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Statistical inference for generalized Ornstein-Uhlenbeck processes*

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Abstract: In this paper, we consider the problem of statistical inference for generalized Ornstein-Uhlenbeck processes of the type

$$X_t = e^{-\xi_t}\left(X_0 + \int_0^t e^{\xi_{u-}}\,du\right),$$

where ξ_s is a Lévy process. Our primal goal is to estimate the characteristics of the Lévy process ξ from the low-frequency observations of the process X. We present a novel approach towards estimating the Lévy triplet of ξ, which is based on the Mellin transform technique. It is shown that the resulting estimates attain optimal minimax convergence rates. The suggested algorithms are illustrated by numerical simulations.

Keywords and phrases: Lévy process, exponential functional, generalized Ornstein-Uhlenbeck process, Mellin transform.

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Contents

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1. Introduction

Let $(\xi_t)_{t\geq 0}$ be a compound Poisson process $(CPP_t)_{t\geq 0}$ with drift $\mu \in \mathbb{R}$, that is,

$$\xi_t = \mu t + CPP_t, \quad CPP_t := \sum_{k=1}^{N_t} Y_k,$$

where N_t is a Poisson process with intensity λ, and $Y_1, Y_2, \dots$ are i.i.d. r.v's independent of N_t. The process ξ_t is a Lévy process with triplet $(\mu, 0, \nu)$, where the Lévy measure for any subset $B \in \mathscr{B}(\mathbb{R})$ is equal to $\nu(B) := \lambda \cdot \mathbb{P}\{Y_1 \in B\}$. The main object of our study is the so-called generalized Ornstein-Uhlenbeck (GOU) process defined as

$$X_t = e^{-\xi_t}\left(X_0 + \int_0^t e^{\xi_{u-}}\, du\right), \quad t \geq 0. \tag{1}$$

The GOU processes have recently got much attention in the literature. A comprehensive study of the GOU processes and an extended list of references can be found in the thesis of Behme [2], where, in particular, it is shown that X_t satisfies the following SDE:

$$dX_t = X_{t-}dU_t + dt, \quad \text{where} \quad U_t := -\xi_t + \sum_{0<s\leq t}\left(e^{-\Delta\xi_s} - 1 + \Delta\xi_s\right).$$

The popularity of GOU processes is related to the fact they appear to be useful in several applications. For instance, the process (1) determines the volatility process in the COGARCH (COntinuous Generalized AutoRegressive Conditionally Heteroscedastic) model introduced in Klüppelberg et al. [19]. One important result from the theory of GOU processes is that, under some conditions, the process (1) is stationary with invariant stationary distribution given by the distribution of the following exponential functional of ξ :

$$A_\infty := \int_0^\infty e^{-\xi_t}\, dt. \tag{2}$$

For instance, if $Y_k, k = 1, 2, \dots$ are drawn from the exponential distribution, and $\mu > 0$, then A_∞ has a Beta-distribution, see Carmona, Petit and Yor [9]. This partial case is illustrated by Figure 1.

Let us note that the functional A_∞ appeared in such application areas as finance (see, e.g. the monograph by Yor [34]), carousel systems (see Litvak and

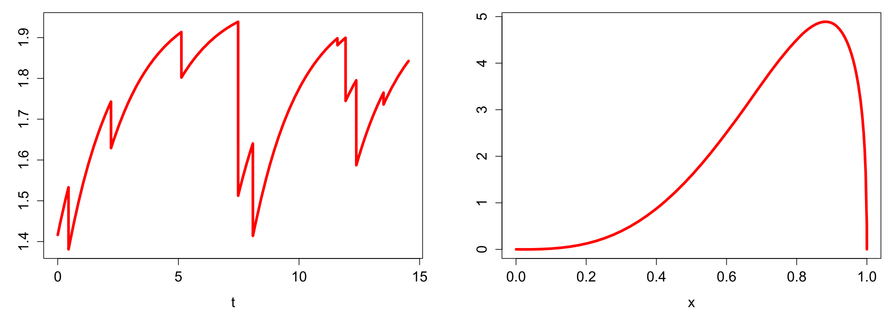

FIG 1. *Left picture: a realization of the generalized Ornstein-Uhlenbeck process for the case when* $Y_k, k = 1, 2, \dots$ *are drawn from the exponential distribution with density* $p(x) = be^{-bx}\mathbb{I}\{x > 0\}$, $b = 3$, $\mu = 0.5$ *and intensity parameter* $\lambda = 0.7$. *Right picture: density of the invariant measure of this process, which corresponds to* $B(b+1, \lambda/\mu)/\mu$ *distribution.*

Adan [24], Litvak and van Zwet [25]), self-similar fragmentations (see Bertoin and Yor [7]), and information transmission problems (especially TCP/IP protocol, see Guillemin, Robert and Zwart [14]). For the detailed discussion on the physical interpretations, we refer to Comtet, Monthus and Yor [10] and the dissertation by Monthus [27].

The properties of A_∞ have been widely studied in the literature and we refer to the survey by Bertoin and Yor [7] for a theoretical background of the exponential functionals. In particular, it is known that the Mellin transform of the density π of the exponential functional A_∞,

$$\mathcal{M}(z) := \mathbb{E}\left[A_\infty^{z-1}\right] = \int_0^\infty x^{z-1}\pi(x)\,dx,$$

satisfies the following recursive formula

$$\mathcal{M}(z) = \frac{\phi(z)}{z}\,\mathcal{M}(z+1), \tag{3}$$

where $\phi(z)$ is a Laplace exponent of the process ξ, i.e., $\phi(z) := -\log\mathbb{E}\left[e^{-z\xi_1}\right]$, and complex z is taken from the strip

$$\Upsilon := \Big\{z \in R :\ 0 < \mathrm{Re}(z) < \theta\Big\} \quad \text{with} \quad \theta := \sup\left\{x \geq 0 : \mathbb{E}[e^{-x\xi_1}] \leq 1\right\}. \tag{4}$$

The recursive formula (3) first appeared for real z in the paper by Maulik and Zwart [26]. The validity of (3) for complex z was recently shown by Kuznetsov, Pardo and Savov [21].

In this paper, we focus on the case when ξ_t is a subordinator, and the distribution of $Y_1, Y_2, \dots$ is absolutely continuous with respect to Lebesque measure on $\mathbb{R}_+$. Suppose that the process (1) is observed at equidistant time points $0 = t_0 < t_1 < \dots < t_n$. Since under some mild assumptions the process is stationary and the invariant distribution is given by the distribution of the exponential functional A_∞ (see Behme [2] and Fasen [12]), we assume that the random variables $X_k := X_{t_k},\ k = 1, \ldots, n$, have all the same distribution, which coincides with the distribution of A_∞. Our main goal is statistical inference on the Lévy triplet $(\mu, 0, \nu)$ based on the observations $X_{t_0}, \ldots, X_{t_n}$. More precisely, we will pursue the following two aims: (1) estimation of the drift term μ and the intensity parameter λ; (2) estimation of the jump size density of the compound Poisson process.

To the best of our knowledge, the statistical inference for GOU processes of the form (1) from their low-frequency observations has not been yet studied in the literature. In fact the resulting statistical problem is quite challenging and needs a careful treatment. Indeed, the only connection between the stationary distribution of a GOU process, which can be estimated from the data, and the parameters of the underlying Lévy process is given by the recurrent relation (3) which is rather implicit. The main idea of our procedure for estimating the parameters of the process ξ can be described as follows. First, by making use of (3), we estimate the Laplace exponent $\phi(z)$ at the points $z = u^\circ + \mathrm{i}v \in \Upsilon$, where $u^\circ > 0$ is fixed and v varies on the equidistant grid between εV_n and V_n (with $\varepsilon > 0$ and $V_n \to \infty$ as $n \to \infty$). Afterwards, we use the representation

$$\phi(u^\circ + \mathrm{i}v) \quad = \quad \lambda + \mu\,(u^\circ + \mathrm{i}v) - \mathcal{F}[\bar{\nu}](-v), \qquad v \in \mathbb{R}, \tag{5}$$

where $\bar{\nu}(dx) := e^{-u^\circ x}\nu(dx)$, and $\mathcal{F}[\bar{\nu}](v)$ stands for the Fourier transform of the measure $\bar{\nu}$, i.e., $\mathcal{F}[\bar{\nu}](v) := \int_{\mathbb{R}_+} e^{\mathrm{i}vx}\bar{\nu}(dx)$. Since $\mathcal{F}[\bar{\nu}](v) \to 0$ as $v \to \infty$ by the Riemann-Lebesgue lemma, upon taking real and imaginary parts of the left and right hand sides of (5), we are able to estimate the parameters μ and λ. With no doubt, the second aim, a complete recovering of the Lévy measure ν, is the most difficult task. Since the estimates of the parameters μ and λ are already obtained, we can estimate by (5) the Fourier transform $\mathcal{F}[\bar{\nu}](v)$ of $\bar{\nu}$ for v from $[-V_n, V_n]$. The last step of this procedure, the estimation of the Lévy measure ν, is based on the regularised inverse Fourier transform formula.

The above estimation algorithm bears some similarity to the spectral estimation algorithm introduced by Belomestny and Reiss [4, 5]. Let us also mention that the problem of statistical inference for Lévy processes (or some of their generalizations) observed at low frequency was the subject of many studies, see, e.g. Neumann and Reiß [28], Reiß [29], Kappus [17], Trabs [32] and Jongbloed et al. [16]. Note that the last reference deals with the Lévy-driven Ornstein-Uhlenbeck processes, which are not of the form (1).

The paper is organized as follows. In the next section, we formulate our main assumptions and give some examples. In Section 3, the main estimation algorithm is presented and discussed in details. Next, we analyze the convergence rates of the proposed algorithms in Section 4 and provide some numerical examples in Section 5. The proofs of our theoretical results are collected in Section 6.

2. Main setup

As it was explained in the introduction, we study the class of subordinators with finite Lévy measures as possible choice for the Lévy process (ξ_t). In terms of the Lévy triplet $(\mu, 0, \nu)$, this means that

$$\mu \geq 0, \qquad \nu(\mathbb{R}_-) = 0, \qquad \lambda := \nu(\mathbb{R}_+) < \infty. \tag{6}$$

A detailed discussion of the subordination theory as well as various examples of such processes (Gamma, Poisson, tempered stable, inverse Gaussian, Meixner processes, etc.), are given in [1, 6, 11, 30, 31]. Note that in the case of subordinators, the truncation function in the Lévy-Khinchine formula can be omitted, that is, the characteristic exponent of ξ is equal to

$$\psi(z) = \log \mathbb{E}\left[e^{\mathrm{i}z\xi_1}\right] = \mathrm{i}\mu z + \int_0^\infty \left(e^{\mathrm{i}zx} - 1\right)\nu(dx). \tag{7}$$

Later on, we also need the Laplace exponent of ξ, which is defined as

$$\phi(z) := -\log \mathbb{E}\left[e^{-z\xi_1}\right] = -\psi\left(\mathrm{i}z\right).$$

In the next proposition, we summarize the main properties of the functional $A_\infty = \int_0^\infty e^{-\xi_t} dt$ and the Laplace exponent $\phi(z)$ in our case.

Proposition 2.1. *(i) The random variable A_∞ admits a bounded density π and fulfills $\mathrm{E}\left[A_\infty^{s-1}\right] < \infty$ for all $s > 0$. If $\mu > 0$, then $0 < A_\infty \leq 1/\mu$ a.s.*

(ii) Moreover, the following relation holds for $\mathrm{Re}[z] > 0$,

$$\phi(z) = z\frac{\mathcal{M}(z)}{\mathcal{M}(z+1)}, \tag{8}$$

where $\mathcal{M}(z)$ is the Mellin transform of π.

(iii) The Laplace exponent $\phi(\cdot)$ has the following representation:

$$\phi(z) = \mu z + \int_0^\infty \left(1 - e^{-zu}\right)\nu(du) \tag{9}$$

$$= \lambda + \mu z - \int_0^\infty e^{-zx}\nu(dx). \tag{10}$$

In particular, taking $z = u^\circ + \mathrm{i}v$ with any fixed u° , we get

$$\phi(u^\circ + \mathrm{i}v) = \lambda + \mu \cdot (u^\circ + \mathrm{i}v) - \mathcal{F}[\bar{\nu}](-v), \qquad v \in \mathbb{R}, \tag{11}$$

where $\mathcal{F}[\bar{\nu}](v) := \int_{\mathbb{R}_+} e^{\mathrm{i}vx}\bar{\nu}(dx)$ is the Fourier transform of the measure $\bar{\nu}(dx) := e^{-u^\circ x}\nu(dx)$.

