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# Secondary Resonances in Penning Traps. Non-Lie Symmetry Algebras and Quantum States

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**Abstract.** The Penning trap Hamiltonian (hyperbolic oscillator in a homogeneous magnetic field) is considered in the basic three-frequency resonance regime. We describe its non-Lie algebra of symmetries. By perturbing the homogeneous magnetic field, we discover that, for special directions of the perturbation, a secondary hyperbolic resonance appears in the trap. For corresponding secondary resonance algebra, we describe its non-Lie permutation relations and irreducible representations realized by ordinary differential operators. Under an additional (Ioffe) inhomogeneous perturbation of the magnetic field, we derive an effective Hamiltonian over the secondary symmetry algebra. In an irreducible representation, this Hamiltonian is a model second-order differential operator. The spectral asymptotics is derived, and an integral formula for the asymptotic eigenstates of the entire perturbed trap Hamiltonian is obtained via coherent states of the secondary symmetry algebra.

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## 1. INTRODUCTION

Penning traps are devices based on the use of the electric field created by a cylinder-like condenser and an axially directed magnetic field in order to hold an electric charge in a compact domain, inside the condenser. The micro- and nano-Penning traps applied, for instance, in fine detectors [1–5] and artificial "atoms" [6] are of special interest. The nano-scale Penning traps already have a visible structure of spectral lines and can be used as controllable quantum devices, say, for quantum computers [7, 8].

Note that, although the Penning traps present one of the fundamental examples of mechanical systems (both classical and quantum), it is not easy to find the mathematical theory of these systems in standard textbooks, see [9–11]. Possibly, the reason is that these systems are hyperbolic rather than elliptic, and therefore, they do not belong to the classical framework of "compact" dynamics and "matrix" perturbation theory. The compact and finite matrix framework is simpler and more stable; for this reason, dynamical systems of elliptic type play a very important role and are studied at the first place. On the other hand, hyperbolic systems have attracted great interest in relativistic mechanics, i.e., in the four-dimensional case. The micro- and nano-technologies now suggest us the two-dimensional case, and this generates an interest in $(2+1)$-hyperbolic systems in 3D-Euclidean space.

The mathematical model of an "ideal" Penning trap is equivalent to a harmonic 3D-oscillator of hyperbolic type (with a saddle point). Three normal frequencies of this oscillator depend on external parameters, namely, on the magnetic field magnitude and the electric voltage on the condenser.

The trapping condition reads

$$\omega > \omega_0, \tag{1.1}$$

where $\omega$ is the magnetic Larmor frequency and $\omega_0$ the frequency of the condenser's quadratic potential in directions transversal to the magnetic field. If one chooses $\omega = \frac{3}{2\sqrt{2}}\omega_0 \approx 1.06\cdot\omega_0$, then the normal frequencies of the trapping 3D-oscillator are in the stable resonance $2 : (-1) : 2$.

In this three-frequency resonance, the spectrum of the ideal Penning trap becomes an arithmetic progression with common difference $\hbar\omega_0/\sqrt{2}$. This equidistance of the spectrum is important, e.g., in quantum computations (quantum arithmetics); note also that the resonance spectral lines are

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well separated from each other (at distance $O(\hbar)$); otherwise, out of resonance, they are disposed much more dense (the distance is of order $O(\hbar^3)$), and therefore less visible and less controllable under small perturbations. Below we consider the Penning trap with this resonance condition.

Now one has to take into account that the 3D-oscillator mentioned above (the ideal trap) is only the leading term of the entire Penning trap Hamiltonian. There are also some perturbations originating from several sources. First, there can be deviations of the homogeneous magnetic field from the axial direction. Second, there are inhomogeneous magnetic corrections, the so-called Ioffe field [12, 13]. Third, there are anharmonic corrections to the electric potential of the condenser, see, e.g., [14].

All these types of perturbations could easily be taken into account if the frequencies of the leading harmonic oscillator were not resonant. However, in our case, they are in the resonance $2 : (-1) : 2$, and a stumbling stone for the application of standard perturbation theory arises, namely, the infinite degeneracy of the spectrum of the hyperbolic resonance oscillator. The “finite matrix” perturbation methods do not effectively work any more.

The spectral degeneracy is a consequence of the fact that the symmetry algebra of the resonance oscillator is not commutative. The noncommutativity implies a nontrivial symplectic geometry. Continuous geometric objects (like symplectic leaves of the symmetry algebra) enable one to model and effectively replace the discrete infinite matrix structure, and integration with respect to a continuous measure can effectively replace the summation of infinite series of matrix elements in perturbation theory. Thus, there is a way to avoid the “stumbling stone” by using novel methods of quantum geometry.

From the geometrical point of view, the occurrence of resonance noncommutativity implies the transformation of the usual Liouville tori into a family of submanifolds, which are generally not isotropic but coisotropic. The geometry of these submanifolds is no longer controlled by the leading oscillator only; the control involves the perturbing part of the entire Penning trap Hamiltonian and the symmetry algebra structure.

Here we meet another interesting mathematical fact: the symmetry algebra of the resonance oscillator is of non-Lie type (except for the trivial resonances all of whose frequencies are mutually equal). This means that the algebra cannot be described by linear commutation relations. Even in the simplest two-frequency resonance, the symmetry algebra is described by nonlinear commutation relations [15, 16] (see also [17], especially for the Penning trap). The case of three-frequency resonance treated in this paper is much richer than the two-frequency case. For a general description of the symmetry algebra in the elliptic three-frequency case, see [18].

The Penning trap presents a highly interesting example of actual physical system in which a hyperbolic three-frequency resonance can occur. Below, we describe the symmetry algebra with non-Lie permutation relations for the resonance $2 : (-1) : 2$ realized in the trap.

The occurrence of non-Lie algebras requires the development of an analog of the entire framework of geometric quantization and representation theory for such algebras. In our work, we apply methods developed in [16, 19, 20].

Along with the algebraic structure, an important role is also played by the averaged perturbing Hamiltonian. This Hamiltonian is obtained using a general algebraic procedure [16, 21] by projecting to the symmetry algebra. This Hamiltonian can be viewed as a system with reduced number of degrees of freedom, i.e., 2 instead of 3.

For the main (or primary) perturbing Hamiltonian, we consider the deviation of the homogeneous magnetic field from its resonance value and the axial direction. The Hamiltonian of this perturbation is very simple: it is just a quadratic form in canonical phase variables. The averaged system is certainly integrable in this case. Thus, at first glance, this perturbation is trivial. However, there is an interesting effect we have observed. The projection of the perturbing Hamiltonian to the (primary) symmetry algebra can again have a resonance regime with its own noncommutative symmetry algebra.

Thus, a secondary nonobvious resonance and a secondary symmetry algebra arise. This algebra is again of non-Lie type. Below we specify its generators, commutation relations, and irreducible representations.

Thus, in the Penning trap, we obtain a double resonance system, i.e., the leading resonance oscillator + the primary perturbing quadratic Hamiltonian in a resonance regime. The classical trajectories of this system are presented as a combination of two cyclic rotations, a fast (by the leading oscillator) and a slower (by the quadratic perturbation); these trajectories span two-dimensional tori. This double resonant system has a noncommutative secondary symmetry algebra. Its irreducible representations control the spectral degeneracy of the double resonant system.

