THE MAIN CUBIOID

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Abstract. The connectedness locus in the parameter space of quadratic polynomials is called the Mandelbrot set. A good combinatorial model of this set is due to Thurston. By definition, the Principal Hyperbolic Domain of the Mandelbrot set consists of parameter values, for which the corresponding quadratic polynomials have an attracting fixed point. The closure of the Principal Hyperbolic Domain of the Mandelbrot set is called the main cardioid. Its topology is completely described by Thurston's model. Less is known about the connectedness locus in the parameter space of cubic polynomials. In this paper, we discuss cubic analogs of the main cardioid and establish relationships between them.

1. Introduction

Studying parameter spaces of polynomials is a major task of complex dynamics. The main achievements here concern quadratic polynomials. One of them is Thurston's beautiful description of a combinatorial model for the parameter space of quadratic polynomials. Motivated by it, we want to study the structure of the closure of the Principal Hyperbolic Domain in the parameter space of cubic polynomials. To this end we study topological dynamics of cubic polynomials, building our investigation on a partial similarity with the dynamics of quadratic polynomials.

For a complex polynomial f, let $K(f)$ be the *filled Julia set* of f consisting of all complex numbers z, whose f-orbits do not escape to

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infinity. We write $J(f)$ for the *Julia set* of f, which is the boundary of $K(f)$. An equivalent definition of the Julia set is that it consists of points z such that the forward f-orbits fail to be Lyapunov stable in a neighborhood of z (equivalently, the sequence of iterates $f^{\circ n}$ fails to be equicontinuous in a neighborhood of z).

Quadratic polynomials depend essentially on one complex parameter c, since every quadratic polynomial is affinely conjugate to a polynomial of the form $f_c(z) = z^2 + c$. The *Mandelbrot set* $\mathcal{M}_2$ is a subset of the parameter plane, the c-plane, consisting of all values $c \in \mathbb{C}$, for which $K(f_c)$ is connected. The geometric shape of $\mathcal{M}_2$ gives a very informative overview of how the dynamics of f_c depends on c. In particular, as c varies within one interior component of $\mathcal{M}_2$, the topological dynamics of the map $f_c : J(f_c) \to J(f_c)$ does not change, i.e., the maps f_c and $f_{c'}$ are topologically conjugate on their Julia sets if c and c' belong to the same interior component of $\mathcal{M}_2$, see [MSS83]. As c crosses the boundary of $\mathcal{M}_2$, bifurcations happen. For example, going from one interior component of $\mathcal{M}_2$ to an adjacent component corresponds to a bifurcation, during which a periodic cycle changes its period.

The central part of $\mathcal{M}_2$ is the so called *main cardioid* consisting of all parameter values c, for which f_c has an attracting or neutral fixed point. It can also be defined as the closure of the *principal hyperbolic domain* PHD_2, the interior component of the Mandelbrot set such that for all $c \in \mathrm{PHD}_2$, the Julia set $J(f_c)$ is homeomorphic to the circle, and the dynamics of $f_c : J(f_c) \to J(f_c)$ is topologically conjugate to the dynamics of the angle doubling map on $\mathbb{R}/\mathbb{Z}$. It is easy to see that polynomials in $\overline{\mathrm{PHD}}_2$ have no periodic cutpoints in $J(f)$ except for at most one neutral fixed point. Recall that a *cutpoint* of a topological space X is a point $\alpha \in X$ such that $X \setminus \{\alpha\}$ is disconnected.

As follows from the Douady–Hubbard–Sullivan–Yoccoz landing theorem [DH8485, Hub93], the Mandelbrot set itself can be thought of as the union of the main cardioid and *limbs* (connected components of $\mathcal{M}_2 \setminus \overline{\mathrm{PHD}}_2$) parameterized by reduced rational fractions $p/q \in (0, 1)$. The limb corresponding to p/q consists of c such that K_c is connected, and there is a repelling fixed cutpoint α of K_c of combinatorial rotation number p/q (this description of limbs is given in [Mil00a]). The point α being repelling means that $|f_c'(\alpha)| > 1$, then indeed α repels all nearby points. Finally, having combinatorial rotation number p/q means the following: the set $K_c \setminus \{\alpha\}$ has q connected components, whose germs at α are mapped to each other like rotation by p/q, i.e., for every component X of $K_c \setminus \{\alpha\}$, there is a small neighborhood U of

α with the property that $f_c(X \cap U)$ is mapped to the pth component from X in the counterclockwise order.

This description of the Mandelbrot set serves as a motivation for our study of higher degree analogs of PHD_2. By *classes* of degree d polynomials, we mean *affine conjugacy classes*. Denote by $[f]$ the class of a polynomial f. Note that, for a quadratic polynomial f_c, the class $[f_c]$ can be identified with c. The degree d *connectedness locus* $\mathcal{M}_d$ (the higher degree generalization of the Mandelbrot set) is the set of classes of degree d polynomials f with connected $K(f)$ (equivalently, $[f] \in \mathcal{M}_d$ if all critical points of f belong to $K(f)$). A polynomial of any degree is said to be *hyperbolic* if the orbits of all its critical points converge to attracting cycles. The set of all classes $[f] \in \mathcal{M}_d$ such that f is hyperbolic splits into the so called *hyperbolic components* of $\mathcal{M}_d$. The degree d *Principal Hyperbolic Domain* PHD_d is the hyperbolic component of $\mathcal{M}_d$ consisting of classes $[f]$ such that $K(f)$ is a Jordan disk. Equivalently, the class $[f]$ of a degree d polynomial f belongs to PHD_d if all critical points of f are in the immediate attracting basin of the same attracting (or super-attracting) fixed point.

Thus, the study of $\overline{\mathrm{PHD}_3}$ in the context of complex dynamics, is the next logical step. Working in this direction and similar to the quadratic case, we establish properties of polynomials in $\overline{\mathrm{PHD}_d}$ in Theorem A which lists necessary conditions for $[f]$ to belong to $\overline{\mathrm{PHD}_d}$.

Theorem A. *Let f be a polynomial, whose class belongs to $\overline{\mathrm{PHD}_d}$. Then f has a fixed non-repelling point and no repelling periodic cutpoints in the Julia set of f. Moreover, all non-repelling periodic cutpoints, except at most one fixed point, have multiplier 1.*

This motivates the following definition of a cubic analog of the main cardioid.

Definition 1.1 (Main Cubioid)**.** The *Main Cubioid* is the set CU of classes of cubic polynomials f with connected $J(f)$ such that:

(1) the polynomial f has at least one non-repelling fixed point,
(2) there are no repelling periodic cutpoints in $J(f)$, and
(3) all non-repelling periodic cutpoints of $J(f)$, except at most one fixed point, have multiplier 1.

The following corollary is immediate from Theorem A.

Corollary 1.2. $\overline{\mathrm{PHD}_3} \subset \mathrm{CU}$.