Proof. The proof of (i) is given in [9] as Proposition 2.1 and Remark 2.2. Formula (8) was firstly proved in [26] for real positive z such that $\phi(z) > 0$ and $\mathcal{M}(z+1) < \infty$. The case of complex z is considered in [21], where one can also find some generalizations of the formula (8) to the integrals with respect to Brownian motion with drift. In particular, applying Theorem 2 from [21], we get that (8) holds for any $z \in \Upsilon$. It would be a worth mentioning that in our case, the set Υ coincides with the positive half-plane (equivalently, the parameter θ is equal to infinity) due to

$$\mathbb{E}\left[e^{-x\xi_1}\right] = -\phi(x) = -\mu x - x\int_{\mathbb{R}_+} e^{-xu}\nu\left((u,+\infty)\right)du < 0, \quad \forall\, x > 0.$$

The last item (iii) directly follows from the Lévy-Khintchine formula. □

3. Estimation of the Lévy triplet

The first step of our estimation procedure consists in the estimation of the Laplace exponent $\phi(z)$ for $z = u^\circ + \mathrm{i}v$, where $u^\circ > 0$ is fixed and v varies. An estimator of $\phi(z)$ can be obtained from the recursive formula (8) for the Mellin transform of π. In fact, motivated by (10), we first estimate the Mellin transform $\mathcal{M}(z)$ via its empirical counterpart

$$\mathcal{M}_n(z) := \frac{1}{n}\sum_{k=1}^{n} X_k^{z-1} \tag{12}$$

and then define an estimate of the Laplace exponent $\phi(z)$ by

$$Y_n(z) = z\frac{\mathcal{M}_n(z)}{\mathcal{M}_n(z+1)}. \tag{13}$$

If the sequence $X_1, \ldots, X_n$ has some mixing properties, then we can expect that $Y_n(z) \to \phi(z)$ in probability.

3.1. Estimation of λ and μ

The general idea of the procedure described below is to estimate the Laplace exponent $\phi(\cdot)$ at the points $z = u^\circ + \mathrm{i}v$ with $v \in \mathbb{R}$ and then use the relation (11) for the estimation of parameters. Assuming that the measure $\bar{\nu}$ has a density, we can apply the Riemann-Lebesque lemma, which states that $\mathcal{F}[\bar{\nu}](-v) \to 0$ as $v \to +\infty$ (see, e.g.[18]). Therefore, we conclude from (11) that $\phi(u^\circ + \mathrm{i}v)$ is approximately (at least for large v) a linear function in v with the slope μ and the intercept term λ. This observation suggests that a properly weighted least-squares approach can be applied to estimate μ and λ. Let V_n be a sequence of positive real numbers and $w(\cdot)$ be a nonnegative weight function supported on $[\varepsilon, 1]$ with some small $\varepsilon > 0$. Define a scaled weight function $w_n(v) = V_n^{-1} w(v/V_n)$ and introduce the estimator of the parameter μ as the solution of the following optimization problem:

$$\mu_n := \arg\min_{\mu} \int_{\varepsilon V_n}^{V_n} w_n(v)\cdot\left(\operatorname{Im}\left[Y_n(u^\circ + \mathrm{i}v)\right] - \mu v\right)^2 dv,$$

with $Y_n(z)$ defined by (13). This estimator naturally arises by taking the imaginary parts from both sides of (11). Afterwards, we take real parts in (11), and define the estimate of the parameter λ by the solution of another optimization problem

$$\lambda_n := \arg\min_{\lambda} \int_{\varepsilon V_n}^{V_n} w_n(v) \cdot \left(\mathrm{Re}\left[Y_n(u^\circ + \mathrm{i}v)\right] - \lambda - \mu_n u^\circ\right)^2 \, dv.$$

The introduced estimators can be also represented in the following form:

$$\begin{aligned}
\mu_n &= \frac{\int_{\varepsilon V_n}^{V_n} w_n(v) \mathrm{Im}\left[Y_n(u^\circ + \mathrm{i}v)\right] \, dv}{\int_{\varepsilon V_n}^{V_n} v w_n(v) \, dv} = \int_{\varepsilon V_n}^{V_n} w_{\mu,n}(v) \mathrm{Im}\left[Y_n(u^\circ + \mathrm{i}v)\right] \, dv \\
\lambda_n &= \frac{\int_{\varepsilon V_n}^{V_n} w_n(v) \mathrm{Re}\left[Y_n(u^\circ + \mathrm{i}v)\right] \, dv}{\int_{\varepsilon V_n}^{V_n} w_n(v) \, dv} - \mu_n u^\circ \\
&= \int_{\varepsilon V_n}^{V_n} w_{\lambda,n}(v) \mathrm{Re}\left[Y_n(u^\circ + \mathrm{i}v)\right] \, dv - \mu_n u^\circ
\end{aligned}$$

with $w_{\mu,n}(v) := V_n^{-2} w_\mu(v/V_n)$ and $w_{\lambda,n}(v) := V_n^{-1} w_\lambda(v/V_n)$, where

$$w_\mu(\cdot) = c_{1,w}^{-1} w(\cdot), \qquad w_\lambda(\cdot) = c_{0,w}^{-1} w(\cdot), \qquad c_{i,w} = \int_\varepsilon^1 v^i w(v) \, dv, \quad i = 0, 1.$$

Taking into account the definition of the weight function $w_n(\cdot)$, we get also some equivalent representations of the estimators μ_n and λ_n

$$\begin{aligned}
\mu_n &= \arg\min_{\mu} \int_\varepsilon^1 w(\alpha) \Big(\mathrm{Im}[Y_n(u + \mathrm{i}\alpha V_n)] - \mu \alpha V_n\Big)^2 d\alpha \\
\lambda_n &:= \arg\min_{\lambda} \int_\varepsilon^1 w(\alpha) \Big(\mathrm{Re}[Y_n(u + \mathrm{i}\alpha V_n)] - \mu_n u - \lambda\Big)^2 d\alpha.
\end{aligned}$$

In practice, we need to replace the above integrals by sums. To this end, let the numbers $\alpha_1, \ldots, \alpha_M$ constitute an equidistant grid on the set $[\varepsilon, 1]$ for some $\varepsilon > 0$. We estimate the Mellin transform $\mathcal{M}(z)$ for all $z \in \{u^\circ + \mathrm{i}\alpha_m V_n,\ m = 1, \ldots, M\}$ and $z \in \{u^\circ - 1 + \mathrm{i}\alpha_m V_n,\ m = 1, \ldots, M\}$ and so get the estimates of the Laplace exponent at the discrete points $z = u^\circ + \mathrm{i}\alpha_m V_n$ (see above). Now we define an estimate of the parameter μ via

$$\begin{aligned}
\hat{\mu}_n &:= \arg\min_{\mu} \sum_{m=1}^{M} w(\alpha_m) \Big(\mathrm{Im}[Y_n(u^\circ + \mathrm{i}\alpha_m V_n)] - \mu \alpha_m V_n\Big)^2 && (14) \\
&= \frac{\sum_{m=1}^{M} w(\alpha_m) \alpha_m \, \mathrm{Im}[Y_n(u^\circ + \mathrm{i}\alpha_m V_n)]}{V_n \cdot \sum_{m=1}^{M} w(\alpha_m) \alpha_m^2}. && (15)
\end{aligned}$$

Afterwards, we estimate the parameter λ by

$$\begin{aligned}
\hat{\lambda}_n &:= \arg\min_{\lambda} \sum_{m=1}^{M} w(\alpha_m) \Big(\mathrm{Re}[Y_n(u^\circ + \mathrm{i}\alpha_m V_n)] - \hat{\mu}_n u - \lambda\Big)^2 && (16) \\
&= \frac{\sum_{m=1}^{M} w(\alpha_m) \mathrm{Re}[Y_n(u^\circ + \mathrm{i}\alpha_m V_n)]}{\sum_{m=1}^{M} w(\alpha_m)} - \hat{\mu}_n u^\circ. && (17)
\end{aligned}$$

The whole algorithm is described below.

Algorithm 1: Estimation of λ and μ

Data: n observations $X_1, \dots, X_n$ of the GOU process (X_t) observed at equidistant grid $j \cdot \Delta$, $j = 1, \dots, n$.

Initiate: Fix $V_n \to \infty$, $\varepsilon \in (0, 1)$ and $u^\circ > -1$.
Set $\alpha_j = \varepsilon + j \cdot (1 - \varepsilon)/M$, $j = 1, \dots, M$.
Fix a function $w(\cdot) \geq 0$ supported on $[\varepsilon, 1]$.
Denote $v_{m,n} := \alpha_m V_n$.

Algorithm:

1. Estimate the Mellin transform $\mathcal{M}(z) := \mathbb{E}\left[A_\infty^{z-1}\right]$ for $z \in \{u^\circ + \mathrm{i}v_{m,n}, 1 + u^\circ + \mathrm{i}v_{m,n}, m = 1, \dots, M\}$ via
$$\mathcal{M}_n(z) = \frac{1}{n} \sum_{k=1}^{n} X_k^{z-1}.$$
2. Estimate the Laplace exponent $\phi(z) := -\log \mathbb{E}\left[e^{-z\xi_1}\right]$ at the points $z \in \{u^\circ + \mathrm{i}v_{m,n}, m = 1, \dots, M\}$ by
$$Y_n(z) = z \frac{\mathcal{M}_n(z)}{\mathcal{M}_n(z+1)}.$$
3. Estimate μ by
$$\mu_n := \frac{\sum_{m=1}^{M} w(\alpha_m)\alpha_m \operatorname{Im}[Y_n(u^\circ + \mathrm{i}v_{m,n})]}{V_n \cdot \sum_{m=1}^{M} w(\alpha_m)\, \alpha_m^2}.$$
4. Estimate λ by
$$\lambda_n := \frac{\sum_{m=1}^{M} w(\alpha_m) \operatorname{Re}[Y_n(u^\circ + \mathrm{i}v_{m,n})]}{\sum_{m=1}^{M} w(\alpha_m)} - \mu_n u^\circ.$$

3.2. Estimation of the Lévy measure ν

As a result of Algorithm 1, we obtain the estimates μ_n and λ_n of the parameters μ and λ, respectively. Based on (11), we first define an estimate for the Fourier transform of $\bar{\nu}$ via

$$\hat{\mathcal{F}}[\bar{\nu}](-v) = -Y_n(u^\circ + \mathrm{i}v) + \mu_n \cdot (u^\circ + \mathrm{i}v) + \lambda_n. \tag{18}$$

Next we estimate the measure ν by a regularised Fourier inversion formula

$$\nu_n(x) = \frac{e^{u^\circ x}}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}vx} \hat{\mathcal{F}}[\bar{\nu}](-v) \mathcal{K}(-v/V_n)\, dv, \tag{19}$$

where $\mathcal{K}$ is a regularizing symmetric kernel supported on $[-1,1]$. Note that with a slight abuse of notation, we use ν also for the density of the Lévy measure, and ν_n for an estimate of this density. In what follows, we also use the notation $\bar{\nu}_n = e^{-u^\circ x}\nu_n$. The formal description of the algorithm is given below.

Algorithm 2: Estimation of ν

Data: n observations $X_1, \ldots, X_n$ of the GOU process (X_t) observed at equidistant grid points $j \cdot \Delta$, $j = 1, \ldots, n$.

Initiate: Fix $V_n \to \infty$ and $u^\circ > -1$.
Set $\alpha_m = -1 + 2 \cdot j/M$, $m = 0, \ldots, M$.
Fix a regularizing kernel $\mathcal{K}$ supported on $[-1,1]$.
Denote $v_{m,n} := \alpha_m V_n$.

Algorithm:

1-2 The first two steps coincide with ones of Algorithm 1.

3. Estimate $\mathcal{F}[\bar{\nu}](-v_{m,n})$ for $\bar{\nu}(dx) = e^{-u^\circ x}\nu(dx)$ by

$$\hat{\mathcal{F}}[\bar{\nu}](-v_{m,n}) = -Y_n(u + \mathrm{i}v_{m,n}) + \mu_n \cdot (u + \mathrm{i}v_{m,n}) + \lambda_n$$

for $m = 0, \ldots, M$.