At the geometric level, one can say that the symplectic leaves of the secondary symmetry algebra determine the reduced phase spaces of the double resonant system. The effective Hamiltonian on these phase spaces is generated by the projection of the secondary perturbation onto the secondary symmetry algebra. In this way, we again reduce the number of degrees of freedom, namely, to 1 instead of 2.

For the secondary perturbation, we choose a linear disturbance of the constant (homogeneous) magnetic field, i.e., the so-called Ioffe correction. The linear disturbance of a field generates a quadratic disturbance of the magnetic potential, and therefore gives a cubic (in the phase coordinates) contribution to the Hamiltonian. Near the center of the Penning trap, this cubic piece is a small perturbation indeed, in view of the coordinate scaling.

The final effective Hamiltonian is obtained by the projection of this secondary Ioffe perturbation onto the secondary symmetry algebra. As is shown below, this Hamiltonian, in an irreducible representation, is realized by a second-order ordinary differential operator. The nondegenerate spectrum and the eigenfunctions of this final operator are just what one needs to compute the approximate spectral data for the original Hamiltonian of the resonance Penning trap.

Thus, our algebraic method approximately reduces the original 3D-differential (Schrödinger) operator with double resonance to a 1D-differential operator. In a sense, this procedure can be regarded as an analog of the method of separation of variables. Instead of finding such variables, we use irreducible representations of the primary and secondary non-Lie algebras of symmetries. When computing irreducible representations in the non-Lie case, we follow [19].

At the geometric level, this algebraic "separation of variables" presents the classical trajectories of the original Hamiltonians as a combination of three cyclic rotations: fast, slower, and the slowest (by the averaged Ioffe correction). This double resonance reduction, using a secondary symmetry algebra in a hyperbolic physical system, is a rare example. Other examples of similar type (but elliptic) appear in the Zeeman–Stark effect [22].

It should be noted that the hyperbolic double resonance effect happens in a very simple physical system, namely, in a homogeneous magnetic field plus a saddle potential.

Approximate eigenstates of the original quantum Penning trap (in the double resonance case) can be presented as follows: the eigenfunctions of the reduced ordinary differential operator together with the "squeezed" coherent states for the secondary symmetry algebra are integrated over 2D-symplectic leaves with respect to a special reproducing measure.

In the semiclassical approach, i.e., when exited states are considered, this integral can be transformed and explicitly represented as the integral of the squeezed coherent states over the periodic trajectory of the averaged secondary (Ioffe) effective Hamiltonian. Such an integral representation of semiclassical eigenfunctions, which avoids all the usual difficulties with focal points, follows the general approach [23], and combines it with quantum geometry and representation theory for non-Lie symmetry algebras in a specific way (for details, see also [16, 24]).

## 2. HAMILTONIAN OF THE PENNING TRAP

The Hamiltonian of the Penning trap with perturbations can be written as

$$\widehat{H} = \widehat{H}_0 + \varepsilon \widehat{H}_1 + \varepsilon^2 \widehat{H}_2 + O(\varepsilon^3). \tag{2.1}$$

Here $\widehat{H}_0$ is the Hamiltonian of the ideal trap

$$\widehat{H}_0 = (1/2)\big[\widehat{p}_1^2 + \widehat{p}_2^2 + \widehat{p}_3^2 + 2\omega(\widehat{p}_1 q_2 - \widehat{p}_2 q_1) + (\omega^2 - \omega_0^2)(q_1^2 + q_2^2) + 2\omega_0^2 q_3^2\big], \tag{2.2}$$

where by $\widehat{p}_j = -i\hbar\partial/\partial q_j$ we denote the momentum operators corresponding to Cartesian coordinates $q_j$ $(j = 1, 2, 3)$, and $\omega$, $\omega_0$ are positive parameters obeying condition $\omega > \omega_0$. The magnetic field of the ideal trap is homogeneous and directed along the third coordinate axis and has the magnitude $2\omega$. The electric potential of the ideal trap has the form $U_0 = (\omega_0^2/2)(2q_3^2 - q_1^2 - q_2^2)$; it obeys the Laplace equation $\Delta U_0 = 0$ and represents the model electric field between two hyperbolic cups. The real electric condenser of the trap is usually not of such hyperbolic shape, but it is cylindric or cubic; its potential $U$ obeys the Laplace equation $\Delta U = 0$ and is approximated by the function $U_0$ near the center $q = 0$ of the trap: $U = U_0 +$ fourth-degree terms $+ \cdots$ . After rescaling the coordinates in a small domain near the center, all terms of the fourth- and higher degrees become inessential and can be moved to the remainder $O(\varepsilon^3)$ in (2.1).

The term $\widehat{H}_1$ in (2.1) is due to a deviation of the homogeneous magnetic field from its "ideal" direction along the third axis. The magnitude of this deviation is assumed to be small (the parameter $\varepsilon$ in (2.1)); the direction of the deviation is given by a vector $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$. Thus,

$$\widehat{H}_1 = \tfrac{1}{2}\widehat{k} \cdot [q \times \mathcal{B}], \qquad k \stackrel{\text{def}}{=} (p_1 + \omega q_2, p_2 - \omega q_1, p_3). \tag{2.3}$$

The third term $\widehat{H}_2$ in (2.1) contains the Ioffe inhomogeneous addition to the magnetic field. The magnitude of this addition is taken in (2.1) to be of order $O(\varepsilon^2)$, but this was assumed just for simplicity; the magnitude can be greater, but it is still less than the correction of order $\varepsilon$ in (2.1). The analytic expression for $\widehat{H}_2$ is $\widehat{H}_2 = \widehat{k}_1(\beta_1 q_2 q_3 + \gamma_1(q_2^2 - q_3^2)) + \widehat{k}_2(\beta_2 q_3 q_1 + \gamma_2(q_3^2 - q_1^2)) + \widehat{k}_3(\beta_3 q_1 q_3 + \gamma_3(q_1^2 - q_2^2)) + (1/8)|[q \times \mathcal{B}]|^2$, where $k$ is the kinetic momentum defined in (2.3) and $\beta_j$ and $\gamma_j$ are parameters of the Ioffe field.

Since the inequality in (1.1) is strict, one can make the canonical (i.e., preserving the commutation relations) transformation of phase coordinates $(q_1, q_2, q_3; p_1, p_2, p_3) \to (x_+, x_-, x_0; p_+, p_-, p_0)$ by the following formulas:

$$\begin{aligned} q_1 &= \frac{1}{\sqrt{2}\sqrt[4]{\omega^2 - \omega_0^2}}(x_+ + x_-), \quad p_1 = \frac{\sqrt[4]{\omega^2 - \omega_0^2}}{\sqrt{2}}(p_+ + p_-), \quad q_2 = \frac{1}{\sqrt{2}\sqrt[4]{\omega^2 - \omega_0^2}}(p_+ - p_-), \\ p_2 &= \frac{\sqrt[4]{\omega^2 - \omega_0^2}}{\sqrt{2}}(x_- - x_+), \qquad q_3 = \frac{1}{\sqrt[4]{2}\sqrt{\omega_0}}x_0, \qquad p_3 = \sqrt[4]{2}\sqrt{\omega_0}p_0. \end{aligned} \tag{2.4}$$

Then the Hamiltonian (2.2) takes the normal form

$$\widehat{H}_0 = (1/\sqrt{2})\big[\omega_+(\widehat{p}_+^2 + x_+^2) - \omega_-(\widehat{p}_-^2 + x_-^2) + \omega_0(\widehat{p}_0^2 + x_0^2)\big], \tag{2.5}$$

where $\omega_\pm = \big(\omega^2 - \omega_0^2/2 \pm \omega(\omega^2 - \omega_0^2)^{1/2}\big)^{1/2}$.