If $[f] \in \overline{\mathrm{PHD}_d}$, then f cannot have two attracting periodic points as otherwise any small perturbation of f will also have two attracting periodic points while there exist polynomials with classes from PHD_d

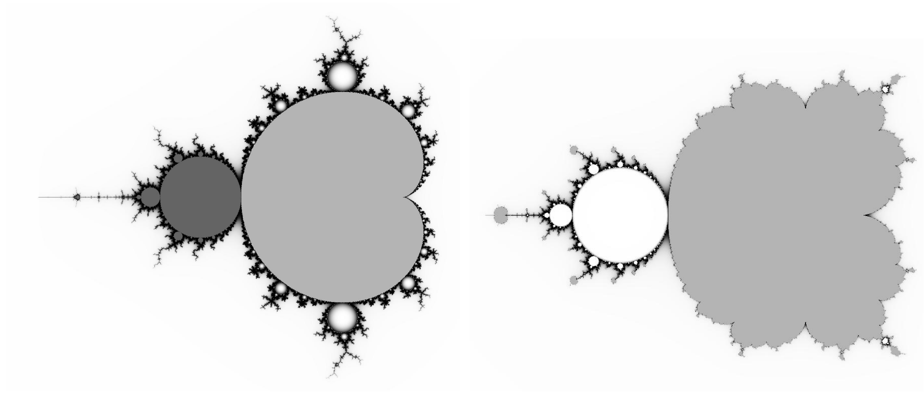

Figure 1. Left: the Mandelbrot set. The main cardioid is light grey; the 1/2-limb is dark grey. Right: the set of parameter values $a = b^2$, for which $[f_{0,b}] \in \mathcal{M}_3$ (note that $[f_{0,b}] = [f_{0,-b}]$ depends only on a). The biggest light grey component consists of points a such that $[f_{0,b}] \in \mathrm{CU}$.

(and hence with only one periodic attracting point) arbitrarily close to f. A part of Theorem A extends this observation to non-repelling periodic points. Observe that, by definition, if $J(f)$ is disconnected, then $[f] \notin \mathrm{CU}$. Also, by the Fatou-Shishikura inequality [Fat20, Shi87], a cubic polynomial f has at most two non-repelling cycles.

Let $\mathcal{F}_\lambda$ be the space of all cubic polynomials of the form

$$f_{\lambda,b}(z) = \lambda z + bz^2 + z^3, \quad b \in \mathbb{C}.$$

This space maps onto the space of classes of all cubic polynomials with a fixed point of multiplier λ as a finite branched covering. This branched covering is equivalent to the map $b \mapsto a = b^2$, i.e., classes of polynomials $f_{\lambda,b} \in \mathcal{F}_\lambda$ are in one-to-one correspondence with the values of a. Thus, if we talk about, say, points $[f]$ of $\mathcal{M}_3$, then it suffices to take $f \in \mathcal{F}_\lambda$ for some λ. A polynomial $f \in \mathcal{F}_\lambda$ is called *stable with respect to* 0 if its Julia set admits an equivariant holomorphic motion over some neighborhood of f in $\mathcal{F}_\lambda$. Intuitively, this means that no bifurcations happen near f in $\mathcal{F}_\lambda$ (precise definitions will be given later). The (λ-) *stable* set $\mathcal{S}_\lambda \subset \mathbb{C}$ is the set of all $b \in \mathbb{C}$ such that $f_{\lambda,b}$ is stable with respect to 0. A (λ-) *stable domain* is a component of $\mathcal{S}_\lambda$. A polynomial g is said to be *stable* if $g \in [f]$ for some polynomial $f \in \mathcal{F}_\lambda$ that is stable with respect to 0.

Definition 1.3. The set $\overline{\mathrm{PHD}}_3^e$ is the union of $\overline{\mathrm{PHD}}_3$ and classes of all polynomials from all λ-stable domains Λ with $|\lambda| \leqslant 1$ such that for all $b \in \mathrm{Bd}(\Lambda)$, we have $[f_{\lambda,b}] \in \overline{\mathrm{PHD}}_3$.

We conjecture that $\overline{\mathrm{PHD}}_3 = \mathrm{CU} = \overline{\mathrm{PHD}}_3^e$. To support this conjecture, we prove Theorem B. Let $\mathcal{LC}$ be the set of classes of all polynomials with locally connected Julia set.

Theorem B. *We have* $\overline{\mathrm{PHD}}_3^e \subset \mathrm{CU}$ *and* $\mathcal{LC} \cap \mathrm{CU} = \mathcal{LC} \cap \overline{\mathrm{PHD}}_3^e$.

To state Theorem C, we need the language of laminations. Laminations have been introduced by Thurston [Thu85] to study topological models of polynomials with locally connected Julia sets. Consider a degree d polynomial f such that $K(f)$ is connected. Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle and $\sigma_d : \mathbb{S}^1 \to \mathbb{S}^1$ be the map defined by $\sigma_d(t) = d \cdot t$ mod 1. If $K(f)$ is locally connected then there exists an equivalence relation $\sim_f$ on $\mathbb{S}^1$ such that σ_d maps $\sim_f$-classes to $\sim_f$-classes, $\mathbb{S}^1/\sim_f$ is homeomorphic to $J(f)$, and $f|_{J(f)}$ is topologically conjugate to the self-mapping of $\mathbb{S}^1/\sim_f$ induced by σ_d. By [Kiw04, BCO11], the definition of $\sim_f$ can be extended to the case, where $K(f)$ is not locally connected. Then the quotient of $\mathbb{S}^1$ by $\sim_f$ is the biggest quotient of $J(f)$ that is a locally connected continuum, and, as before, σ_d maps $\sim_f$-classes to $\sim_f$-classes. If $K(f)$ (equivalently, $J(f)$) is not locally connected, then $\sim_f$ may have infinite classes. The purpose of laminations is to describe the combinatorial structure of $f|_{J(f)}$.

Definition 1.4. A curve Γ in the dynamic plane of f consisting of (pre)periodic dynamic external rays $R_f(\theta_1)$, $R_f(\theta_2)$ and their common landing point $x \in J(f)$ is called a *(pre)periodic cut* with *vertex* x. The set $G(x)$ then is defined as the convex hull of the arguments of all dynamic external rays landing at x.

Define the set $\mathcal{L}_f$ of chords in $\mathbb{D}$ consisting of all edges of convex hulls of all $\sim_f$-classes, all edges of polygons $G(x)$ associated with vertices x of (pre)periodic cuts, and all limits of these edges. A chord $\overline{ab}$ in $\mathcal{L}_f$ is a *leaf* of $\mathcal{L}_f$; set $\sigma_3(\overline{ab}) = \overline{\sigma_3(a)\sigma_3(b)}$. The closure of a complementary domain to $\mathcal{L}_f$ in $\mathbb{D}$ is a *gap* of $\mathcal{L}_f$; we write $\sigma_3(G)$ for the convex hull of $\sigma_3(G \cap \mathbb{S}^1)$. The pair $(\sim_f, \mathcal{L}_f)$ is called the *laminational pair* associated with f. A *rotational set* H of $\sim_f$ is a k-periodic leaf or a gap of $\mathcal{L}_f$ such that the map $\sigma_3^{\circ k} : H \cap \mathbb{S}^1 \to H \cap \mathbb{S}^1$ extends to a monotone map of $\mathbb{S}^1$ topologically semi-conjugate to a *non-trivial* rotation of $\mathbb{S}^1$.