4. Estimate ν by

$$\bar{\nu}_n(x) \quad = \quad e^{u^\circ x} \frac{1}{2\pi \cdot (1+M)} \sum_{m=0}^{M} e^{\mathrm{i}v_{m,n}x} \hat{\mathcal{F}}[\bar{\nu}](-v_{m,n}) \mathcal{K}(\alpha_m).$$

4. Convergence

In order to analyse the convergence properties of the estimates μ_n, λ_n and ν_n we need to further specify the class of Lévy processes (ξ_t).

Definiton 4.1. For $s \in \mathbb{N} \cup \{0\}$ and $R > 0$, let $\mathcal{G}(s, R)$ denote the set of all Lévy triplets $(\mu, 0, \nu)$, such that ν is supported on $\mathbb{R}_+$ and

$$\max\left\{ \nu(\mathbb{R}_+), \int_{\mathbb{R}} |v|^{2s} \left|\mathcal{F}[\bar{\nu}](v)\right|^2 \, dv \right\} \le R, \tag{20}$$

where $\bar{\nu}(dx) = e^{-u^\circ x}\nu(dx)$.

Note that if (20) holds, then $\bar{\nu}$ is s-times (weakly) differentiable with

$$\left\|\bar{\nu}^{(s)}\right\|_{\infty} \leq \frac{1}{2\pi} \int_{\mathbb{R}} |v|^{s} \left|\mathcal{F}[\bar{\nu}](-v)\right| \, dv < \infty. \tag{21}$$

It turns out that the convergence rates of the estimates μ_n, λ_n and ν_n crucially depend on the asymptotic behaviour of the Mellin transform of A_{∞}. In order to specify this behaviour, let us fix some $u^0 > 0$ and introduce two classes of probability densities:

$$\mathcal{P}(\beta) := \left\{ p : \exists L > 0 \text{ s.t. } \liminf_{|v| \to \infty} \left[|v|^{\beta} \left| \mathcal{M}[p](u^{\circ} + \mathrm{i}v) \right| \right] \geq L \right\}, \tag{22}$$

$$\mathcal{E}(\alpha) := \left\{ p : \exists L > 0 \text{ s.t. } \liminf_{|v| \to \infty} \left[e^{\alpha|v|} \left| \mathcal{M}[p](u^{\circ} + \mathrm{i}v) \right| \right] \geq L \right\}, \tag{23}$$

where $\alpha, \beta \in \mathbb{R}$, and for any density p, $\mathcal{M}[p]$ stands for the Mellin transform of p. Trivially, for any β, it holds $\mathcal{P}(\beta) \subset \mathcal{E}(\alpha)$ with any $\alpha \in \mathbb{R}$.

Before we formulate the main convergence results, let us look at some examples.

Example 4.2. Consider the class of Lévy processes with $\mu = 0$, $\sigma = 0$ and the Lévy density ν of the form

$$\nu(x) = \sum_{j=1}^{N} \left[\sum_{k=1}^{m_j} g_{jk} x^{k-1} \right] e^{-\rho_j x} \cdot \mathbb{I}\{x > 0\}$$

with $N, m_j \in \mathbb{N}$, $\rho_j > 0$, $g_{jk} > 0$. We can apply the Erdélyi lemma (see Section 3.2 from [13]) to derive

$$\int_{\mathbb{R}_+} x^{k-1} f(x) e^{ivx} dx \asymp c_1 v^{-k}, \qquad v \to \infty$$

for any exponentially decaying and smooth function f on $\mathbb{R}_+$, and some complex c_1 depending on f. Therefore, we conclude that

$$\left|\mathcal{F}[\bar{\nu}](-v)\right| = \left| \sum_{j=1}^{N} \sum_{k=1}^{m_j} \alpha_{jk} \int_{\mathbb{R}_+} x^{k-1} f_j(x) e^{ivx} dx \right| \asymp c_2 v^{-k^*},$$
$$\text{where} \quad f_j(x) = e^{-(\rho_j + u^{\circ})x}, \quad k^* := \arg\min_{k} \{\exists\, j : \ \alpha_{jk} \neq 0\},$$

where $c_2 > 0$ depends on u°. Hence for any $s < k^* - 1$, the condition (20) holds for some $R > 0$. Furthermore, taking into account the asymptotic behaviour of the Gamma function (see, e.g., formula 8.328 from [15]):

$$|\Gamma(u + \mathrm{i}v)| = \exp\left\{ -\frac{\pi}{2} v + \left(u - \frac{1}{2}\right) \ln v \right\} \cdot \sqrt{2\pi} \left(1 + o(1)\right), \qquad v \to \infty, \tag{24}$$

we derive

$$|\mathcal{M}(u^\circ + \mathrm{i}v)| \asymp \sqrt{2\pi} A^{1-u^\circ} \exp\left\{ -\frac{\pi}{2} v + \left(u^\circ - \frac{1}{2} + \sum_{j=1}^{N} \rho_j m_j + \sum_{j=1}^{K} \mathrm{Re}(\zeta_j) \right) \ln v \right\},$$

where $\zeta_1, \ldots, \zeta_K$ are the roots of the equation

$$\sum_{j=1}^{N} \sum_{k=1}^{m_j} \frac{g_{jk}(k-1)!}{(\rho_j + z)^k} = \lambda - \mu z,$$

see [20]. Therefore, for any $u^\circ > 1/2$, we conclude that $\pi \in \mathcal{E}(\pi/2)$ with any $L > 0$.

Example 4.3. Next, we provide an example of a Lévy process ξ_t with $A_\infty = \int_0^\infty e^{-\xi_t}\, dt$ having a density from $\mathcal{P}(\beta)$. Consider a subordinator $\mathcal{T}$ with drift $\mu > 0$ and the Lévy density

$$\nu(x) = ab \exp\{-bx\}\, I\{x > 0\}, \quad a, b > 0.$$

The exponential functional A_∞ of the process (ξ_t) has a density of the form

$$\pi(x) = C_1 x^b (1 - \mu x)^{(a/\mu)-1}\, I\{0 < x < 1/\mu\}$$

with some $C_1 > 0$, see [9]. In other words, A_∞ has the same distribution as ξ/μ, where the r.v. ξ has the Beta distribution with parameters $\alpha = b + 1$ and $\beta = a/\mu = \lambda/\mu$. The Mellin transform of the function $\pi(x)$ in the half-plane $\mathrm{Re}(s) > -\alpha$ is hence given by

$$\begin{aligned} \mathcal{M}(z) = \frac{\mathbb{E}\left[\xi^{z-1}\right]}{\mu^{z-1}} &= \frac{1}{\mu^{z-1}} \frac{B(z+\alpha-1, \beta)}{B(\alpha, \beta)} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \cdot \frac{1}{\mu^{z-1}} \frac{\Gamma(z+\alpha-1)}{\Gamma(z+\alpha+\beta-1)}. \end{aligned}$$

Using (24), we conclude that the Mellin transform of A_∞ has a polynomial decay in this case. More precisely,

$$|\mathcal{M}(u^\circ + \mathrm{i}v)| \asymp L \cdot |v|^{-\lambda/\mu} \quad \text{with} \quad L = \mu^{-u^\circ+1} \frac{\Gamma(\lambda/\mu + b + 1)}{\Gamma(b+1)},$$

as $|v| \to \infty$ and therefore $\pi \in \mathcal{P}(\lambda/\mu)$.

Example 4.4. Another interesting example arises when we the process ξ_t is proportional to a Poisson process N_t. In this case, we have the integral

$$A_\infty = \int_{\mathbb{R}+} q^{N_t} dt = \int_{\mathbb{R}+} e^{-\log(q) N_t} dt,$$

where $q \in (0,1)$. This integral can be also represented as $A_\infty = \sum_{n=0}^\infty q^n (T_{n+1} - T_n)$, where T_n are the jumping times of the process N_t. In [7], it is shown that

$$\mathcal{M}(z) = \Gamma(z) \cdot \frac{\prod_{j=0}^\infty \left(1 - q^{z+j}\right)}{\prod_{j=0}^\infty \left(1 - q^{1+j}\right)}.$$

Since for $z = u^\circ + \mathrm{i}v$,

$$\begin{aligned}\left|\prod_{j=0}^{\infty}\left(1-q^{z+j}\right)\right| &= \left|\exp\left\{\sum_{j=0}^{\infty}\ln\left(1-q^{z+j}\right)\right\}\right| \\ &= \left|\exp\left\{\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}q^{n(z+j)}\right\}\right| \\ &= \left|\exp\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{q^{nz}}{1-q^{n}}\right\}\right| \\ &= \exp\left\{\cos\left(v\ln(q)\right)\cdot\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{q^{nu^\circ}}{1-q^{n}}\right]\right\},\end{aligned}$$

and taking into account the asymptotic behaviour of the gamma-function (24), we conclude that $\pi \in \mathcal{E}(\pi/2)$.

Let us now formulate the main result concerning the convergence of the estimates μ_n and λ_n.

Theorem 4.5 (upper bounds for μ_n and λ_n)**.** *Let (ξ_t) be a Lévy process with a triplet from $\mathcal{G}(s,R)$. Suppose that the sequence $X_0, X_1, \ldots, X_n$ is α-mixing and strictly stationary. Denote the α-mixing coefficients of the sequence $X_0, X_1, \ldots, X_n$ by $\alpha(s)$.*

(i) Assume that the density π of A_∞ belongs to $\mathcal{P}(\beta)$ with some $\beta \in \mathbb{R}$ and $L > 0$, and moreover

$$\alpha(j) \lesssim e^{-j\alpha^*}, \quad j \in \mathbb{N}, \quad \text{for some} \quad \alpha^* \geq 0. \tag{25}$$

Then the quadratic risks of the estimates μ_n and λ_n, under the choice $V_n = n^{1/(2\beta+2s+3)}$, satisfy the following asymptotic relations

$$\mathrm{E}\left[\left|\mu_n - \mu\right|^2\right] \lesssim n^{-2(s+2)/(2\beta+2s+3)}\log(n)$$

and

$$\mathrm{E}\left[\left|\lambda_n - \lambda\right|^2\right] \lesssim n^{-2(s+1)/(2\beta+2s+3)}\log(n),$$

as $n \to \infty$.

(ii) If $\pi \in \mathcal{E}(\alpha)$ and

$$\alpha(j) \lesssim j^{-\alpha^*}, \quad j \in \mathbb{N}, \quad \text{for some} \quad \alpha^* \geq 2, \tag{26}$$

then the choice

$$V_n = \frac{1}{2\alpha}\log(n) - \frac{s+2}{\alpha}\log(\log(n)),$$

leads to the rates

$$\mathrm{E}\left[\left|\mu_n - \mu\right|^2\right] \lesssim \log^{-2(s+2)}(n),$$
$$\mathrm{E}\left[\left|\lambda_n - \lambda\right|^2\right] \lesssim \log^{-2(s+1)}(n).$$

Discussion. Theorem 4.5 shows that the estimates μ_n and λ_n converge polynomially fast to μ and λ, respectively, provided $\pi \in \mathcal{P}(\beta)$. For example, the rate for λ_n can be written (up to a logarithmic factor) as

$$n^{-2(s+1)/(2\beta+2(s+1)+1)}$$

indicating an inverse problem with regularity of order $s+1$ and ill-posedness of order β, latter being related to the decay of the Mellin transform of π. If $\pi \in \mathcal{E}(\beta)$, then the rates are logarithmic pointing out to a severely ill-posed statistical problem.

Proof. Proof is given in Section 6.1. □

In a similar way, we can establish the upper bounds for the risk of $\bar{\nu}_n$. In the theorem formulated below, the quality of the estimate $\bar{\nu}_n$ is measured in terms of the mean integrated squared error (MISE):

$$\mathrm{MISE}(\bar{\nu}_n) := \mathbb{E}\left[\int_{\mathbb{R}} |\bar{\nu}_n(x) - \bar{\nu}(x)|^2 \, dx\right].$$

Theorem 4.6 (upper bounds for $\bar{\nu}_n$)**.** *Let the assumptions of Theorem 4.5 be fulfilled and let $\mathcal{K}(\cdot)$ be a kernel satisfying*

$$|1 - \mathcal{K}(x)| \leq A|x|^s, \qquad \forall x \in \mathbb{R} \setminus \{0\} \tag{27}$$

with some $A > 0$.

