector in $\mathbb{R}^d$, $d \geq 1$, then for a given matrix $A \in \mathbb{R}^d \otimes \mathbb{R}^d$, $Z := (S^{\alpha/2,+}_1)^{1/2} AN$, where $S^{\alpha/2,+}_1$ stands for the $\alpha/2$ stable subordinator at time 1, has Fourier transform $\mathbb{E}[\exp(i\lambda \cdot Z)] = \exp(-2^{-\alpha/2}|A^*\lambda|^\alpha)$, $\lambda \in \mathbb{R}^d$. The random variable $Z$ is called elliptically symmetric.

Observe that if $AA^*$ is non degenerate then the non-degeneracy condition (2.2) is fulfilled. This condition guarantees the existence of the density of $Z$ in $\mathbb{R}^d$. When, for $d \geq 2$, the symmetric stable driving process cannot be simulated explicitly, we can under the non degeneracy condition (2.2) approximate the spectral measure of $Y_1$ by a discrete sum of Dirac masses of the previous type for which the associated random variable can be simulated exactly. We refer to Nolan and Rajput [32] for practical discussions on the choice of the points and associated error bounds on the difference of densities of the two random variables. Another approach might be to use random variables that can be explicitly simulated and converge, up to a suitable normalization given by a limit theorem, to the random variable with the required spectral measure. This scheme is proposed in Zhang et al. [40] to approximate a spatial Lévy motion (see Section III in that reference). Again, the difficulty in this approach consists in controlling the induced error.

### 3.2. *Diffusive case*

This case is associated with spatial motions with generators of the form $L_1$ in (2.10). We then specifically assume that $\alpha = 2$ in (3.1) and that the $(\eta_j)_{j\geq 1}$ are s.t. their moments coincide with those of the standard Gaussian law in $\mathbb{R}^d$ up to order 2. We consider two kinds of conditions:

($\mathbf{I}_{\text{Eul}}$) The $(\eta_j)_{j\geq 1}$ have standard Gaussian densities (so that the above scheme (3.1) is the usual Euler discretization).

($\mathbf{I}_m$) For a given integer $m \geq 2d+1$, the $(\eta_j)_{j\geq 1}$ have a density $Q_M$ which is $C^4$ and has, together with its derivatives up to order 4, polynomial decay of order $M > d(2m+1)+4$. Namely, for all $z \in \mathbb{R}^d$ and multi-index $\nu$, $|\nu| \leq 4$:

$$|D_\nu Q_M(z)| \leq C_M(1+|z|)^{-M}.$$

We say that assumption ($\mathbf{A}_{D,\text{Eul}}$) (resp. ($\mathbf{A}_{D,m}$)) is in force as soon as (**S**), (**UE**), ($\mathbf{I}_{\text{Eul}}$) (resp. ($\mathbf{I}_m$)) hold. We write ($\mathbf{A}_D$) whenever ($\mathbf{A}_{D,\text{Eul}}$) or ($\mathbf{A}_{D,m}$) holds.

It is well known that under ($\mathbf{A}_D$) the random variables $X_t, X^h_{t_i}$ have densities for all $t > 0, i \in \mathbb{N}^*$ respectively, see e.g. [19]. This property then transfers to $X_{Z_T^\beta}$ and $X^h_{Z_T^{\beta,h}}$ thanks to the convolution equation (2.12) and its discrete analogue for the approximation (see Eq. (3.4)). We will denote by $p_{X_{Z_T^\beta}}$ and $p_{X^h_{Z_T^{\beta,h}}}$ the associated p.d.f. To distinguish precisely the approximations under ($\mathbf{A}_{D,\text{Eul}}$) or ($\mathbf{A}_{D,m}$) we will specifically denote, when needed, by $X^{\text{Eul},h}_{Z_T^{\beta,h}}$, $p_{X^{\text{Eul},h}_{Z_T^{\beta,h}}}$ the Euler approximation and its density.

We first have the following density bounds for the random variables $X_{Z_T^\beta}$ and $X^{\text{Eul},h}_{Z_T^{\beta,h}}, X^h_{Z_T^{\beta,h}}$.

**Proposition 3.2** (*Existence of the Density and Associated Estimates*)**.** *Under* ($\mathbf{A}_D$)*, for a given $T > 0$, the laws of the random variables $X_{Z_T^\beta}$ and $X^{\text{Eul},h}_{Z_T^{\beta,h}}$ starting from $x \in \mathbb{R}^d$ have densities $p_{X_{Z_T^\beta}}(x,\cdot)$, $p_{X^{\text{Eul},h}_{Z_T^{\beta,h}}}(x,\cdot)$ in $\mathbb{R}^d\backslash\{x\}$ for $d \geq 2$, and on $\mathbb{R}$ for $d = 1$. The expression of $p_{X_{Z_T^\beta}}(x,\cdot)$ is given by* (2.12) *whereas*

$$p_{X^{\text{Eul},h}_{Z_T^{\beta,h}}}(x,\cdot) = \sum_{i\geq 1} \mathbb{P}[Z_T^{\beta,h} = t_i]\, p^{\text{Eul},h}(t_i,x,\cdot). \tag{3.4}$$

*Furthermore, there exists $c := c((\mathbf{A}), \beta) \geq 1$ s.t. for all $(x, y) \in (\mathbb{R}^d)^2$ and $x \neq y$ if $d \geq 2$, the following estimates hold:*

$$(p_{X_{Z_T^\beta}} + p_{X^{\mathrm{Eul},h}_{Z_T^{\beta,h}}})(x, y) \leq c(\hat{p}_\beta + \tilde{p}_\beta)(T, x - y) \tag{3.5}$$

*where*

$$\begin{aligned}\hat{p}_\beta(T, x - y) \ &:= \left(\frac{\mathbb{I}_{d\leq 2}}{T^{\beta/2}|x - y|^{\mathbb{I}_{d=2}}} + \frac{\mathbb{I}_{d\geq 3}}{T^\beta |x - y|^{d-2}}\right)\exp(cT^{\beta/2})\\ &\quad \times \exp\left(-c^{-1}\frac{|x - y|^2}{T^\beta}\right),\\ \tilde{p}_\beta(T, x - y) &:= \frac{\exp(cT^{\beta/(1+\beta)})}{T^{\beta d/2}}\exp\left(-c^{-1}\left\{\frac{|x - y|^2}{T^\beta}\right\}^{1/(2-\beta)}\right).\end{aligned} \tag{3.6}$$

*For the Markov Chain approximation, under $(\mathbf{A}_{D,m})$, $p_{X^h_{Z_T^{\beta,h}}}$ expands as in (3.4) replacing $p^{\mathrm{Eul},h}$ by $p^h$, and for all $(x, y) \in (\mathbb{R}^d)^2$, $x \neq y$, $d \geq 2$:*

$$p_{X^h_{Z_T^{\beta,h}}}(x, y) \leq c\{\hat{q}_{2(m-1)-[(d-2)+\mathbb{I}_{d\leq 2}],\beta} + \tilde{q}_{2(m-1),\beta}\}(T, x - y) \tag{3.7}$$

*where for all $l \in \mathbb{N}^*$:*

$$\begin{aligned}\hat{q}_{l,\beta}(T, x - y) &:= \left(\frac{\mathbb{I}_{d\leq 2}}{T^{\beta/2}|x - y|^{\mathbb{I}_{d=2}}} + \frac{\mathbb{I}_{d\geq 3}}{T^\beta |x - y|^{d-2}}\right)\exp(cT^{\beta/2}) \times \left(1 + \frac{|x - y|}{T^{\beta/2}}\right)^{-l},\\ \tilde{q}_{l,\beta}(T, x - y) &:= \frac{\exp(cT^{\beta/(1+\beta)})}{T^{\beta d/2}}\left(1 + \frac{|x - y|}{T^{\beta/2}}\right)^{-\lfloor\frac{l}{2-\beta}\rfloor}.\end{aligned} \tag{3.8}$$

Proposition 3.2 is a direct corollary of our technical estimates (see Lemma 4.11 and its proof). We also establish in Appendix B a similar lower bound for the density of $X_{Z_T^\beta}$, proving that the estimate is sharp.

We now have the following convergence result:

**Theorem 3.1** (*Error Bounds for the Approximation Schemes*)**.**

- ***Euler scheme.*** *Under $(\mathbf{A}_{D,\mathrm{Eul}})$, there exists $c := c((\mathbf{A}_D))$ s.t. for a given time step $h \in (0, 1)$, a deterministic time horizon $T > 0$ s.t. $T > h^{1/2}$ (not necessarily a multiple of $h$), for every $x \neq y$ we have:*

$$\begin{aligned}&|p_{X_{Z_T^\beta}}(x, y) - p_{X^{\mathrm{Eul},h}_{Z_T^{\beta,h}}}(x, y)|\\ &\quad = \left|\int_{\mathbb{R}^+} p_{Z^\beta}(T, u)p(u, x, y)du - \sum_{i\geq 1}\mathbb{P}[Z_T^{\beta,h} = t_i]p^{\mathrm{Eul},h}(t_i, x, y)\right|\\ &\quad \leq ch\Big(\mathcal{E}_{\beta,\mathrm{Time}}(T, x - y) + \mathcal{E}_{\beta,\mathrm{Space}}(T, x - y)\Big),\end{aligned} \tag{3.9}$$

*where:*

$$\mathcal{E}_{\beta,\mathrm{Time}}(T, x - y) := \left\{\left(\frac{\mathbb{I}_{d\leq 2}}{T^{\beta/2}|x - y|} + \frac{\mathbb{I}_{d\geq 3}}{|x - y|^2}\right)\hat{p}_\beta(T, x - y) + \frac{1}{T^\beta}\tilde{p}_\beta(T, x - y)\right\},$$

$$\mathcal{E}_{\beta,\mathrm{Space}}(T,x-y) := \left\{\left(\frac{\mathbb{I}_{d=1}+\mathbb{I}_{d\geq 3}}{|x-y|}+\frac{\mathbb{I}_{d=2}}{T^{\beta/2}}\right)\hat{p}_\beta(T,x-y)+\frac{1}{T^{\beta/2}}\tilde{p}_\beta(T,x-y)\right\},$$

*with $\hat{p}_\beta$, $\tilde{p}_\beta$ as in (3.6).*

– ***Markov chain.*** *Consider now assumption $(\mathbf{A}_{D,m})$ (general Markov Chain). Then, there exists $c := c((\mathbf{A}_{D,m}))$ and for all $\varepsilon \in (0, 1/5)$, $c_\varepsilon := c_\varepsilon((\mathbf{A}_{D,m}))$ s.t. for all given time step $h \in (0, 1)$, $T^\beta \geq h^{1/5-\varepsilon}$, $(x, y) \in (\mathbb{R}^d)^2$, $x \neq y$:*

$$|p_{X_{Z_T^\beta}}(x,y)-p_{X^h_{Z_T^{\beta,h}}}(x,y)| \leq c\big\{h\mathcal{E}_{\beta,\mathrm{Time}}(T,x-y)$$
$$+\{c_\varepsilon h^{1/2}\mathcal{E}^M_{\beta,\mathrm{Space,LLT}}(T,x-y)+\mathcal{E}^{M,\varepsilon}_{\beta,\mathrm{Space,NoLLT}}(T,x-y,h)\}\big\}, \tag{3.10}$$

*where the contribution $\mathcal{E}_{\beta,\mathrm{Time}}(T, x-y)$ due to the time sensitivity, see Eqs. (3.9) and (4.1), is as above and:*

$$\mathcal{E}^M_{\beta,\mathrm{Space,LLT}}(T,x-y)$$
$$:= \left\{\left(\frac{\mathbb{I}_{d=1}+\mathbb{I}_{d\geq 3}}{|x-y|}+\frac{\mathbb{I}_{d=2}}{T^{\beta/2}}\right)\hat{q}_{m-[(d-1)+\mathbb{I}_{d=1}],\beta}(T,x-y)+\frac{1}{T^{\beta/2}}\tilde{q}_{m,\beta}(T,x-y)\right\},$$

*with $\tilde{q}_{m,\beta}$ and $\hat{q}_{m-[(d-1)+\mathbb{I}_{d=1}],\beta}$ as in (3.8). Also:*

$$\mathcal{E}^{M,\varepsilon}_{\beta,\mathrm{Space,NoLLT}}(T,x-y,h) := \frac{(h^{1/5-\varepsilon})^{\frac{1}{2}\mathbb{I}_{d\leq 2}}}{T^\beta|x-y|^{d-2+\mathbb{I}_{d\leq 2}}}\exp(cT^{\beta/2})$$
$$\times\left[\exp\left(-c\frac{|x-y|^2}{h^{1/5-\varepsilon}}\right)+\left(1+\frac{|x-y|}{(h^{1/5-\varepsilon})^{1/2}}\right)^{-2(m-1)+(d-2)+\mathbb{I}_{d\leq 2}}\right].$$

**Comments on the results.**

– **Euler scheme.** Observe that the term $\hat{p}_\beta$ in (3.6) comes from the contribution of small times in formula (2.12). This gives the additional (diagonal) spatial singularity in the fractional case. The term $\tilde{p}_\beta$ in (3.6) comes from the integration over long times in (2.12). It induces a loss of concentration in space w.r.t to the usual case $\beta = 1$, though preserving the usual parabolic equilibrium: the spatial contribution remains comparable to the square root of time.

- In the diagonal region, i.e. for points s.t. $|x-y| \leq \kappa T^{\beta/2}$ for some $\kappa \geq 1$, if we restrict to spatial points that are equivalent to the time contribution for the usual parabolic metric, i.e. $\kappa^{-1}T^{\beta/2} \leq |x-y| \leq \kappa T^{\beta/2}$, the previous error bound reads in *small* time, i.e. for $T \leq 1$:

$$|p_{X_{Z_T^\beta}}(x,y)-p_{X^{\mathrm{Eul},h}_{Z_T^{\beta,h}}}(x,y)| \leq \frac{ch}{T^\beta}(\hat{p}_\beta+\tilde{p}_\beta)(T,x-y). \tag{3.11}$$

  This means that we find the upper bound for the density of the processes involved, see (3.5), multiplied by a factor corresponding to the singularity induced by a time derivative (or equivalently second order derivatives in space) of the non degenerate diffusive heat kernel at time $T^\beta$. There is even here a difference w.r.t. to the usual analysis, see, e.g., [20], which involves a singularity of order one in space, which in the current bound comes from the contribution $\mathcal{E}_{\beta,\mathrm{Space}}(T, x-y)$ in (3.9) and is negligible if $T \leq 1$. This is due to the discrete approximation of the inverse of the subordinator which yields to consider derivatives in time for the densities of the spatial motion (see Eq. (4.1) and Lemma 4.1). If the error bound (3.9) is now considered for $T$ sufficiently large, the contribution $\mathcal{E}_{\beta,\mathrm{Space}}(T, x-y)$ dominates and we are back to the *usual* error rate for the considered region.

Observe as well that the spatial diagonal singularity induced by the fractional time derivative yields additional constraints. Namely, for the Euler approach, to get a convergence rate of order $h^{\varepsilon}$ one needs to take: $|x-y| \geq h^{(1-\varepsilon)/d}$, $d \geq 1$.

- In the off-diagonal region, i.e. $|x-y| \geq \kappa T^{\beta/2}$, we do not have spatial singularities anymore. Indeed, the error bound also reads as in (3.11). Anyhow, the specificity comes from the loss of concentration due to the term $\tilde{p}_\beta$ coming from the long time integration in (2.12) and appearing in the two-sided estimates of Appendix B. We refer to Lemmas 4.8 and 4.11 (see also Proposition 3.2 and Theorem B.1) for a detailed presentation of these aspects.

The previous facts make us feel that the previous bounds are sharp, up to constants.

– **Markov chains.** For the error bound (3.10) associated with the Markov Chain approximation, the term $\mathcal{E}_{\beta,\mathrm{Time}}$ is similar to the Euler case, since it does not depend on the chosen discretization scheme but only on the time sensitivities of the spatial density, see again (4.1). On the other hand two additional contributions appear. The first one, $\mathcal{E}^{M}_{\beta,\mathrm{Space,LLT}}$ enjoys the same convergence rate $h^{1/2}$ as in the usual LLT (see Petrov [34,19]) and the terms $\hat{q}_{m-(d-1)+\mathbb{I}_{d=1},\beta}$, $\tilde{q}_{m,\beta}$ appearing in (3.10) are the *Markov chain analogue* of the contributions $\hat{p}_\beta$ and $\tilde{p}_\beta$ in (3.9).[1] Note first that we keep under ($\mathbf{A}_{D,m}$) the same index $m$ as the one appearing for the weak error at a given fixed time, see (4.28) and [21,19], for the contribution $\tilde{q}_{m,\beta}$ associated with the large time integration and which also yields a loss of concentration. Observe as well that the spatial diagonal singularities in small time also deteriorate the concentration in this framework. We point out that, for the LLT to apply, the considered time has to be sufficiently large, namely $u \geq h^{1/5-\varepsilon}$ in (2.12). The last term $\mathcal{E}^{M,\varepsilon}_{\beta,\mathrm{Space,NoLLT}}$ comes precisely from those *very* small times $u \leq h^{1/5-\varepsilon}$ for which the limit results do not apply. Also, applying the LLT in (very) small time deteriorates the concentration, see the difference between Theorem 1 in [21], which is used to prove Theorem 3.1, and Theorem 1.1. in [19] which gives the LLT with the same concentration rate appearing in the density estimates (3.7) of Proposition 3.2 if $T > 0$ is fixed and $h$ is sufficiently small. Eventually, the constraint $m \geq 2d+1$ in ($\mathbf{I}_m$) appears to keep an integrable bound in (3.10).