Theorem C. *If* $[f] \in \mathrm{CU}$ *then* $\mathcal{L}_f$ *coincides with the set of leaves of* $\sim_f$*, and the following properties hold:*

1. *each periodic leaf of* $\sim_f$ *has an attached to it gap* G *such that* $G \cap \mathbb{S}^1$ *is infinite and is not one* $\sim_f$*-class;*
2. $\sim_f$ *has at most one rotational set.*

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Notation and terminology: we write $\overline{A}$ for the closure of a set A in a topological space and $\mathrm{Bd}(A)$ for the boundary of A; the n-th iterate of a map f is denoted by $f^{\circ n}$. For $A \subset \mathbb{C}$ let $\mathrm{CH}(A)$ be the *convex hull* of A in $\mathbb{C}$. If it does not cause ambiguity, we speak of cutpoints meaning cutpoints of the appropriate Julia sets. We will consistently identify *angles*, i.e. elements of $\mathbb{R}/\mathbb{Z}$, with points of the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. If G is the convex hull of some closed subset $G' \subset \mathbb{S}^1$, then we call G' the *basis* of G. A gap G is said to be *finite* or *infinite* according to whether G' is finite or infinite.

2. Proof of Theorem A

We first recall some terminology and notation.

2.1. **Dynamic rays.** Let $f(z) = z^d + a_{d-1}z^{d-1} + \dots + a_0$ be a monic degree d polynomial. The Green function G_f is defined by the formula

$$G_f(z) = \lim_{n\to\infty} \frac{\log_+ |f^{\circ n}(z)|}{d^n},$$

where $\log_+ r$ equals $\log r$ if $r > 0$ and 0 otherwise. The function G_f is harmonic on the complement of the filled Julia set $K(f)$ of f and is equal to 0 on $K(f)$. Define *dynamic rays* as unbounded trajectories of the gradient flow for G_f (in the introduction, dynamic rays have been defined in the case, where $K(f)$ is connected; this new definition matches the old one in this case). Let $V(f)$ be the union of all dynamic rays of f. Then $V(f)$ is a forward-invariant open set, and there is a conformal isomorphism ϕ_f between $V(f)$ and some open subset of the set $\{|z| > 1\}$ with the following properties:

$$\phi_f(f(z)) = \phi_f(z)^d, \quad G_f(z) = \log|\phi_f(z)|.$$

These properties define the map ϕ_f almost uniquely: the only way to change the map ϕ_f without violating the two properties is to post-compose it with multiplication by a $(d-1)$-st root of unity.

The map ϕ_f is called a *Böttcher coordinate.* It is used to parameterize dynamic rays of f. Every dynamic ray is the preimage of a straight radial ray $\{re^{2\pi i\theta} \,|\, r > r_0\}$, $r_0 \geqslant 1$, under the map ϕ_f (in the case where $K(f)$ is disconnected, we may have $r_0 > 1$ if the dynamic ray contains a pre-critical point in its closure). We will write $R_f(\theta)$ for this ray, and call it the *dynamic ray of argument* θ. If f is a degree d polynomial, not necessarily monic, then we can make it monic by a complex linear change of variables. Thus, it still makes sense to talk about dynamic rays of f.

However, arguments of dynamic rays are not well-defined, since they depend on the choice of a Böttcher coordinate. Hence every time we consider rays in dynamic planes of different polynomials, we must resolve the issue of choosing the arguments consistently. For example, if a sequence f_n of degree d polynomials converges to a degree d polynomial f, then we can choose any Böttcher coordinate for f, and then, for f_n sufficiently close to f, choose the Böttcher coordinate for f_n that is close to the chosen Böttcher coordinate for f.

Suppose that the Julia set $J(f)$ is connected. Consider a periodic repelling cutpoint α of f, and let r be its minimal period. Then, by the Douady–Hubbard–Sullivan–Yoccoz landing theorem [DH8485, Hub93], there are finitely many dynamic rays landing at α; we will assume that the choice of their arguments is fixed, and denote the set of the arguments by $\mathrm{Ar}_f(\alpha)$. Every wedge between consecutive rays landing at α contains exactly one component of $J(f) \setminus \{\alpha\}$. The dynamic rays landing at α may form one or more orbits under the map $f^{\circ r}$. Choose one of the orbits, and let $\theta_0, \dots, \theta_{q-1}$ denote the arguments of all rays in this orbit labeled in the counterclockwise order. Suppose that $d^r \cdot \theta_i = \theta_{i+p \pmod q}$, where d is the degree of the polynomial f. In this case, we say that α has *combinatorial rotation number* p/q. It is easy to see that the combinatorial rotation number is well defined, i.e., does not depend on the choice of the orbit. Observe that p, q are coprime except for the case when $p/q = 0$ and all rays landing at α are invariant. Every repelling fixed point has a well-defined combinatorial rotation number (thus, p/q above does not depend on the choice of θ_0).

2.2. **Polynomials in $\overline{\mathrm{PHD}}_d$.** We now recall Lemma B.1 from [GM93] that goes back to Douady and Hubbard [DH8485].

Lemma 2.1. *Let f be a polynomial, and z be a repelling periodic point of f. If the ray $R_f(\theta)$ lands at z, then, for every polynomial g sufficiently close to f, the ray $R_g(\theta)$ lands at a repelling periodic point w close to z. Moreover, w depends holomorphically on g.*

Choose a polynomial f with $[f] \in \overline{\mathrm{PHD}}_d$. We want to prove the following statements:

(1) the map f has no repelling periodic cutpoints, and
(2) the map f has at most one non-repelling periodic point of multiplier different from 1.

Statement (1) follows from Lemma 2.1. Indeed, suppose that α is a repelling periodic cutpoint of f. Let $\theta_0, \dots, \theta_{q-1}$ be the arguments of dynamic rays landing at α. Since α is a cutpoint, we have $q > 1$. Lemma 2.1 says that, for g sufficiently close to f, the dynamic rays with arguments $\theta_0, \ldots, \theta_{q-1}$ in the dynamic plane of g land at the same periodic point that is obtained from α by analytic continuation. We get a contradiction if we choose g such that $[g] \in \mathrm{PHD}_d$.

The second statement is a consequence of the *Pommerenke–Levin–Yoccoz inequality*, see e.g. [Hub93]. It follows from the Pommerenke–Levin–Yoccoz inequality that, for any sequence of polynomials f_n with repelling fixed points $\alpha_n \to \alpha$, the fact that $f_n'(\alpha_n) \to e^{2\pi i \rho}$ implies that the combinatorial rotation numbers of f_n at α_n converge to ρ. We can now prove Lemma 2.2.

Lemma 2.2. *Consider a polynomial f, whose class belongs to $\overline{\mathrm{PHD}}_d$, and a sequence of polynomials f_n converging to f, whose classes belong to PHD_d. If f has a non-repelling fixed point α, whose multiplier is different from 1, then α is the limit of the attracting fixed points of f_n.*

Proof. Let α be neutral and α_n be a fixed point of f_n such that $\alpha_n \to \alpha$ as $n \to \infty$. If α_n are attracting for arbitrarily large n, then we are done. Otherwise assume that for all large n, the points α_n are repelling fixed points. Then their combinatorial rotation numbers equal 0. By the assumptions, $f'(\alpha) = e^{2\pi i \rho}$, where $\rho \not\equiv 0 \pmod{2\pi}$. On the other hand, by the Pommerenke–Levin–Yoccoz inequality, the combinatorial rotation numbers of f_n at α_n (which are all equal to 0) must converge to ρ, a contradiction. □

We can now complete the proof of Theorem A.