(i) Assume that the density of A_∞ belongs to $\mathcal{P}(\beta)$ with some $\beta \in \mathbb{R}$ and $L > 0$, and moreover

$$\alpha(j) \lesssim e^{-j\alpha^*}, \quad j \in \mathbb{N}, \quad \textit{for some} \quad \alpha^* > 0.$$

Then under the choice $V_n = n^{1/(2\beta+2s+3)}$, the MISE *of the estimator $\bar{\nu}_n$ is bounded as follows:*

$$\mathrm{MISE}(\bar{\nu}_n) \lesssim n^{-2s/(2\beta+2s+3)}, \quad n \to \infty.$$

(ii) If the density of A_∞ belongs to the class $\mathcal{E}(\alpha)$ and

$$\alpha(j) \lesssim j^{-\alpha^*}, \quad j \in \mathbb{N}, \quad \textit{for some} \quad \alpha^* \geq 2,$$

then under the choice

$$V_n = \frac{1}{2\alpha} \log(n) - \frac{s+2}{\alpha} \log(\log(n))$$

we have

$$\mathrm{MISE}(\bar{\nu}_n) \lesssim \log^{-2s}(n), \quad n \to \infty.$$

Proof. Proof is given in Section 6.2. □

The next theorem shows that the rates obtained in the previous theorem are optimal up to a logarithmic factor.

Theorem 4.7 (lower bounds for $\bar{\nu}_n$). *Fix some $s \in \mathbb{N} \cup \{0\}$, $R > 0$, $\alpha > 0$, $\beta > 0$, $L > 0$ and define*

$$\varphi_n(\pi) := \varphi_n(\pi, \rho) = \begin{cases} n^{s/(2\beta+2s+3)} \log^{-\rho}(n), & \text{if } \pi \in \mathcal{P}(\beta), \\ \log^s(n), & \text{if } \pi \in \mathcal{E}(\alpha), \end{cases}$$

for any $\rho > 0$ and any probability density $\pi \in \mathcal{P}(\beta) \cup \mathcal{E}(\alpha)$, Then for some $\rho^* > 0$, it holds

$$\inf_{\bar{\nu}_n} \sup_{\substack{\mathcal{T} \in \mathcal{G}(s,R) \\ \pi_{\mathcal{T}} \in \mathcal{P}(\beta) \cup \mathcal{E}(\alpha)}} \left\{ \varphi_n^2(\pi_{\mathcal{T}}, \rho^*) \cdot \mathbb{E}_{\pi_{\mathcal{T}}^{\otimes n}} \left[\int_{\mathbb{R}} |\bar{\nu}_n(x) - \bar{\nu}(x)|^2 \, dx \right] \right\} > 0, \tag{28}$$

where the infimum is taken over all possible estimates $\bar{\nu}_n$ of the function $\bar{\nu}$ based on an i.i.d. sample $X_1, \ldots, X_n$ from the distribution $\pi_{\mathcal{T}}$ of $A_\infty := \int_0^\infty e^{-\xi_t}\, dt$ such that the Lévy triplet $\mathcal{T}$ of (ξ_t) belongs to $\mathcal{G}(s, R)$.

Proof. Proof is given in Section 6.3. □

An important condition of Theorems 4.5 and 4.6 is (25), which means that the sequence $X_0, X_1, \ldots, X_n$ is exponentially α-mixing. Since β-mixing coefficient between two sigma-algebras is larger than or equal to the corresponding α-mixing coefficient, it is sufficient to show that $X_0, X_1, \ldots, X_n$ is an exponentially β-mixing sequence (see Section 1.1 from [8]). For the case of the GOU processes (1), the latter question was addressed in [12]. The sufficient conditions for exponential β-mixing given in [12] are:

1. the distribution of A_∞ has a Pareto-like asymptotic behaviour, that is,
$$\mathbb{P}\{A_\infty > x\} \asymp C x^{-\alpha} \quad \text{as} \quad x \to \infty$$
with some $\alpha > 0$ and $C > 0$;
2. there exist $A > 0$, $B > A$ and $h > 0$ such that $\psi(A) = 0, \psi(B) < \infty$ with ψ given in (7), and
$$\mathbb{E} \left| e^{-\xi_h} \int_0^h e^{\xi_{u-}} du \right|^B < \infty.$$

As it is proved in [23], both conditions are guaranteed by the positiveness of μ and the existence of a positive zero of the function $\psi(\cdot)$. We refer also to [22] for some further results in this direction.

5. Simulation study

Example 1. Consider the subordinator τ_t with the Lévy density

$$\nu(x) = ab \exp\{-bx\} \, \mathbb{I}\{x > 0\}, \quad a, b > 0. \tag{29}$$

Note that in this case, $\lambda = \int_{\mathbb{R}_+} \nu(u)du = a$. Define a Lévy process

$$\xi_t = \mu t + \sigma W_t + \tau_t, \tag{30}$$

where W_t is a Brownian motion. The Laplace exponent of ξ_t is given by

$$\phi(z) = z\left(\mu - \frac{1}{2}\sigma^2 z + \frac{a}{b+z}\right). \tag{31}$$

In [9], it is shown that the exponential functional $A_\infty = \int_0^\infty e^{-\xi_t}\, dt$ is finite for any μ and σ, and moreover the density function π of A_∞ satisfies the following differential equation

$$-\frac{\sigma^2}{2}x^2\pi''(x) + \left[\left(\frac{\sigma^2}{2}(3-b)+\mu\right)x - 1\right]\pi'(x) + \left[(1-b)\left(\frac{\sigma^2}{2}+\mu\right) - a + \frac{b}{x}\right]\pi(x) = 0. \tag{32}$$

Some special cases are considered below:

1. In the case $\mu = 0$, $\sigma = 0$ (pure jump process), this equation has a solution

$$\pi_1(x) = Cx^b e^{-ax}\, I\{x > 0\}, \tag{33}$$

   and therefore $A_\infty \stackrel{d}{=} G(b+1, a)$, where $G(\alpha, \beta)$ is a Gamma distribution with shape parameter α and rate β.
2. If $\mu > 0$, $\sigma = 0$ (pure jump process with drift), then

$$\pi_2(x) = Cx^b(1-\mu x)^{(a/\mu)-1}\, I\{0 < x < 1/\mu\}. \tag{34}$$

   In this situation $A_\infty \stackrel{d}{=} B(b+1, a/\mu)/\mu$, where $B(\alpha, \beta)$ is a Beta - distribution.
3. In the case $\mu \neq 0$, $\sigma \neq 0$, the equation (32) also allows for the closed form solution. Assuming for simplicity $\sigma^2/2 = 1$, $\mu = -(b+1)$, we get the solution of (32) in the following form:

$$\pi_3(x) = C\, x^{b-1/2} \exp\left\{\frac{1}{2x}\right\} I_\mu\left(\frac{1}{2x}\right), \tag{35}$$

   where we denote by I_μ the modified Bessel function of the first kind, $\mu = \sqrt{a+1/4}$, and the constant C is later chosen to guarantee the condition $\int_0^\infty \pi_3(x)dx = 1$.

For our numerical study, we assume that the data are generated from the distribution of (2), where the process (ξ_t) is defined by (30) with $\mu = 1.8, \sigma = 0$, and the subordinator τ_t in the form (29) with $a = 0.7$, $b = 0.2$. A sample from the distribution of the integral A_∞ can be simulated from the corresponding

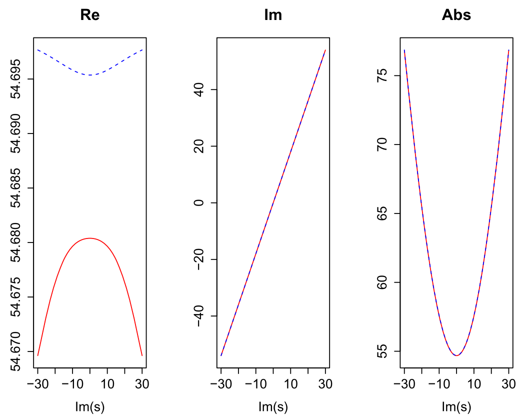

FIG 2. *Plots of theoretical (blue dashed) and empirical (red solid) Laplace exponents in Example 1. Real, imaginary parts and absolute values are presented. Note that the difference between dotted blue curve and red curve is quite small on all plots.*

Beta-distribution, see (34). The aim of the estimation procedure is to estimate the parameters μ and $\lambda = a$.

We apply Algorithm 1 with $u^\circ = 29$, $V_n = 30$, and $w(x) = \mathbb{I}\{x \in (0.001, 1)\}$. Figure 2 graphically compares the proposed estimator of the Laplace exponent $\phi(u^\circ + \mathrm{i}v)$ with its theoretical values $(\mu + a/(b + u^\circ + \mathrm{i}v)) \cdot (u^\circ + \mathrm{i}v)$. Estimates for the parameters μ and $\lambda = a$ are given in (15) and (17), respectively. The boxplots of this estimates based on 25 simulation runs are presented on Figure 3.

Remark 5.1. Let us comment on the choice of the tuning parameters u° and V_n. In fact the estimates for the parameters μ and $\lambda = a$ are not sensitive to the choice of the parameter u°. To see this, we run the estimation algorithm for different values of u°. The results in form of the corresponding boxplots are shown in Figure 4. While the quality of the estimate μ_n can be slightly improved by increasing u°, the estimate λ_n behaves in a similar way for different values of u°. For the choice of the cut-off parameter V_n one can use either the asymptotic formulas of Theorem 4.5 or some adaptive procedures.

Example 2. Consider the compound Poisson process

$$\xi_t = -\log q \left(\sum_{k=1}^{N_t} \eta_k \right),$$

where $q \in (0,1)$ is fixed, N_t is a Poisson process with intensity λ and η_k are i.i.d. random variables with a distribution $\mathcal{L}$. The integral A_∞ admits the representation

$$A_\infty = \int_0^\infty q^{-\xi_t} dt = \sum_{n=0}^{\infty} q^{S_n} \left(T_{n+1} - T_n \right),$$

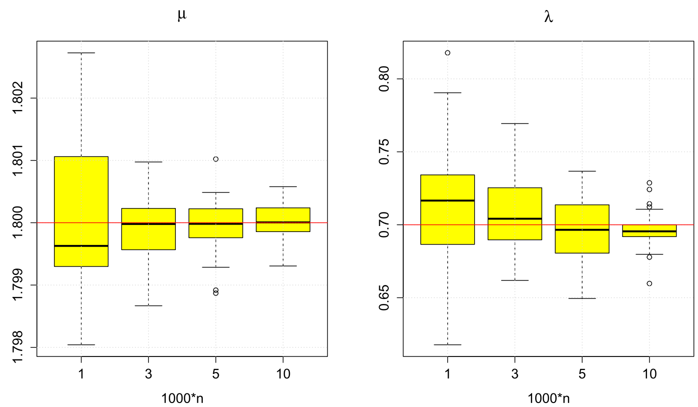

FIG 3. *Boxplots for the estimates of μ and λ for different sample sizes n based on 25 simulation runs.*

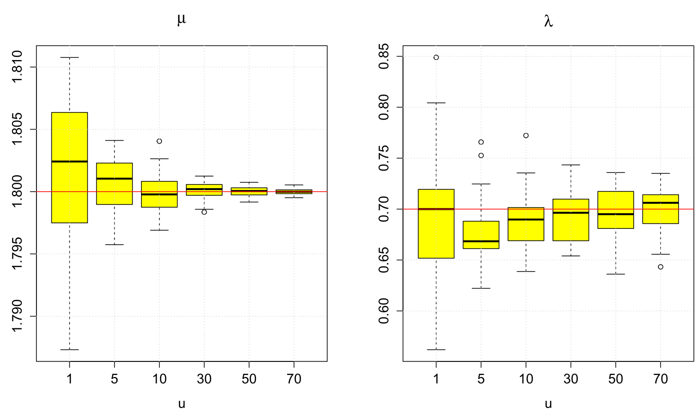

FIG 4. *Boxplots for the estimates of μ and λ for different values of the parameter u° based on 25 simulation runs and sample size $n = 3000$.*

where T_n is the jump time of N, i.e., $T_n = \inf\{t : N_t = n\}$, and $S_n = \sum_{k=1}^n \eta_k$. Note that if η_k take only positive values, then ξ_t is a subordinator.