– **Continuity of the estimates when** $\beta \uparrow 1$. In this case, the Caputo derivative in (2.9) tends to the usual derivative. It is therefore natural to ask whether our results match the ones previously obtained in that case (see [19,20,22]). Let us now emphasize that when $\beta \uparrow 1$ then $\psi_{S^{\beta,+}_u}(\lambda) \to \exp(-\lambda u), \lambda \geq 0$. Thus, the subordinator $S^{\beta,+}$, and its inverse $Z^\beta$ both tend to the deterministic drift with slope 1. This means that for a given $T > 0$, $\mathbb{P}[Z^\beta_T \in du] \underset{\beta\to 1}{\longrightarrow} \delta_T(du)$. Hence, the time integral in (2.12) disappears at the limit, i.e. $p_{X_{Z^\beta_T}}(x,y) \underset{\beta\to 1}{\longrightarrow} p(T,x,y)$. From (3.9) and (4.1), the same phenomenon occurs for the scheme provided $T = Kh,\ K \in \mathbb{N}^*$ and the error associated with the time sensitivity, term $\mathcal{E}_1$ in (4.1) yielding the contribution $\mathcal{E}_{\beta,\mathrm{Time}}$ would vanish in the previous error bounds. Since the previous Dirac convergence would also kill the additional contributions in small and long time, we finally derive that the error bounds obtained in (3.9) and (3.10) are coherent with those of the previous works at the limit.

### 3.3. Strictly stable case

Additionally to assumptions (**A**), (**UE**) we will assume that :

[1] We recall from Proposition 3.2 that $(\hat{p}_\beta + \tilde{p}_\beta)(T, x-y)$ serves as an upper bound for $(p_{X_{Z^\beta_T}})(x,y)$.

(**ND**) The spherical part $\nu$ of the measure $\mu$ in (2.10) satisfies the non-degeneracy condition (2.2). Moreover, $\nu$ has a $C^3$ density w.r.t. the Lebesgue measure of the sphere.

(**B**) The drift $b = 0$ if $\alpha \le 1$.

This last condition is rather usual to have pointwise bounds on the density of the SDE, see [17]. We will consider similarly to the diffusive case the approximation (3.1) for $\alpha \in (0, 2)$ restricting ourselves to the Euler scheme case, i.e. the $(\eta_j)_{j\ge1}$ are i.i.d. symmetric stable random variables with common law $Y_1$ (driving process at time 1). Thus, from Remark 3.1, the only driving process satisfying the previous assumptions that can be exactly simulated is the elliptically symmetric one. It would be interesting to investigate how the error discussed in Nolan and Rajput [32], due to the discrete approximation of the spectral measure, propagates through the scheme. This is beyond the scope of the present work.

We say that $(\mathbf{A}_S)$ holds if (**S**), (**UE**), (**ND**), (**B**) are in force. Recall from Kolokoltsov [17,22] that the random variables $X_t, X^{\mathrm{Eul},h}_{t_i}$ have a density for all $t > 0$, $i \in \mathbb{N}^*$ respectively. From the controls of Lemma 4.11 we are able to establish that this property transfers to the random variables $X_{Z_T^\beta}, X^{\mathrm{Eul},h}_{Z_T^{\beta,h}}$ from (2.12). We have the following result.

**Proposition 3.3.** *Under $(\mathbf{A}_S)$, for a given $T > 0$, the laws of the random variables $X_{Z_T^\beta}$ and $X^{\mathrm{Eul},h}_{Z_T^{\beta,h}}$ starting from $x \in \mathbb{R}^d$ have densities in $\mathbb{R}^d\backslash\{x\}$ for $d \ge 2$ or $d = 1, \alpha \le 1$, and on $\mathbb{R}$ for $d = 1, \alpha > 1$. The expression of these densities is given by Eqs. (2.12) and (3.4) respectively. Furthermore, there exists $c := c((\mathbf{A}), \beta) \ge 1$ s.t. for all $(x, y) \in (\mathbb{R}^d)^2$ and $x \ne y$ if $d \ge 2$ or $d = 1, \alpha \le 1$, the following estimates hold:*

$$(p_{X_{Z_T^\beta}} + p_{X^{\mathrm{Eul},h}_{Z_T^{\beta,h}}})(x, y) \le c(\hat{p}_\beta + \tilde{p}_\beta)(T, x - y), \quad c := c(\beta, (\mathbf{A}_S)),$$

*with*

$$\begin{aligned}\hat{p}_\beta(T, x - y) &:= \exp(cT^{\beta\omega})\left\{\frac{1}{T^\beta|x - y|^{d-\alpha}}\mathbb{I}_{|x-y|\le T^{\beta/\alpha}} + \frac{T^\beta}{|x - y|^{d+\alpha}}\mathbb{I}_{|x-y|>T^{\beta/\alpha}}\right\},\\ \tilde{p}_\beta(T, x - y) &:= \frac{\exp(cT^{\beta\omega/(1-\omega(1-\beta))})}{T^{\beta d/\alpha}\left(1 + \frac{|x-y|}{T^{\beta/\alpha}}\right)^{d+\alpha}}, \quad \omega := \frac{1}{\alpha} \wedge 1.\end{aligned} \tag{3.12}$$

**Theorem 3.2.** *Under $(\mathbf{A}_S)$, there exists a constant $c$ s.t. for a given time step $h \in (0, 1)$ and a fixed time horizon $T > h^{1/\beta}$, and any $(x, y) \in (\mathbb{R}^d)^2$ s.t. $|x - y| \ge h^{1/\alpha}$ we have the following convergence result:*

$$\begin{aligned}&|p_{X_{Z_T^\beta}}(x, y) - p_{X^{\mathrm{Eul},h}_{Z_T^{\beta,h}}}(x, y)|\\ &\le ch\left\{\left(\frac{\mathbb{I}_{|x-y|\le T^{\beta/\alpha}}}{|x - y|^\alpha} + \frac{\mathbb{I}_{|x-y|>T^{\beta/\alpha}}}{T^\beta}\right)\hat{p}_\beta(T, x - y) + \frac{1}{T^\beta}\tilde{p}_\beta(T, x - y)\right\},\end{aligned} \tag{3.13}$$

*where $\hat{p}_\beta, \tilde{p}_\beta$ are as in (3.12).*

**Remark 3.4.** We observe a phenomenon which is similar to the diffusive case, i.e. the error bound has the expected rate of order $h$ up to additional singularities. In the strictly stable case, i.e. $\alpha \in (0, 2)$, two contributions give additional singularities w.r.t. those already appearing in the sharp density bounds (see Theorem B.1). Those contributions write $T^{-\beta}$, $|x-y|^{-\alpha}$ and come from the time derivative of the density in (2.12).

On the other hand, since the spatial motion already has heavy tails, we do not observe here a loss of concentration phenomenon as we did in the diffusive case. Eventually, we still have diagonal spatial singularities, which are in the strictly stable case more direct to formulate. They appear precisely when $|x-y| \le T^{\beta/\alpha}$, that is when the diagonal regime holds w.r.t. to the usual stable parabolic scaling. This is again the effect of the fractional in time derivative analyzed in small time, see (2.12). Let us mention that there is some continuity w.r.t. to the stability index for the induced singularity (see (3.6) and (3.12)), at least when $d \ge 3$ since the diffusive case provides a factor $(T^{\beta}|x-y|^{d-2})^{-1}$ for that contribution in $\hat{p}_\beta(T, x-y)$.

## 4. Proof of the main results

In the following we denote by $c$ a generic positive constant that may change from line to line. Explicit dependences for those constants $c$, mainly on assumptions ($\mathbf{A}_D$) or ($\mathbf{A}_S$), $\beta$, are specified as well.

### 4.1. Decomposition of the error

For given points $(T, x, y) \in \mathbb{R}_+^* \times (\mathbb{R}^d)^2$, we get from (2.12) and (3.3) that the error writes:

$$\begin{aligned}
\mathcal{E}(T,x,y) &:= (p_{X_{Z_T^\beta}} - p_{X^h_{Z_T^{\beta,h}}})(x,y)\\
&= \int_{\mathbb{R}^+} p_{Z^\beta}(T,u)p(u,x,y)du - \sum_{i\ge 1}\mathbb{P}[Z_T^{\beta,h} = t_i]p^h(t_i,x,y),\\
&= \int_{\mathbb{R}^+} p_{Z^\beta}(T,u)(p(u,x,y) - p(\phi(u),x,y))du\\
&\quad + \left\{\sum_{i\ge 1}\mathbb{P}[Z_T^{\beta,h} = t_i](p(t_i,x,y) - p^h(t_i,x,y))\right\},\\
&:= \mathcal{E}_1(T,x,y) + \mathcal{E}_2(T,x,y),
\end{aligned} \tag{4.1}$$

denoting for all $u \in \mathbb{R}_+^*$, $\phi(u) := \inf\{t_i := ih : t_i > u\}$, where we have used (3.3) for the last but one equality. Let us specify that $\mathcal{E}_1$ is a term involving the time sensitivities of the density of the SDE (2.11), which does not depend on the approximation scheme. The contribution $\mathcal{E}_2$ is associated with the spatial sensitivity involving the discretization error for the considered scheme.

The idea is then to use some known asymptotics of the density $S^{\beta,+}$ (see e.g. Zolotarev [41] or Hahn et al. [14]) which provide those for $p_{Z^\beta}$. On the other hand, the quantities $|p(t_i,x,y) - p^h(t_i,x,y)|$ have been thoroughly investigated in the literature for the diffusive case, see [19, 13]. The results established therein give that for a general approximation scheme of type (3.1) the difference is controlled at order $h^{1/2}$ which is the convergence rate in the Gaussian LLT, see, e.g., Petrov [34]. If the scheme already involves Gaussian increments (usual Euler scheme of the diffusion) the control is improved and the convergence rate is $h$, see also [20]. In the strictly stable case, i.e. $\alpha < 2$, the convergence rate of the Euler scheme, for which the exact stable increments are considered in (3.1), has been investigated in [22]. The proof of Theorems 3.1 and 3.2 immediately follows from Eq. (4.1) and the following lemmas.

**Lemma 4.1** (*Time Sensitivity in the Error*). *Under assumption* $(\mathbf{A}_D)$ *(Diffusive case) we have that there exists* $c := c(\beta, (\mathbf{A}_D)) \geq 1$, *s.t. for all* $T > 0$, $(x, y) \in (\mathbb{R}^d)^2$, $x \neq y$:

$$\begin{aligned}|\mathcal{E}_1(T, x, y)| &\leq ch\Bigg\{\frac{\exp(cT^{\beta/2})}{T^\beta|x-y|^d}\exp\left(-\frac{c^{-1}}{2}\frac{|x-y|^2}{T^\beta}\right)\\
&\quad+\frac{\exp(cT^{\beta/(1+\beta)})}{T^{\beta(1+d/2)}}\exp\left(-c^{-1}\left\{\frac{|x-y|^2}{T^\beta}\right\}^{1/(2-\beta)}\right)\Bigg\}\\
&\leq ch\left\{\left(\frac{\mathbb{I}_{d\leq 2}}{T^{\beta/2}|x-y|}+\frac{\mathbb{I}_{d\geq 3}}{|x-y|^2}\right)\hat{p}_\beta(T, x-y)+\frac{1}{T^\beta}\tilde{p}_\beta(T, x-y)\right\}.\end{aligned}$$

*Under Assumption* $(\mathbf{A}_S)$ *(Strictly stable case), there exists* $c := c(\beta, (\mathbf{A}_S))$ *s.t. for all* $T > 0, (x, y) \in (\mathbb{R}^d)^2$, $x \neq y$:

$$\begin{aligned}|\mathcal{E}_1(T, x, y)| &\leq ch\left\{\exp(cT^{\beta\omega})\left(\frac{\mathbb{I}_{|x-y|\leq T^{\beta/\alpha}}}{T^\beta|x-y|^d}+\frac{\mathbb{I}_{|x-y|>T^{\beta/\alpha}}}{|x-y|^{d+\alpha}}\right)+\frac{\exp(cT^{\beta\omega/(1-\omega(1-\beta))})}{T^{\beta(1+d/\alpha)}\vee|x-y|^{d+\alpha}}\right\}\\
&\leq ch\left\{\left(\frac{\mathbb{I}_{|x-y|\leq T^{\beta/\alpha}}}{|x-y|^\alpha}+\frac{\mathbb{I}_{|x-y|>T^{\beta/\alpha}}}{T^\beta}\right)\hat{p}_\beta(T, x-y)+\frac{1}{T^\beta}\tilde{p}_\beta(T, x-y)\right\},\end{aligned}$$

*where* $\omega = \frac{1}{\alpha} \wedge 1$.

**Remark 4.2.** Observe that in the stable case there is no loss of concentration as in the diffusive one. The second term of the stable bound in Lemma 4.1 corresponds to the *usual* estimate for the time derivative of the stable density, see Lemma 4.10 and [17]. The first one corresponds to the additional spatial singularity induced by the fractional derivative in time in the diagonal regime $|x - y| \leq T^{\beta/\alpha}$ (stable parabolic scaling at time $T^\beta$).

We now give two Lemmas concerning the spatial sensitivity that involve the analysis of the discretization error and the time randomization.

**Lemma 4.3** (*Spatial Sensitivity: Diffusive Case*). *Under assumption* $(\mathbf{A}_D)$ *there exists* $c := c(\beta, (\mathbf{A}_D))$ *s.t. for all* $T > 0$, $(x, y) \in (\mathbb{R}^d)^2$, $x \neq y$:

– *For the Euler scheme:*

$$\begin{aligned}|\mathcal{E}_2^{\mathrm{Eul}}(T, x, y)| &\leq ch\left\{\left\{\frac{\mathbb{I}_{d=1}+\mathbb{I}_{d\geq 3}}{|x-y|}+\frac{\mathbb{I}_{d=2}}{T^{\beta/2}}\right\}\hat{p}_\beta(T, x-y)+\frac{1}{T^{\beta/2}}\tilde{p}_\beta(T, x-y)\right\}\\
&:= ch\mathcal{E}_{\beta,\mathrm{Space}}(T, x-y)\end{aligned}\tag{4.2}$$

*where we write* $\mathcal{E}_2^{\mathrm{Eul}}$, *with a slight difference w.r.t. the definition in (4.1), to specifically make the distinction with general Markov Chains.*

– *For a general Markov Chain, we have that for all* $\varepsilon \in (0, 1/5)$ *there exist* $c_\varepsilon := c_\varepsilon((\mathbf{A}_{D,m}))$ *and* $c := c((\mathbf{A}_{D,m}))$ *s.t. for all* $T > 0$, $(x, y) \in (\mathbb{R}^d)^2$, $x \neq y$:

$$\begin{aligned}|\mathcal{E}_2(T, x, y)| &\leq \Bigg[c_\varepsilon h^{1/2}\Bigg\{\left\{\frac{\mathbb{I}_{d=1}+\mathbb{I}_{d\geq 3}}{|x-y|}+\frac{\mathbb{I}_{d=2}}{T^{\beta/2}}\right\}\hat{q}_{m-(d-1)+\mathbb{I}_{d=1},\beta}(T, x-y)\\
&\quad+\frac{1}{T^{\beta/2}}\tilde{q}_{m,\beta}(T, x-y)\Bigg\}+\frac{c(h^{1/5-\varepsilon})^{\frac{1}{2}\mathbb{I}_{d\leq 2}}}{T^\beta|x-y|^{d-2+\mathbb{I}_{d\leq 2}}}\exp(cT^{\beta/2})\\
&\quad\times\left[\exp\left(-c\frac{|x-y|^2}{h^{1/5-\varepsilon}}\right)+\left(1+\frac{|x-y|}{(h^{1/5-\varepsilon})^{1/2}}\right)^{-2(m-1)+(d-2)+\mathbb{I}_{d\leq 2}}\right]\Bigg]\\
&:= \{c_\varepsilon h^{1/2}\mathcal{E}^M_{\beta,\mathrm{Space,LLT}}(T, x-y)+\mathcal{E}^{M,\varepsilon}_{\beta,\mathrm{Space,NoLLT}}(T, x-y, h)\}.\end{aligned}\tag{4.3}$$

**Lemma 4.4** (*Spatial Sensitivity: Strictly Stable Case*). *Under assumption* $(\mathbf{A}_S)$, *there exists* $c := c(\beta, (\mathbf{A}_S)) \ge 1$ *s.t. for all* $T > 0$, $(x, y) \in (\mathbb{R}^d)^2$, $x \ne y$:

$$
\begin{aligned}
&|\mathcal{E}_2^{\mathrm{Eul}}(T, x, y)|\\
&\le ch\left\{\exp(cT^{\beta\omega})\left(\frac{\mathbb{I}_{|x-y|\le T^{\beta/\alpha}}}{T^\beta|x-y|^d} + \frac{\mathbb{I}_{|x-y|>T^{\beta/\alpha}}}{|x-y|^{d+\alpha}}\right) + \frac{\exp(cT^{\beta\omega/(1-\omega(1-\beta))})}{T^{\beta(1+d/\alpha)} \vee |x-y|^{d+\alpha}}\right\}\\
&\le ch\left\{\left(\frac{\mathbb{I}_{|x-y|\le T^{\beta/\alpha}}}{|x-y|^\alpha} + \frac{\mathbb{I}_{|x-y|>T^{\beta/\alpha}}}{T^\beta}\right)\hat{p}_\beta(T, x-y) + \frac{1}{T^\beta}\tilde{p}_\beta(T, x-y)\right\}.
\end{aligned}
$$

**Remark 4.5.** We emphasize that in the strictly stable case, both time and spatial sensitivities yield the same error. This comes from the fact that the short time singularity for the error expansion of the Euler scheme, exactly matches the time singularity of the derivative of the stable heat kernel, see Eqs. (4.32) and (4.12).