Proof of Theorem A. Observe that if $[f] \in \overline{\mathrm{PHD}}_d$ then one of the non-repelling cycles of f must be a fixed point (indeed, as we approximate f with polynomials g, whose classes belong to PHD_d, the attracting fixed points of g converge to a non-repelling fixed point of f). Let $[f] \in \overline{\mathrm{PHD}}_d$. By way of contradiction, suppose that α and β are two non-repelling periodic points, whose multipliers are different from one. Replacing f with a suitable iterate, we may assume that α is fixed but has multiplier different from 1. By Lemma 2.2, α is the limit of the attracting fixed points of polynomials g with $[g] \in \mathrm{PHD}_d$ approximating the polynomial f. However, the same must be true for the point β, a contradiction. □

3. Proof of the first part of Theorem B

We will write $\mathcal{F}$ for the union of all $\mathcal{F}_\lambda$, and $\mathcal{F}_{nr}$ for the union of all $\mathcal{F}_\lambda$ with $|\lambda| \leqslant 1$. Fix λ with $|\lambda| \leqslant 1$. Let $\mathfrak{g}_{\lambda,b}$ be the Green function for $K(f_{\lambda,b})$. Let $V_{\lambda,b}$ be the union of all unbounded trajectories of the gradient flow generated by $\mathfrak{g}_{\lambda,b}$. The Böttcher coordinate is an analytic map $\phi_{\lambda,b} : V_{\lambda,b} \to \mathbb{C}$ with $\phi_{\lambda,b} \circ f_{\lambda,b} = \phi_{\lambda,b}^3$ and $\phi_{\lambda,b}(z) =$

$z + o(z)$ as $z \to \infty$. The following theorem is an easy consequence of the analytic dependence of the Böttcher coordinate on parameters [DH8485, BrHu88].

Theorem 3.1 ([BuHe01], Proposition 2)**.** *Let $\mathcal{V}_\lambda$ be the union of $\{b\} \times V_{\lambda,b}$ over all $b \in \mathbb{C}$. This set is open in $\mathbb{C}^2$. The map $\Phi_\lambda : \mathcal{V}_\lambda \to \mathbb{C}^2$ given by the formula $\Phi_\lambda(b, z) = (b, \phi_{\lambda,b}(z))$ is an analytic embedding of $\mathcal{V}_\lambda$ into $\mathbb{C}^2$.*

We will write $R_{\lambda,b}(\theta)$ for the dynamic ray $R_{f_{\lambda,b}}(\theta)$.

3.1. **Polynomials with parabolic points and their petals.** Let g be a polynomial of arbitrary degree such that 0 is a fixed parabolic point of g of multiplier 1. Suppose that $g(z) = z + az^{q+1} + o(z^{q+1})$, where q is a positive integer. Below, we recall some basic terminology, which is used when working with parabolic points; a general reference is [Mil06]. An *attracting vector* for g is defined as a vector (=complex number) v such that av^q is a negative real number, i.e. v and av^{q+1} have opposite directions. Clearly, there are q straight rays consisting of attracting vectors, which divide the plane of complex numbers into q *repelling sectors*.

Consider a repelling sector S. Note that the set $S^{-q} = \{z \in \mathbb{C} \,|\, z^{-q} \in S\}$ is the complement of the ray $\{-ta \,|\, t > 0\}$ in $\mathbb{C}$. We will write $F : S^{-q} \to \mathbb{C}$ for the composition of the function $w \mapsto w^{-1/q}$ mapping S^{-q} onto S, the function g mapping S onto $g(S)$, and the function $z \mapsto z^{-q}$ mapping $g(S)$ to $\mathbb{C}$. We have $F(w) = w - qa + \alpha(w)$, where $\alpha(w)$ denotes a power series in $w^{-1/q}$ that converges in a neighborhood of infinity, and whose free term is zero (note that the function $w \mapsto w^{-1/q}$ is single valued and holomorphic on S^{-q}). It follows that there exists a positive real number r with the property that $|\alpha(w)| < \frac{|a|}{2}$ whenever $|w| \geqslant r|a|$. Consider the half-plane Π given by the inequality $\mathrm{Re}(w/a) \geqslant r$. We include the point ∞ into Π so that Π is a compact subset of the Riemann sphere. Since $w \in \Pi$ implies that $|w| \geqslant r|a|$, we have $F(\Pi) \supset \Pi$, and also that the shortest Euclidean distance from a finite point on the boundary of Π to a point on the boundary of $F(\Pi)$ is at least $(q - \frac{1}{2})|a|$. The preimage RP of the half-plane Π under the map $z \mapsto z^{-q}$ from $S \cup \{\infty\}$ to $S^{-q} \cup \{0\}$ is called a (closed) *repelling petal* of g.

We choose one repelling petal in every repelling sector; thus, our polynomial $g(z) = z + az^{q+1} + o(z^{q+1})$ has q repelling petals. A repelling petal RP of g is such that $g(RP) \supset RP$. Let us discuss the dependence of the repelling petals on parameters. The argument of the following lemma is rather standard, see e.g. the proof of Lemma 5 in [BuHe01].

Lemma 3.2. *Let $g_t(z) = z + a_t z^{q+1} + o(z^{q+1})$ be a continuous family of polynomials, in which a_t never vanishes, and the parameter t runs through some locally compact metric space I that is a union of countably many compact spaces. Then all q repelling petals of g_t can be chosen to vary continuously with respect to the parameter.*

Proof. First note that, since the attracting vectors depend continuously on the parameter t, so do the repelling sectors (to be more precise, the intersections of the repelling sectors with a finite disk about 0 depend continuously on t with respect to the Hausdorff metric). We will choose S_t to be a continuous family of repelling sectors for g_t.

Consider some particular parameter value t. The map

$$F_t(w) = w - qa_t + \alpha_t(w)$$

of S_t^{-q} to $\mathbb{C}$ is obtained from the map $g_t : S_t \to g_t(S_t)$ by the conjugation $z \mapsto z^{-q}$, as above. Let r_t be a positive real number with the property that $|\alpha_t(w)| < \frac{|a_t|}{4}$ whenever $|w| \geqslant r_t|a_t|$. Set $\Pi_r(a)$ to be the half plane given by the inequality $\mathrm{Re}(w/a) \geqslant r$. We include the point ∞ to $\Pi_r(a)$. Thus $\Pi_r(a)$ is compact. Note that $\alpha_{t'}(w')$ is a continuous function of (t', w') defined at least for $w' \in \Pi_{r_t}(a_{t'})$ and t' sufficiently close to t. It follows that there is some open neighborhood E_t of t in I such that $|\alpha_{t'}(w)| < \frac{|a_{t'}|}{2}$ for all $t' \in E_t$ and all $w \in \Pi_{r_t}(a_{t'})$ (we use the compactness of $\Pi_r(a)$ and its continuous dependence on a in the Hausdorff metric associated with the spherical metric).

The family of neighborhoods E_t is an open covering of I. Choose a countable and locally finite subcovering E_{t_n}. Let $\varphi_n : I \to \mathbb{R}$ be the associated partition of unity, so that the support of the continuous function $\varphi_n \geqslant 0$ is contained in E_{t_n}, and $\sum_n \varphi_n = 1$. Set $R_t = \sum_n \varphi_n(t) r_{t_n}$. Then R_t depends continuously on t, and we have $|\alpha_t(w)| < \frac{|a_t|}{2}$ for all $w \in \Pi_{R_t}(a_t)$ and $t \in I$. The half-plane $\Pi_{R_t}(a_t)$ depends continuously on t in the Hausdorff metric. Then the image of $\Pi_{R_t}(a_t)$ under the map $w \mapsto w^{-1/q}$ from $S_t^{-q} \cup \{\infty\}$ to $S_t \cup \{0\}$ is a repelling petal depending continuously on t. □

3.2. **Stability of rays and their perturbations.** Throughout this subsection, we fix λ that is a root of unity, i.e. $\lambda = \exp(2\pi i p/q)$ for some relatively prime p and q. Since λ is fixed, we will skip λ from the notation f_b, $R_b(\theta)$ etc. We discuss conditions that guarantee that a dynamical ray $R_b(\theta)$ landing at 0 is stable, i.e., for b' sufficiently close to b, the ray $R_{b'}(\theta)$ also lands at 0.