Fix some positive α and consider the case when $\mathcal{L}$ is the standard normal distribution truncated on the interval $(\alpha, +\infty)$. The density function of $\mathcal{L}$ is given by

$$p_{\mathcal{L}}(x) = p(x)/(1 - F(\alpha)),$$

where $p(\cdot)$ and $F(\cdot)$ are the density and the distribution functions of the standard Normal distribution. In this case, the Laplace exponent of ξ_t is equal to

$$\phi(z) = \lambda \left[1 - \frac{1 - F\left(\alpha + (\log q) z\right)}{1 - F\left(\alpha\right)} \, \exp\left\{ -\frac{\left(\log q\right)^2 z^2}{2} \right\} \right],$$

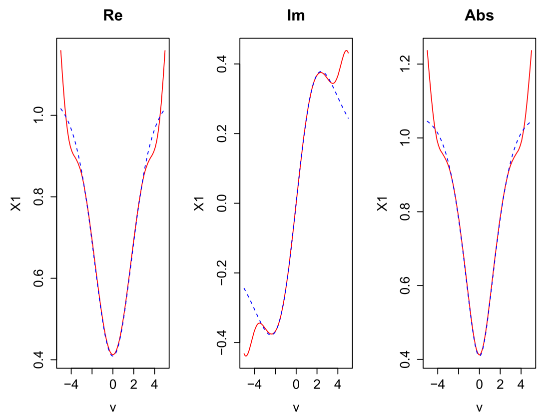

FIG 5. *Plots of theoretical (blue dashed) and empirical (red solid) Laplace exponents for Example 2. Graphs present real, imaginary and absolute values. For $v \in [-3, 3]$ the curves are visually indistinguishable.*

where the function $F(\cdot)$ can be calculated for complex arguments from the error function:

$$F(z) := \frac{1}{2}\left(\operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + 1\right), \quad \text{where} \quad \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds.$$

In this example, we aim to estimate the Lévy measure of the process (ξ_t), which is given by

$$\nu(dx) = \frac{\lambda}{1 - F(\alpha)} p(x) \mathbb{I}\{x > \alpha\} dx.$$

For our numerical study, we take $q = 0.5, \alpha = 0.1$, and $\lambda = 1$. First, we estimate the Laplace exponent by (13). The quality of the corresponding estimate at the complex points $z = u^\circ + \mathrm{i}v$ with $u^\circ = 1$ and $v \in [-5, 5]$ can be visually seen in Figure 5.

Next, we proceed with the estimation of the Fourier transform of the measure $\bar{\nu}(x) := e^{-u^\circ x}\nu(x)$ by applying (18). For the last step of the Algorithm 2, i.e. the reconstruction of the Lévy measure by (19), we follow [3] and use the so-called flat-top kernel, which is defined as follows:

$$\mathcal{K}(x) = \begin{cases} 1, & |x| \le 0.05, \\ \exp\left(-\frac{e^{-1/(|x|-0.05)}}{1-|x|}\right), & 0.05 < |x| < 1, \\ 0, & |x| \ge 1. \end{cases}$$

The quality of the resulting estimate $\bar{\nu}_n$ is shown in Figure 6.

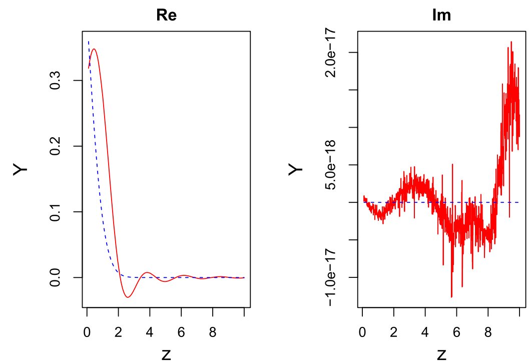

FIG 6. *Left: plots of the function $\bar{\nu}(x)$ (blue dashed line) and its estimate $\mathrm{Re}(\bar{\nu}_n(x))$ (red solid). Right: the imaginary part of the estimate $\bar{\nu}_n(x)$ (red solid) and the line $Y = 0$ (blue dashed line). The right plot shows that the imaginary part of our estimate is quite small.*

6. Proofs

6.1. Upper bounds for the quadratic risks of μ_n and λ_n

The next proposition is the main technical result for this section.

Proposition 6.1. *Let ξ_t be a Lévy triplet from $\mathcal{G}(s, R)$. Suppose that the sequence $X_0, X_1, \ldots, X_n$ of observations of the exponential functional $A_\infty := \lim_{T\to\infty} A_T = \int_0^\infty e^{-\xi_t}\, dt$ is α-mixing and strictly stationary. Denote the mixing coefficients of the sequence $X_0, X_1, \ldots, X_n$ by $\alpha(s)$.*

Then for any $p \in \{0, 1, \ldots, n\}$ we have

$$\mathrm{E}\left[|\mu_n - \mu|^2\right] \lesssim \frac{p}{n}\int_0^\infty \frac{|u^\circ + \mathrm{i}v|^2}{|\mathcal{M}(u^\circ + \mathrm{i}v + 1)|^2}\,|w_{\mu,n}(v)|^2\, dv + \sum_{j=p+1}^{n} \alpha(j)\left[\int_0^\infty \frac{|u^\circ + \mathrm{i}v|\,|w_{\mu,n}(v)|}{|\mathcal{M}(u^\circ + \mathrm{i}v + 1)|}\, dv\right]^2 + \|\bar{\nu}^{(s)}\|_\infty^2 \|\mathcal{F}^{-1}[w_{\mu,n}(\cdot)/(-i\cdot)^s]\|_{L^1}^2,$$

$$\mathrm{E}\left[|\lambda_n - \lambda|^2\right] \lesssim \frac{p}{n}\int_0^\infty \frac{|u^\circ + \mathrm{i}v|^2}{|\mathcal{M}(u^\circ + \mathrm{i}v + 1)|^2}\,|w_{\lambda,n}(v)|^2\, dv + \sum_{j=p+1}^{n} \alpha(j)\left[\int_0^\infty \frac{|u^\circ + \mathrm{i}v|\,|w_{\lambda,n}(v)|}{|\mathcal{M}(u^\circ + \mathrm{i}v + 1)|}\, dv\right]^2 + \|\bar{\nu}^{(s)}\|_\infty^2 \|\mathcal{F}^{-1}[w_{\lambda,n}(\cdot)/(-i\cdot)^s]\|_{L^1}^2,$$

provided $\sum_{j=1}^\infty \alpha^{1-\epsilon}(j) < \infty$ for some $\epsilon > 0$ and the sequence V_n satisfies

$$\sup_{v\in[0,V_n]} \frac{1}{|\mathcal{M}(u^\circ + \mathrm{i}v + 1)|} = o(n^{1/2}). \tag{36}$$

Proof. **1.** Denote $Y(z) := \phi(z) = z \cdot \mathcal{M}(z)/\mathcal{M}(z+1)$, then

$$\mu = \int_0^\infty w_{\mu,n}(v) \mathrm{Im}\left[Y(u^\circ + \mathrm{i}v)\right] dv + \int_0^\infty w_{\mu,n}(v) \mathrm{Im}[\mathcal{F}[\bar{\nu}](-v)]\, dv$$

and we have

$$\begin{aligned} \mu_n - \mu &= \int_0^\infty w_{\mu,n}(v) \mathrm{Im}\left[Y_n(u^\circ + \mathrm{i}v) - Y(u^\circ + \mathrm{i}v)\right] dv \\ &\qquad\qquad - \int_0^\infty w_{\mu,n}(v) \mathrm{Im}[\mathcal{F}[\bar{\nu}](-v)]\, dv \\ &= \mathrm{Im}\left[\int_0^\infty w_{\mu,n}(v) S_n(u^\circ + \mathrm{i}v)\, dv\right] - \mathrm{Im}[D_n(u^\circ)] \end{aligned}$$

with

$$S_n(u^\circ + \mathrm{i}v) = Y_n(u^\circ + \mathrm{i}v) - Y(u^\circ + \mathrm{i}v), \quad D_n(u^\circ) = \int_0^\infty w_{\mu,n}(v) \mathcal{F}[\bar{\nu}](-v) dv.$$

Note that

$$\mathbb{E}\left[(\mu_n - \mu)^2\right] \le 2 \cdot \mathbb{E}\left[\left(\mathrm{Im}\left[\int_0^\infty w_{\mu,n}(v) S_n(u^\circ + \mathrm{i}v)\, dv\right]\right)^2\right] + 2\left|D_n(u^\circ)\right|^2.$$

2. Since

$$\begin{aligned} \frac{S_n(z)}{z} &= \frac{\mathcal{M}_n(z)}{\mathcal{M}_n(z+1)} - \frac{\mathcal{M}(z)}{\mathcal{M}(z+1)} \\ &= \frac{\mathcal{M}_n(z)\mathcal{M}(z+1) - \mathcal{M}(z)\mathcal{M}_n(z+1)}{\mathcal{M}_n(z+1)\mathcal{M}(z+1)} \\ &= \frac{\left[\mathcal{M}_n(z) - \mathcal{M}(z)\right]\mathcal{M}_n(z+1) - \left[\mathcal{M}_n(z+1) - \mathcal{M}(z+1)\right]\mathcal{M}_n(z)}{\mathcal{M}_n(z+1)\mathcal{M}(z+1)} \\ &= \frac{\left[\mathcal{M}_n(z) - \mathcal{M}(z)\right]}{\mathcal{M}(z+1)} - \frac{Y_n(z)}{z}\frac{\left[\mathcal{M}_n(z+1) - \mathcal{M}(z+1)\right]}{\mathcal{M}(z+1)} \\ &= \frac{\left[\mathcal{M}_n(z) - \mathcal{M}(z)\right]}{\mathcal{M}(z+1)} - \frac{S_n(z)}{z}\frac{\left[\mathcal{M}_n(z+1) - \mathcal{M}(z+1)\right]}{\mathcal{M}(z+1)} \\ &\quad - \frac{Y(z)}{z}\frac{\left[\mathcal{M}_n(z+1) - \mathcal{M}(z+1)\right]}{\mathcal{M}(z+1)}, \end{aligned}$$

we get

$$S_n \cdot (1 + R_{2,n}) = -Y \cdot R_{2,n} + R_{1,n}$$

with

$$R_{1,n}(z) = z\,\frac{\left[\mathcal{M}_n(z) - \mathcal{M}(z)\right]}{\mathcal{M}(z+1)}, \quad R_{2,n}(z) = \frac{\left[\mathcal{M}_n(z+1) - \mathcal{M}(z+1)\right]}{\mathcal{M}(z+1)}.$$

Following the lines of the proof of Theorem 1.5 from [8], we get

$$\mathrm{E}\left[|\mathcal{M}_n(z)-\mathcal{M}(z)|^2\right]=\frac{1}{n^2}\sum_{0\le k,j\le n-1}\mathrm{Cov}\left(X_k^{z-1},X_j^{z-1}\right)$$
$$=\frac{1}{n}\mathrm{Var}\left(X_0^{z-1}\right)+\frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\mathrm{Cov}\left(X_0^{z-1},X_k^{z-1}\right). \quad (37)$$

Note that the sum in the last representation converges as $n\to\infty$, because by Davydov's inequality (see Corollary 1.1. from [8])

$$\left|\mathrm{Cov}\left(X_0^{z-1},X_k^{z-1}\right)\right|\le\frac{2r}{r-2}\left(2\alpha(k)\right)^{(r-2)/r}\left(\mathbb{E}\left[X_0^{(u^\circ-1)r}\right]\right)^{2/r}, \quad (38)$$

and therefore the series $\sum\left(X_0^{z-1},X_k^{z-1}\right)$ is convergent if $r=2/\varepsilon$.