Now assume that

$$\omega = (3\omega_0)/(2\sqrt{2}), \qquad \text{or} \qquad \omega^2 = (9/8)\omega_0^2. \tag{2.6}$$

Then $\omega_+ = \omega_0$ and $\omega_- = \omega_0/2$, and the Hamiltonian (2.5) reads

$$\widehat{H}_0 = \frac{\omega_0}{2\sqrt{2}}\Big[2(\widehat{p}_+^2 + x_+^2) - (\widehat{p}_-^2 + x_-^2) + 2(\widehat{p}_0^2 + x_0^2)\Big] = \frac{\omega_0}{\sqrt{2}}[2\widehat{z}_+^*\widehat{z}_+ - \widehat{z}_-^*\widehat{z}_- + 2\widehat{z}_0^*\widehat{z}_0] + \frac{3\hbar\omega_0}{2\sqrt{2}}, \tag{2.7}$$

where

$$\widehat{z}_\pm = (x_\pm + i\widehat{p}_\pm)/\sqrt{2}, \qquad \widehat{z}_0 = (x_0 + i\widehat{p}_0)/\sqrt{2}. \tag{2.8}$$

As we see from (2.7) under condition (2.6), the Hamiltonian of the ideal Penning trap is a linear combination of three oscillators whose frequencies are in the resonance $2 : (-1) : 2$.

The spectrum of $\widehat{H}_0$ is discrete but infinitely degenerate under the resonance condition (2.6). The problem is: How to take into account the perturbing terms $\widehat{H}_1, \widehat{H}_2, \ldots$ in the entire Hamiltonian (2.1) of the trap?

## 3. SYMMETRY ALGEBRA OF THE 2 : (−1) : 2 TRAP

The degeneracy of the spectrum is controlled by the symmetry algebra, i.e., by the algebra of all operators commuting with $\widehat{H}_0$. This algebra is nontrivial (noncommutative) if and only if the frequencies $\omega_+, \omega_-, \omega_0$ in (2.5) are in resonance. We deal with (2.6) and, hence, with the resonance $2 : (-1) : 2$. The algebra of symmetries of the Hamiltonian (2.7) is referred to as *resonance algebra*.

The following operators can be chosen as generators of the symmetry algebra:

$$S_\pm = \widehat{z}_\pm^*\widehat{z}_\pm, \qquad S_0 = \widehat{z}_0^*\widehat{z}_0, \tag{3.1}$$

$$A_\rho = \widehat{z}_+^*\widehat{z}_0, \qquad A_\sigma = \widehat{z}_+^*(\widehat{z}_-^*)^2, \qquad A_\theta = (\widehat{z}_-^*)^2\widehat{z}_0^*. \tag{3.2}$$

All these operators commute with $\widehat{H}_0$ (2.7). The operators (3.1) are self-adjoint, but the operators (3.2) are not, and so the conjugate operators $A_\rho^*$, $A_\sigma^*$, $A_\theta^*$ have to be included into the set of generators of the symmetry algebra.

The commutation relations between these generators are the following ones:

$$\begin{aligned} &[S_+, A_\rho] = \hbar A_\rho, \quad [S_0, A_\rho] = -\hbar A_\rho, \quad [S_+, A_\sigma] = \hbar A_\sigma, \quad [S_-, A_\sigma] = 2\hbar A_\sigma, \quad [S_-, A_\theta] = 2\hbar A_\theta, \\ &[S_0, A_\theta] = \hbar A_\theta, \qquad [A_\rho, A_\sigma^*] = -\hbar A_\theta^*, \qquad [A_\rho, A_\theta] = \hbar A_\sigma, \qquad [A_\sigma, A_\theta^*] = -4\hbar\Big(S_- + \frac{\hbar}{2}\Big)A_\rho, \\ &[A_\rho^*, A_\rho] = \hbar(S_0 - S_+), \qquad [A_\sigma^*, A_\sigma] = \hbar(4S_+S_- + S_-^2 + 2\hbar S_+ + 3\hbar S_- + 2\hbar^2), \\ &[A_\theta^*, A_\theta] = \hbar(S_-^2 + 4S_-S_0 + 3\hbar S_- + 2\hbar S_0 + 2\hbar^2). \end{aligned} \tag{3.3}$$

The other commutators can be obtained either by conjugation of the above-written commutators or are equal to zero.

Certainly, the operator (2.7) realized over the 3D-space can have the symmetry algebra with the maximal number of independent generators $2\times3-1=5$. That is, not all nine operators $S_+$, $S_-$, $S_0$, $A_\rho$, $A_\sigma$, $A_\theta$, $A_\rho^*$, $A_\sigma^*$, $A_\theta^*$ are independent in this realization. Therefore, there must be constraints reducing the dimension from 9 to 5.

First, we note that relations (3.3) admit three Casimir operators (which commute with all the generators):

$$C_\rho = A_\rho A_\rho^* - S_+(S_0+\hbar), \qquad C_\sigma = A_\sigma A_\sigma^* - S_+S_-(S_- - \hbar), \qquad C_\theta = A_\theta A_\theta^* - S_0S_-(S_- - \hbar). \tag{3.4}$$

In the realization (3.1), (3.2), these three operators vanish.

Also there are "quasi-Casimir" operators

$$C_+ = A_\rho A_\sigma^* - S_+A_\theta^*, \qquad C_- = A_\sigma A_\theta^* - S_-(S_- - \hbar)A_\rho, \qquad C_0 = A_\rho A_\theta - (S_0+\hbar)A_\sigma. \tag{3.5}$$

The commutators of these operators with all generators are proportional to the operators (3.5). In the realization (3.1), (3.2), the operators (3.5) also vanish.

Thus, the symmetry algebra of the resonance oscillator (2.6) can be defined as an algebra with nine generators and relations (3.3) factorized by the ideal generated by the elements (3.4) and (3.5).

In this quotient algebra, we still have certainly a single additional Casimir element

$$C = 2S_+ - S_- + 2S_0, \tag{3.6}$$

which is in fact the operator $\widehat{H}_0$ (2.7), namely, $\widehat{H}_0 = \frac{\omega_0}{\sqrt{2}}\big(C + \frac{3\hbar}{2}\big)$.

## 4. ALGEBRAIC AVERAGING

To study the perturbed operator (2.1), we follow the general scheme of algebraic averaging [16, 21]. Namely, we perform a unitary transformation which "kills" the part of the perturbation in (2.1) that does not commute with the leading term $\widehat{H}_0$. The new perturbation commutes with $\widehat{H}_0$, and therefore, belongs to the algebra of symmetries described in the preceding section.