*4.2. Asymptotics of the PDF of the inverse subordinator*

As a preliminary remark, let us mention that in the particular case $\beta = 2^{-n}$, $n \in \mathbb{N}^*$, for all $T > 0$, the random variable $Z_T^\beta$, inverse at the level $T$ of the subordinator $S^{\beta,+}$ can be easily and explicitly simulated from the following Proposition. In particular, the auxiliary discrete approximation $Z_T^{\beta,h}$ is not needed in those cases, and the time sensitivity error $\mathcal{E}_1(T, x, y)$ in (4.1) would vanish (considering a straightforward extension of the discretization scheme for the last time step).

**Proposition 4.6** (*Identity in Law for* $Z_T^\beta$, $\beta = 2^{-n}$, $n \in \mathbb{N}^*$, $T > 0$).

$$
\forall n \in \mathbb{N}^*,\ \beta = 2^{-n}, \quad Z_T^\beta \overset{\text{(law)}}{=} \sqrt{2}|B^1(\sqrt{2}|B^2(\sqrt{2}|B^3(\cdots(\sqrt{2}|B^n(T)|)\cdots)|)|)|, \tag{4.4}
$$

*where the* $(B^i)_{i\in\{1,\dots,n\}}$ *are independent Brownian motions.*

**Proof.** Let us start with $n = 1$. In this case, it is well known that, recalling that $(S_t^{\frac12,+})_{t\ge0}$ is the standard subordinator with index $\frac12$, one has $\left(S_t^{\frac12,+}\right)_{t\ge0} \overset{\text{(law)}}{=} \left(\tau_{\frac{t}{\sqrt2}}\right)_{t\ge0}$, where $\tau_{\frac{t}{\sqrt2}} := \inf\{s \ge 0 : B_s = \frac{t}{\sqrt2}\}$, stands for the hitting time of level $\frac{t}{\sqrt2}$ for the standard Brownian motion $B$. Indeed, considering for a given $\lambda \ge 0$ the exponential martingale $\left(\exp(\sqrt{2\lambda}B_s - \lambda s)\right)_{s\ge0}$, the stopping theorem gives that, for all $s \ge 0$, $\mathbb{E}[\exp(\sqrt{2\lambda}B_{s\wedge\tau_{\frac{t}{\sqrt2}}} - \lambda(s\wedge\tau_{\frac{t}{\sqrt2}}))] = 1$ which implies, as $s \to +\infty$, $\mathbb{E}[\exp(-\lambda\tau_{\frac{t}{\sqrt2}})] = \exp(-t\lambda^{1/2})$. Write now for all $t \ge 0$:

$$
\mathbb{P}[Z_T^{\frac12} \le t] = \mathbb{P}[S_t^{\frac12,+} > T] = \mathbb{P}[\tau_{\frac{t}{\sqrt2}} > T] = \mathbb{P}\left[\sup_{s\in[0,T]} B_s \le \frac{t}{\sqrt2}\right] = \mathbb{P}\left[\sqrt2|B_T| \le t\right],
$$

using the well known Lévy identity $\sup_{s\in[0,T]} B_s \overset{\text{(law)}}{=} |B_T|$, $T > 0$, for the last equality, see [35]. This proves (4.4) for $n = 1$. Introduce now a family of i.i.d. random processes $\left((S_i(t))_{t\ge0}\right)_{i\ge1}$ with law $(S_t^{\frac12,+})_{t\ge0}$. It is then easily seen that $\bar{S}_n(t) := S_n(S_{n-1}(\dots S_1(t)))$ is a standard stable subordinator of index $\beta = \frac{1}{2^n}$. An immediate induction indeed gives:

$$\forall \lambda \geq 0, \quad \mathbb{E}[\exp(-\lambda S_n(S_{n-1}(\dots S_1(t))))] = \mathbb{E}[\exp(-\lambda^{1/2} S_{n-1}(\dots S_1(t)))] = \cdots \\ = \exp(-\lambda^{1/(2^n)} t).$$

Assume that identity (4.4) holds for a given $n \in \mathbb{N}^*$ and let $B^{n+1}$ be an additional independent Brownian motion independent of the $(B^i)_{i \in [\![1,n]\!]}$. Write:

$$\begin{aligned} \mathbb{P}[\bar{S}_{n+1}(t) > T] = \mathbb{P}[S_{n+1}(\bar{S}_n(t)) > T] &= \mathbb{P}\left[\sup_{s \in [0,T]} B^{n+1}(s) \leq \frac{\bar{S}_n(t)}{\sqrt{2}}\right] \\ &= \mathbb{P}\left[\bar{C}_n\left(\sqrt{2} \sup_{s \in [0,T]} B^{n+1}(s)\right) \leq t\right], \end{aligned} \tag{4.5}$$

where $\bar{C}_n$ stands for the inverse of $\bar{S}_n$. Now, the induction hypothesis exactly gives that for all $u \geq 0$

$$\bar{C}_n(u) \overset{(\text{law})}{=} \sqrt{2}|B^1(\sqrt{2}|B^2(\sqrt{2}|B^3(\cdots(\sqrt{2}|B^n(u)|)\cdots)|)|)|.$$

The proof then follows from the Lévy identity and (4.5) since $\bar{S}_{n+1}$ is a subordinator of index $1/2^{n+1}$. □

**Remark 4.7.** It can be shown similarly that for given $(\beta_i)_{i \in [\![1,n]\!]} \in (0,1)^n$, $n \in \mathbb{N}^*$, if $\big((S_i(t))_{t \geq 0}\big)_{i \in [\![1,n]\!]}$ are independent subordinators s.t for $i \in [\![1,n]\!]$, $(S_i(t))_{t \geq 0} \overset{(\text{law})}{=} (S_t^{\beta_i,+})$, then $\bar{S}_n(t) := S_n(S_{n-1}(\dots S_1(t)))$ is a subordinator of index $\beta := \prod_{i=1}^n \beta_i$ and

$$Z_T^\beta \overset{(\text{law})}{=} Z^{\beta_n}_{Z^{\beta_{n-1}}_{\ddots_{Z_T^{\beta_1}}}}.$$

The delicate part is that this relation cannot be easily exploited from the practical viewpoint, except if $\beta_i = \frac{1}{2}, \forall i \in [\![1,n]\!]$.

In particular, it follows from Proposition 4.6 that, for a given spatial Markov process $(X_t)_{t \geq 0}$ with generator $L$ of the form (2.10), the solution of Eq. (1.1) with $\beta = 2^{-n}$ writes for a continuous function $f$ and for all $(T,x) \in \mathbb{R}^+ \times \mathbb{R}^d$:

$$u(T,x) := \mathbb{E}[f(X^{0,x}_{Z_T^{2^{-n}}})] := \mathbb{E}[f(X^{0,x}_{\sqrt{2}|B^1(\sqrt{2}|B^2(\sqrt{2}|B^3(\cdots(\sqrt{2}|B^n(T)|)\cdots)|)|)|})],$$

generalizing Theorem 2.2 in Beghin and Orsingher [6].

We now state the Lemma concerning the required asymptotics for the laws of $Z_T^\beta, Z_T^{\beta,h}$.

**Lemma 4.8** (*Bounds for the Density of the Inverse Subordinator and Its Discrete Approximation*)**.** *For all $T > 0$,*

$$p_{Z^\beta}(T,u) \leq \theta(T,u), \quad \forall i \in \mathbb{N}^*, \qquad \mathbb{P}[Z_T^{\beta,h} = t_i] \leq C_\beta h \theta(T,t_i), \tag{4.6}$$

*where:*

$$\theta(T,u) := \frac{c_\beta}{T^\beta} \exp\left(-c_\beta^{-1}\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right), \tag{4.7}$$

*for some constants $c_\beta,\ C_\beta \geq 1$.*

**Remark 4.9.** The above control is the generalization of the well known Gaussian bound for $\beta = 1/2$ recalled in Proposition 4.6.

**Proof.** Let us first recall the following asymptotics for $p_{S^{\beta,+}}(1,\cdot)$ from equation (2.2), (2.3) in [14], see also [41]:

$$\begin{aligned} &p_{S^{\beta,+}}(1,v) \sim K(\beta/v)^{(1-\beta/2)/(1-\beta)} \exp\left(-[1-\beta](v/\beta)^{\beta/(\beta-1)}\right), \\ &\quad v \to 0^+, \quad K = 1/\sqrt{2\pi\beta(1-\beta)}, \\ &p_{S^{\beta,+}}(1,v) \sim \frac{\beta}{\Gamma(1-\beta)} v^{-\beta-1}, \quad v \to +\infty. \end{aligned} \tag{4.8}$$

Write now from (2.5) recalling the identity $S_u^{\beta,+} \overset{\text{(law)}}{=} u^{1/\beta} S_1^{\beta,+}$, $u > 0$:

$$\begin{aligned} p_{Z^\beta}(T,u) &= -\partial_u \left[\int_0^T \frac{1}{u^{1/\beta}} p_{S^{\beta,+}}\left(1, \frac{y}{u^{1/\beta}}\right)\right] dy \\ &= \frac{1}{\beta u^{1+1/\beta}} \left\{\int_0^T p_{S^{\beta,+}}\left(1,\frac{y}{u^{1/\beta}}\right) dy + \int_0^T \partial_y p_{S^{\beta,+}}\left(1,\frac{y}{u^{1/\beta}}\right) y dy\right\} \\ &= \frac{T}{\beta u^{1+1/\beta}} p_{S^{\beta,+}}\left(1, \frac{T}{u^{1/\beta}}\right), \end{aligned} \tag{4.9}$$

integrating by parts and using (4.8).

Hence,

$$\begin{aligned} &p_{Z^\beta}(T,u) \sim \frac{T^{-\beta}}{\Gamma(1-\beta)}, \; u \to 0^+, \\ &p_{Z^\beta}(T,u) \sim \left(\frac{\beta}{T}\right)^{\beta/(2(1-\beta))} K u^{\frac{\beta-1/2}{1-\beta}} \exp\left(-[1-\beta]\,(\beta)^{\beta/(1-\beta)} \left(\frac{u}{T^\beta}\right)^{1/(1-\beta)}\right) \\ &u \to +\infty. \end{aligned}$$

The above controls and elementary computations give the first bound in (4.6). Indeed, there exists $c_0 := c_0(\beta) \geq 1$ s.t.,

$$\begin{aligned} p_{Z_T^\beta}(T,u) &\leq c \frac{T^{-\beta}}{\Gamma(1-\beta)}, \quad u \leq \left(\frac{T}{c_0}\right)^\beta, \\ p_{Z_T^\beta}(T,u) &\leq cK \left(\frac{\beta}{T}\right)^{\beta/(2(1-\beta))} u^{\frac{\beta-1/2}{1-\beta}} \exp\left(-[1-\beta]\,(\beta)^{\beta/(1-\beta)} \left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right) \\ &\leq \frac{c_\beta}{T^\beta} \exp\left(-c_\beta^{-1} \left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right), \quad u \geq (c_0 T)^\beta. \end{aligned}$$

For the last bound we observe that for $\beta \leq \frac{1}{2}$ and $u \geq (c_0T)^\beta$, $[u^{\frac{\beta-\frac{1}{2}}{1-\beta}}]/[T^{\frac{\beta}{2(1-\beta)}}] \leq \bar{c}T^{\frac{\beta}{1-\beta}([\beta-\frac{1}{2}]-\frac{1}{2})} = \bar{c}T^{-\beta}$, with a constant $\bar{c} := \bar{c}(\beta)$. For $\beta > \frac{1}{2}$, we obtain similarly that $[u^{\frac{\beta-\frac{1}{2}}{1-\beta}}]/[T^{\frac{\beta}{2(1-\beta)}}] \leq T^{-\beta}[\frac{u}{T^\beta}]^{\frac{\beta-\frac{1}{2}}{1-\beta}}$. The last contribution can be absorbed by the exponential so that we are back to the previous case. For $u \in [(c_0^{-1}T)^\beta, (c_0T)^\beta]$, the statement follows from (4.9) and the positivity of $p_{S^{\beta+}}(1,\cdot)$ on compact sets.

From the control $p_{Z^\beta}(T,u) \le \theta(T,u)$ and the relation (3.3), we indeed derive that for all $i \in \mathbb{N}^*$, $\mathbb{P}[Z_T^{\beta,h} = t_i] \le C_\beta h\theta(T,t_i)$ for some $C_\beta \ge 1$. □

For the rest of the analysis we will need some controls on the p.d.f. of $X_t$ and their time derivatives in both the diffusive and strictly stable case. Such results can be found in Friedman [12] under $(\mathbf{A}_D)$ or [17] under $(\mathbf{A}_S)$. The discrete counterpart for the Euler discretization schemes $X_{t_i}^{\mathrm{Eul},h}$ are explicitly given in Theorem 2.1 of [24] under $(\mathbf{A}_{D,\mathrm{Eul}})$ and would follow from [22] under $(\mathbf{S})$. For the diffusive Markov Chain, the controls can be derived from Lemmas 3.6 and 3.11 in [19].

All those controls are derived from the parametrix expansion of the densities. Some points about these technique are recalled in the Appendix A for the reader's convenience. Let us emphasize that, for our analysis, we need to specify explicitly the dependence of the constants in time since we are led to integrate over arbitrarily large time intervals in (2.12).

**Lemma 4.10** (*Non Degenerate Heat Kernel Bounds*)**.**

– ***Diffusive case.*** *Under Assumption $(\mathbf{A}_D)$ (diffusive case), there exists $(c,C) := (c,C)((\mathbf{A}_D),d) \ge 1$ s.t. introducing for all $(u,x,y) \in \mathbb{R}_+^* \times (\mathbb{R}^d)^2$,*

$$g_c(u,x-y) := \frac{1}{(2\pi cu)^{d/2}} \exp\left(-\frac{|x-y|^2}{2cu}\right),$$

*we have for $\gamma \in \{0,1\}$:*

$$\begin{aligned} &|\partial_u^\gamma p(u,x,y)| \le \frac{C}{u^\gamma} \exp(Cu^{1/2}) g_c(u,x-y), \\ &p^{\mathrm{Eul},h}(t_i,x,y) \le C\exp(Ct_i^{1/2}) g_c(t_i,x-y), \quad i \in \mathbb{N}^*. \end{aligned} \tag{4.10}$$

*Under $(\mathbf{A}_{D,m})$ there exists $c := c((\mathbf{A}_{D,m}),d) \ge 1$ s.t. for all $i \in [\![1,i_C]\!]$, $(x,y) \in (\mathbb{R}^d)^2$:*

$$p^h(t_i,x,y) \le \frac{c\exp(ct_i^{1/2})}{t_i^{d/2}} \left(1 + \frac{|x-y|}{t_i^{1/2}}\right)^{-2(m-1)}. \tag{4.11}$$

– ***Strictly stable case.***

*Under Assumption $(\mathbf{A}_S)$ (strictly stable case), there exists $C := C((\mathbf{A}_S),d) \ge 1$, $\omega := \frac{1}{\alpha} \wedge 1$ s.t. setting for all $(u,x,y) \in \mathbb{R}_+^* \times (\mathbb{R}^d)^2$,*

$$p_S(u,x-y) = \frac{c_\alpha}{u^{d/\alpha}} \left(1 + \frac{|x-y|}{u^{1/\alpha}}\right)^{-(d+\alpha)}$$

*where the constant $c_\alpha$ is chosen to have $\int_{\mathbb{R}^d} p_S(u,z)dz = 1$, we obtain for $\gamma \in \{0,1\}$:*

$$\begin{aligned} &|\partial_u^\gamma p(u,x,y)| \le \frac{C}{u^\gamma} \exp(Cu^\omega) p_S(u,x-y), \\ &p^{\mathrm{Eul},h}(t_i,x,y) \le C\exp(Ct_i^\omega) p_S(t_i,x-y), \quad i \in \mathbb{N}^*. \end{aligned} \tag{4.12}$$

*Observe that the above controls are exponentially explosive in time.*

We now give a fundamental lemma for the error analysis. The quantities below derive from the time sensitivities or the previous error bounds from [19,20] used to control the terms $\mathcal{E}_1, \mathcal{E}_2$ in (4.1) respectively.