Proposition 3.3. *We have $f_b^{\circ q}(z) = z + T_{p/q}(b) z^{q+1} + o(z^{q+1})$, where $T_{p/q}(b)$ is a non-zero polynomial in b.*

Proof. By the Petal Theorem [Bea00, Theorem 6.5.10], we have

$$f_b^{\circ q}(z) = z + T_{p/q}(b)z^{q+1} + o(z^{q+1}).$$

It remains to prove that the polynomial $T_{p/q}(b)$ cannot be identically equal to zero. For any b such that $T_{p/q}(b) = 0$, the polynomial f_b has at least two cycles of attracting petals at 0. Each of the associated cycles of Fatou domains must contain a critical point of f_b. Thus both critical orbits of f_b converge to 0. However, for large b, one of the critical points escapes. Therefore, for such b, we have $T_{p/q}(b) \neq 0$. □

Proposition 3.4. *Suppose that a dynamic ray $R_{b_*}(\theta)$ with periodic θ lands at* 0*, and $T_{p/q}(b_*) \neq 0$. Then, for all b sufficiently close to b_*, the ray $R_b(\theta)$ lands at* 0*.*

Proof. By Lemma 10.1 of [Mil06], the ray $R_{b_*}(\theta)$ must be tangent to some repelling vector of $f_{b_*}^{\circ q}$ at 0. Let RP_{b_*} be the corresponding repelling petal of $f_{b_*}^{\circ q}$. The period of θ is equal to q by Theorem 18.13 of [Mil06]. There are two points z_* and $f_{b_*}^{\circ q}(z_*)$ in $R_{b_*}(\theta)$ that lie in the interior of RP_{b_*}. By Lemma 3.2 for all b sufficiently close to b_*, we can define a repelling petal RP_b of $f_b^{\circ q}$ that is close to RP_{b_*} in the Hausdorff metric.

Let L_* denote the subray of the ray $R_{b_*}(\theta)$ from z_* to infinity. By Theorem 3.1, for every $\varepsilon > 0$, we can choose a neighborhood U of b_* such that, for all $b \in U$, the corresponding piece L of R_b is ε-close to L_* in the Hausdorff metric. The number ε can be chosen so that this implies that L enters the corresponding petal RP_b. Dynamics inside RP_b implies that $R_b(\theta)$ lands at 0. □

3.3. **The proof of inclusion $\overline{\mathrm{PHD}}_3^e \subset \mathrm{CU}$.** We first recall the necessary terminology. Let Λ be a Riemann surface, and $Z \subset \mathbb{C}$ any (!) subset. A *holomorphic motion* of the set Z is a map $\mu : Z \times \Lambda \to \mathbb{C}$ with the following properties:

- for every $z \in Z$, the map $\mu(z, \cdot) : \Lambda \to \mathbb{C}$ is holomorphic;
- for $z \neq z'$ and every $\nu \in \Lambda$, we have $\mu(z, \nu) \neq \mu(z', \nu)$;
- there is a point ν_0 such that $\mu(z, \nu_0) = z$ for all $z \in Z$.

A crucial result about holomorphic motions is the λ-*lemma* of Mañé, Sad and Sullivan [MSS83]: *a holomorphic motion of a set Z extends to a unique holomorphic motion of the closure $\overline{Z}$; moreover, this extension is a continuous function in two variables.* Suppose that for each $\nu \in \Lambda$ a map $h_\nu : Z \to \mathbb{C}$ is given. A holomorphic motion $\mu : Z \times \Lambda \to \mathbb{C}$ is called *equivariant* (*with respect to the family of maps h_ν*) if for every $\nu \in \Lambda$ and every $z \in Z$ with $h_{\nu_0}(z) \in Z$ we have $h_\nu(\mu(z, \nu)) = \mu(h_{\nu_0}(z), \nu)$. In the introduction, we have defined the λ-stable set $\mathcal{S}_\lambda$ consisting of parameter values b_* such that $J(f_{\lambda,b})$ moves holomorphically with respect to b in some neighborhood of b_*, and the corresponding holomorphic motion is equivariant. We have also defined λ-stable domains as components of $\mathcal{S}_\lambda$. Often, by stable domains we also mean the corresponding subsets of $\mathcal{F}_\lambda$.

By Theorem A, we have $\mathrm{PHD}_3 \subset \mathrm{CU}$. Consider a stable domain $\mathcal{U} \subset \mathcal{F}_\lambda$ such that for all $f \in \mathrm{Bd}(\mathcal{U})$, we have $[f] \in \overline{\mathrm{PHD}}_3$. We need to prove that classes of all polynomials in $\mathcal{U}$ belong to CU. Suppose that $f_* = f_{\lambda,b_*} \in \mathcal{U}$, and show that then f_* has properties (1)–(3) from Definition 1.1.

Property (1). Clearly, f_* has a fixed non-repelling point 0, thus property (1) is fulfilled.

Property (2). Let us prove that f_* has no repelling cutpoints. Assume that f_* has a repelling periodic cutpoint z_{b_*}. The set $\mathrm{Ar}_{f_*}(z_{b_*})$ of arguments of external rays of f_* landing at z_{b_*} consists of at least two angles. Since all maps in $\mathcal{U}$ are quasi-symmetrically conjugate, it is easy to see (e.g., by Lemma 3.5 [BOPT13b]) that all maps $f_{\lambda,b} \in \mathcal{U}$ have repelling periodic cutpoints z_b corresponding to z_{b_*}. By Lemma 2.1 $\mathrm{Ar}_{f_{\lambda,b}}(z_b) = \mathrm{Ar}_{f_*}(z_{b_*})$. Suppose that $\{\alpha, \beta\} \subset \mathrm{Ar}_{f_*}(z_{b_*})$.

Let Λ be the set of all parameter values b with $f_{\lambda,b} \in \mathcal{U}$, and choose a sequence $b_n \to b' \in \mathrm{Bd}(\Lambda)$. We may assume that $z_{b_n} \to z_{b'}$, where $z_{b'}$ is a non-attracting periodic point of $f_{\lambda,b'}$. If both rays $R_{b'}(\alpha)$, $R_{b'}(\beta)$ land at *repelling* periodic points, then these landing points must coincide as otherwise by Lemma 2.1 we obtain a contradiction with the fact that $R_{b_n}(\alpha)$, $R_{b_n}(\beta)$ land at z_{b_n} and $z_{b_n} \to z_{b'}$. However, by Theorem A, the map $f_{\lambda,b'}$ does not have repelling periodic cutpoints. Hence one of the rays $R_{b'}(\alpha)$, $R_{b'}(\beta)$ lands at a parabolic periodic point. Clearly, for at most finitely many parameter values $b' \in \mathrm{Bd}(\Lambda)$ the rays $R_{b'}(\alpha)$ or $R_{b'}(\beta)$ land at a parabolic point distinct from 0. Assume that for infinitely many $b' \in \mathrm{Bd}(\Lambda)$ the rays $R_{b'}(\alpha)$ land at 0 which is a parabolic fixed point: $\lambda = \exp(2\pi i p/q)$ for some relatively prime p and q.