The formulas are as follows. We seek a unitary $V=\exp\{-iR\varepsilon/\hbar\}$ such that

$$V^{-1}\cdot\widehat{H}\cdot V = \widehat{H}_0 + \varepsilon\widehat{H}_{10} + \varepsilon^2\widehat{H}_{20} + O(\varepsilon^3), \tag{4.1}$$

where

$$[\widehat{H}_0, \widehat{H}_{10}] = [\widehat{H}_0, \widehat{H}_{20}] = 0. \tag{4.2}$$

The operator $R$ generating the unitary family $V$ has the form $R = R_0 + \varepsilon R_1$, where

$$\frac{i}{\hbar}[\widehat{H}_0, R_0] = \widehat{H}_1 - \widehat{H}_{10}, \qquad \frac{i}{\hbar}[\widehat{H}_0, R_1] = \widehat{H}_2 + \frac{i}{2\hbar}[R_0, \widehat{H}_1 + \widehat{H}_{10}] - \widehat{H}_{20}. \tag{4.3}$$

Since $\widehat{H}_0 = \frac{\omega_0}{\sqrt{2}}\big(C+\frac{3\hbar}{2}\big)$, where the operator $C$ (3.6) has the spectrum $\{\hbar n \mid n = 0, \pm1, \pm2, \dots\}$, the homological equations (4.3) can be solved easily:

$$\widehat{H}_{10} = \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{it}{\hbar}C}\widehat{H}_1 e^{\frac{it}{\hbar}C}\,dt, \quad \widehat{H}_{20} = \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{it}{\hbar}C}\Big(\widehat{H}_2 + \frac{i}{2\hbar}[R_0, \widehat{H}_1 + \widehat{H}_{10}]\Big)e^{\frac{it}{\hbar}C}\,dt, \tag{4.4}$$

and the generators $R_0$, $R_1$ are given by

$$R_0 = \frac{\sqrt{2}}{2\pi\omega_0}\int_0^{2\pi} e^{-\frac{it}{\hbar}C}\widehat{H}_1 e^{\frac{it}{\hbar}C}t\,dt, \quad R_1 = \frac{\sqrt{2}}{2\pi\omega_0}\int_0^{2\pi} e^{-\frac{it}{\hbar}C}\Big(\widehat{H}_2 + \frac{i}{2\hbar}[R_0, \widehat{H}_1 + \widehat{H}_{10}]\Big)e^{\frac{it}{\hbar}C}t\,dt. \tag{4.5}$$

The right-hand sides in (4.4), (4.5) can be computed explicitly, because the operator $C$ is just the linear combination (3.6) of the "action" operators $S_\pm$, $S_0$ and the evolution of all phase space coordinates $\widehat{z}_\pm$, $\widehat{z}_0$ with respect to each action can be derived easily and explicitly.

**Theorem 4.1.** *The Penning trap Hamiltonian* (2.1) *is unitary equivalent, up to* $O(\varepsilon^3)$, *to the Hamiltonian* (4.1) *with the perturbing terms* $\widehat{H}_{10}$, $\widehat{H}_{20}$ *expressed via the generators of the symmetry algebra* (3.3) *by the following formulas*:

$$\widehat{H}_{10} = \mathcal{B}_3\Big(2S_+ + S_- + \frac{3\hbar}{2}\Big) - \frac{\mathcal{B}_1 + i\mathcal{B}_2}{\sqrt{2}}A_\rho - \frac{\mathcal{B}_1 - i\mathcal{B}_2}{\sqrt{2}}A_\rho^*, \tag{4.6}$$

$$\widehat{H}_{20} = f_+S_+ + f_-S_- + f_0S_0 + (g_\rho A_\rho + \overline{g}_\rho A_\rho^*) + (g_\sigma A_\sigma + \overline{g}_\sigma A_\sigma^*) + (g_\theta A_\theta + \overline{g}_\theta A_\theta^*) + r. \tag{4.7}$$

In formula (4.7), the scalar coefficients are given by the formulas $f_+ = \xi^2(1-9\eta^2)\dfrac{\sqrt{2}}{8\omega_0}$, $f_- = \xi^2\Big(1-\dfrac{7+20\eta^2}{3}\Big)\dfrac{\sqrt{2}}{8\omega_0}$, $f_0 = -\xi^2(1-\eta^2)\dfrac{5\sqrt{2}}{24\omega_0}$, $g_\rho = \xi^2\eta\sqrt{1-\eta^2}\dfrac{e^{i\varphi}}{\sqrt{2}\omega_0}$, $g_\sigma = \dfrac{2^{3/4}}{\sqrt{\omega_0}}(\gamma_2 - i\gamma_1)$, $g_\theta = \dfrac{1}{2^{7/4}\sqrt{\omega_0}}(2\beta_3-\beta_1-\beta_2+4i\gamma_3)$, and $r = -\hbar\xi^2(1+7\eta^2)\cdot\dfrac{\sqrt{2}}{8\omega_0}$, where we use the notation

$$\xi^2 = \mathcal{B}_1^2+\mathcal{B}_2^2+2\mathcal{B}_3^2, \qquad \eta = \sqrt{2}\mathcal{B}_3/\xi, \tag{4.8}$$

and the angle $\varphi$ is given by

$$\mathcal{B}_1 + i\mathcal{B}_2 = \xi\sqrt{1-\eta^2}e^{i\varphi}. \tag{4.9}$$

## 5. SECONDARY RESONANCE

The perturbing terms $\widehat{H}_{10}, \widehat{H}_{20}, \dots$ in (4.1) commute with the leading term $\widehat{H}_0$. Therefore, we must now study the Hamiltonian

$$\widehat{H}_{10} + \varepsilon\widehat{H}_{20} + O(\varepsilon^2) \tag{5.1}$$

on the eigenspaces of the resonance oscillator $\widehat{H}_0$. These eigenspaces can be determined by computing the irreducible representations of the primary resonance algebra described in Section 3. The expression for $\widehat{H}_{10}$ and $\widehat{H}_{20}$ in terms of generators of this algebra was given in Theorem 4.1, see 4.6.

Note that the generators $A_\rho$, $A_\rho^*$ in (4.6) commute with $S_-$, and therefore $[\widehat{H}_{10}, S_-] = 0$. Thus, the Hamiltonian (5.1) under consideration is a perturbation of this integrable system (mostly by the Ioffe field contribution).

The symmetry algebra of the operator $\widehat{H}_{10}$ (4.6) is trivial in general position (commutative and generated by $S_-$ and $\widehat{H}_{10}$ itself), and its spectrum is nondegenerate. However, under a special resonance condition imposed on the components of the perturbing magnetic field $\mathcal{B}$, this algebra becomes noncommutative, and a spectral degeneracy appears.

The condition for a secondary resonance in our trap is:

$$(k-l)^2(\mathcal{B}_1^2+\mathcal{B}_2^2) = 16((k+l)^2 + kl/2)\mathcal{B}_3^2, \tag{5.2}$$

where $k$, $l$ are some positive coprime integers. Assume that $k > l$ if $\mathcal{B}_3 > 0$ and $k < l$ if $\mathcal{B}_3 < 0$ in (5.2).

**Theorem 5.1.** *Under condition* (5.2), *the symmetry algebra of the operator* $\widehat{H}_{10}$ (4.6) *is noncommutative. The generators of this secondary resonance algebra are*

$$A_0 = S_-, \quad A_\pm \stackrel{\text{def}}{=} (1\pm\eta)S_+ + (1\mp\eta)S_0 \mp \sqrt{1-\eta^2}(e^{i\varphi}A_\rho + e^{-i\varphi}A_\rho^*), \tag{5.3}$$

$$B = B_+^l B_-^k, \tag{5.4}$$

*where*

$$B_\pm \stackrel{\text{def}}{=} \sqrt{1\pm\eta}\, e^{i\varphi/2}A_\sigma \mp \sqrt{1\mp\eta}\, e^{-i\varphi/2}A_\theta, \tag{5.5}$$

*the parameter* $\eta$ *is taken from* (4.8),

$$\eta = \frac{k-l}{3(k+l)}, \tag{5.6}$$

*and the angle* $\varphi$ *is taken from* (4.9).