**Lemma 4.11** (*Fundamental Bounds*)**.** *Let $T > 0$ be given. The following controls hold:*

– ***Diffusive case***: *There exists* $\bar c := \bar c((\mathbf{A}_D))$ *s.t. for* $\gamma \in \{0, \frac{1}{2}, 1\}$ *and for all* $(x, y) \in (\mathbb{R}^d)^2,\ x \neq y$:

$$\begin{aligned}&\int_0^{+\infty} u^{-\gamma} p_{Z^\beta}(T, u) g_c(u, x-y)\exp(Cu^{1/2})du \\ &\quad + \sum_{i\geq 1} t_i^{-\gamma}\mathbb{P}[Z_T^{\beta,h} = t_i] g_c(t_i, x-y)\exp(Ct_i^{1/2}) \\ &\leq \bar c\Bigg\{\Bigg(\frac{\mathbb{I}_{d\geq 3}}{|x-y|^{2\gamma}} + \frac{\mathbb{I}_{d=2}}{|x-y|^{\mathbb{I}_{\gamma=1}} T^{\frac{\beta}{2}\mathbb{I}_{\gamma>0}}} + \frac{\mathbb{I}_{d=1}}{|x-y|^{\mathbb{I}_{\gamma>0}} T^{\frac{\beta}{2}\mathbb{I}_{\gamma=1}}}\Bigg)\hat p_\beta(T, x-y) \\ &\quad + \frac{1}{T^{\beta\gamma}}\tilde p_\beta(T, x-y)\Bigg\}. \end{aligned} \tag{4.13}$$

*For the Markov Chain approximation, for a given* $l \in \mathbb{N}^*, l \geq 2d+1$ *there exists* $c := c(l, \beta)$ *s.t. denoting for all* $(x, y) \in (\mathbb{R}^d)^2$,

$$q_l(t_i, x-y) := \frac{c_l}{t_i^{d/2}}\left(1 + \frac{|x-y|}{t_i^{1/2}}\right)^{-l},$$

*where* $c_l$ *is s.t.* $\int_{\mathbb{R}^d} q_l(t_i, z)dz = 1$, *we have for all* $\gamma \in \{0, 1/2\}$ *and* $x \neq y$:

$$\begin{aligned}&\sum_{i\geq 1} q_l(t_i, x-y) t_i^{-\gamma}\exp(Ct_i^{1/2})\mathbb{P}[Z_T^{\beta,h} = t_i] \\ &\leq c\Bigg\{\Bigg\{\frac{\mathbb{I}_{d=1} + \mathbb{I}_{d\geq 3}}{|x-y|^{2\gamma}} + \frac{\mathbb{I}_{d=2}}{T^{\frac{\beta}{2}\mathbb{I}_{\gamma>0}}}\Bigg\}\hat q_{l-[d+2(\gamma-1)+\mathbb{I}_{d=1}+\mathbb{I}_{d=2,\gamma=0}],\beta}(T, x-y) \\ &\quad + \frac{1}{T^{\beta\gamma}}\tilde q_{l,\beta}(T, x-y)\Bigg\}. \end{aligned} \tag{4.14}$$

*Note that, for* $l = m$ *and* $\gamma = 1/2$, *the above contribution corresponds to the term* $\mathcal{E}^M_{\beta,\mathrm{Space,LLT}}(T, x-y)$ *in Theorem 3.1.*

– ***Strictly stable case***: *There exists* $\bar c := \bar c((\mathbf{A}_S))$ *s.t. for* $\gamma \in \{0, 1\}$ *and for all* $(x, y) \in (\mathbb{R}^d)^2,\ x \neq y$:

$$\begin{aligned}&\int_0^{+\infty} u^{-\gamma} p_{Z^\beta}(T, u) p_S(u, x-y)\exp(Cu^{\omega})du \\ &\quad + \sum_{i\geq 1} t_i^{-\gamma}\mathbb{P}[Z_T^{\beta,h} = t_i] p_S(t_i, x-y)\exp(Ct_i^{\omega}) \\ &\leq \bar c\Bigg\{\Bigg(\frac{\mathbb{I}_{|x-y|\leq T^{\beta/\alpha}}}{|x-y|^{\gamma\alpha}} + \frac{\mathbb{I}_{|x-y|>T^{\beta/\alpha}}}{T^{\beta\gamma}}\Bigg)\hat p_\beta(T, x-y) + \frac{1}{T^{\beta\gamma}}\tilde p_\beta(T, x-y)\Bigg\}. \end{aligned} \tag{4.15}$$

**Remark 4.12.** Observe that the controls (4.13), (4.14) remain valid if $d = 1, \gamma = 0$ for all $(x, y) \in \mathbb{R}^2$. Also, (4.15) still holds in this case if $\alpha > 1$.

**Proof of Propositions 3.2 and 3.3.** Now, from the independence of $Z_T^\beta$ and $(X_t)_{t\geq 0}$, using a simple conditioning argument, we obtain as a direct corollary of Lemmas 4.10, 4.11 and Remark 4.12, taking $\gamma = 0$ in (4.13), (4.15), the existence of the density for the random variables $X_{Z_T^\beta}$ in both the diffusive and strictly stable case as well as the indicated associated controls. The arguments can be reproduced for $X^{\mathrm{Eul},h}_{Z^{\beta}_t}$, since $Z^{\beta,h}_T$ and $(X^{\mathrm{Eul},h}_{t_i})_{i\in\mathbb{N}}$ are still independent and the Euler schemes enjoy for fixed times similar bounds than the initial process, see Eqs. (4.10) and (4.12). This completes the proof of Propositions 3.2 and 3.3 for the Euler schemes. For the Markov Chain approximation, the control (3.7) is derived similarly from (4.11), (4.14) taking $l = 2(m-1)$, $\gamma = 0$ and Remark 4.12. □

**Remark 4.13.** In the above bounds, from the definitions of $\hat{p}_\beta$, $\tilde{p}_\beta$, we have in some sense the *usual scaling* in time for the Gaussian and stable heat-kernels at time $T^\beta$. We indeed observe that the dichotomy between the diagonal and off-diagonal regime is performed depending on $|x-y| \le K(T^\beta)^{1/\alpha}$ or $|x-y| > K(T^\beta)^{1/\alpha}$ for some positive constant $K$. In the diagonal regime we also have, as indicated previously, an additional spatial singularity whereas in the off-diagonal one, the concentration is modified in the diffusive case due to the integration of the density $p_{Z^\beta}(T,u)$ for large $u$ in (2.12) (resp. $\mathbb{P}[Z^{\beta,h}_T = t_i]$ for large $i$ in (3.4)).

In the Gaussian case, these bounds are coherent with those of [11] and our approach can be viewed as an alternative probabilistic strategy to derive estimates for the fundamental solution of fractional in time diffusive PDEs. For the strictly stable case, to the best of our knowledge these results seem to be new. Some lower bounds are derived in Appendix B.

**Proof of Lemma 4.11.** – **Diffusive case**. We focus on the first integral in the l.h.s. of (4.13) since the other term can be handled similarly (Riemann sum approximation). Exploiting the bounds from Lemma 4.8, we split the time integration domain at the characteristic time scale $T^\beta$ and write:

$$\begin{aligned} Q_{\gamma,\beta}(T,x-y) &:= \int_0^{+\infty} u^{-\gamma} p_{Z^\beta}(T,u) g_c(u,x-y)\exp(Cu^{1/2})du \\ &\le \frac{c_\beta \exp(CT^{\beta/2})}{T^\beta}\exp\left(-\frac{c^{-1}}{4}\frac{|x-y|^2}{T^\beta}\right)\int_0^{T^\beta}\frac{du}{u^{\gamma+d/2}}\exp\left(-\frac{c^{-1}}{4}\frac{|x-y|^2}{u}\right) \\ &\quad + \frac{c_\beta}{T^\beta}\int_{T^\beta}^{+\infty}\frac{du}{u^{\gamma+d/2}}\exp(Cu^{1/2})\exp\left(-c^{-1}\frac{|x-y|^2}{2u}\right)\exp\left(-c_\beta^{-1}\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right) \\ &:= Q_{\gamma,\beta,S}(T,x-y) + Q_{\gamma,\beta,L}(T,x-y), \end{aligned} \tag{4.16}$$

where the subscripts $S, L$ respectively denote the contributions in *short* and *long* time. For the contribution $Q_{\gamma,\beta,S}(T,x-y)$, we have to analyze the integral $I(T^\beta,\gamma,|x-y|,d) := \int_0^{T^\beta}\frac{du}{u^{\gamma+d/2}}\exp(-\frac{c^{-1}}{4}\frac{|x-y|^2}{u})$. Observe that except if $d=1, \gamma=0$ there is a singularity in small time in the integral which needs to be compensated by the spatial component $|x-y|\neq 0$ in the exponential. This yields the spatial (diagonal) singularity.

To equilibrate the time singularity we will thoroughly rely on the fact that the density of the hitting time of level $a>0$ for the scalar Brownian motion writes (see e.g. [35]):

$$\forall u>0, \quad f_a(u) := \frac{2a}{(2\pi u^3)^{1/2}}\exp\left(-\frac{a^2}{2u}\right). \tag{4.17}$$

– If $\gamma + d/2 \ge 3/2$ we write for all $|x-y|\neq 0$:

$$\begin{aligned} &I(T^\beta,\gamma,|x-y|,d) \\ &= \int_0^{T^\beta}\frac{du|x-y|}{u^{3/2}}\frac{|x-y|^{d+2\gamma-3}}{u^{(d+2\gamma-3)/2}}\exp\left(-\frac{c^{-1}}{4}\frac{|x-y|^2}{u}\right)\frac{1}{|x-y|^{d+2\gamma-2}} \\ &\le \frac{\tilde{c}}{|x-y|^{d+2(\gamma-1)}}\int_0^{T^\beta}\frac{du|x-y|}{(2\pi u^3)^{1/2}}\exp\left(-\tilde{c}^{-1}\frac{|x-y|^2}{2u}\right) \le \frac{\tilde{c}}{|x-y|^{d+2(\gamma-1)}}. \end{aligned}$$

– If $\gamma + d/2 < 3/2$ which actually happens for $d = 1, \gamma \in \{0, \frac{1}{2}\}$ or $d = 2$ and $\gamma = 0$ we write:

$$
\begin{aligned}
&I(T^\beta, 0, |x-y|, 1) \le cT^{\beta/2},\\
&I(T^\beta, 0, |x-y|, 2) \;=\; I(T^\beta, 1/2, |x-y|, 1)\\
&\qquad\le \frac{T^{\beta/2}}{|x-y|}\int_0^{T^\beta} \frac{du|x-y|}{u^{3/2}} \exp\left(-\frac{c^{-1}}{4}\frac{|x-y|^2}{u}\right)\\
&\qquad\le \frac{\tilde{c}T^{\beta/2}}{|x-y|}.
\end{aligned}
$$

The previous results can be summarized as follows. There exists $c := c((\mathbf{A}_D), \beta)$ s.t. for all $(T, x, y) \in \mathbb{R}_+^* \times (\mathbb{R}^d)^2$, $|x-y| \neq 0$:

$$
\begin{aligned}
Q_{\gamma,\beta,S}(T, x-y) &\le c\exp(cT^{\beta/2})\exp\left(-c^{-1}\frac{|x-y|^2}{2T^\beta}\right)\\
&\quad\times\left\{\frac{\mathbb{I}_{d=1,\gamma=0}}{T^{\beta/2}} + \frac{\mathbb{I}_{d=2,\gamma=0} + \mathbb{I}_{d=1,\gamma=1/2}}{T^{\beta/2}|x-y|} + \frac{\mathbb{I}_{d+2\gamma\ge 3}}{T^\beta|x-y|^{d+2(\gamma-1)}}\right\}.
\end{aligned}
\tag{4.18}
$$

The restriction $|x-y| \neq 0$ is not necessary if $d = 1,\ \gamma = 0$.

On the other hand, from Young's inequality, there exists $\bar{c}_\beta > 0$ s.t. for all $u, \varepsilon > 0$:

$$
u^{1/2} = \left(\varepsilon\frac{u}{T^\beta}\right)^{1/2}(\varepsilon^{-1}T^\beta)^{1/2} \le \bar{c}_\beta\left\{\left(\varepsilon\frac{u}{T^\beta}\right)^{1/(1-\beta)} + \varepsilon^{-1/(1+\beta)}T^{\beta/(1+\beta)}\right\}.
$$

We thus get from (4.16) that, for $\varepsilon$ sufficiently small, there exists $\tilde{c}_\beta$ s.t.:

$$
\begin{aligned}
Q_{\gamma,\beta,L}(T, x-y) &\le \frac{cc_\beta}{T^{\beta(1+\gamma+d/2)}}\exp(\tilde{c}_\beta T^{\beta/(1+\beta)})\\
&\quad\times\int_{T^\beta}^{+\infty} du\exp\left(-c^{-1}\frac{|x-y|^2}{2u}\right)\exp\left(-\frac{c_\beta^{-1}}{2}\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right).
\end{aligned}
$$

Now two cases occur according to the dichotomy indicated in Remark 4.13. If $|x-y| \le T^{\beta/2}$ then setting $\tilde{u} := u/T^\beta$ we get $Q_{\gamma,\beta,L}(T, x-y) \le \frac{\bar{c}_\beta}{T^{\beta(\gamma+d/2)}}\exp(\bar{c}_\beta T^{\beta/(1+\beta)})$. If now $|x-y| > T^{\beta/2}$, we see that the two exponential contributions in the integral are *equivalent* if $\kappa_1\{|x-y|^2T^{\beta/(1-\beta)}\}^{\frac{1-\beta}{2-\beta}} \le u \le \kappa_2\{|x-y|^2T^{\beta/(1-\beta)}\}^{\frac{1-\beta}{2-\beta}},\ 1 < \kappa_1 < \kappa_2 < +\infty$, for which

$$
\begin{aligned}
&\exp\left(-\frac{c_\beta^{-1}}{2}\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right)\\
&\quad\le \exp\left(-\frac{c_\beta^{-1}\kappa_1^{1/(1-\beta)}}{4}\left[\frac{|x-y|^2}{T^\beta}\right]^{1/(2-\beta)}\right)\exp\left(-\frac{c_\beta^{-1}}{4}\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right).
\end{aligned}
$$

On the other hand, for $u \in [T^\beta, \kappa_1\{|x-y|^2T^{\beta/(1-\beta)}\}^{\frac{1-\beta}{2-\beta}}]$ we have:

$$
\exp\left(-c^{-1}\frac{|x-y|^2}{2u}\right) \le \exp\left(-(2c\kappa_1)^{-1}\left[\frac{|x-y|^2}{T^\beta}\right]^{1/(2-\beta)}\right).
$$

For $u \in [\kappa_2\{|x-y|^2T^{\beta/(1-\beta)}\}^{\frac{1-\beta}{2-\beta}}, +\infty)$, it suffices to bound

$$
\exp\left(-\frac{c_\beta^{-1}}{4}\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right) \le \exp\left(-\frac{c_\beta^{-1}}{4}\kappa_2^{1/(1-\beta)}\left(\frac{|x-y|^2}{T^\beta}\right)^{1/(2-\beta)}\right).
$$

Thus, setting as in the diagonal case $\tilde{u} = u/T^\beta$, we derive that there exists $\bar{c}_\beta := \bar{c}_\beta((\mathbf{A}_D), \kappa_1, \kappa_2) \geq 1$, s.t. for all $(T, x, y) \in \mathbb{R}_+^* \times (\mathbb{R}^d)^2$:

$$\begin{aligned} &Q_{\gamma,\beta,L}(T, x-y) \\ &\quad \leq \frac{\bar{c}_\beta}{T^{\beta(\gamma+d/2)}} \exp(\bar{c}_\beta T^{\beta/(1+\beta)}) \exp\left(-\bar{c}_\beta^{-1}\left[\frac{|x-y|^2}{T^\beta}\right]^{1/(2-\beta)}\right). \end{aligned} \tag{4.19}$$

The statement (4.13) follows from (4.16), (4.18) and (4.19).

– **Markov chain.**

Setting $\underline{i} := \lceil T^\beta/h \rceil$, let us write from Lemma 4.8:

$$\begin{aligned} Q_{\gamma,\beta}(T, x-y) &:= \sum_{i\geq 1} \mathbb{P}[Z_T^{\beta,h} = t_i] \exp(C t_i^{1/2}) t_i^{-\gamma} q_l(t_i, x-y) \\ &\leq \frac{c}{T^\beta} \exp(C T^{\beta/2}) \int_h^{t_{\underline{i}}} u^{-(d/2+\gamma)} \left(1 + \frac{|x-y|}{u^{1/2}}\right)^{-l} du \\ &\quad + \frac{c}{T^{\beta(1+(d/2+\gamma))}} \exp(c T^{\beta/(1+\beta)}) \\ &\quad \times \int_{t_{\underline{i}}}^{+\infty} \left(1 + \frac{|x-y|}{u^{1/2}}\right)^{-l} \exp\left(-c_\beta^{-1}\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right) du \\ &:= Q_{\gamma,\beta,S}(T, x-y) + Q_{\gamma,\beta,L}(T, x-y), \end{aligned} \tag{4.20}$$

using as above Young's inequality to get the upper bound and replacing, up to modifications of the constants, the Riemann sums by integrals.