Let us show that in the above case $T_{p/q}(b') = 0$. Indeed, arbitrarily close to b', there are parameter values b, for which $R_b(\alpha)$ does not land at 0. It follows from Proposition 3.4 that $T_{p/q}(b') = 0$. However, the polynomial $T_{p/q}$ has only finitely many roots, a contradiction.

Property (3). Suppose that f_* has a non-repelling n-periodic point $z_{b_*} \neq 0$ with multiplier not equal to 1. Since f_* is stable, the corresponding periodic point $z_b \neq 0$ of f_b, $b \in \Lambda$, is n-periodic and non-repelling. If $b \to b' \in \mathrm{Bd}(\Lambda)$, then $z_b \to z_{b'}$ where $z_{b'}$ is a non-repelling $f_{b'}^{\circ n}$-fixed point. Consider two cases. First, suppose that $z_{b'} \neq 0$. Then by Theorem A the multiplier at $z_{b'}$ is 1. There are only finitely many values of b', for which this can happen. Second, suppose that $z_{b'} = 0$. We have $f_b^{\circ n}(z) - z = z(z - z_b)Q_b(z)$ for some polynomial Q_b, whose coefficients are algebraic functions of b that have no poles in $\mathbb{C}$ (indeed, all roots of the left-hand side have this property). We obtain in the limit as $b \to b'$ that $f_{b'}^{\circ n}(z) - z = z^2 Q_{b'}(z)$, hence 0 is a parabolic fixed point of $f_{b'}$. We may assume that the multiplier at 0 is $e^{2\pi i p/q}$. Let us show that then 0 is a degenerate parabolic point (i.e., that $T_{p/q}(b') = 0$ where $T = T_{p/q}$ is the polynomial introduced in Proposition 3.3).

Indeed, $n = mq$ is a multiple of q, and as in Proposition 3.3 by the Petal Theorem $f_b^{\circ q}(z) = z + T(b)z^{q+1} + o(z^{q+1})$. It is easy to see by induction that then for any k we have $f_b^{\circ kq}(z) = z + kT(b)z^{q+1} + o(z^{q+1})$. On the other hand, as above $f_b^{\circ mq}(z) - z = z^{q+1}(z - z_b)R_b(z)$ where $R_b(z)$ is a polynomial of z whose coefficients are algebraic functions of b that have no poles in $\mathbb{C}$. Hence in the limit we have $f_{b'}^{\circ mq}(z) - z = z^{q+2}R_{b'}(z)$. It follows that $mT(b')z^{q+1} + o(z^{q+1}) = z^{q+2}R_{b'}(z)$ and hence $mT(b') + o(1) = zR_{b'}(z)$, which implies that $T(b') = 0$ as desired. Clearly, there are finitely many such values of b'. Thus, we showed that overall there are only finitely many values of b' to which b may converge, a contradiction with $\mathrm{Bd}(\Lambda)$ being infinite.

4. Laminations associated to polynomials

We first recall the language of laminations. A Riemann map $\psi_f : \mathbb{D} \to \overline{\mathbb{C}} \setminus K(f)$ from the unit disk $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ in $\mathbb{C}$ to the complement of $K(f)$ in the Riemann sphere $\overline{\mathbb{C}}$ can be chosen so that $\psi_f(0) = \infty$ and $\psi_f(z^d) = f(\psi_f(z))$ for all $z \in \mathbb{D}$. A *dynamic ray* $R_f(\theta)$ of argument θ is by definition the set $\psi_f((0,1)e^{2\pi i\theta})$. We say that $R_f(\theta)$ *lands* at a point $z \in J(f)$ if $z = \lim_{t\to 1^-} \psi_f(te^{2\pi i\theta})$. By the classical Carathéodory theorem, if $K(f)$ is locally connected, then there exists a continuous extension $\overline{\psi}_f : \overline{\mathbb{D}} \to \overline{\mathbb{C}}$, which maps the unit circle $\mathbb{S}^1$ onto $J(f)$. In particular, all rays land. Let $\sim_f$ be the equivalence relation on $\mathbb{S}^1$, whose classes are fibers of $\overline{\psi}_f$. It is called the (Thurston) *invariant lamination* associated with f. *Leaves* of $\sim_f$ are defined as edges of the convex hulls of all $\sim_f$-classes. It is not hard to see that leaves of $\sim_f$ are disjoint in $\mathbb{D}$.

Laminations can also be defined abstractly, i.e., without reference to a polynomial.

Definition 4.1 (Laminations)**.** An equivalence relation $\sim$ on the unit circle $\mathbb{S}^1$ is called a *lamination* if either $\mathbb{S}^1$ is one $\sim$-class (such laminations are called *degenerate*), or the following holds:

(E1) the graph of $\sim$ is a closed subset in $\mathbb{S}^1 \times \mathbb{S}^1$;
(E2) if $t_1 \sim t_2 \in \mathbb{S}^1$ and $t_3 \sim t_4 \in \mathbb{S}^1$, but $t_2 \not\sim t_3$, then the open straight line segments in $\mathbb{C}$ with endpoints t_1, t_2 and t_3, t_4 are disjoint;
(E3) each equivalence class of $\sim$ is totally disconnected.

A lamination $\sim$ admits a *canonical extension onto* $\mathbb{C}$: its classes are either convex hulls of classes of $\sim$, or points which do not belong to such convex hulls. By Moore's Theorem the space $\mathbb{C}/\sim$ is homeomorphic to $\mathbb{C}$. The quotient map $p_\sim : \mathbb{S}^1 \to \mathbb{S}^1/\sim$ extends to the plane with the only non-trivial point-preimages (*fibers*) being the convex hulls of $\sim$-classes. From now on we will always consider such extensions of the quotient map.

Definition 4.2 (Laminations and dynamics)**.** A lamination $\sim$ is called (σ_d-)*invariant* if:
(D1) $\sim$ is *forward invariant*: for a $\sim$-class $\mathbf{g}$, the set $\sigma_d(\mathbf{g})$ is a $\sim$-class;
(D2) for any $\sim$-class $\mathbf{g}$, the map $\sigma_d : \mathbf{g} \to \sigma_d(\mathbf{g})$ extends to $\mathbb{S}^1$ as an orientation preserving covering map such that $\mathbf{g}$ is the full preimage of $\sigma_d(\mathbf{g})$ under this covering map.

For a σ_d-invariant lamination $\sim$ consider the *topological Julia set* $\mathbb{S}^1/\sim = J_\sim$ and the *topological polynomial* $f_\sim : J_\sim \to J_\sim$ induced by σ_d. One can extend $f_\sim$ to a branched-covering map $f_\sim : \mathbb{C} \to \mathbb{C}$ of degree d called a *topological polynomial* too. The map $p_\sim$ semi-conjugates σ_d with $f_\sim$, at least on the unit circle and all leaves of $\sim$. Unlike complex polynomials, topological polynomials can have periodic critical points in their topological Julia sets. The complement $K_\sim$ of the unique unbounded component $U_\infty(J_\sim)$ of $\mathbb{C} \setminus J_\sim$ is called the *filled topological Julia set*. For $a, b \in \mathbb{S}^1$, let $\overline{ab}$ be the *chord* with endpoints a and b. If $A \subset \mathbb{S}^1$ is closed, boundary chords of the convex hull $\mathrm{CH}(A)$ of A are called *edges* of $\mathrm{CH}(A)$.