We have $\mathrm{E}[|\mathcal{M}_n(u^\circ+\mathrm{i}v)-\mathcal{M}(u^\circ+\mathrm{i}v)|^2]\lesssim n^{-1}$ uniformly in $v\in\mathbb{R}$. As a result

$$\mathrm{E}\left[|R_{2,n}(u^\circ+\mathrm{i}v)|^2\right]\lesssim\frac{1}{n\cdot|\mathcal{M}(u^\circ+\mathrm{i}v+1)|^2}.$$

The condition (36) implies now that $\sup_{v\in[0,V_n]}|R_{2,n}(u^\circ+\mathrm{i}v)|^2=o_P(1)$. Furthermore, we have

$$\mathrm{Var}\left[\int_0^\infty R_{1,n}(u^\circ+\mathrm{i}v)w_{\mu,n}(v)\,dv\right]$$
$$=\int_0^\infty\int_0^\infty\frac{\mathrm{Cov}(\mathcal{M}_n(u^\circ+\mathrm{i}v_1),\mathcal{M}_n(u^\circ+\mathrm{i}v_2))}{\mathcal{M}(u^\circ+\mathrm{i}v_1+1)\overline{\mathcal{M}(u^\circ+\mathrm{i}v_2+1)}}$$
$$\cdot(u^\circ+\mathrm{i}v_1)(u^\circ-\mathrm{i}v_2)w_{\mu,n}(v_1)\,w_{\mu,n}(v_2)\,dv_1dv_2.$$

Similar to (37), we consider a representation

$$\mathrm{Cov}(\mathcal{M}_n(u^\circ+\mathrm{i}v_1),\mathcal{M}_n(u^\circ+\mathrm{i}v_2))$$
$$=\frac{1}{n}\left[g_0(v_1,v_2)+2\sum_{j=1}^{p}g_j(v_1,v_2)+2\sum_{j=p+1}^{n-1}g_j(v_1,v_2)\right],$$

where $g_j(v_1,v_2):=(1-j/n)\cdot\mathrm{Cov}(X_0^{u^\circ+\mathrm{i}v_1-1},X_j^{u^\circ+\mathrm{i}v_2-1}),\ j=0\ldots(n-1)$. Applying once more Davydov's inequality, we get

$$|g_j(v_1,v_2)|\le\frac{2r}{r-2}\left(2\alpha(j)\right)^{(r-2)/r}\left(\mathbb{E}\left[X_0^{(u^\circ-1)r}\right]\right)^{2/r}, \quad (39)$$

Now using the Cauchy-Schwarz inequality we get

$$\mathrm{Var}\left[\int_0^\infty R_{1,n}(u^\circ+\mathrm{i}v)w_{\mu,n}(v)\,dv\right]\lesssim p\int_0^\infty\frac{|u^\circ+\mathrm{i}v|^2|w_{\mu,n}(v)|^2}{|\mathcal{M}(u^\circ+\mathrm{i}v+1)|^2}\,dv$$
$$+\sum_{j=p+1}^{n}\alpha(j)\left[\int_0^\infty\frac{|u^\circ+\mathrm{i}v||w_{\mu,n}(v)|}{|\mathcal{M}(u^\circ+\mathrm{i}v+1)|}\,dv\right]^2.$$

Finally using the fact $\sup_{v\in[0,V_n]}|R_{2,n}(u^\circ+\mathrm{i}v)|^2=o_P(1)$, we derive

$$\mathrm{Var}\left[\int_0^\infty S_n(u^\circ+\mathrm{i}v)w_{\mu,n}(v)\,dv\right]\lesssim p\int_0^\infty \frac{|u^\circ+\mathrm{i}v|^2|w_{\mu,n}(v)|^2}{|\mathcal{M}(u^\circ+\mathrm{i}v+1)|^2}\,dv$$
$$+\sum_{j=p+1}^{n}\alpha(j)\left[\int_0^\infty \frac{|u^\circ+\mathrm{i}v||w_{\mu,n}(v)|}{|\mathcal{M}(u^\circ+\mathrm{i}v+1)|}\,dv\right]^2.$$

3. Turn now to the term D_n. By Plancherel's identity

$$\begin{aligned}\left|\int_0^\infty w_{\mu,n}(v)\mathcal{F}[\bar\nu](-v)dv\right| &= \left|\int_0^\infty \frac{w_{\mu,n}(v)}{(-\mathrm{i}v)^s}\left[(-\mathrm{i}v)^s\mathcal{F}[\bar\nu](-v)\right]dv\right|\\ &= \left|\int_0^\infty \frac{w_{\mu,n}(v)}{(-\mathrm{i}v)^s}\left[\mathcal{F}[\bar\nu^{(s)}](-v)\right]dv\right|\\ &= 2\pi\left|\int_{-\infty}^\infty \bar\nu^{(s)}(x)\overline{\mathcal{F}^{-1}[w_{\mu,n}(\cdot)/(-i\cdot)^s](x)}dx\right|\\ &\le 2\pi\|\bar\nu^{(s)}\|_\infty\|\mathcal{F}^{-1}[w_{\mu,n}(\cdot)/(-i\cdot)^s]\|_{L^1}.\end{aligned}$$

□

Proof of Theorem 4.5

(i) Suppose that $\pi\in\mathcal{P}(\beta)$ and $\alpha(j)\lesssim e^{-j\alpha^*}$, then by taking $p=c\log(n)$ for c large enough, we arrive at

$$\begin{aligned}\mathrm{E}\left[|\mu_n-\mu|^2\right] &\lesssim \frac{V_n^{-4}\log(n)}{n}\int_0^{V_n}|v|^{2\beta+2}\left|w_\mu(v/V_n)\right|^2\,dv+V_n^{-2(s+2)}\\ &\lesssim n^{-1}\log(n){V_n}^{2\beta-1}+V_n^{-2(s+2)},\\ \mathrm{E}\left[|\lambda_n-\lambda|^2\right] &\lesssim \frac{V_n^{-2}\log(n)}{n}\int_0^{V_n}|v|^{2\beta+2}\left|w_\lambda(v/V_n)\right|^2\,dv+V_n^{-2(s+1)}\\ &\lesssim n^{-1}\log(n){V_n}^{2\beta+1}+V_n^{-2(s+1)}\end{aligned}$$

By taking $V_n=n^{1/(2\beta+2s+3)}$, we get

$$\mathrm{E}\left[|\mu_n-\mu|^2\right]\lesssim n^{-2(s+2)/(2\beta+2s+3)}\log(n)$$

and

$$\mathrm{E}\left[|\lambda_n-\lambda|^2\right]\lesssim n^{-2(s+1)/(2\beta+2s+3)}\log(n).$$

(ii) Suppose that $\pi\in\mathcal{E}(\alpha)$, then by taking $p=0$, we get

$$\begin{aligned}\mathrm{E}\left[|\mu_n-\mu|^2\right] &\lesssim \frac{V_n^{-4}}{n}\left[\int_0^{V_n}\frac{|u^\circ+\mathrm{i}v|\,|w_\mu(v/V_n)|}{\exp(-\alpha|v|)}\,dv\right]^2+V_n^{-2(s+2)}\\ &\lesssim \frac{1}{n}\exp(2\alpha V_n)+V_n^{-2(s+2)},\end{aligned}$$

$$\mathrm{E}\left[|\lambda_n - \lambda|^2\right] \lesssim \frac{V_n^{-2}}{n}\left[\int_0^{V_n} \frac{|u^\circ + \mathrm{i}v|\,|w_\lambda(v/V_n)|}{\exp(-\alpha|v|)}\,dv\right]^2 + V_n^{-2(s+1)}$$
$$\lesssim \frac{V_n^2}{n}\exp(2\alpha V_n) + V_n^{-2(s+1)}.$$

Under the choice $V_n = \frac{1}{2\alpha}\log(n) - \frac{s+2}{\alpha}\log(\log(n))$, one derives

$$\mathrm{E}\left[|\mu_n - \mu|^2\right] \lesssim \log^{-2(s+2)}(n)$$

and

$$\mathrm{E}\left[|\lambda_n - \lambda|^2\right] \lesssim \log^{-2(s+1)}(n).$$

6.2. Upper bounds for MISE($\bar{\nu}_n$)

Proposition 6.2. *Let the assumptions of the Proposition 6.1 be fulfilled and let the kernel $\mathcal{K}(\cdot)$ satisfy the assumption (27). Then the mean integrated squared error of the estimator $\bar{\nu}_n(x)$ satisfies the following asymptotic relation*

$$\mathrm{MISE}(\bar{\nu}_n) \lesssim \frac{1}{n}\int_{\mathbb{R}} \frac{|u^\circ + \mathrm{i}v|^2\,[\mathcal{K}(v/V_n)]^2}{|\mathcal{M}(u^\circ + \mathrm{i}v + 1)|^2}\,dv$$
$$+ C_1 V_n^3 \cdot \mathbb{E}\left[(\mu_n - \mu)^2\right] + C_2 V_n \cdot \mathbb{E}\left[(\lambda_n - \lambda)^2\right] + C_3\frac{AL}{V_n^{2s}}$$

with some $C_1, C_2, C_3 > 0$.

Proof. Recall that

$$\bar{\nu}_n(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}vx}\hat{\mathcal{F}}[\bar{\nu}](-v)\mathcal{K}(-v/V_n)dv = \mathcal{F}^{-1}[\hat{\mathcal{F}}_{\bar{\nu}}(\cdot)\mathcal{K}(\cdot/V_n)](x),$$

and

$$\hat{\mathcal{F}}[\bar{\nu}](-v) = -Y_n(u^\circ + \mathrm{i}v) + \mu_n \cdot (u^\circ + \mathrm{i}v) + \lambda_n,$$
$$\mathcal{F}[\bar{\nu}](-v) = -Y(u^\circ + \mathrm{i}v) + \mu \cdot (u^\circ + \mathrm{i}v) + \lambda.$$

By the Parsenval's identity,

$$\mathrm{MISE} = \frac{1}{2\pi}\mathbb{E}\left[\int_{\mathbb{R}} |\mathcal{F}[\bar{\nu}_n](v) - \mathcal{F}[\bar{\nu}](v)|^2\,dv\right]$$
$$= \frac{1}{2\pi}\mathbb{E}\left[\int_{\mathbb{R}} \left|\hat{\mathcal{F}}[\bar{\nu}](v)\mathcal{K}(v/V_n) - \mathcal{F}[\bar{\nu}](v)\right|^2\,dv\right]$$
$$= \frac{1}{2\pi}\mathbb{E}\left[\int_{\mathbb{R}} \left|\left(\hat{\mathcal{F}}[\bar{\nu}](v) - \mathcal{F}[\bar{\nu}](v)\right)\mathcal{K}(v/V_n) + \left(\mathcal{K}(v/V_n) - 1\right)\mathcal{F}[\bar{\nu}](v)\right|^2\,dv\right]$$
$$\leq \frac{1}{\pi}\mathbb{E}\left[\int_{\mathbb{R}} \left|\left(\hat{\mathcal{F}}[\bar{\nu}](v) - \mathcal{F}[\bar{\nu}](v)\right)\mathcal{K}(v/V_n)\right|^2\,dv\right]$$

$$+\frac{1}{\pi}\mathbb{E}\left[\int_{\mathbb{R}}\left|\left(\mathcal{K}(v/V_n)-1\right)\mathcal{F}[\bar{\nu}](v)\right|^2 dv\right]$$
$$\leq \frac{3}{\pi}\left(J_1+J_2+J_3\right)+\frac{1}{\pi}J_4,$$

where

$$\begin{aligned}
J_1 &:= \mathbb{E}\left[\int_{\mathbb{R}}\left|Y_n(u^\circ+\mathrm{i}v)-Y(u^\circ+\mathrm{i}v)\right|^2\left[\mathcal{K}(v/V_n)\right]^2 dv\right],\\
J_2 &:= A_n\cdot\mathbb{E}\left[(\mu_n-\mu)^2\right] \qquad \text{with } A_n:=\int_{\mathbb{R}}|u^\circ+\mathrm{i}v|^2\cdot\left[\mathcal{K}(v/V_n)\right]^2 dv,\\
J_3 &:= B_n\cdot\mathbb{E}\left[(\lambda_n-\lambda)^2\right] \qquad \text{with } B_n:=\int_{\mathbb{R}}\left[\mathcal{K}(v/V_n)\right]^2 dv,\\
J_4 &:= \int_{\mathbb{R}}\left|\left(\mathcal{K}(v/V_n)-1\right)\mathcal{F}[\bar{\nu}](v)\right|^2 dv.
\end{aligned}$$

The treatment of J_1 is based on the observation that

$$Y_n(z)-Y(z)\asymp R_{1,n}=z\,\frac{[\mathcal{M}_n(z)-\mathcal{M}(z)]}{\mathcal{M}(z+1)}.$$

We get that

$$J_1\asymp\int_{\mathbb{R}}\mathbb{E}\left[\left|\mathcal{M}_n(u^\circ+\mathrm{i}v)-\mathcal{M}(u^\circ+\mathrm{i}v)\right|^2\right]\frac{|u^\circ+\mathrm{i}v|^2\left[\mathcal{K}(v/V_n)\right]^2}{\left|\mathcal{M}(u^\circ+\mathrm{i}v+1)\right|^2}dv.$$

As it was shown before, $\mathbb{E}[|\mathcal{M}_n(u^\circ+\mathrm{i}v)-\mathcal{M}(u^\circ+\mathrm{i}v)|^2]\lesssim n^{-1}$, see (37)–(39). Therefore,

$$J_1\lesssim\frac{1}{n}\cdot\int_{\mathbb{R}}\frac{|u^\circ+\mathrm{i}v|^2\left[\mathcal{K}(v/V_n)\right]^2}{\left|\mathcal{M}(u^\circ+\mathrm{i}v+1)\right|^2}dv.$$