**Theorem 5.2.** *The commutation relations between generators* (5.3), (5.4) *read*

$$[A_0, B] = 2(k+l)\hbar B, \quad [A_+, B] = 2l\hbar B, \quad [A_-, B] = 2k\hbar B, \quad [B^*, B] = \hbar f^\hbar(A_0, A_+, A_-). \tag{5.7}$$

*The other commutators can be obtained either by conjugation of the above-written commutators or are equal to zero. In formula* (5.7),[1]

[1] The product $\prod$ is chosen to be equal to 1 if the upper limit in it is less than the lower limit.

$$f^\hbar(A_0,A_+,A_-) \stackrel{\text{def}}{=} \frac{1}{\hbar}\Big\{ \prod_{r=1}^{2(k+l)} (A_0+r\hbar)\prod_{q=1}^{k}(A_-+2q\hbar)\prod_{p=1}^{l}(A_++2p\hbar) - \prod_{r=1}^{2(k+l)} (A_0-r\hbar+\hbar)\prod_{q=1}^{k}(A_--2q\hbar+2\hbar)\prod_{p=1}^{l}(A_+-2p\hbar+2\hbar)\Big\}. \tag{5.8}$$

*The non-Lie algebra* (5.7) *has three Casimir* (*central*) *elements*

$$M \stackrel{\text{def}}{=} A_+ - A_- + \frac{k-l}{k+l}A_0, \qquad C = A_+ + A_- - A_0, \qquad K \stackrel{\text{def}}{=} BB^* - \rho^\hbar(A_0,A_+,A_-), \tag{5.9}$$

*where* $\rho^\hbar(A_0,A_+,A_-) \stackrel{\text{def}}{=} \prod_{r=1}^{2(k+l)}(A_0-r\hbar+\hbar)\prod_{q=1}^{k}(A_--2q\hbar+2\hbar)\prod_{p=1}^{l}(A_+-2p\hbar+2\hbar)$.

**Theorem 5.3.** *In the realization* (3.1), (3.2), *the Casimir elements* (5.9) *read*

$$M = \frac{2\sqrt{2}}{\xi}\widehat{H}_{10} - \eta(C+3\hbar), \qquad M = \frac{\sqrt{2}}{\omega_0}\widehat{H}_0 - \frac{3\hbar}{2}, \qquad K = 0, \tag{5.10}$$

*where* $\eta$ *is taken from* (5.6) *and* $\xi = \sqrt{2}\mathcal{B}_3/\eta$. *On the* $n$*th eigenspace of the leading Hamiltonian* $\widehat{H}_0$, *where the Casimir* $C$ *takes the value* $n\hbar$, *the spectrum of the Casimir* $M$ *is* $\{\frac{4}{k+l}\cdot m\hbar - \frac{k-l}{k+l}\cdot n\hbar \mid m \in L\}$. *Here the subset* $L \subset \mathbb{Z}$ *is defined as follows*:

if $k=1$, $l=0$, then $L=\mathbb{Z}_+$,

if $k=0$, $l=1$, then $L=\mathbb{Z}_-$,

if $k\neq 0$, $l\neq 0$, then $L=\{kt_+ - lt_- + klt_0 \mid 0\leqslant t_+\leqslant l-1, 0\leqslant t_-\leqslant k-1, t_0\in\mathbb{Z}\}$.

**Corollary 5.4.** *Under the secondary resonance condition* (5.2), *the spectrum of the Hamiltonian* (4.1) *reads*

$$\frac{\omega_0}{\sqrt{2}}\Big(n+\frac{3}{2}\Big)\hbar + \varepsilon\Big(\frac{6m}{k-l} - n + \frac{3}{2}\Big)\hbar\mathcal{B}_3 + O(\varepsilon^2), \tag{5.11}$$

*where* $n\in\mathbb{Z}$, $m\in L$.

The corrections $O(\varepsilon^2)$ in this spectrum are determined by the secondary perturbation $\widehat{H}_{20}$ (4.7) in (5.1). In order to take them into account, one has to perform a secondary averaging operation similar to those made in Section 4. By a unitary transform we obtain, instead of (5.1), the new Hamiltonian

$$\widehat{H}_{10} + \varepsilon\widehat{H}_{200} + O(\varepsilon^2) \tag{5.12}$$

with the new perturbing term commuting with the leading part: $[\widehat{H}_{10},\widehat{H}_{200}]=0$. The explicit formula for $\widehat{H}_{200}$ is obtained by the same integral operation as in (4.4) but, instead of the Casimir operator $C$ in the exponent, one now has to use the Casimir operator $\frac{1}{4}(k+l)M$ whose spectrum belongs to the arithmetic progression $\{m\hbar + a\}$.

For simplicity, consider now only the simplest resonance $k:l = 1:0$ in (5.2), i.e., assume that

$$\mathcal{B}_1^2 + \mathcal{B}_2^2 = 16\mathcal{B}_3^2. \tag{5.13}$$

This condition means that the angle between the trap axis and the perturbing magnetic field $\mathcal{B}$ is about 76°. In this case, the secondary resonance algebra (5.7) reads

$$[A_0,B] = 2\hbar B, \qquad [A_-,B] = 2\hbar B, \qquad [B^*,B] = 2\hbar(A_0^2 + 2A_0A_- + 3\hbar A_0 + \hbar A_- + 2\hbar^2). \tag{5.14}$$

The Casimir elements are

$$M = A_+ - A_- + A_0, \quad C = A_+ + A_- - A_0, \quad K = BB^* - A_0(A_0-\hbar)A_-. \tag{5.15}$$

**Theorem 5.5.** *Under the secondary resonance condition* (5.13), *the secondary averaged Hamiltonian* $\widehat{H}_{200}$ *in* (5.12) *is expressed via generators of the secondary symmetry algebra* (5.14) *by the formula*

$$\widehat{H}_{200} = \varkappa\cdot B + \bar{\varkappa}\cdot B^* + \mu_0A_0 + \mu_+A_+ + \mu_-A_- + \nu, \tag{5.16}$$

*where the scalar coefficients are*

$$\varkappa = \frac{1}{2^{7/4}\sqrt{3\omega_0}}\Big[(2\beta_3 - \beta_1 - \beta_2 + 4i\gamma_3)e^{i\varphi/2} + 4(\gamma_2 - i\gamma_1)e^{-i\varphi/2}\Big], \\ \mu_0 = -\frac{14\sqrt{2}}{3\omega_0}\mathcal{B}_3^2, \qquad \mu_+ = -\frac{17\sqrt{2}}{9\omega_0}\mathcal{B}_3^2, \qquad \mu_- = \frac{2\sqrt{2}}{9\omega_0}\mathcal{B}_3^2, \qquad \nu = -\hbar\frac{97\sqrt{2}}{16\omega_0}\mathcal{B}_3^2, \tag{5.17}$$

*and the angle* $\varphi$ *is derived from the relation* $\cos\varphi = \mathcal{B}_1/4\mathcal{B}_3$.