Definition 4.3 (Leaves and gaps)**.** If A is a $\sim$-class, call an edge $\overline{ab}$ of $\mathrm{Bd}(\mathrm{CH}(A))$ a *leaf*. All points of $\mathbb{S}^1$ are also called (*degenerate*) *leaves*. The family $\mathcal{L}_\sim$ of all leaves of $\sim$ is called the *geometric lamination* (*geolamination*) *generated by* $\sim$. Let $\mathcal{L}^+_\sim$ be the union of all leaves of $\mathcal{L}_\sim$. The closure of a non-empty component of $\mathbb{D} \setminus \mathcal{L}^+_\sim$ is called a *gap* of $\sim$. Leaves and gaps of $\mathcal{L}_\sim$ are called $\mathcal{L}_\sim$*-sets*; a leaf which is not an edge of a finite gap is called *independent*. If G is a gap or leaf, we call the set $G' = \mathbb{S}^1 \cap G$ the *basis of* G.

Extend σ_d (keeping the notation) linearly over all *individual chords* in $\overline{\mathbb{D}}$ (e.g., over leaves of $\mathcal{L}_\sim$); even though the extended σ_d is not well-defined on the entire disk, it is well-defined on $\mathcal{L}^+_\sim$. A gap or leaf U

is said to be *(pre)periodic* if $\sigma_d^{\circ m+k}(U') = \sigma_d^{\circ m}(U')$ for some $m \geqslant 0$, $k > 0$. If m above can be chosen to be 0, then U is called *periodic*; the minimal number k above is called the *period* of U. If U is (pre)periodic but not periodic then it is called *preperiodic*.

Definition 4.4 (Rotational sets and numbers)**.** If $\mathbf{g}$ is a periodic non-degenerate finite $\sim$-class of period n, the map $\sigma_d^{\circ n}|_{\mathbf{g}}$ is conjugate (by a conjugacy that preserves the cyclic order) to a rigid rotation R_ρ by a rational angle ρ on a finite R_ρ-invariant subset of $\mathbb{S}^1$. The number ρ is then called the *rotation number of* $\mathbf{g}$. A gap G such that its basis G' is infinite is called a *Fatou gap*. A periodic Fatou gap G of period n such that $f_\sim^{\circ n}|_{\mathrm{Bd}(p_\sim(G))}$ is conjugate to an irrational rotation by an angle ρ, is called a *Siegel* gap while ρ is called the *rotation number of* G. Otherwise $f_\sim^{\circ n}|_{\mathrm{Bd}(p_\sim(G))}$ is conjugate to a map σ_k with some $k > 1$ and G is called a *Fatou gap of degree* k. Siegel gaps and finite $\sim$-classes with non-zero rotation number are called *rotational sets*.

Let $X \subset \mathbb{C}$ be a continuum, and let $U_\infty(X)$ be the unbounded component of $\mathbb{C} \setminus X$. If $X = \mathrm{Bd}(U_\infty(X))$, we call X *unshielded*. A continuous map $\varphi : Y \to Z$ is *monotone* if all fibers are continua. Let A be a continuum. A monotone onto map $\varphi : A \to Y_{\varphi,A}$ with locally connected $Y_{\varphi,A}$ is called a *finest* (*monotone*) *map* if for any monotone map $\psi : A \to L$ onto a locally connected continuum L there is a map $h : Y_{\varphi,A} \to L$ with $\psi = h \circ \varphi$ (then h is monotone because for $x \in L$, we have $h^{-1}(x) = \varphi(\psi^{-1}(x))$). If $\varphi : A \to B$, $\varphi' : A \to B'$ are two finest maps, then the map associating points $\varphi(x) \in B$ and $\varphi'(x) \in B'$ for every $x \in A$ is a homeomorphism between B and B'. Hence all sets $Y_{\varphi,A}$ are homeomorphic, all finest maps φ are the same up to a homeomorphism, and we can talk of ***the*** *finest model* $Y_A = Y$ *of* A and ***the*** *finest map* $\varphi_A = \varphi$ *of* A *onto* Y.

Theorem 4.5 (Theorem 1 [BCO11])**.** *Let Q be an unshielded continuum. Then there exist the finest map φ and the finest model Y of Q given by a lamination $\sim_Q$ on $\mathbb{S}^1$ so that $Y = \mathbb{S}^1/\sim_Q$; moreover, φ can be extended to a map $\varphi : \mathbb{C} \to \mathbb{C}$ which collapses only those complementary domains to Q whose boundaries are collapsed by φ, and is a homeomorphism elsewhere in $\mathbb{C} \setminus Q$. For $y \in Y$ the fiber $\varphi^{-1}(y)$ coincides with the topological hull of the union of impressions of all external to Q rays with arguments from the set $p_{\sim_Q}^{-1}(y)$.*

Here $p_{\sim_Q}$ is the quotient map of $\mathbb{S}^1$ onto Y; we call $\sim_Q$ the *finest lamination*. By a *finest map* we mean any extension of the finest map of Q over $\mathbb{C}$.

Definition 4.6 (Critical leaves and gaps)**.** A leaf of a lamination $\sim$ is called *critical* if its endpoints have the same image. A gap G is said to be *critical* if $\sigma_d|_{G'}$ is at least k-to-1 for some $k > 1$.

Lemma 4.7 is well known; we state it here without a proof.

Lemma 4.7. *An edge of a periodic gap is either* (*pre*)*critical or* (*pre*)*periodic.*

In [BCO11], Theorem 4.5 is applied to polynomials with connected Julia set which yields Theorem 4.8 (a similar earlier result is due to Kiwi [Kiw04]).

Theorem 4.8 ([BCO11], Theorem 2)**.** *Let f be a complex polynomial with connected Julia set and finest lamination $\sim_f = \sim_{J(f)}$. Then there exists a topological polynomial $f_{\sim_f} : \mathbb{C} \to \mathbb{C}$ and a finest map $\varphi_f : \mathbb{C} \to \mathbb{C}$ which semiconjugates f and $f_{\sim_f}$. If $x \in J_{\sim_f}$ corresponds to a finite periodic $\sim_f$-class $p^{-1}_{\sim_f}(x)$ then the fiber $\varphi_f^{-1}(x)$ is a point. No periodic Fatou domain of f of degree greater than 1 is collapsed by φ_f.*

We need the following definition.

Definition 4.9. Call gaps *finite* or *infinite* if their bases are finite or infinite; infinite $\sim$-classes have infinite gaps as their convex hulls (such gaps will be called *infinite gap-classes*). By [BL02] all such gaps are (pre)periodic, and periodic infinite gap-classes are Fatou gaps of degree greater than 1. Call the corresponding fibers *CS-fiber.* Thus, if $x \in J_{\sim_f}$ corresponds to an infinite gap-class $p^{-1}_{\sim_f}(x)$ then the x-fiber $\varphi_f^{-1}(x)$ is said to be a *CS-fiber.*

The following lemma explains the terminology.

Lemma 4.10 ([BOPT13b], Proposition 4.4)**.** *A periodic CS-fiber contains either a Cremer point or a Siegel point.*

The drawback of using the lamination $\sim_f$ to model the dynamics of f is that $\sim_f$ may incompletely reflect the properties of repelling periodic cutpoints of $J(f)$. In a lot of cases this does not happen. Indeed, if all $\sim_f$-classes are finite then by Theorem 4.8 there is a one-to-one correspondence between repelling and parabolic periodic cutpoints of f and their preimages on the one hand and the (pre)periodic non-degenerate classes of $\sim_f$. However in the case when some $\sim_f$-classes are infinite this may no longer be the case.