For $Q_{\gamma,\beta,S}$ we proceed as in the derivation of (4.18), depending on $\gamma + d/2 \geq 3/2$ or $\gamma + d/2 < 3/2$. We derive, that for $\gamma + d/2 \geq 3/2$,

$$\begin{aligned} Q_{\gamma,\beta,S}(T, x-y) &\leq \frac{c}{T^\beta} \exp(c T^{\beta/2}) \frac{1}{|x-y|^{d+2\gamma-2}} \\ &\quad \times \int_h^{t_{\underline{i}}} \frac{|x-y|}{u^{3/2}} \left(1 + \frac{|x-y|}{u^{1/2}}\right)^{-l+(d+2\gamma-3)} du \\ &\leq \frac{c}{T^\beta} \exp(c T^{\beta/2}) \frac{1}{|x-y|^{d+2(\gamma-1)}} \left(1 + \frac{|x-y|}{T^{\beta/2}}\right)^{-l+d+2(\gamma-1)}. \end{aligned}$$

Similarly, for $\gamma + d/2 < 3/2$,

$$\begin{aligned} &Q_{\gamma,\beta,S}(T, x-y) \\ &\quad \leq c \exp(c T^{\beta/2}) \left[\frac{\mathbb{I}_{d=1,\gamma=0}}{T^{\beta/2}} \left(1 + \frac{|x-y|}{T^{\beta/2}}\right)^{-l} \right. \\ &\quad \left. + \frac{\mathbb{I}_{d=1,\gamma=\frac{1}{2}} + \mathbb{I}_{d=2,\gamma=0}}{|x-y| T^{\beta/2}} \left(1 + \frac{|x-y|}{T^{\beta/2}}\right)^{-l+1}\right]. \end{aligned}$$

We finally obtain:

$$\begin{aligned} Q_{\gamma,\beta,S}(T, x-y) \leq c \exp(c T^{\beta/2}) \Bigg\{ &\frac{\mathbb{I}_{d=1,\gamma=0}}{T^{\beta/2}} \left(1 + \frac{|x-y|}{T^{\beta/2}}\right)^{-l} \\ &+ \frac{\mathbb{I}_{d=1,\gamma=\frac{1}{2}} + \mathbb{I}_{d=2,\gamma=0}}{|x-y| T^{\beta/2}} \left(1 + \frac{|x-y|}{T^{\beta/2}}\right)^{-l+1} \\ &+ \frac{1}{T^\beta} \frac{\mathbb{I}_{d+2\gamma\geq 3}}{|x-y|^{d+2(\gamma-1)}} \left(1 + \frac{|x-y|}{T^{\beta/2}}\right)^{-l+d+2(\gamma-1)} \Bigg\}. \end{aligned} \tag{4.21}$$

On the other hand if $|x-y| \le T^{\beta/2}$ (diagonal regime) we readily get $Q_{\gamma,\beta,L}(T,x-y) \le \frac{c\exp(cT^{\beta/(1+\beta)})}{T^{\beta(d/2+\gamma)}}$. If now $|x-y| > T^{\beta/2}$, splitting as above for $u \in [t_{\underline{i}}, \kappa_1\{|x-y|^2 T^{\beta/(1-\beta)}\}^{\frac{1-\beta}{2-\beta}}]$ and $u \ge \kappa_1\{|x-y|^2 T^{\beta/(1-\beta)}\}^{\frac{1-\beta}{2-\beta}}$ yields :

$$Q_{\gamma,\beta,L}(T,x-y) \le \frac{c}{T^{\beta(d/2+\gamma)}} \exp(cT^{\beta/(1+\beta)}) \left(1 + \left[\frac{|x-y|}{T^{\beta/2}}\right]^{1/(2-\beta)}\right)^{-l}. \qquad (4.22)$$

Eq. (4.14) follows putting (4.22) and (4.21) in (4.20).

– **Strictly stable case.** We again focus on the first term in the l.h.s. of (4.15). The discrete sum can be handled similarly. Analogously to (4.16), replacing $g_c$ by $p_S$ and the exponent $1/2$ in the exponential by $\omega$, we write from Lemmas 4.8 and 4.10:

$$Q_{\gamma,\beta,S}(T,x-y) \le \frac{c_\beta c_\alpha \exp(CT^{\beta\omega})}{T^\beta} \int_0^{T^\beta} \frac{du}{u^{\gamma+d/\alpha}} \frac{1}{\left(1+\frac{|x-y|}{u^{1/\alpha}}\right)^{d+\alpha}}.$$

Two cases are to be distinguished.

– If $d=1, \gamma=0, \alpha>1$ then the above diagonal singularity is integrable and one gets:

$$\begin{aligned} Q_{0,\beta,S}(T,x-y) &\le \frac{c_\beta c_\alpha \exp(CT^{\beta\omega})}{T^\beta} \frac{1}{\left(1+\frac{|x-y|}{T^{\beta/\alpha}}\right)^{1+\alpha}} \int_0^{T^\beta} \frac{du}{u^{1/\alpha}} \\ &\le \frac{cc_\alpha \exp(cT^{\beta\omega})}{T^{\beta/\alpha}} \frac{1}{\left(1+\frac{|x-y|}{T^{\beta/\alpha}}\right)^{1+\alpha}}. \end{aligned} \qquad (4.23)$$

In particular, this estimate holds even for $x=y$. There is no spatial singularity.

– For all the other cases, i.e. $d=1, \gamma=1, \alpha>1$ or $d=1, \alpha\le 1, d\ge 2$ for $\gamma\in\{0,1\}$, we need to consider $x\neq y$ in order to equilibrate the time singularity. In small time we still distinguish the diagonal and off-diagonal regimes. Precisely,

– If $|x-y| \le T^{\beta/\alpha}$, then:

$$\begin{aligned} Q_{\gamma,\beta,S}(T,x-y) &\le \frac{c_\beta c_\alpha \exp(CT^{\beta\omega})}{T^\beta} \left\{ \frac{1}{|x-y|^{d+\alpha}} \int_0^{|x-y|^\alpha} du u^{1-\gamma} \right. \\ &\quad \left. + \int_{|x-y|^\alpha}^{T^\beta} \frac{du}{u^{\gamma+d/\alpha}} \right\} \\ &\le c_\beta c \exp(cT^{\beta\omega}) \left\{ \frac{1}{T^\beta |x-y|^{d-\alpha(1-\gamma)}} + \frac{1}{T^{\beta(\gamma+d/\alpha)}} \right\}. \end{aligned} \qquad (4.24)$$

– If $|x-y| > T^{\beta/\alpha}$

$$\begin{aligned} Q_{\gamma,\beta,S}(T,x-y) &\le \frac{c_\beta c_\alpha \exp(CT^{\beta\omega})}{T^\beta} \frac{1}{|x-y|^{d+\alpha}} \int_0^{T^\beta} du u^{1-\gamma} \\ &\le c_\beta c \exp(cT^{\beta\omega}) \frac{T^{\beta(1-\gamma)}}{|x-y|^{d+\alpha}}. \end{aligned} \qquad (4.25)$$

Let us mention that, in the diagonal regime, we get as in the diffusive case an additional spatial singularity.

Let us now deal with $Q_{\gamma,\beta,L}(T,x-y)$. We first write from Young's inequality that there exists $\bar c_\beta$ s.t. for all $u, \varepsilon > 0$:

$$u^\omega = \left(\varepsilon \frac{u}{T^\beta}\right)^\omega (\varepsilon^{-1}T^\beta)^\omega \le \bar c_\beta \left\{ \left(\varepsilon \frac{u}{T^\beta}\right)^{1/(1-\beta)} + (\varepsilon^{-1}T^\beta)^{\omega/(1-\omega(1-\beta))} \right\}.$$

Thus, taking $\varepsilon$ small enough yields:

$$
\begin{aligned}
Q_{\gamma,\beta,L}(T,x-y) &\le c\frac{\exp(cT^{\beta\omega/(1-\omega(1-\beta))})}{T^{\beta}}\int_{T^\beta}^{+\infty} du\left(\frac{u}{|x-y|^{d+\alpha}}\wedge u^{-d\beta/\alpha}\right)u^{-\gamma}\\
&\quad\times\exp\left(-\frac{c_\beta^{-1}}{2}\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right)\\
&\le c\exp(cT^{\beta\omega/(1-\omega(1-\beta))})\left(\frac{T^\beta}{|x-y|^{d+\alpha}}\wedge T^{-\beta d/\alpha}\right)T^{-\beta\gamma}. \qquad (4.26)
\end{aligned}
$$

The result follows from (4.23)–(4.26). □

### 4.3. Time sensitivity: Proof of Lemma 4.1

The lemma, which gives the control on the time sensitivity term $\mathcal{E}_1(T,x,y)$ in the decomposition error (4.1), is proved using the bounds of Lemma 4.11. We simply write from Lemma 4.10, and Lemma 4.8 for the first time step, that in the diffusive case, there exists $\bar c := \bar c((\mathbf{A}_D))$ s.t. :

$$
\begin{aligned}
|\mathcal{E}_1(T,x,y)| \le \bar c\Bigg\{&\frac{h\exp(Ch^{1/2})}{T^\beta|x-y|^d}\exp\left(-c^{-1}\frac{|x-y|^2}{h}\right)\\
&+h\int_h^{+\infty}p_{Z^\beta}(T,u)u^{-1}\exp(Cu^{1/2})g_c(u,x-y)du\Bigg\},
\end{aligned}
$$

whereas in the strictly stable case, there exists $\bar c := \bar c((\mathbf{A}_S))$ s.t.:

$$
|\mathcal{E}_1(T,x,y)| \le \bar c\left\{\frac{\exp(Ch^\omega)h^2}{T^\beta|x-y|^{d+\alpha}}+h\int_h^{+\infty}p_{Z^\beta}(T,u)u^{-1}\exp(Cu^\omega)p_S(u,x-y)du\right\}.
$$

Recalling that in the strictly stable case we assumed $h^{1/\beta}\le T$ and $h^{1/\alpha}\le|x-y|$, the result then follows from the above controls and equations (4.13), (4.15) in Lemma 4.11.

### 4.4. Sensitivity in space: Proof of Lemmas 4.3 and 4.4

#### 4.4.1. The case of a diffusive spatial motion

The key control for the analysis of the term $\mathcal{E}_2(T,x,y)$ in (4.1) is provided by the following Lemma.

**Lemma 4.14** (*Control for the Spatial Error*). *For our current approximation schemes in (3.1) we have:*

- *Under $(\mathbf{A}_{D,\mathrm{Eul}})$, i.e. for the Euler scheme, there exists $(C,c) := (C,c)((\mathbf{A}_D),d)\ge 1$ s.t. for a given $h>0$ and for all $i\in\mathbb{N}^*$:*

$$
\begin{aligned}
|(p-p^h_{\mathrm{Eul}})(t_i,x,y)| &\le Ch\left\{\frac{1}{t_i^{1/2}}\vee\exp(Ct_i^{1/2})\right\}g_c(t_i,x-y),\\
g_c(t_i,z) &= \frac{1}{(2\pi ct_i)^{d/2}}\exp\left(-\frac{|z|^2}{2ct_i}\right),\quad z\in\mathbb{R}^d,
\end{aligned}
\qquad (4.27)
$$

*standing for the usual d-dimensional Gaussian density with variance $ct_i$.*

– *Under* $(\mathbf{A}_{D,m})$, *i.e. general Markov Chain approximation, for* $\delta < 1/5$ *there exists* $C := C((\mathbf{A}_{D,m}), d, \delta) \geq 1$ *s.t. if* $t_i \geq h^\delta$:

$$\begin{aligned} |(p-p^h)(t_i,x,y)| &\leq Ch^{1/2}\left\{\frac{1}{t_i^{1/2}} \vee \exp(Ct_i^{1/2})\right\} q_m(t_i,x-y),\\ 0 \leq q_m(t_i,x-y) &:= c_m t_i^{-d/2}\left(1+\frac{|x-y|}{t_i^{1/2}}\right)^{-m}. \end{aligned} \tag{4.28}$$

Let us mention that $q_m$ enjoys the same parabolic scaling as $g_c$ but has polynomial decay. We observe the usual convergence rates in $h$ and $h^{1/2}$ respectively for the Euler scheme and the Markov Chain approximation. For general Markov Chains, this is the rate of the Gaussian LLT (see [34,19,21]). For the Euler scheme, the innovations are already Gaussian so that the first term disappears in the previously mentioned LLT, yielding a contribution of order $h$.

The important point of the previous Lemma is that it specifies the behavior of the constants in *short* and *long* time. This allows us to balance, as for the time sensitivity, those constants with the controls on the laws of $Z_T^\beta$, $Z_T^{\beta,h}$ given in Lemma 4.8 thanks to Lemma 4.11.

**Proof.** For the Euler scheme, the short time behavior of the error has already been established in [13, see Theorem 2.3 therein] whereas the exponential bounds can once again be derived from the explicit form of the parametrix series used to investigate the error in [20]. For the sake of completeness, we recall these aspects in Appendix A.1.

For a general Markov Chain approaching the diffusion, i.e. for which the innovations are not necessarily Gaussian, to enter a Gaussian asymptotics specified by the Gaussian LLT, a certain number of time steps is needed. This is why we impose the condition $t_i \geq h^\delta, \delta < 1/5$ which is the one required in [21, see assumption (B2) therein] to establish the Gaussian LLT in short time. The behavior of the constants in large time can also be derived from [19] (similarly to the procedure presented in Appendix A.1). For the control in short time we refer to Remark 1 after Theorem 1 in [21]. Anyhow, for the reader's convenience, we recall in Appendix A.1 the approach developed therein. □

We thus directly get from (4.1) and (4.27) in Lemma 4.14 that there exists $(c, C) := c((\mathbf{A}_D)) \geq 1$ s.t. for all $T > 0$, $(x, y) \in (\mathbb{R}^d)^2$, $x \neq y$:

$$|\mathcal{E}_2^{\mathrm{Eul}}(T,x,y)| \leq Ch\sum_{i\geq 1}\mathbb{P}[Z_T^{\beta,h}=t_i]\exp(Ct_i^{1/2})t_i^{-1/2}g_c(t_i,x-y). \tag{4.29}$$

The result (4.2) for the Euler scheme now follows from Eq. (4.13) in Lemma 4.11.

For the Markov Chain approximation, setting $i_C := \lceil h^{-4/5-\varepsilon}\rceil$ we write using (4.28):

$$\begin{aligned} |\mathcal{E}_2(T,x,y)| &\leq \Bigg\{\sum_{i\in[\![1,i_C]\!]}\mathbb{P}[Z_T^h=t_i]|(p-p^h)(t_i,x,y)|\\ &\quad + C\sqrt{h}\sum_{i>i_C}\mathbb{P}[Z_T^{\beta,h}=t_i]\exp(Ct_i^{1/2})\frac{1}{t_i^{1/2}}q_m(t_i,x-y)\Bigg\}\\ &\leq \{\mathcal{E}_{21}(T,x,y)+c_\varepsilon\sqrt{h}\mathcal{E}^M_{\beta,\mathrm{Space,LLT}}(T,x-y)\}, \end{aligned} \tag{4.30}$$

exploiting Eq. (4.14) in Lemma 4.11 for the last inequality in the previous r.h.s. It therefore remains to control $\mathcal{E}_{21}(T,x,y)$.

For this contribution, we do not have comparison results between the two densities. The technique thus consists in controlling each of the terms. We thus derive from the definition in (4.30) and Eqs. (4.6), (4.10), (4.11) that:

$$
\begin{aligned}
\mathcal{E}_{21}(T,x,y) \le\ & \frac{c}{T^{\beta}} h \exp(cT^{\beta/2}) \\
& \times \left\{ \sum_{i=1}^{i_C} t_i^{-d/2} \left[ \exp\left( -\frac{c^{-1}}{2} \frac{|x-y|^2}{t_i} \right) + \left( 1 + \frac{|x-y|}{t_i^{1/2}} \right)^{-2(m-1)} \right] \right\}.
\end{aligned}
$$

From the arguments in the proof of Lemma 4.11, we can write for all $d \ge 1$:

$$
\begin{aligned}
\mathcal{E}_{21}(T,x,y) \le\ & \frac{c}{T^{\beta}} \exp(cT^{\beta/2}) \frac{t_{i_C}^{\frac{1}{2}\mathbb{I}_{d\le 2}}}{|x-y|^{d-2+\mathbb{I}_{d\le 2}}} \\
& \times \left[ \exp\left( -c \frac{|x-y|^2}{t_{i_C}} \right) + \left( 1 + \frac{|x-y|}{t_{i_C}^{1/2}} \right)^{-2(m-1)+(d-2)+\mathbb{I}_{d\le 2}} \right] \\
\le\ & \mathcal{E}^{M,\varepsilon}_{\beta,\mathrm{Space},\mathrm{NoLLT}}(T, x-y, h),
\end{aligned}
\tag{4.31}
$$

recalling $t_{i_C} = h^{1/5-\varepsilon}$. Putting together estimates (4.31) into (4.30) for the Markov Chain approximation concludes the proof of Lemma 4.3.

*4.4.2. The case of strictly stable driven SDE*

Similarly to the diffusive case, the crucial point is the following Lemma.