To complete the proof, it is sufficient to note that

$$A_n\asymp V_n^3\cdot\int_{\mathbb{R}}y^2\left[\mathcal{K}(y)\right]^2dy, \qquad B_n=V_n\cdot\int_{\mathbb{R}}\left[\mathcal{K}(y)\right]^2dy,$$

and

$$J_4\leq A\int_{\mathbb{R}}\left|\frac{v}{V_n}\right|^{2s}\left|\mathcal{F}[\bar{\nu}](v)\right|^2dv\leq\frac{AL}{V_n^{2s}}. \qquad \square$$

Proof of Theorem 4.6

(i) Recall that if $\pi\in\mathcal{P}(\beta)$, then

$$\begin{aligned}
\mathrm{E}\left[|\mu_n-\mu|^2\right] &\lesssim n^{-1}\log(n){V_n}^{2\beta-1}+V_n^{-2(s+2)},\\
\mathrm{E}\left[|\lambda_n-\lambda|^2\right] &\lesssim n^{-1}\log(n)V_n^{2\beta+1}+V_n^{-2(s+1)},
\end{aligned}$$

see the proof of Theorem 4.5. Taking into account that $J_1 \lesssim n^{-1}V_n^{2\beta+3}$, we arrive at

$$\mathrm{MISE}(\bar{\nu}_n) \lesssim n^{-1}V_n^{3+2\beta} + n^{-1}\log(n)V_n^{2\beta+1} + V_n^{-2(s+1)} + V_n^{-2s}.$$

Choosing $V_n = n^{1/(2\beta+2s+3)}$, we get

$$n^{-1}V_n^{3+2\beta} = V_n^{-2s} \gtrsim V_n^{-2(s+1)},$$

and therefore

$$\mathrm{MISE}(\bar{\nu}_n) \lesssim n^{-1}\log(n)V_n^{2\beta+1} + V_n^{-2s} \lesssim n^{-2s/(2\beta+2s+3)}.$$

(ii) Similarly, we derive the upper bound for the class $\mathcal{E}(\alpha)$. Recall that

$$\begin{aligned} \mathrm{E}\left[|\mu_n - \mu|^2\right] &\lesssim n^{-1}\exp(2\alpha V_n) + V_n^{-2(s+2)}, \\ \mathrm{E}\left[|\lambda_n - \lambda|^2\right] &\lesssim n^{-1}V_n^2\exp(2\alpha V_n) + V_n^{-2(s+1)} \end{aligned}$$

and therefore

$$\mathrm{MISE}(\bar{\nu}_n) \lesssim n^{-1}\log(n)V_n^3 e^{2\alpha V_n} + V_n^{-2s} \lesssim (\log n)^{-2s}.$$

6.3. *Lower bounds for* MISE

Proof of Theorem 4.7. The general idea of the proof is to apply Theorem 2.7 from [33]. This theorem yields that (28) holds, if there exists a parameterized set of Lévy triplets

$$\mathcal{T}_\theta = (1, 0, \nu_\theta) \subset \mathcal{G}(s, R), \quad \theta \in \{0,1\}^L$$

for some $s \in \mathbb{N} \cup 0, R > 0$, $L > 0$ and a set of parameters $\{\theta^{(j)},\ j = 0, \ldots, M\}$ such that the following two properties hold.

(i) For any $0 \le j < k \le M$,

$$\int_{\mathbb{R}} |\nu_{\theta^{(j)}}(x) - \nu_{\theta^{(k)}}(x)|^2\,dx \ge 2\varphi_n. \tag{40}$$

(ii) Denote by $\pi_{\theta_j}, j = 0, \ldots, M$, the probability distribution of the exponential Lévy model $A_{j,\infty} = \int_0^\infty e^{-\xi_{j,s}}ds$, where $\xi_{j,s}$ is a Lévy subordinator with triplet $\mathcal{T}_{\theta_j}$. Then

$$\frac{n}{M}\sum_{j=1}^{M} \chi^2\left(\pi_{\theta^{(j)}}, \pi_{\theta^{(0)}}\right) \le \varkappa\log(M), \tag{41}$$

for n large enough, where for any probability measures $\mathbb{P}$ and $\mathbb{Q}$ we denote their χ^2-*divergence* by

$$\chi^2(\mathbb{P}|\mathbb{Q}) := \begin{cases} \int \left(\frac{d\mathbb{P}}{d\mathbb{Q}} - 1\right)^2 d\mathbb{Q}, & \text{if } \mathbb{P} \ll \mathbb{Q}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Below we present a detailed proof for the polynomial case.

1. Presentation of the models. Consider an exponential Lévy model $A_{0,\infty} = \int_0^\infty e^{-\xi_{0,s}} ds$, where $\xi_{0,s}$ is a Lévy subordinator with a triplet $(1, 0, \nu_0)$ and $\nu_0(x) = abe^{-bx}$ for some $0 < a \leq 1$, $0 < b < 1$. It is clear that $(1, 0, \nu_0) \in \mathcal{G}(0, R)$ for some $R > 0$ and the Laplace exponent of $\xi_{0,s}$ is given by

$$\phi_0(z) = z + \int_0^\infty (1 - e^{-xz})\, \nu_0(x)\, dx = z \left[1 + \frac{a}{z + b}\right], \qquad \mathrm{Re}(z) > -b,$$

see Example 1 from Section 5. For the case of general classes $\mathcal{G}(s, R)$ with $s > 0$, we could take a Lévy density of the form $\nu_0(x) = b^{1+s} x^s e^{-bx}/\Gamma(s + 1)$.

Fix some $L > 0$ and let us construct now a parameterized set of Lévy triplets $\mathcal{T}_\theta = (1, 0, \nu_\theta)$, $\theta \in \{0, 1\}^L$, with Lévy measure ν_θ defined by

$$\nu_\theta(x) := \nu_0(x) + \delta \cdot \Delta_\theta(x),$$

where $\delta > 0$ small enough,

$$\begin{aligned} \Delta_\theta(x) &:= \big(g_\theta(x) + a(g_\theta \star \exp(-b\cdot)(x))\big)', \\ g_\theta(x) &:= \sum_{k=L+1}^{2L} \theta_{k-L} \cos(k\gamma_L x) g_0(x), \end{aligned}$$

θ_{k-L} stands for the $(k - L)$-th component of the vector θ, $\gamma_L \to \infty$ as $L \to \infty$, and

$$g_0(x) := x^{-3/2} \exp(-1/x), \quad x > 0.$$

2. Distributional properties of the models. In this step, we perform some technical calculations, which will be used later. It holds

$$\begin{aligned} \mathcal{L}[\Delta_\theta](z) &= \int_0^\infty e^{-zx} \Delta_\theta(x)\, dx \\ &= z\left[1 + \frac{a}{z + b}\right] \left[\int_0^\infty e^{-zx} g_\theta(x)\, dx\right] = \phi_0(z) \cdot \mathcal{L}[g_\theta](z), \end{aligned}$$

where $\mathcal{L}[g_\theta](z)$ is the Laplace transform of the function $g_0(\cdot)$, which is equal to

$$\mathcal{L}[g_\theta](z) = \frac{1}{2} \sum_{k=L+1}^{2L} \theta_{k-L} \left[\mathcal{L}[g_0](z + \mathrm{i}\gamma_L k) + \mathcal{L}[g_0](z - \mathrm{i}\gamma_L k)\right].$$

We see that $\int_0^\infty \Delta_\theta(x)\, dx = 0$ and

$$\phi_\theta(z) - \phi_0(z) = \delta \phi_0(z) \mathcal{L}[g_\theta](z),$$

where $\phi_\theta(\cdot)$ is the Laplace exponent of a Lévy process $\xi_{\theta,s}$ with the Lévy triplet $\mathcal{T}_\theta$. Furthermore, the Laplace transform of g_0 is given by

$$\mathcal{L}[g_0](u + \mathrm{i}v) = \sqrt{\pi} e^{-2(z_+ + iz_-)}$$

with $2z_{\pm}^2 = \sqrt{u^2+v^2} \pm u$. The Mellin transform of the density π_θ corresponding to the Lévy model $\mathcal{T}_\theta$ satisfies the following functional equation

$$\frac{\mathcal{M}_\theta(z)}{\mathcal{M}_0(z)} = \frac{\phi_\theta(z)}{\phi_0(z)} \frac{\mathcal{M}_\theta(z+1)}{\mathcal{M}_0(z+1)}.$$

Since

$$\frac{\phi_\theta(z)}{\phi_0(z)} - 1 = \delta \mathcal{L}[g_\theta](z),$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} |\mathcal{L}[g_\theta](z+k)| &\leq C \exp\left(-\sqrt{2\mathrm{Re}(z)} - \sqrt{2\,|\mathrm{Im}(z)|}\right) \\ &\qquad \cdot \sum_{k=1}^{\infty} \sum_{j=L+1}^{2L} \exp(-\sqrt{2\gamma_L j} - \sqrt{2k}) \\ &\leq C' \exp\left(-\sqrt{2\mathrm{Re}(z)} - \sqrt{2\,|\mathrm{Im}(z)|}\right), \quad \mathrm{Re}(z) \geq 0, \end{aligned}$$

we derive the following infinite product representation for the ratio $\mathcal{M}_\theta(z)/\mathcal{M}_0(z)$

$$\frac{\mathcal{M}_\theta(z)}{\mathcal{M}_0(z)} = \prod_{k=0}^{\infty} (1 + \delta \mathcal{L}[g_\theta](z+k)).$$

Furthermore, it can be proved that

$$\left| \frac{\mathcal{M}_\theta(u+\mathrm{i}v)}{\mathcal{M}_0(u+\mathrm{i}v)} - 1 \right| \leq c\delta \, |\mathcal{L}[g_\theta](u+\mathrm{i}v)|$$

for some absolute constant $c > 0$. Note that the random variables $A_{\theta,\infty} = \int_0^\infty e^{-\xi_{\theta,s}} ds$ with $\xi_{\theta,s}$ being a Lévy process with the triplet $\mathcal{T}_\theta$, satisfies $0 < A_{\theta,\infty} < 1$ a.s. Moreover the density p_0 of the r.v. $A_{0,\infty}$ has the form

$$\pi_0(x) = \frac{1}{B(b-1,a)} x^b (1-x)^{a-1} 1_{\{0<x<1\}}$$

and the Mellin transform $\mathcal{M}_0(z)$ of $A_{0,\infty}$ is given by

$$\mathcal{M}_0(z) = \frac{B(z+b,a)}{B(b-1,a)}, \tag{42}$$

see Example 4.3.

3. Class $\mathcal{G}(s,R)$. In this step, we check that constructed models $\mathcal{T}_{\theta^{(j)}}$, $j = 1, \ldots, M$ belong to class $\mathcal{G}(s,R)$ with $s = 0$ and some $R > 0$. We have for any $\theta \in \{0,1\}^L$,

$$\int_{\mathbb{R}} |v|^{2s} \left| \mathcal{F}[\nu_\theta](v) \right|^2 dx \quad \leq \quad \int_{\mathbb{R}} |v|^{2s} \left| \mathcal{F}[\nu_0](v) \right|^2 dv$$

$$
\begin{aligned}
&+\int_{\mathbb{R}} |v|^{2s} \left|\mathcal{F}\left[\nu_\theta\right](v) - \mathcal{F}\left[\nu_0\right](v)\right|^2 \, dv \\
\leq \quad & \int_{\mathbb{R}} |v|^{2s} \left|\mathcal{F}\left[\nu_0\right](v)\right|^2 \, dv + \delta^2 \int_{\mathbb{R}} |v|^{2s} \left|\mathcal{F}[\Delta_\theta](v)\right|^2 \, dv.
\end{aligned}
$$