**Corollary 5.6.** *Under the primary resonance condition* (2.6) *and the secondary resonance condition* (5.13), *the spectrum of the Hamiltonian* (2.1) *of the perturbed Penning trap has the following asymptotics*:

$$\frac{\omega_0}{\sqrt{2}}\Big(n+\frac{3}{2}\Big)\hbar+\varepsilon\Big(6m-n+\frac{3}{2}\Big)\hbar\mathcal{B}_3+\varepsilon^2\lambda_{n,m,k}+O(\varepsilon^3), \tag{5.18}$$

where $n\in\mathbb{Z}$, $m\in L$, and $\lambda_{n,m,k}$ are the eigenvalues of the operator (5.16) over the secondary resonance algebra (5.14) in its irreducible representation in which the Casimir elements (5.15) take the values

$$K=0, \qquad C=n\hbar, \qquad M=(4m-n)\hbar. \tag{5.19}$$

## 6. IRREDUCIBLE REPRESENTATIONS OF THE SECONDARY RESONANCE ALGEBRA

Now we describe the irreducible representations of the algebra (5.7) (in particular, of (5.14)).

For any $n\in\mathbb{Z}$, introduce a subset $T_n\subset\mathbb{Z}^2_+$ as $T_n\overset{\text{def}}{=}\{(t_+,t_-)\in\mathbb{Z}^2_+\mid t_++t_-\geqslant n/2\}$. This subset has a natural partial order,

$$(a_+,a_-)\leqslant(b_+,b_-) \quad\Longleftrightarrow\quad a_+\leqslant b_+ \quad\text{and}\quad a_-\leqslant b_-. \tag{6.1}$$

**Lemma 6.1.** *For any* $m\in L$, *there is a unique solution* $(m_+,m_-)\in T_n$ *of the equation*

$$km_+-lm_-=m, \tag{6.2}$$

*which is minimal with respect to the partial order* (6.1).

*In particular, if* $k=1$, $l=0$, *then the minimal solution of* (6.2) *is given by the formula*

$$m_+=m, \qquad m_-=[(1+\max\{n-2m,0\})/2], \tag{6.3}$$

*where* $[\dots]$ *stands for the integer part of a number.*

Define now the following function $\mathcal{F}_{n,m}$ of the integer arguments,

$$\mathcal{F}_{n,m}(j)\overset{\text{def}}{=}2^{k+l}\hbar^{3(k+l)}\big(2(m_++m_-)-n+2(k+l)(j-1)+1\big)_{2(k+l)}\big(m_-+k(j-1)+1\big)_k\big(m_++l(j-1)+1\big)_l, \tag{6.4}$$

where the Pochhammer symbols are defined by $(c)_0=1$ and $(c)_j=c(c+1)\cdots(c+j-1)$ for $j\geqslant1$.

In particular, if $k=1$, $l=0$, then

$$\mathcal{F}_{n,m}(j)=2\hbar^3(2(m+m_-)-n+2j)(2(m+m_-)-n-1+2j)(m_-+j). \tag{6.5}$$

**Theorem 6.1.** *The irreducible representations of the secondary resonance algebra* (5.7) *are given by the ordinary differential operators*

$$\mathbb{A}_0=a_0(D), \qquad \mathbb{A}_\pm=a_\pm(D), \qquad \mathbb{B}=\bar z, \qquad \mathbb{B}^*=\bar b(\overset{2}{\bar z},\overset{1}{D}), \tag{6.6}$$

*where* $D\overset{\text{def}}{=}\bar z\frac{d}{d\bar z}$ *and*

$$\begin{aligned} a_0(d)&\overset{\text{def}}{=}2\hbar(m_++m_--n/2+(k+l)d), & a_+(d)&\overset{\text{def}}{=}2\hbar(m_++ld),\\ a_-(d)&\overset{\text{def}}{=}2\hbar(m_-+kd), & \bar b(\bar z,d)&\overset{\text{def}}{=}\mathcal{F}_{n,m}(d)/\bar z. \end{aligned} \tag{6.7}$$

*In these representations, the values of the Casimir elements* (5.9) *are as follows*:

$$\mathbb{M}\equiv\mathbb{A}_+-\mathbb{A}_-+\tfrac{k-l}{k+l}\mathbb{A}_0=\tfrac{4}{k+l}m\hbar-\tfrac{k-l}{k+l}n\hbar,\ \mathbb{C}\equiv\mathbb{A}_++\mathbb{A}_-+\mathbb{A}_0=n\hbar,\ \mathbb{K}\equiv\mathbb{B}\mathbb{B}^*-\rho^\hbar(\mathbb{A}_0,\mathbb{A}_+,\mathbb{A}_-)=0. \tag{6.8}$$

The conjugation of operators in the above formulas is taken with respect to the inner product in the space of holomorphic functions in the variable $\bar z$ given by

$$(\bar z^j,\bar z^r)\overset{\text{def}}{=}s_{n,m}(j)\cdot\delta_{jr}, \tag{6.9}$$

where

$$s_{n,m}(j)=\mathcal{F}_{n,m}(1)\cdots\mathcal{F}_{n,m}(j)=2^{(k+l)j}\hbar^{3(k+l)j}\big(2(m_++m_-)-n+1\big)_{2(k+l)j}(m_-+1)_{kj}(m_++1)_{lj}.$$

**Remark 6.1.** Consider the case $k=1$, $l=0$. In this case, since the function $\mathcal{F}_{n,m}$ (6.4) is cubic, the operator $\mathbb{B}^*$ in the representation (6.6) is a third-order differential operator. Thus, the Hamiltonian (5.16) in the irreducible representation realized by (6.6) is a third-order differential operator as well. To reduce the order of this operator, one needs to find another realization of the irreducible representations of the algebra (5.7) using second-order operators instead of third-order ones.

Note that the pair of integer numbers $(m_+,m_-)$ defined in Lemma 6.1 satisfies at least one of the following conditions:

(i) $m_+ < l$ or $m_- < k$,

(ii) $2(m_+ + m_-) - n < 2(k+l)$.

In case (i), write

$$\mathbb{B} = \bar{z}\prod_{r=1}^{k+l}(\mathbb{A}_0 + r\hbar), \qquad \mathbb{B}^* = \frac{1}{\bar{z}}\prod_{r=0}^{k+l-1}(\mathbb{A}_0 - r\hbar)\prod_{q=0}^{k-1}(\mathbb{A}_- - 2q\hbar)\prod_{p=0}^{l-1}(\mathbb{A}_+ - 2p\hbar). \tag{6.10}$$

In case (ii), write

$$\mathbb{B} = \prod_{q=0}^{k-1}(\mathbb{A}_- - 2q\hbar)\prod_{p=0}^{l-1}(\mathbb{A}_+ - 2p\hbar)\cdot\bar{z}, \qquad \mathbb{B}^* = \frac{1}{\bar{z}}\prod_{r=0}^{2(k+l)-1}(\mathbb{A}_0 - r\hbar). \tag{6.11}$$

**Theorem 6.2.** *Formulas* (6.10), (6.11) *realize the irreducible representation of the secondary resonance algebra* (5.7) *corresponding to the same values* (6.8) *of the Casimir elements as the representation constructed in Theorem* 6.1 (*an equivalent representation*).