**Lemma 4.15.** *Under* $(\mathbf{A}_S)$ *there exists* $C := C((\mathbf{A}_S))$, *s.t. for a given* $t_i := ih$ *we have:*

$$
\begin{aligned}
& |(p - p^h_{\mathrm{Eul}})(t_i, x, y)| \le Ch \left\{ \frac{1}{t_i} \vee \exp(Ct_i^{\omega}) \right\} p_S(t_i, x-y), \\
& p_S(t_i, z) := \frac{c_\alpha}{t_i^{d/\alpha}} \frac{1}{\left( 1 + \frac{|z|}{t_i^{1/\alpha}} \right)^{d+\alpha}}, \quad \forall z \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} p_S(t_i, z) dz = 1.
\end{aligned}
\tag{4.32}
$$

**Proof.** The exponential control is derived similarly to the diffusive case, and comes from the specific form of the parametrix expansion used to analyze the error. The control in small time can also be derived from this representation, similarly to what occurs in [22]. To make this last point clear we give some details in Appendix A. □

Lemma 4.4 now follows from Lemma 4.15 and Eq. (4.15) in Lemma 4.11.

## 5. Perspectives

We did not consider in this work the case of a general Markov Chain approximating stable driven SDEs. To do so the first step would consist in establishing a LLT for sums of i.i.d random variables belonging to the domain of attraction of a stable law. Such results have been thoroughly studied in the scalar case, see e.g. Mitalauskas and Statuljavičjus [30] or the monograph by Christoph and Wolf [10]. We also refer to the PhD. of Squartini [38] and to [31] (Cauchy case), for recent developments. We believe those results can be extended to the multidimensional case and would also *transmit* to a Markov Chain approximation through a continuity technique like the parametrix. Now such limit theorems could also allow to consider more general *fractional like* time derivatives that would be associated with the inverse of inhomogeneous stable subordinators $(S_t^+)_{t\geq 0}$ with generators

$$L_t\phi(s) = \int_{\mathbb{R}_+^*} (\phi(s+u) - \phi(s))g(t,u)\frac{du}{u^{1+\alpha}}.$$

Eventually, an analysis of generalized *fractional like* derivatives involving as well the spatial variable like in Kolokoltsov's work, see e.g. Proposition 4.4 in [18], would require to establish LLT for approximations of the couple $(S_t^+, X_t)$ with generator

$$\begin{aligned}\mathcal{L}_t\phi(s,x) &= \int_{\mathbb{R}_+^*} (\phi(s+u,x) - \phi(s,x))g(t,u,x)\frac{du}{u^{1+\alpha}}\\ &\quad + \int_{\mathbb{R}^d} (\phi(s,x+z) - \phi(s,x))\, f(t,x,z)\frac{dz}{|z|^{d+\alpha}}.\end{aligned}$$

This case is much more delicate in the sense that the complete coupling breaks the independence in (2.13) between the spatial motion and the process associated with the fractional like time derivative. We anyhow believe that the arguments of [18] can be adapted provided the LLT holds. This will concern further research. We eventually mention the work of Becker-Kern et al. [5] concerning some limit theorems in this coupled case for marginals. Again, the process convergence remains an open problem.


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## Appendix A. Derivation of the error bounds in small and large time

We briefly explain in this section the bounds appearing in (4.27), (4.28) and (4.32) that are crucial for our analysis in order to balance the indicated explosive behaviors with the decays of the density of the inverse subordinator, see Lemma 4.8. Actually, the only control that needs to be fully justified is (4.32) in small time. The other bounds in small time are already established in the previously quoted papers. Also, the exponential bounds are in some sense *classical*.

Let us consider the case of the Euler scheme associated with (2.11) first. The crucial point is that the densities $p(t_i, x, \cdot)$, $p^h_{\mathrm{Eul}}(t_i, x, \cdot)$, of respectively $X_{t_i}$ in (2.11) and its Euler scheme $X^h_{t_i}$ in (3.1) starting at $x$, enjoy a *parametrix* expansion, see again [12,26,19,20] for the diffusive case and [17,22] for the stable one. We proceed with the simplest example of a non degenerate diffusion, but the results for the other cases can be derived similarly. Consider the dynamics (2.11) under $(\mathbf{A}_D)$. The p.d.f $p(t, x, \cdot)$ of $X_t^x$ exists for every $t > 0$ and writes:

$$p(t,x,y) = \tilde{p}(t,x,y) + \sum_{i\geq 1} \tilde{p} \otimes H^{(i)}(t,x,y), \tag{A.1}$$

where for all $(t, x, y) \in \mathbb{R}_+^* \times \mathbb{R}^{2d}$, $H(t, x, y) = (L - \tilde{L})\tilde{p}(t, x, y)$ where $\tilde{p}(t, x, y)$ stands for the p.d.f. at point $y$ of the *frozen* Gaussian or stable process $\tilde{X}_t^{x,y} = x + \sigma(y)Y_t$, with generator $\tilde{L}$. We point out here that, since we are considering bounded drifts, it is better to take a zero drift in the parametrix instead of the "usual approach" consisting in choosing $\bar{X}_t^{x,y} = x + b(y)t + \sigma(y)Y_t$ as frozen process. This choice gives the upper bounds of Lemma 4.10 improving the constants in large time. Note that the frozen density is always considered at the freezing point. In (A.1) for two integrable functions $f, g : \mathbb{R}_+^* \times (\mathbb{R}^d)^2 \to \mathbb{R}$, we denote $f \otimes g(t, x, y) := \int_0^t du \int_{\mathbb{R}^d} f(u, x, z) g(t - u, z, y) dz$ and also for all $i \geq 1$, $H^{(i)} := H^{(i-1)} \otimes H$, $f \otimes H^{(0)} = f$.

## A.1. Diffusion case

It is known (see [12,20]) that, under $(\mathbf{A}_D)$, there exist $(c_1, c_2) := (c_1, c_2)((\mathbf{A}_D)) \geq 1$ s.t. for all $i \geq 1, t > 0$,

$$|\tilde{p} \otimes H^{(i)}(t, x, y)| \leq \frac{(c_1 t)^{i/2}}{\Gamma(i - 1/2)} g_{c_2}(t, x - y). \tag{A.2}$$

From Eqs. (A.2) and (A.1), it is easily seen that there exists $(c_1, c_2) := (c_1, c_2)((\mathbf{A}_D))$ s.t.

$$p(t, x, y) \leq c_1 \exp(c_1 t^{1/2}) g_{c_2}(t, x - y). \tag{A.3}$$

This explains the exponential control. This approach is due to McKean and Singer [26]. It can be easily transposed to the approximation schemes. The p.d.f. of Euler approximations enjoy similar properties as the one of the initial SDE (see the quoted references).

The control in small time comes from the explicit form of the error decomposition which is similar to (A.1). Namely, one has (see [20] in the considered case)

$$(p - p_{\mathrm{Eul}}^h)(t_i, x, y) = \frac{h}{2} \left( p \otimes (L^2 - L_*^2) p \right) (t_i, x, y) + h^2 R(t_i, x, y), \tag{A.4}$$

where the remainder term satisfies as well $|R(t_i, x, y)| \leq c_1 \exp(c_1 t_i^{1/2}) t_i^{-1/2} g_{c_2}(t_i, x - y)$. Also, we denote for $i \in \{1, 2\}$, $L_*^i \phi(x) := (L_\xi^i \phi(x))|_{\xi = x}$, $L_\xi \phi(x) = \langle b(\xi), \nabla \phi(x) \rangle + \frac{1}{2}\mathrm{Tr}(a(\xi) D_x^2 \phi(x))$. Observe that $L\phi(x) = L_*\phi(x)$, but more generally the operators do not coincide anymore when iterated. Precisely considering $d = 1$ to alleviate the notations, we get for all $t \in (0, t_i)$:

$$\begin{aligned} (L^2 - L_*^2) p(t, x, y) = {} & \{b(x)b'(x) + \frac{1}{2} a(x) b''(x)\} \partial_x p(t, x, y) \\ & + \left\{ \frac{1}{2} b(x) a'(x) + a(x) b'(x) + \frac{1}{4} a(x) a''(x) \right\} \partial_x^2 p(t, x, y) \\ & + \frac{1}{2} a(x) a'(x) \partial_x^3 p(t, x, y). \end{aligned} \tag{A.5}$$

Now, the controls on the density and its derivatives imply under the smoothness assumptions in $(\mathbf{A}_D)$ similarly to (A.3) that there exists $\bar{c} := \bar{c}((\mathbf{A}_D))$ s.t. for all multi-indexes $\alpha, \beta$, $|\alpha| + |\beta| \leq 3$ for all $t > 0$:

$$|\partial_x^\alpha \partial_y^\beta \partial p(t, x, y)| \leq \frac{\bar{c}}{t^{\frac{|\alpha| + |\beta|}{2}}} \exp(\bar{c} t^{1/2}) g_{c_2}(t, x - y). \tag{A.6}$$

We refer to Friedman [12] for details. Hence, the most *singular* term in (A.5) is the last one. Precisely,

$$|a(x)a'(x)\partial_x^3 p(t,x,y)| \le \frac{c}{t^{3/2}} \exp(\bar c t^{1/2}) g_{c_2}(t, x-y). \tag{A.7}$$

Now, recalling from (A.4) that this control needs to be plugged in the time–space convolution $p \otimes [(L^2 - L_*^2)]p(t_i,x,y) = \int_0^{t_i} ds \int_{\mathbb{R}^d} p(s,x,z)(L^2 - L_*^2)p(t_i - s, z, y)dz$, two cases can occur: if $s \in [0, t_i/2]$ the control in (A.7) is not singular and gives the stated bound of $t_i^{-1/2}$ once integrated in time. On the other hand when $s \in [t_i/2, t_i]$, some integration by parts need to be performed for the singular term yielding a third order spatial derivative in $z$ of $p(s,x,z)$. Thanks to (A.6), which gives $|\partial_z^3 p(s,x,z)| \le \frac{\bar c}{s^{\frac{3}{2}}} \exp(\bar c s^{1/2}) g_{c_2}(s, x-z)$, and the integration in time, we get again a time singularity in $t_i^{-1/2}$ for the main contribution in (A.4). The remainder can be handled similarly (see [20]). This analysis can be performed as well for the Markov Chain approximation, see [21].

### A.2. *Strictly stable case*

The expansion (A.1) also holds under $(\mathbf{A}_S)$, see [17] from which we have

$$|\tilde p \otimes H^{(i)}(t,x,y)| \le \frac{(c_1 t)^{i\omega}}{\Gamma(i-\omega)} p_S(t, x-y), \quad \omega := \left(\frac{1}{\alpha} \wedge 1\right) \in (1/2, 1].$$

This therefore gives the indicated exponential bound $p(t,x,y) \le c\exp(ct^\omega)p_S(t,x-y)$ for $c := c((\mathbf{A}_S))$. Now, let us denote by $f$ the spherical density of the measure $\nu$ (see assumption (**ND**)). We can rewrite:

$$\begin{aligned} L\varphi(x) &= L_*\varphi(x) = \langle b(x), \nabla_x\varphi(x)\rangle \\ &\quad + \int_{\mathbb{R}^d} \{\varphi(x+\sigma(x)z) - \varphi(x) - \langle \nabla_x \varphi(x), \sigma(x)z\rangle \mathbb{I}_{|\sigma(x)z|\le 1}\} f\left(\frac{z}{|z|}\right)\frac{dz}{|z|^{d+\alpha}} \\ &= \langle b(x), \nabla_x\varphi(x)\rangle + \int_{\mathbb{R}^d} \{\varphi(x+z) - \varphi(x) - \langle \nabla_x\varphi(x), z\rangle \mathbb{I}_{|z|\le 1}\}\Theta(x,z)dz, \end{aligned}$$

where we denoted for all $\zeta \in \mathbb{R}^d$,

$$\Theta(\zeta, z) := \frac{f\left(\frac{\sigma^{-1}(\zeta)z}{|\sigma^{-1}(\zeta)z|}\right)}{|\sigma^{-1}(\zeta)z|^{d+\alpha}\det(\sigma(\zeta))}. \tag{A.8}$$

With the notations of the previous section (considering $\alpha < 1$ to avoid the cut-off and alleviate the notations) we get:

$$\begin{aligned} L^2 p(t,x,y) &= \langle b(x), \nabla_x b(x) \nabla_x p(t,x,y)\rangle + \langle b(x), D_x^2 p(t,x,y) b(x)\rangle \\ &\quad + \left\langle b(x), \int_{\mathbb{R}^d} (\nabla_x p(t,x+z,y) - \nabla_x p(t,x,y))\Theta(x,z)dz\right\rangle \\ &\quad + \left\langle b(x), \int_{\mathbb{R}^d} (p(t,x+z,y) - p(t,x,y))\nabla_x \Theta(x,z)dz\right\rangle \\ &\quad + \int_{\mathbb{R}^d}\left\{\left[\int_{\mathbb{R}^d}\{p(t,x+z'+z,y) - p(t,x+z',y)\}\Theta(x+z',z)dz\right]\right. \\ &\quad \left. - \left[\int_{\mathbb{R}^d}\{p(t,x+z,y) - p(t,x,y)\}\Theta(x,z)dz\right]\right\}\Theta(x,z')dz', \end{aligned}$$

$$
\begin{aligned}
L_*^2 p(t,x,y) = \Bigg[ &\langle b(x), D_x^2 p(t,x,y) b(\xi)\rangle \\
&+ \left\langle b(x), \int_{\mathbb{R}^d} (\nabla_x p(t,x+z,y) - \nabla_x p(t,x,y)) \Theta(\xi,z) dz \right\rangle \\
&+ \int_{\mathbb{R}^d} \Bigg\{ \Bigg[ \int_{\mathbb{R}^d} \{p(t,x+z'+z,y) - p(t,x+z',y)\} \Theta(\xi,z) dz \Bigg] \\
&- \Bigg[ \int_{\mathbb{R}^d} \{p(t,x+z,y) - p(t,x,y)\} \Theta(x,z) dz \Bigg] \Bigg\} \Theta(\xi,z') dz' \Bigg] \Bigg|_{\xi=x}.
\end{aligned}
$$

Thus, the difference writes:

$$
\begin{aligned}
(L^2 - L_*^2) p(t,x,y) = &\langle b(x), \nabla_x b(x) \nabla_x p(t,x,y)\rangle \\
&+ \left\langle b(x), \int_{\mathbb{R}^d} (p(t,x+z,y) - p(t,x,y)) \nabla_x \Theta(x,z) dz \right\rangle \\
&+ \int_{\mathbb{R}^d} \Bigg\{ \Bigg[ \int_{\mathbb{R}^d} \{p(t,x+z'+z,y) - p(t,x+z',y)\} \\
&\times \{\Theta(x+z',z) - \Theta(x,z)\} dz \Bigg] \Bigg\} \times \Theta(x,z') dz'.
\end{aligned}
$$

We mention that, even though we assumed $b = 0$ for $\alpha < 1$ we kept the explicit dependence on $b$ in the above computations for the sake of completeness. Let us now focus on the last term in the above equation:

$$
\begin{aligned}
D(t,x,y) := &\int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} (p(t,x+z+z',y) - p(t,x+z',y)) \times \{\Theta(x+z',z) - \Theta(x,z)\} dz \right\} \\
&\times \Theta(x,z') dz',
\end{aligned}
$$

which is the only contribution if $\alpha < 1$ and which would be the most singular for $\alpha \geq 1$. The smoothness and non-degeneracy assumptions (**S**), (**UE**), (**ND**) on the coefficients yield:

$$
\begin{aligned}
|\nabla_x \Theta(x,z)| &\leq \frac{c}{|z|^{d+\alpha}}, \\
|D(t,x,y)| &\leq c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |p(t,x+z+z',y) - p(t,x+z',y)| \\
&\quad \times \frac{\mathbb{I}_{|z'| > \frac{1}{4} t^{1/\alpha}} + |z'| \mathbb{I}_{|z'| \leq \frac{1}{4} t^{1/\alpha}}}{|z|^{d+\alpha}} \times \frac{1}{|z'|^{d+\alpha}} dz' dz,
\end{aligned}
$$

where we have truncated with respect to the characteristic time-scale $t^{1/\alpha}$ (up to a constant) in the variable $z'$. Indeed, if $|z'| \leq \frac{1}{4} t^{1/\alpha}$ we perform a Taylor expansion and exploit the previous bound on $\nabla_x \Theta$ whereas if $|z'| > \frac{1}{4} t^{1/\alpha}$ we simply use the uniform bound of $\Theta$ deriving from its definition in (A.8) and the non degeneracy assumptions. Write now:

$$
\begin{aligned}
|D(t,x,y)| \leq c &\int_{|z'| \leq \frac{1}{4} t^{1/\alpha}} \int_{\mathbb{R}^d} |p(t,x+z+z',y) - p(t,x+z',y)| \\
&\times \frac{|z'|}{|z'|^{d+\alpha}} \times \frac{1}{|z|^{d+\alpha}} dz' dz
\end{aligned}
$$

$$+c\int_{|z'|>\frac14 t^{1/\alpha}}\int_{\mathbb{R}^d}|p(t,x+z+z',y)-p(t,x+z',y)|$$
$$\times\frac{1}{|z'|^{d+\alpha}}\times\frac{1}{|z|^{d+\alpha}}dz'dz:=(D_1+D_2)(t,x,y).$$