The inequality $|\phi_0(-\mathrm{i}v)| \leq c \cdot |v|$ for $v \in \mathbb{R}$, where $c = 1 + a/b$, implies

$$
\begin{aligned}
\int_{\mathbb{R}} |v|^{2s} \left|\mathcal{F}[\Delta_\theta](v)\right|^2 \, dv \quad &\leq \quad c \int_{\mathbb{R}} |v|^{2(s+1)} \left|\mathcal{L}[g_\theta](-\mathrm{i}v)\right|^2 \, dv \\
&= \quad \frac{c}{2} \int_{\mathbb{R}} |v|^{2(s+1)} \cdot \Bigg| \sum_{k=L+1}^{2L} \theta_{k-L} \left(\mathcal{L}[g_0](-\mathrm{i}v + \mathrm{i}\gamma_L k)\right. \\
&\qquad\qquad \left. + \mathcal{L}[g_0](-\mathrm{i}v - \mathrm{i}\gamma_L k)\right)\Bigg|^2 \, dv \\
&\leq \quad \frac{c}{2} \sum_{k=L+1}^{2L} \int_{\mathbb{R}} |v|^{2(s+1)} \left|\mathcal{L}[g_0](-\mathrm{i}v + \mathrm{i}\gamma_L k)\right|^2 \, dv \\
&\qquad + \frac{c}{2} \sum_{k=L+1}^{2L} \int_{\mathbb{R}} |v|^{2(s+1)} \left|\mathcal{L}[g_0](-\mathrm{i}v - \mathrm{i}\gamma_L k)\right|^2 \, dv \\
&\qquad + R_L,
\end{aligned}
$$

where

$$
\begin{aligned}
R_L \quad = \quad & 2 \sum_{k \neq j} \int_{\mathbb{R}} |v|^4 \mathcal{L}[g_0](-\mathrm{i}v - \mathrm{i}j\gamma_L) \overline{\mathcal{L}[g_0](-\mathrm{i}v - \mathrm{i}k\gamma_L)} \, dv \\
& + 2 \sum_{k \neq j} \int_{\mathbb{R}} |v|^4 \mathcal{L}[g_0](-\mathrm{i}v + \mathrm{i}j\gamma_L) \overline{\mathcal{L}[g_0](-\mathrm{i}v + \mathrm{i}k\gamma_L)} \, dv
\end{aligned}
$$

It holds

$$
\begin{aligned}
|R_L| \quad &\leq \quad CL \sum_{j=1}^{2L} (j\gamma_L)^{2(s+1)} \exp(-\sqrt{2\gamma_L j}) \\
&\leq \quad CL^{2(s+1)+2} \gamma_L^{2(s+1)} \exp(-\sqrt{2\gamma_L}) \\
&= \quad o\left(L^{2(s+1)+1}\right),
\end{aligned}
$$

provided $\gamma_L = c \log^2(L)$ for large enough $c > 0$. Hence $\int_{\mathbb{R}} |v|^{2s} \left|\mathcal{F}[\Delta_\theta](v)\right|^2 \, dv$ is bounded if $\delta^2 \gamma_L^{2(s+1)} L^{2s+3} = O(1)$.

4. Upper bound for the L^2-distance between elements of $\{\nu_\theta\}$.
Fix two vectors $\theta, \theta' \in \{0,1\}^L$. We have

$$
\begin{aligned}
\int_{\mathbb{R}} |\nu_\theta(x) - \nu_{\theta'}(x)|^2 \, dx \quad &= \quad \frac{1}{2\pi} \delta^2 \int_{\mathbb{R}} \left|\phi_0(-\mathrm{i}v) \mathcal{L}[g_\theta - g_{\theta'}](-\mathrm{i}v)\right|^2 \, dv \\
&= \quad \frac{1}{2\pi} \delta^2 \sum_{k=L+1}^{2L} \left(\theta_{k-L} - \theta'_{k-L}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \int_{\mathbb{R}} \left|\phi_0(-\mathrm{i}v)\mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}\gamma_L k)\right|^2 \, dv \\
&+ \frac{1}{2\pi}\delta^2 \sum_{k=L+1}^{2L} \left(\theta_{k-L}-\theta'_{k-L}\right)^2 \\
&\cdot \int_{\mathbb{R}} \left|\phi_0(-\mathrm{i}v)\mathcal{L}[g_0](-\mathrm{i}v-\mathrm{i}\gamma_L k)\right|^2 \, dv \\
&+ \frac{1}{2\pi}\delta^2 R_L,
\end{aligned}
$$

where

$$
\begin{aligned}
R_L \quad \leq \quad & 2\sum_{k\neq j} \int_{\mathbb{R}} \left|\phi_0(-\mathrm{i}v)\right|^2 \mathcal{L}[g_0](-\mathrm{i}v-\mathrm{i}j\gamma_L)\overline{\mathcal{L}[g_0](-\mathrm{i}v-\mathrm{i}k\gamma_L)}\, dv \\
& + 2\sum_{k\neq j} \int_{\mathbb{R}} \left|\phi_0(-\mathrm{i}v)\right|^2 \mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}j\gamma_L)\overline{\mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}k\gamma_L)}\, dv.
\end{aligned}
$$

Consider, for example,

$$
\begin{aligned}
\int_{\mathbb{R}} \left|\phi_0(-\mathrm{i}v)\mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}\gamma_L k)\right|^2 \, dv &= \int_{\mathbb{R}} \left|\phi_0(-i(v+\gamma_L k))\mathcal{L}[g_0](-\mathrm{i}v)\right|^2 \, dv \\
&= \int_{\mathbb{R}} |v+\gamma_L k|^2 \left|1+\frac{a}{b-i(v+\gamma_L k)}\right|^2 e^{-2\sqrt{2|v|}}\, dv \\
&= \gamma_L^2 k^2 \int_{\mathbb{R}} \left|1+\frac{a}{b-i(v+\gamma_L k)}\right|^2 e^{-2\sqrt{2|v|}}\, dv + O(\gamma_L k).
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\sum_{k=L+1}^{2L} \left(\theta_{k-L}-\theta'_{k-L}\right)^2 \int_{\mathbb{R}} \left|\phi_0(-\mathrm{i}v)\mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}\gamma_L k)\right|^2 \, dv \\
&\quad = C\gamma_L^2 \sum_{k=L+1}^{2L} \left(\theta_{k-L}-\theta'_{k-L}\right)^2 k^2 + o\left(\gamma_L^2 \sum_{k=L+1}^{2L} \left(\theta_{k-L}-\theta'_{k-L}\right)^2 k^2\right) \\
&\quad \geq C'\gamma_L^2 L^2 \sum_{k=1}^{L} I(\theta_k \neq \theta'_k),
\end{aligned}
$$

as $L \to \infty$ and $\rho(\theta,\theta') = \sum_{k=1}^{L} I(\theta_k \neq \theta'_k) > 0$. Analogously,

$$
\sum_{k=L+1}^{2L} \left(\theta_{k-L}-\theta'_{k-L}\right)^2 \int_{\mathbb{R}} \left|\phi_0(-\mathrm{i}v)\mathcal{L}[g_0](-\mathrm{i}v-\mathrm{i}\gamma_L k)\right|^2 \, dv \quad = \quad C''\gamma_L^2 L^2 \rho(\theta,\theta').
$$

Furtheremore, one shows (see above) that

$$
|R_L| = o\left(L^3\right).
$$

5. Choice of $\theta^{(0)}, \ldots, \theta^{(M)}$.

Our choice is based on the well-known Varshamov-Gilbert bound (see [33], Lemma 2.9), which implies that there are $M > 2^{L/8}$ vectors $\theta^{(0)}, \ldots, \theta^{(M)} \in \{0,1\}^L$ such that

$$\rho(\theta^{(j)}, \theta^{(k)}) \geq L/8.$$

6. Upper bound for $K(\pi_0, \pi_\theta)$.

By Parseval identity for Mellin transforms, we get

$$\begin{aligned} K(\pi_0,\pi_\theta) &= \int_0^1 \frac{|\pi_\theta(x)-\pi_0(x)|^2}{\pi_0(x)}\,dx \\ &= \int_0^1 x^{-b}(1-x)^{1-a}\,|\pi_\theta(x)-\pi_0(x)|^2\,dx \\ &\leq \int_0^1 x^{-b}\,|\pi_\theta(x)-\pi_0(x)|^2\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{M}_\theta((1-b)/2+\mathrm{i}v)-\mathcal{M}_0((1-b)/2+\mathrm{i}v)|^2\,dv \\ &\leq \frac{c\delta^2}{2\pi}\int_{-\infty}^{\infty} |\mathcal{M}_0((1-b)/2+\mathrm{i}v)|^2\,|\mathcal{L}[g_\theta](u+\mathrm{i}v)|^2\,dv. \end{aligned}$$

So we get

$$\begin{aligned} K(\pi_0,\pi_\theta) \;&\leq\; \frac{c\delta^2}{2\pi}\sum_{k=L+1}^{2L}\int_{\mathbb{R}} |\mathcal{M}_0((1-b)/2+\mathrm{i}v)|^2\,|\mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}\gamma_L k)|^2\,dv, \\ &\quad+\frac{c\delta^2}{2\pi}\sum_{k=L+1}^{2L}\int_{\mathbb{R}} |\mathcal{M}_0((1-b)/2+\mathrm{i}v)|^2\,|\mathcal{L}[g_0](-\mathrm{i}v-\mathrm{i}\gamma_L k)|^2\,dv \\ &\quad+\frac{c\delta^2}{2\pi}R_L, \end{aligned}$$

where

$$\begin{aligned} R_L \leq 2\sum_{k\neq j}\int_{\mathbb{R}} &|\mathcal{M}_0((1-b)/2+\mathrm{i}v)|^2\,\mathcal{L}[g_0](-\mathrm{i}v-\mathrm{i}j\gamma_L)\overline{\mathcal{L}[g_0](-\mathrm{i}v-\mathrm{i}k\gamma_L)}\,dv \\ &+2\sum_{k\neq j}\int_{\mathbb{R}} |\mathcal{M}_0((1-b)/2+\mathrm{i}v)|^2\,\mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}j\gamma_L)\overline{\mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}k\gamma_L)}\,dv. \end{aligned}$$

The equation (42) implies that $\mathcal{M}_0(z)$ is finite for all z with $\mathrm{Re}(z) \geq 0$ and

$$\begin{aligned} \mathcal{M}_0(u+\mathrm{i}v) &= C(a,b)\frac{\Gamma(u+\mathrm{i}v+b)}{\Gamma(u+\mathrm{i}v+b+a)} \\ &\asymp C(a,b)e^{-a\log(u+\mathrm{i}v+b)} \\ &= C(a,b)\left((u+b)^2+v^2\right)^{-a/2}e^{i\mathrm{Arg}(u+\mathrm{i}v+b)}, \quad u^2+v^2\to\infty. \end{aligned}$$

Hence

$$|\mathcal{M}_0(u+\mathrm{i}v)| \asymp C(a,b)\left((u+b)^2+v^2\right)^{-a/2}, \quad u^2+v^2 \to \infty$$

and the density π_0 of $A_{0,\infty}$ belongs to the class $\mathcal{P}(a)$ with $L = C(a,b)$ (see also Example 4.3). We have

$$\sum_{k=L+1}^{2L} \int_{\mathbb{R}} |\mathcal{M}_0((1-b)/2+\mathrm{i}v)|^2 \, |\mathcal{L}[g_0](-\mathrm{i}v+\mathrm{i}\gamma_L k)|^2 \, dv = O(L^{-2a+1})$$

and

$$\sum_{k=L+1}^{2L} \int_{\mathbb{R}} |\mathcal{M}_0((1-b)/2+\mathrm{i}v)|^2 \, |\mathcal{L}[g_0](-\mathrm{i}v-\mathrm{i}\gamma_L k)|^2 \, dv = O(L^{-2a+1}).$$

Hence

$$\frac{n}{M}\sum_{m=1}^{M} \chi^2(\pi_0, \pi_{\theta^{(m)}}) \le n\delta^2 L^{-2a}\log(M), \quad L \to \infty \tag{43}$$

for large n.

7. Choice of L. To complete the proof, we choose L such that the conditions (40) and (41) are fulfilled. First note that since our model belongs to the class $\mathcal{G}(s,R)$, we can take $\gamma_L = c\log^2(L)$ and $\delta^2 = \gamma_L^{-2(s+1)} L^{-2s-3} \cdot O(1)$, see Step 3 of the proof for details. Second, comparing (43) with (41), we fix $\varkappa = n\delta^2 L^{-2a}$. This leads to the choice of L as the solution of the equation

$$L^{2a+2s+3}\log^{4(s+1)}(L) = nO(1)$$

Combination of the results from Steps 4 and 5 yields the condition (40), because

$$\begin{aligned}\int_{\mathbb{R}} |\nu_\theta(x) - \nu_{\theta'}(x)|^2 \, dx &\ge C_1\delta^2\gamma_L^2 L^3 \\ &= C_2(\log L)^{-4s} L^{-2s} \\ &= C_3\,(\log L)^{4s\frac{-2a-1}{2a+2s+3}}\, n^{-2s/(2a+2s+3)}\end{aligned}$$

for some $C_1, C_2, C_3 > 0$ and L large enough. This observation completes the proof.

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