*The operator conjugation in* (6.10), (6.11) *is given by the same formula* (6.9) *as in Theorem* 6.1, *and the numbers* $s_{n,m}(j)$ *are now defined as follows*:

— *in case* (i),

$$s_{n,m}(j) = 2^{(k+l)j}\hbar^{(k+l)j}\frac{\big(2(m_+ + m_-) - n + 1\big)_{2(k+l)j}(m_- + 1)_{kj}(m_+ + 1)_{lj}}{\Big(\prod_{r=1}^{j}\big(2(m_+ + m_-) - n + 2(k+l)(r-1) + 1\big)_{k+l}\Big)^2},$$

— *in case* (ii),

$$s_{n,m}(j) = \left(\frac{\hbar}{2}\right)^{(k+l)j}\frac{\big(2(m_+ + m_-) - n + 1\big)_{2(k+l)j}}{(m_- + 1)_{kj}(m_+ + 1)_{lj}}.$$

It is possible to write out other useful variants of the realization of irreducible representations of the algebra (5.7). The general classification is given in [19].

In particular, if $k = 1$ and $l = 0$, we derive from formulas of type (6.10) and (6.11) the following irreducible representation of the secondary resonance algebra (5.14) by (at most second-order) differential operators:

(i) if $n \leqslant 2m$, then

$$\mathbb{B} = 2\hbar\bar{z}\Big(\bar{z}\frac{d}{d\bar{z}} + m - \frac{n}{2} + \frac{3}{4} - \frac{(-1)^n}{4}\Big), \qquad \mathbb{B}^* = 4\hbar^2\Big(\bar{z}\frac{d}{d\bar{z}} + m - \frac{n}{2} + \frac{3}{4} + \frac{(-1)^n}{4}\Big)\frac{d}{d\bar{z}}, \tag{6.12}$$

(ii) if $n > 2m$, then

$$\mathbb{B} = 2\hbar\bar{z}\Big(\bar{z}\frac{d}{d\bar{z}} + 1 - \frac{(-1)^n}{2}\Big), \quad \mathbb{B}^* = 4\hbar^2\Big(\bar{z}\frac{d}{d\bar{z}} - m + \Big[\frac{n+1}{2}\Big] + 1\Big)\frac{d}{d\bar{z}}, \tag{6.13}$$

and $\mathbb{A}_\pm$ and $\mathbb{A}_0$ are first-order operators given by the following formulas:

(i) if $n \leqslant 2m$, then $\mathbb{A}_0 = 2\hbar\big(m - \frac{n}{2} + \bar{z}\frac{d}{d\bar{z}}\big)$, $\mathbb{A}_+ = 2\hbar m$, and $\mathbb{A}_- = 2\hbar\bar{z}\frac{d}{d\bar{z}}$;

(ii) if $n > 2m$, then $\mathbb{A}_0 = 2\hbar\big(\frac{1-(-1)^n}{4} + \bar{z}\frac{d}{d\bar{z}}\big)$, $\mathbb{A}_+ = 2\hbar m$, and $\mathbb{A}_- = 2\hbar\big(\big[\frac{n+1}{2}\big] - m + \bar{z}\frac{d}{d\bar{z}}\big)$.

The inner product for the operator conjugation in (6.12), (6.13) is given by (6.9), where the numbers $s_{n,m}(j)$ are defined by

— in case (i),

$$s_{n,m}(j) = (2\hbar)^j j!\frac{\big(m - \frac{n}{2} + \frac{3}{4} + \frac{(-1)^n}{4}\big)_j}{\big(m - \frac{n}{2} + \frac{3}{4} - \frac{(-1)^n}{4}\big)_j},$$

— in case (ii),

$$s_{n,m}(j) = (2\hbar)^j j!\frac{\big(\big[\frac{n+1}{2}\big] - m + 1\big)_j}{\big(1 - \frac{(-1)^n}{2}\big)_j}.$$

## 7. INTEGRAL REPRESENTATIONS OF EIGENFUNCTIONS VIA COHERENT STATES OF THE SECONDARY RESONANCE ALGEBRA

We have determined the inner product in the space of antiholomorphic functions over $\mathbb{C}$ by the formula

$$(u, v) = \sum_{j\geqslant 0} s_{n,m}(j)u_j\bar{v}_j. \tag{7.1}$$

Here $u(\bar z)=\sum_{j\geqslant 0}u_j\bar z^j$, $v(\bar z)=\sum_{j\geqslant 0}v_j\bar z^j$, and the numbers $s_{n,m}(j)$, for fixed $n$ and $m$ and variable $j=0,1,2,\dots$, are given by (6.9). These numbers are related to the choice of an irreducible representation of the given algebra.

The reproducing kernel in the space with the inner product (7.1) is

$$\mathcal{K}_{n,m}(\bar w,z)\overset{\text{def}}{=\!=}\sum_{j\geqslant 0}\frac{(\bar w z)^j}{s_{n,m}(j)}. \tag{7.2}$$

The term "reproducing" is applied here because of the following property: if one takes the inner product (7.1) of two functions $\mathcal{K}_{n,m}(\cdot,z)$ and $\mathcal{K}_{n,m}(\cdot,w)$, then the result is equal to $\mathcal{K}_{n,m}(\bar w,z)$.

One can now define a family of coherent states $\mathfrak{p}_{n,m}(z)$ of the given algebra so that

$$(\mathfrak{p}_{n,m}(z),\mathfrak{p}_{n,m}(w))_{L^2}=\mathcal{K}_{n,m}(\bar w,z). \tag{7.3}$$

The inner product on the left-hand side of (7.3) is taken in the space of the given representation of the algebra. In our case, this space is $L^2=L^2(\mathbb{R}^3)$, i.e., the original Hilbert space for the Schrödinger operator (2.1).

The family of coherent states obeying (7.3) can be determined by the formula

$$\mathfrak{p}_{n,m}(z)=F_{n,m}(z,B)\mathfrak{p}_{n,m}(0). \tag{7.4}$$

Here the function $F_{n,m}$ is given by $F_{n,m}(z,\bar w)=\sum_{j\geqslant 0}(z\bar w)^j/\beta(j)$, where

$$\beta(j)=\Big(j!s_{n,m}(j)\mathcal{F}_{n,m}(j-1)\cdots\mathcal{F}_{n,m}(0)\Big)^{1/2}$$

and $\mathcal{F}_{n,m}$ is taken from (6.4). The "vacuum vector" $\mathfrak{p}_{n,m}(0)$ in (7.4) is an eigenvector of all Cartan generators $A_0$, $A_+$, $A_-$ of the algebra (5.7) and it is annihilated by the generator $B^*$.