Let us treat those two terms separately. Recalling from [17] and the above bounds that $|\nabla_x p(t,x,y)|\le\frac{c\exp(ct^\omega)}{t^{1/\alpha}}p_S(t,x-y)$ we derive:

$$\begin{aligned}D_1(t,x,y)\le{}&\frac{c\exp(ct^\omega)}{t^{1/\alpha}}\int_{|z'|\le\frac14 t^{1/\alpha},|z|\le\frac14 t^{1/\alpha}}p_S(t,x+z'+\theta z-y)|z|\\&\times\frac{|z'|}{|z'|^{d+\alpha}}\times\frac{1}{|z|^{d+\alpha}}dz'dz\\&+\int_{|z'|\le\frac14 t^{1/\alpha},|z|>\frac14 t^{1/\alpha}}|p(t,x+z+z',y)-p(t,x+z',y)|\\&\times\frac{|z'|}{|z'|^{d+\alpha}}\times\frac{1}{|z|^{d+\alpha}}dz'dz:=(D_{11}+D_{12})(t,x,y),\end{aligned}$$

for some $\theta:=\theta(x,z,z',y)\in[0,1]$ in $D_{11}$. In that contribution, since both $z,z'$ are small w.r.t. the characteristic time $t^{1/\alpha}$, we have:

$$\begin{aligned}p_S(t,x+z'+\theta z-y)&\le\frac{c}{t^{d/\alpha}}\frac{1}{\left(1+\frac{|x+z'+\theta z-y|}{t^{1/\alpha}}\right)^{d+\alpha}}\le\frac{c}{t^{d/\alpha}}\frac{1}{\left(\frac12+\frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}}\\&\le c2^{d+\alpha}p_S(t,x-y).\end{aligned}\tag{A.9}$$

Hence:

$$D_{11}(t,x,y)\le\frac{c\exp(ct^\omega)}{t^{1/\alpha}}p_S(t,x-y)\left(\int_{r\le\frac14 t^{1/\alpha}}\frac{dr}{r^\alpha}\right)^2\le\frac{c\exp(ct^\omega)}{t^{2-\frac1\alpha}}p_S(t,x-y).$$

For the contribution $|p(t,x+z+z',y)-p(t,x+z',y)|$ in $D_{12}(t,x,y)$, since $|z|$ can be large, we write directly, recalling as well that $|z'|$ is small and proceeding as in (A.9):

$$\begin{aligned}|p(t,x+z+z',y)-p(t,x+z',y)|&\le|p(t,x+z+z',y)|+|p(t,x+z',y)|\\&\le c(p_S(t,x+z-y)+p_S(t,x-y)).\end{aligned}\tag{A.10}$$

Observe that if $|x-y|\le t^{1/\alpha}$, i.e. the diagonal regime holds for $p_S(t,x-y)$, we can then use the global upper bound $(p_S(t,x+z-y)+p_S(t,x-y))\le\frac{c}{t^{d/\alpha}}$, in the control (A.10) so that

$$\begin{aligned}D_{12}(t,x,y)&\le c\exp(ct^\omega)p_S(t,x-y)\int_{|z'|\le\frac14 t^{1/\alpha}}\frac{dz'|z'|}{|z'|^{d+\alpha}}\int_{|z|>\frac14 t^{1/\alpha}}\frac{dz}{|z|^{d+\alpha}}\\&\le c\exp(ct^\omega)p_S(t,x-y)t^{-2+\frac1\alpha}.\end{aligned}$$

If now $|x-y|>t^{1/\alpha}$, we have for all given $\varepsilon\in(0,1)$, $p_S(t,x+z-y)\le c_\varepsilon p_S(t,x-y)$ if $z\notin B(x-y,\varepsilon|x-y|)$. Indeed, $|x-y+z|\ge\big||x-y|-|z|\big|\ge\varepsilon|x-y|$. On the other hand, if $z\in B(x-y,\varepsilon|x-y|)$, $|z|\ge(1-\varepsilon)|x-y|$ and $|z|^{-(d+\alpha)}\le((1-\varepsilon)|x-y|)^{-(d+\alpha)}\le$

$c_\varepsilon t^{-1} p_S(t, x-y)$, up to a possible modification of $c_\varepsilon$. Thus,

$$\begin{aligned} D_{12}(t,x,y) &\le c\exp(ct^\omega)\Bigg\{ t^{-2+\frac{1}{\alpha}} p_S(t,x-y) \\ &\quad + c_\varepsilon p_S(t,x-y) \int_{(B(x-y,\varepsilon|x-y|))^c \cap \{|z|>\frac14 t^{1/\alpha}\}} \frac{dz}{|z|^{d+\alpha}} \int_{|z'|\le \frac14 t^{1/\alpha}} \frac{|z'|dz'}{|z'|^{d+\alpha}} \\ &\quad + \frac{1}{\{(1-\varepsilon)|x-y|\}^{d+\alpha}} \int_{B(x-y,\varepsilon|x-y|)} p_S(t,x+z-y)dz \int_{|z'|\le\frac14 t^{1/\alpha}} \frac{|z'|dz'}{|z'|^{d+\alpha}} \Bigg\} \\ &\le c\exp(ct^\omega) p_S(t,x-y) t^{-2+\frac{1}{\alpha}}. \end{aligned}$$

This proves that:

$$D_1(t,x,y) \le c\exp(ct^\omega) p_S(t,x-y) t^{-2+\frac{1}{\alpha}}. \tag{A.11}$$

Let us observe that we have some continuity for the singularity w.r.t. the stability index, w.r.t. to the diffusive case, as far as the small jumps are concerned (see Eq. (A.7)). The large jumps deteriorate the singularity. Precisely,

$$|D_2(t,x,y)| \le \int_{|z'|>\frac14 t^{1/\alpha}} \frac{dz'}{|z'|^{d+\alpha}} \times \int_{\mathbb{R}^d} |p(t,x+z'+z,y) - p(t,x+z',y)| \frac{dz}{|z|^{d+\alpha}}.$$

From the previous arguments, splitting again the small and large jumps in the $z$ variable w.r.t. the characteristic time scale $t^{1/\alpha}$, we get:

$$\int_{\mathbb{R}^d} |p(t,x+z'+z,y) - p(t,x+z',y)| \frac{dz}{|z|^{d+\alpha}} \le \frac{c}{t}\exp(ct^\omega) p_S(t,x+z'-y).$$

This therefore gives following the dichotomy used for the term $D_{12}(t,x,y)$:

$$|D_2(t,x,y)| \le \frac{c}{t^2}\exp(ct^\omega) p_S(t,x-y),$$

which together with (A.11) indeed gives the bound $|D(t,x,y)| \le \frac{c}{t^2}\exp(ct^\omega) p_S(t,x-y)$.

Let us emphasize from [22] that the expansion (A.4) also holds in the strictly stable case. The previous control then gives the result, similarly to the discussion in the previous paragraph if $s \in [0, t_i/2]$. For $s \in [t_i/2, t_i]$ we take the spatial adjoint and the associated contribution can be analyzed similarly. Eventually, this analysis extends to the remainder terms in (A.4).

## Appendix B. Two-sided heat kernel estimates

We derive in this section two-sided estimates for the density of $X_{Z_T^\beta}$. Precisely we have the following theorem.

**Theorem B.1** (*Two-Sided Heat Kernel Bounds*)**.**

– ***Diffusive case:*** *Under* $(\mathbf{A}_D)$*, there exists* $c := c(\beta, (\mathbf{A}_D)) \ge 1$ *s.t. for a given* $T > 0$ *and for all* $(x,y) \in (\mathbb{R}^d)^2$*, one has:*

- *For $d = 1$:*

$$\frac{c^{-1}}{T^{\beta/2}}\left\{\exp(-cT^{\beta})\exp\left(-c\frac{|x-y|^2}{T^{\beta}}\right)+\exp(-cT)\exp\left(-c\left\{\frac{|x-y|^2}{T^{\beta}}\right\}^{\frac{1}{2-\beta}}\right)\right\}$$
$$\le p_{X_{Z_T^{\beta}}}(x,y)$$
$$\le \frac{c}{T^{\beta/2}}\left\{\exp(cT^{\frac{\beta}{2}})\exp\left(-c^{-1}\frac{|x-y|^2}{T^{\beta}}\right)\right.$$
$$\left.+\exp(cT^{\frac{\beta}{1+\beta}})\exp\left(-c^{-1}\left\{\frac{|x-y|^2}{T^{\beta}}\right\}^{\frac{1}{2-\beta}}\right)\right\}. \tag{B.1}$$

- *For $d = 2$ and $x \neq y$:*

$$\frac{c^{-1}}{T^{\beta}}\left\{\exp(-cT^{\beta})\exp\left(-c\frac{|x-y|^2}{T^{\beta}}\right)\left(\left|\log\left(c^{1/2}\frac{|x-y|}{T^{\beta/2}}\right)\right|\mathbb{I}_{|x-y|\le c^{-1/2}T^{\beta/2}}+1\right)\right.$$
$$\left.+\exp(-cT)\exp\left(-c\left\{\frac{|x-y|^2}{T^{\beta}}\right\}^{\frac{1}{2-\beta}}\right)\right\}$$
$$\le p_{X_{Z_T^{\beta}}}(x,y)$$
$$\le \frac{c}{T^{\beta}}\left\{\exp(cT^{\frac{\beta}{2}})\exp\left(-c^{-1}\frac{|x-y|^2}{T^{\beta}}\right)\left(\left|\log\left(c^{-1/2}\frac{|x-y|}{T^{\beta/2}}\right)\right|\mathbb{I}_{|x-y|\le c^{1/2}T^{\beta/2}}+1\right)\right.$$
$$\left.+\exp(cT^{\frac{\beta}{1+\beta}})\exp\left(-c^{-1}\left\{\frac{|x-y|^2}{T^{\beta}}\right\}^{\frac{1}{2-\beta}}\right)\right\}. \tag{B.2}$$

- *For $d \ge 3$ and $x \neq y$:*

$$c^{-1}\left\{\frac{\exp(-cT^{\beta})}{T^{\beta}|x-y|^{d-2}}\exp\left(-c\frac{|x-y|^2}{T^{\beta}}\right)+\frac{\exp(-cT)}{T^{\beta d/2}}\exp\left(-c\left\{\frac{|x-y|^2}{T^{\beta}}\right\}^{\frac{1}{2-\beta}}\right)\right\}$$
$$\le p_{X_{Z_T^{\beta}}}(x,y)$$
$$\le c\left\{\frac{\exp(cT^{\frac{\beta}{2}})}{T^{\beta}|x-y|^{d-2}}\exp\left(-c^{-1}\frac{|x-y|^2}{T^{\beta}}\right)\right.$$
$$\left.+\frac{\exp(cT^{\frac{\beta}{1+\beta}})}{T^{\beta d/2}}\exp\left(-c^{-1}\left\{\frac{|x-y|^2}{T^{\beta}}\right\}^{\frac{1}{2-\beta}}\right)\right\}. \tag{B.3}$$

– ***Strictly stable case:*** *Under $(\mathbf{A}_S)$, there exists $c := c(\beta, (\mathbf{A}_S)) \ge 1$ s.t. for $d \ge 2$ and $\alpha \in (0, 2)$ or $d = 1$, $\alpha \le 1$, a given $T > 0$ and for all $(x, y) \in (\mathbb{R}^d)^2$, $x \neq y$, one has:*

$$c^{-1}\left\{\frac{\exp(-cT^{\beta})}{T^{\beta}|x-y|^{d-\alpha}}\mathbb{I}_{|x-y|\le T^{\beta/\alpha}}+\frac{1}{T^{\beta d/\alpha}}\frac{\exp(-cT^{\beta})}{\left(1+\frac{|x-y|}{T^{\beta/\alpha}}\right)^{d+\alpha}}\right\}\le p_{X_{Z_T^{\beta}}}(x,y)$$
$$\le c\left\{\frac{\exp(cT^{\beta\omega})}{T^{\beta}|x-y|^{d-\alpha}}\mathbb{I}_{|x-y|\le T^{\beta/\alpha}}+\frac{1}{T^{\beta d/\alpha}}\frac{\exp(cT^{\frac{\beta\omega}{1-\omega(1-\beta)}})}{\left(1+\frac{|x-y|}{T^{\beta/\alpha}}\right)^{d+\alpha}}\right\},\quad \omega := \frac{1}{\alpha}\wedge 1. \tag{B.4}$$

*If* $d=1, \alpha>1$ *we get for all* $(x,y)\in\mathbb{R}^2$*:*

$$c^{-1}\left\{\frac{1}{T^{\beta d/\alpha}}\frac{\exp(-cT^{\beta})}{\left(1+\frac{|x-y|}{T^{\beta/\alpha}}\right)^{d+\alpha}}\right\}$$

$$\le p_{X_{Z_T^{\beta}}}(x,y)\le c\left\{\frac{1}{T^{\beta d/\alpha}}\frac{\exp(cT^{\frac{\beta\omega}{1-\omega(1-\beta)}})}{\left(1+\frac{|x-y|}{T^{\beta/\alpha}}\right)^{d+\alpha}}\right\},\quad \omega:=\frac{1}{\alpha}. \tag{B.5}$$

**Proof.** We first mention that the arguments in the proof of Lemma 4.8 also give that, a lower bound homogeneous to (4.6) holds for $p_{Z^\beta}(T,u)$. Precisely:

$$\frac{c_\beta^{-1}}{T^\beta}\exp\left(-c_\beta\left\{\frac{u}{T^\beta}\right\}^{1/(1-\beta)}\right)\le p_{Z^\beta}(T,u). \tag{B.6}$$

– **Diffusive case.** Let us concentrate on the lower bounds, since the upper bounds are already derived in Proposition 3.2, and on the two-sided bounds for the two dimensional case. Note that we obtain in Eqs. (B.2), (B.3), for dimensions $d\ge 2$, the same singularities that we observe for the Poisson kernel of the associated dimension.

We focus below on the lower bound for $d\ge 3$ and the two-sided bounds for $d=2$. The other controls can be derived from arguments similar to those developed below. One of the key points for the proof is the lower bound for the density of the spatial motion. Namely a lower bound homogeneous to the one in Lemma 4.10 holds for the density of the spatial motion. It is classical, from chaining arguments, see e.g. Chapter 7 in Bass [4], to derive from the parametrix expansions presented above that there exists $c:=c((\mathbf{A}_D))\ge 1$ s.t. for all $(x,y)\in(\mathbb{R}^d)^2$:

$$p(t,x,y)\ge c^{-1}\frac{\exp(-ct)}{t^{d/2}}\exp\left(-c\frac{|x-y|^2}{t}\right). \tag{B.7}$$

*Lower bound for* $d\ge 3$. We now start from (2.12), (B.6) and (B.7) to derive:

$$\begin{aligned}p_{X_{Z_T^{\beta}}}(x,y)&\ge\frac{c^{-1}}{T^\beta}\left\{\exp(-cT^\beta)\int_0^{T^\beta}\frac{du}{u^{d/2}}\exp\left(-c\frac{|x-y|^2}{u}\right)\right.\\&\left.\quad+\int_{T^\beta}^{+\infty}\frac{du}{u^{d/2}}\exp\left(-c\frac{|x-y|^2}{u}\right)\exp(-cu)\exp\left(-c\left[\frac{u}{T^\beta}\right]^{1/(1-\beta)}\right)\right\}\\&:=(m_1+m_2)(T^\beta,x-y).\end{aligned} \tag{B.8}$$

To control $m_1(T^\beta,x-y)$ we consider again the previous, diagonal/off-diagonal dichotomy:

– For $|x-y|/T^{\beta/2}\le 1$ (diagonal regime) write:

$$\begin{aligned}m_1(T^\beta,x-y)&\ge\frac{c^{-1}\exp(-cT^\beta)}{T^\beta}\int_{|x-y|^2/2}^{|x-y|^2}\frac{du}{u^{d/2}}\\&\ge\frac{c^{-1}\exp(-cT^\beta)}{T^\beta}\int_{|x-y|^2/2}^{|x-y|^2}\frac{du}{|x-y|^d}\ge\frac{c^{-1}\exp(-cT^\beta)}{T^\beta|x-y|^{d-2}}.\end{aligned}$$

– For $|x-y|/T^{\beta/2} > 1$ (off-diagonal regime) write:

$$m_1(T^\beta, x-y) \geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta |x-y|^{d-2}} \int_{T^\beta/2}^{T^\beta} \frac{du}{u} \left(\frac{|x-y|}{u^{1/2}}\right)^{d-2} \exp\left(-c\frac{|x-y|^2}{u}\right)$$
$$\geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta |x-y|^{d-2}} \exp\left(-c\frac{|x-y|^2}{T^\beta}\right) \int_{T^\beta/2}^{T^\beta} \frac{du}{u}$$
$$\geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta |x-y|^{d-2}} \exp\left(-c\frac{|x-y|^2}{T^\beta}\right).$$

We have thus established that:

$$m_1(T^\beta, x-y) \geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta |x-y|^{d-2}} \exp\left(-c\frac{|x-y|^2}{T^\beta}\right). \tag{B.9}$$

On the other hand for the contribution $m_2(T^\beta, x-y)$ in (B.8) we get:

– For $|x-y|/T^{\beta/2} \leq 1$ (diagonal regime) write:

$$m_2(T^\beta, x-y) \geq \frac{c^{-1}\exp(-cT^\beta)}{T^{\beta(1+d/2)}} \int_{T^\beta}^{2T^\beta} du \exp\left(-c\left[\frac{u}{T^\beta}\right]^{1/(1-\beta)}\right)$$
$$\geq \frac{c^{-1}\exp(-cT^\beta)}{T^{\beta d/2}}.$$

– For $|x-y|/T^{\beta/2} > 1$ (off-diagonal regime), write first from Young's inequality,

$$u \leq c_\beta\Big[\Big(\frac{u}{T^\beta}\Big)^{1/(1-\beta)} + (T^\beta)^{1/\beta}\Big].$$

Computations similar to those in Lemma 4.11 then yield:

$$m_2(T^\beta, x-y) \geq \frac{c^{-1}\exp(-cT)}{T^\beta}$$
$$\times \int_{[|x-y|^2 T^{\frac{\beta}{1-\beta}}]^{\frac{1-\beta}{2-\beta}}}^{2[|x-y|^2 T^{\frac{\beta}{1-\beta}}]^{\frac{1-\beta}{2-\beta}}} \frac{du}{u^{d/2}} \exp\left(-c\frac{|x-y|^2}{u}\right) \exp\left(-c\left[\frac{u}{T^\beta}\right]^{1/(1-\beta)}\right)$$
$$\geq \frac{c^{-1}\exp(-cT)}{T^\beta} \exp\left(-c\left[\frac{|x-y|^2}{T^\beta}\right]^{\frac{1}{2-\beta}}\right) \frac{1}{[|x-y|^2 T^{\frac{\beta}{1-\beta}}]^{\frac{1-\beta}{2-\beta}(d/2-1)}}$$
$$\geq \frac{c^{-1}\exp(-cT)}{T^\beta} \exp\left(-c\left[\frac{|x-y|^2}{T^\beta}\right]^{\frac{1}{2-\beta}}\right) \frac{1}{[T^{\beta(1+\frac{1}{1-\beta})}]^{\frac{1-\beta}{2-\beta}(d/2-1)}}$$
$$\geq \frac{c^{-1}\exp(-cT)}{T^{\beta d/2}} \exp\left(-c\left[\frac{|x-y|^2}{T^\beta}\right]^{\frac{1}{2-\beta}}\right),$$

recalling that $\forall z \geq 1,\ \exp(-z^{1/(2-\beta)}) z^{-\frac{1-\beta}{2-\beta}(d/2-1)} \geq c^{-1}\exp(-cz^{1/(2-\beta)})$ for the last but one inequality. We have thus proved in all cases:

$$m_2(T^\beta, x-y) \geq \frac{c^{-1}\exp(-cT)}{T^{\beta d/2}} \exp\left(-c\left[\frac{|x-y|^2}{T^\beta}\right]^{\frac{1}{2-\beta}}\right),$$

which together with (B.9) and (B.8) completes the proof of the lower bound for $d \geq 3$.

*Bounds for $d = 2$.* The lower bound (B.8) and an homogeneous upper bound still hold, with obvious modifications of the constants. We focus on the contribution $m_1(T^\beta, x-y)$ yielding the additional spatial diagonal singularity. The term $m_2(T^\beta, x-y)$ can be analyzed as above. Let us write:

$$m_1(T^\beta, x-y) \geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta}\int_0^{T^\beta}\frac{du}{u}\exp\left(-c\frac{|x-y|^2}{u}\right).$$

Assume first that $c^{1/2}|x-y|/T^{\beta/2} \leq 1$. Setting $v := \exp\left(-c\frac{|x-y|^2}{u}\right)$ in the above integral we derive:

$$\begin{aligned}
m_1(T^\beta, x-y) &\geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta}\int_0^{\exp(-c\frac{|x-y|^2}{T^\beta})}\frac{dv}{-\log(v)}\\
&\geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta}\left\{-v\log(-\log(v))|_0^{\exp(-c\frac{|x-y|^2}{T^\beta})} + \int_0^{\exp(-c\frac{|x-y|^2}{T^\beta})}\log(-\log(v))dv\right\}\\
&\geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta}\Big\{\exp\Big(-c\frac{|x-y|^2}{T^\beta}\Big)\Big(-\log\Big(c\frac{|x-y|^2}{T^\beta}\Big)\Big)\\
&\quad + \int_{e^{-1}}^{\exp(-c\frac{|x-y|^2}{T^\beta})}\log(-\log(v))dv\Big\}\\
&\geq \frac{c^{-1}\exp(-cT^\beta)}{T^\beta}\Big\{\exp\Big(-c\frac{|x-y|^2}{T^\beta}\Big)\Big(-\log\Big(c\frac{|x-y|^2}{T^\beta}\Big)\Big)\\
&\quad + \log\Big(c\frac{|x-y|^2}{T^\beta}\Big)\Big(\exp\Big(-c\frac{|x-y|^2}{T^\beta}\Big) - e^{-1}\Big)\Big\}.
\end{aligned}$$

Hence:

$$\begin{aligned}
m_1(T^\beta, x-y) &\geq \frac{c^{-1}\exp(-cT^\beta)e^{-1}}{T^\beta}\Big(-\log\Big(c\frac{|x-y|^2}{T^\beta}\Big)\Big)\\
&\geq \frac{c^{-1}e^{-1}\exp(-cT^\beta)\exp\left(-c\frac{|x-y|^2}{T^\beta}\right)}{T^\beta}\left|\log\left(c\frac{|x-y|^2}{T^\beta}\right)\right|.
\end{aligned}$$

The upper bound could be derived similarly.

If now $c^{1/2}|x-y|/T^{\beta/2} > 1$, we write:

$$\begin{aligned}
m_1(T^\beta, x-y) &\geq \frac{c^{-1}\exp(-cT^\beta)\exp\left(-c\frac{|x-y|^2}{T^\beta}\right)}{T^\beta}\int_{T^\beta/2}^{T^\beta}\frac{du}{u}\\
&\geq \frac{c^{-1}\exp(-cT^\beta)\exp\left(-c\frac{|x-y|^2}{T^\beta}\right)}{T^\beta}.
\end{aligned}$$

The previous bounds give the result.

– **Strictly stable case.** We focus here on the lower bound for $d \geq 2$. The other cases can be handled similarly. From the lower bound in Kolokoltsov [17] for the density and a chaining argument one derives that there exists $c := c((\mathbf{A}_S)) \geq 1$ s.t. for all $t > 0$, $(x, y) \in (\mathbb{R}^d)^2$:

$$p(t, x, y) \geq \frac{c^{-1} \exp(-ct)}{t^{d/\alpha}} \frac{1}{\left(1 + \frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}}. \tag{B.10}$$

From (2.12), (B.6) and (B.10) we now get:

$$\begin{aligned} p_{X_{Z_T^\beta}}(x, y) &\geq \frac{c^{-1}}{T^\beta} \left\{ \exp(-cT^\beta) \int_0^{T^\beta} \frac{du}{u^{d/\alpha}} \frac{1}{\left(1 + \frac{|x-y|}{u^{1/\alpha}}\right)^{d+\alpha}} \right. \\ &\quad \left. + \int_{T^\beta}^{+\infty} \frac{du}{u^{d/\alpha}} \frac{1}{\left(1 + \frac{|x-y|}{u^{1/\alpha}}\right)^{d+\alpha}} \exp(-cu) \exp\left(-c\left[\frac{u}{T^\beta}\right]^{1/(1-\beta)}\right) \right\} \\ &:= (m_1 + m_2)(T^\beta, x - y). \end{aligned} \tag{B.11}$$

Let us first control $m_1(T^\beta, x - y)$ exploiting again the diagonal/off-diagonal dichotomy.

– If $|x - y| \leq T^{\beta/\alpha}$ then

$$m_1(T^\beta, x - y) \geq \frac{c^{-1} \exp(-cT^\beta)}{T^\beta} \int_{|x-y|^\alpha/2}^{|x-y|^\alpha} \frac{du}{u^{d/\alpha}} \geq \frac{c^{-1} \exp(-cT^\beta)}{T^\beta |x - y|^{d-\alpha}}.$$

– If $|x - y| > T^{\beta/\alpha}$ we get:

$$m_1(T^\beta, x - y) \geq \frac{c^{-1}}{T^\beta} \exp(-cT^\beta) \int_{T^\beta/2}^{T^\beta} du \frac{u}{|x - y|^{d+\alpha}} \geq \frac{c^{-1} T^\beta}{|x - y|^{d+\alpha}} \exp(-cT^\beta).$$

We have thus established

$$\begin{aligned} &m_1(T^\beta, x - y) \\ &\geq c^{-1} \exp(-cT^\beta) \left\{ \frac{1}{T^\beta |x - y|^{d-\alpha}} \mathbb{I}_{|x-y| \leq T^{\beta/\alpha}} + \frac{T^\beta}{|x - y|^{d+\alpha}} \mathbb{I}_{|x-y| > T^{\beta/\alpha}} \right\}. \end{aligned} \tag{B.12}$$

Let us now turn to $m_2(T^\beta, x - y)$. We get:

– If $|x - y| \leq T^{\beta/\alpha}$,

$$m_2(T^\beta, x - y) \geq \frac{c^{-1}}{T^\beta} \exp(-cT^\beta) \int_{T^\beta}^{2T^\beta} \frac{du}{u^{d/\alpha}} \geq \frac{c^{-1}}{T^{d\beta/\alpha}} \exp(-cT^\beta).$$

– If $|x - y| > T^{\beta/\alpha}$,

$$m_2(T^\beta, x - y) \geq \frac{c^{-1}}{T^\beta |x - y|^{d+\alpha}} \exp(-cT^\beta) \int_{T^\beta}^{2T^\beta} u du \geq \frac{c^{-1} T^\beta}{|x - y|^{d+\alpha}} \exp(-cT^\beta).$$

Plugging the above estimates and (B.12) into (B.11) gives the result. □

**Remark B.1.** Let us emphasize that the bounds of Theorem B.1 would hold under the weaker assumptions that the coefficients $b, \sigma$ are measurable and s.t. $\sigma\sigma^*$ is Hölder continuous and $b$ is bounded. Indeed, the two-sided heat kernel bounds for the spatial motion hold in that case. We can for instance refer to Sheu [37] in the diffusive case or to Huang [15] in the strictly stable case.

Let us point out as well that the two sided bounds hold in the diffusive case under those assumptions for the density of $X^{h,\mathrm{Eul}}_{Z_T^{\beta,h}}$ associated with the Euler scheme approximation for the spatial motion. Again, the key estimate is a two-sided bound for the Euler scheme which can be found in Lemaire and Menozzi [24]. In the strictly stable case, the upper bound holds under the indicated assumptions. This is a consequence of the parametrix expansion for the density of the scheme, see [22]. The lower bound is more delicate to obtain since even in the diffusive case, the localization arguments which are standard for the SDE need to be carefully adapted for the scheme.

**Remark B.2.** We conclude saying that the results in Theorem 3.1 could be slightly improved in light of the sharp estimates of Theorem B.1 for $d = 2$. We did not exploit those controls in the presentation of the main results mainly for notational coherence and simplicity.

## References

[1] B. Baeumer, M.M. Meerschaert, Stochastic solutions for fractional Cauchy problems, Fract. Calc. Appl. Anal. 4 (4) (2001) 481–500.
[2] V. Bally, D. Talay, The law of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function, Probab. Theory Related Fields 104 (1) (1996) 43–60.
[3] V. Bally, D. Talay, The law of the Euler scheme for stochastic differential equations, II. Convergence rate of the density, Monte Carlo Methods Appl. 2 (1996) 93–128.
[4] R.F. Bass, Diffusions and Elliptic Operators, Springer, 1997.
[5] P. Becker Kern, M.M. Meerschaert, H.P. Scheffler, Limit theorems for coupled continuous time random walks, Ann. Probab. 32 (1B) (2004) 730–756.
[6] L. Beghin, E. Orsingher, Fractional diffusion equations and processes with randomly varying time, Ann. Probab. 37 (1) (2009) 206–249.
[7] K. Bichteler, J.B. Gravereaux, J. Jacod, Malliavin Calculus for Processes With Jumps, in: Stochastics Monographs, vol. 2, 1987.
[8] P. Chakraborty, A stochastic differential equation model with jumps for fractional advection and dispersion, J. Stat. Phys. 136 (2009) 527–551.
[9] J.M. Chambers, C.L. Mallows, B.W. Stuck, A method for simulating stable random variables, J. Amer. Statist. Assoc. 71 (354) (1976) 340–344.
[10] G. Christoph, W. Wolf, Convergence Theorems with a Stable Limit Law, in: Mathematical Research, vol. 70, Akademie-Verlag, Berlin, 1992.
[11] S.D. Eĭdelman, A.N. Kochubeĭ, Cauchy problem for fractional diffusion equations, J. Differential Equations 199 (2) (2004) 211–255.
[12] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, 1964.
[13] E. Gobet, C. Labart, Short time asymptotics of the density of the Euler scheme, Electron. Commun. Probab. 23 (2008).
[14] M. Hahn, K. Kobayashi, S. Umarov, Fokker–Planck–Kolmogorov equations associated with time-changed fractional Brownian motion, Proc. Amer. Math. Soc. 139 (2011) 691–705.
[15] L. Huang, Two-sided bounds for SDEs driven by tempered stable processes, 2015. http://arxiv.org/abs/1504.04183.
[16] A.N. Kochubei, Cauchy problem for fractional diffusion-wave equations with variable coefficients, Appl. Anal. 93 (10) (2014) 2211–2242.
[17] V. Kolokoltsov, Symmetric stable laws and stable-like jump diffusions, Proc. Lond. Math. Soc. 80 (2000) 725–768.
[18] V. Kolokoltsov, Generalized Continuous-Time Random Walks (CTRW), Theory Probab. Appl. 53 (4) (2009) 594–609.
[19] V. Konakov, E. Mammen, Local limit theorems for transition densities of Markov chains converging to diffusions, Probab. Theory Related Fields 117 (2000) 551–587.
[20] V. Konakov, E. Mammen, Edgeworth type expansions for Euler schemes for stochastic differential equations, Monte Carlo Methods Appl. 8 (3) (2002) 271–285.
[21] V. Konakov, E. Mammen, Small time Edgeworth-type expansions for weakly convergent nonhomogeneous Markov chains, Probab. Theory Related Fields 143 (1–2) (2009) 137–176.
[22] V. Konakov, S. Menozzi, Weak error for stable driven stochastic differential equations: Expansion of the densities, J. Theoret. Probab. 24 (2) (2010) 554–578.
[23] V. Konakov, S. Menozzi, S. Molchanov, Explicit parametrix and local limit theorems for some degenerate diffusion processes, Ann. Inst. Henri Poincaré, Série B 46 (4) (2010) 908–923.
[24] V. Lemaire, S. Menozzi, On some non asymptotic bounds for the Euler scheme, Electron. J. Probab. 15 (2010) 1645–1681.

[25] M. Magdziarz, A. Weron, K. Weron, Fractional Fokker–Planck dynamics: Stochastic representation and computer simulation, Phys. Rev. E 75 (2007).
[26] H.P. McKean, I.M. Singer, Curvature and the eigenvalues of the Laplacian, J. Differential Geom. 1 (1967) 43–69.
[27] M.M. Meerschaert, H.P. Scheffler, Limit theorems for continuous-time random walks with infinite mean waiting times, J. Appl. Probab. 41 (3) (2004) 623–638.
[28] M.M. Meerschaert, A. Sikorskii, Stochastic Models for Fractional Calculus, in: de Gruyter Studies in Mathematics, vol. 43, Walter de Gruyter & Co., Berlin, 2012.
[29] M.M. Meerschaert, P. Straka, Inverse Stable Subordinators, Math. Model. Nat. Phenom. 8 (2) (2013) 1–16.
[30] A. Mitalauskas, V. Statuljavičjus, An asymptotic expansion in the case of a stable approximating law, Litov. Mat. Sb. 16 (4) (1976) 149–166. 255.
[31] S.A. Molchanov, V.V. Petrov, N. Squartini, Quasicumulants and limit theorems in case of the Cauchy limiting law, Markov Process. Related Fields 13 (3) (2007) 597–624.
[32] J.P. Nolan, B. Rajput, Calculation of multidimensional stable densities, Comm. Statist. Simulation Comput. 24 (3) (1995) 551–566.
[33] D. Nualart, Analysis on Wiener space and anticipating stochastic calculus, in: Lectures on Probability Theory and Statistics, Saint-Flour, 1995, in: LNM, vol. 1690, Springer, Berlin, 1998, pp. 123–227.
[34] V. Petrov, Sum of Independent Random Variables, Cambridge University Press, 2005.
[35] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, third ed., in: Grundlehren der Mathematischen Wissenschaften, 293, Springer, Berlin, 1999.
[36] K. Sato, Lévy Processes and Infinitely divisible Distributions, Cambridge University Press, 2005.
[37] S.J. Sheu, Some estimates of the transition density of a nondegenerate diffusion Markov process, Ann. Probab. 19 (2) (1991) 538–561.
[38] N. Squartini, Global limit theorems for sums of independent identically distributed random variables using quasicumulants. ProQuest LLC, Ann Arbor, MI (Ph.D. thesis), The University of North Carolina at Charlotte, 2007.
[39] P. Straka, B.I. Henry, Lagging and leading coupled continuous time random walks, renewal times and their joint limits, Stochastic Process. Appl. 121 (2) (2011) 324–336.
[40] Y. Zhang, D.A. Benson, M.M. Meerschaert, E.M. Labolle, Hans-Peter Scheffler, Random walk approximation of fractional-order multiscaling anomalous diffusion, Phys. Rev. E 74 (2006).
[41] V.M. Zolotarev, One dimensional Stable Processes, Transl. Amer. Math. Soc. (1992).