Consider now the simplest secondary resonance $k=1$, $l=0$ and the resonance algebra (5.14). Then the equations for the vacuum vector are as follows:

$$\begin{gathered} B^*\mathfrak{p}_{n,m}(0)=0,\qquad A_0\mathfrak{p}_{n,m}(0)=\hbar(2(m+m_-)-n)\mathfrak{p}_{n,m}(0),\\ A_+\mathfrak{p}_{n,m}(0)=2\hbar m\mathfrak{p}_{n,m}(0),\qquad A_-\mathfrak{p}_{n,m}(0)=2\hbar m_-\mathfrak{p}_{n,m}(0). \end{gathered} \tag{7.5}$$

The normalized solution of the system of equations (7.5) is

$$\begin{aligned} \mathfrak{p}_{n,m}(0)&=c_{n,m}(\hat z_-^*)^{2(m+m_-)-n}(\sqrt{2}e^{i\varphi/2}\hat z_+^*-e^{-i\varphi/2}\hat z_0^*)^m\\ &\quad\times(e^{i\varphi/2}\hat z_+^*+\sqrt{2}e^{-i\varphi/2}\hat z_0^*)^{m_-}\exp\big(-(x_+^2+x_-^2+x_0^2)/2\hbar\big), \end{aligned}$$

where

$$c_{n,m}=\Big(3^{m+m_-}\pi^{3/2}\hbar^{3(m+m_-)-n+3/2}\big(2(m+m_-)-n\big)!m!m_-!\Big)^{-1/2}.$$

By using coherent states, one can intertwine the original representation of the algebra (5.14) with its irreducible representations. The intertwining mapping is

$$g\to I_{n,m}(g)\overset{\text{def}}{=\!=}\frac{1}{2\pi\hbar}\int_{\mathbb{C}}g(\bar z)\mathfrak{p}_{n,m}(z)l(|z|^2)\,d\bar z\,dz. \tag{7.6}$$

Here $l$ stands for the density of the "reproducing measure" with respect to which the following reproducing property holds:

$$\int_{\mathbb{C}}\mathcal{K}_{n,m}(\bar w,z)\mathcal{K}_{n,m}(\bar z,w)l(|z|^2)\,d\bar z\,dz=\mathcal{K}(\bar w,w). \tag{7.7}$$

The inner product (7.1) in terms of this measure reads

$$(u,v)=\int_{\mathbb{C}}u(\bar z)\overline{v(\bar z)}l(|z|^2)\,d\bar z\,dz. \tag{7.8}$$

The mapping (7.6) obeys

$$A_{0,\pm}I_{n,m}(g)=I_{n,m}(\mathbb{A}_{0,\pm}(g)),\quad BI_{n,m}(g)=I_{n,m}(\mathbb{B}(g)),\quad B^*I_{n,m}(g)=I_{n,m}(\mathbb{B}^*(g)), \tag{7.9}$$

where $A_0$, $A_+$, $A_-$, $B$, $B^*$ are the original generators of the algebra and $\mathbb{A}_0$, $\mathbb{A}_+$, $\mathbb{A}_-$, $\mathbb{B}$, $\mathbb{B}^*$ are the generators of the irreducible representation (6.12) or (6.13).

Applying (7.6), (7.9) to the operator $\widehat{H}_{200}$ (5.16), we derive $\widehat{H}_{200}I_{n,m}(g)=I_{n,m}(\mathbb{H}(g))$, where

$$\mathbb{H} \stackrel{\text{def}}{=} \varkappa\mathbb{B} + \bar{\varkappa}\mathbb{B}^* + \mu_0\mathbb{A}_0 + \mu_+\mathbb{A}_+ + \mu_-\mathbb{A}_- + \nu. \tag{7.10}$$

Thus, in order to compute the spectrum and eigenvectors of $\widehat{\mathbb{H}}_{200}$ in the eigenspace of $\widehat{\mathbb{H}}_0$ and $\widehat{\mathbb{H}}_{10}$, one needs to consider the eigenvalue problem in the space of antiholomorphic functions with the inner product (7.8) for the operator $\mathbb{H}$ (7.10):

$$\mathbb{H}g = \lambda g. \tag{7.11}$$

From (6.12) or (6.13), it follows that (7.11) is the second-order ordinary differential equation with coefficients that are linear or quadratic in $\bar{z}$.

Denote the eigenvalues and the eigenfunctions of (7.11) by $\lambda = \lambda_{n,m,k}$, $g = g_{n,m,k}$, where $k = 0, 1, 2, \dots$ is the new "quantum number" indexing the eigenvalues. Then the vectors $I_{n,m}(g_{n,m,k})$ are the eigenvectors of $\widehat{H}_{200}$.

The reconstruction of eigenstates of the original Hamiltonian $\widehat{H}$ can now be obtained by applying the unitary operator $V$ (4.1) and another unitary operator $V_1$ transforming (5.1) into (5.12).

**Theorem 7.1.** *Let the frequencies $\omega$ and $\omega_0$ satisfy the resonance condition* (2.6), *and let the components of the perturbing homogeneous magnetic field $\mathcal{B}$ satisfy the secondary resonance condition* (5.13). *Then the eigenstates of the Penning trap Hamiltonian $\widehat{H}$* (2.1) *are given, up to $O(\varepsilon)$, by the integral formula*

$$\psi_{n,m,k} = \frac{1}{2\pi\hbar}\int_{\mathbb{C}} g_{n,m,k}(\bar{z}) \cdot VV_1\mathfrak{p}_{n,m}(z) \cdot l(|z|^2)\, d\bar{z}\, dz, \tag{7.12}$$

*where $g_{n,m,k}$ are the eigenfunctions of the second-order ordinary differential operator $\mathbb{H}$* (7.10) *with generators $\mathbb{A}_{0,\pm}$, $\mathbb{B}$, $\mathbb{B}^*$ given in* (6.12) *or* (6.13).

*The corresponding eigenvalues of $\widehat{H}$ read*

$$\frac{\omega_0}{\sqrt{2}}\Big(n + \frac{3}{2}\Big)\hbar + \varepsilon\Big(6m - n + \frac{3}{2}\Big)\hbar\mathcal{B}_3 + \varepsilon^2\lambda_{n,m,k} + O(\varepsilon^3),$$

*where $n \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, and $\lambda_{n,m,k}$ are the eigenvalues of $\mathbb{H}$.*

**Remark 7.1.** The family of vectors $VV_1\mathfrak{p}_{n,m}(z)$ in (7.12) can be viewed as a kind of "squeezed" coherent state of the secondary resonance algebra of the Penning trap. The squeezing is made by unitary transformations $V$, $V_1$, which reduce the original Hamiltonian (2.1) to the integrable form (5.12). The basic coherent states $\mathfrak{p}_{n,m}(z)$ in (7.12) correspond to irreducible representations of the non-Lie secondary resonance algebra (5.14).

Note that some coherent states of the Penning trap were studied in [25, 26].

**Remark 7.2.** In the integrand of (7.12), one still needs to compute the solutions $g_{n,m,k}$ of the ordinary differential equation (7.11). However, in the semiclassical approximation, this can be done explicitly, and (7.12) can be reduced to an integral of the form

$$\psi_{n,m,k} \simeq \frac{1}{\sqrt{2\pi\hbar}}\int_{\Lambda_{n,m,k}} e^{\frac{i}{\hbar}\{\text{Action}\}}\sqrt{\{\text{Jacobian}\}} \cdot VV_1\mathfrak{p}_{n,m} \cdot \{\text{Measure}\} + O(\hbar),$$

where $\Lambda_{n,m,k}$ are level lines, of the Wick symbol of the operator $\mathbb{H}$, that are subjected to a quantization condition and {Action}, {Jacobian}, {Measure} are taken on $\Lambda_{n,m,k}$ by a canonical procedure [23, 24]. The eigenvalues $\lambda_{n,m,k}$ are given, up to $O(\hbar^2)$, by the values of the Wick symbol on the level lines $\Lambda_{n,m,k}$.

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