E.g., suppose that a cubic polynomial f has a fixed repelling point 0 at which $R_f(0)$ and $R_f(\frac{1}{2})$ land, and no more repelling periodic cut-points. Moreover, suppose that in each "half-plane" created by the cut

$R_f(0) \cup \{0\} \cup R_f(\frac{1}{2})$ there is a fixed Cremer point. Denote these fixed Cremer points by a and b. Then the thickening construction of Douady and Hubbard [DH85] (see also [Mil00b] and, more specifically, [EY99]) shows that there are two quadratic-like Julia sets in $J(f)$, namely J_a (containing a) and J_b (containing b). Each of them corresponds to a quadratic Julia set with a Cremer fixed point, and by [BO06] the only monotone map of J_a (J_b) onto a locally connected continuum is a collapse to one point. It follows that the only monotone map of J onto a locally connected continuum is a collapse to one point. Hence the lamination $\sim_f$ identifies all points of the circle and misses the fact that f has a fixed repelling cutpoint 0.

Let F be a fiber associated with an infinite $\sim_f$-class. We saw that F may contain periodic repelling points cutting F such that the corresponding leaves are not included in $\mathcal{L}_{\sim_f}$. By Proposition 40 [BCO11] there are at most finitely many repelling or parabolic cutpoints in F. To each such point x we associate the convex hull of the set $\mathrm{Ar}_f(x)$. We add the edges of such convex hulls to $\mathcal{L}_{\sim_f}$. Then we add to $\mathcal{L}_{\sim_f}$ the edges of gaps corresponding to preimages of such points. Finally, we take the limit leaves of this family of leaves and add them to $\mathcal{L}_{\sim_f}$. This creates a new geolamination $\mathcal{L}_f$ called the *geolamination generated by* f. In this way we combine $\mathcal{L}_{\sim_f}$ with the *rational lamination* defined by Kiwi in [Kiw04]. In $\mathcal{L}_f$ we will distinguish between Fatou gaps corresponding to non-degenerate Fatou domains of f, infinite gaps of $\mathcal{L}_f$ that are gap-classes of $\mathcal{L}_{\sim_f}$, and infinite gaps H of $\sim_f$ such that $H \cap \mathbb{S}^1$ is one $\sim_f$-class subdivided by finitely many finite gaps or leaves and their preimages as in the definition of $\mathcal{L}_f$.

Definition 4.11. A *laminational pair* is a pair $\{\sim, \mathcal{L}\}$ where $\mathcal{L} \supset \mathcal{L}_\sim$ is a geolamination obtained by adding to $\mathcal{L}_\sim$ finitely many finite periodic gaps or leaves inside the convex hull of each infinite $\sim$-class as well as all their pullbacks and limits so that $\mathcal{L}$ is a geolamination.

Definition 4.12. A cubic laminational pair $\{\sim, \mathcal{L}\}$ is *cubioidal* if $\mathcal{L}$ has at most one rotational set and each periodic non-degenerate leaf of $\mathcal{L}$ has an attached to it Fatou gap whose basis is not contained in one $\sim$-class.

There are two extreme cases for $\{\sim_f, \mathcal{L}_f\}$. First, $\sim_f$ may identify no two points. Then $J(f)$ is a Jordan curve, $\mathcal{L}_f$ has no leaves, and $[f] \in \mathrm{CU}$. We call such laminational pair *empty*. Second, $\sim_f$ may identify all points of $\mathbb{S}^1$ while $\mathcal{L}_f$ contains no leaves. By our Lemma 5.1 then again $[f] \in \mathrm{CU}$. We call such laminational pair *degenerate*. The degenerate and the empty laminational pairs share the same geolamination, are cubioidal, and correspond to polynomials f with $[f] \in \mathrm{CU}$, yet correspond to two very different types of dynamics. In all other cases $\mathcal{L}_f$ includes some non-degenerate leaves.

5. Proof of Theorem C

Lemma 5.1 deals with the case when $\mathcal{L}_f$ has only degenerate leaves. We will assume that 0 is a fixed point of f.

Lemma 5.1. *If all leaves of $\mathcal{L}_f$ are degenerate then $[f] \in \mathrm{CU}$, and 0 is the unique non-repelling periodic point of f. Moreover, if $\sim_f$ consists of one class, then 0 is a Cremer or Siegel fixed point.*

Proof. We may assume that $\sim_f$ consists of one class coinciding with $\mathbb{S}^1$. By definition of $\mathcal{L}_f$ the map f has no repelling periodic cutpoints, and, by Theorem 4.8, the polynomial f has no attracting or parabolic periodic points. By Lemma 4.10, the point 0 is a fixed Cremer or Siegel point. Suppose that there is a non-repelling periodic point $x \neq 0$ of f. Similar to the above x is also a Cremer or a Siegel periodic point. Then by [Kiw00] there exists a repelling periodic point separating x and 0, a contradiction. □

Lemma 5.2. *If $[f] \in \mathrm{CU}$ then $\mathcal{L}_{\sim_f} = \mathcal{L}_f$.*

Before we prove Lemma 5.2, we need to recall a description of quadratic (i.e., degree 2) invariant gaps given in [BOPT13a]. Let G be a quadratic invariant gap of some σ_3-invariant lamination. Then there is a unique edge M of G (the *major* of G) separating the circle into two arcs, one of which contains all vertices of G and is of length at most $\frac{2}{3}$; the leaf M must be critical or periodic. Moreover, all edges of G are iterated σ_3-preimages of M. Suppose that a quadratic invariant gap G is a gap of $\sim_f$. By [BOPT13b, Theorem 7.7], if $M = \overline{\theta_1\theta_2}$ is a periodic major of G, then the external rays $R_f(\theta_1)$, $R_f(\theta_2)$ land at the same point. This implies that if $\overline{\alpha\beta}$ is a (pre)periodic edge of G then the external rays $R_f(\alpha)$, $R_f(\beta)$ land at the same point.

Proof of Lemma 5.2. Suppose that $[f] \in \mathrm{CU}$ and $\mathcal{L}_{\sim_f} \neq \mathcal{L}_f$. Then f has a periodic CS-fiber F. By Lemma 4.10, there exists a Cremer or Siegel periodic point $y \in F$. If F is not invariant then y is not fixed contradicting Definition 1.1. Thus F is invariant, and we may assume that $y = 0 \in F$ is a fixed Cremer or Siegel point. As above, the corresponding to F invariant Fatou gap G is of degree greater than 1. If G is of degree 3 then $G' = \mathbb{S}^1$. Since $[f] \in \mathrm{CU}$, the map f does not have repelling periodic cutpoints. Since $G' = \mathbb{S}^1$, hence $F = J(f)$, by Theorem 4.8, the map f cannot have parabolic periodic points. Hence by definition in this case all leaves of $\mathcal{L}_f = \mathcal{L}_{\sim_f}$ are degenerate. Assume that G is of degree 2. Since $[f] \in \mathrm{CU}$ has no repelling cutpoints, then $\mathcal{L}_{\sim_f} \neq \mathcal{L}_f$ implies that there is a parabolic periodic cutpoint x of F. Since f is cubic, by the Fatou-Shishikura inequality, the union of the orbit of x and the point 0 is the set of all non-repelling periodic points of f. In particular, there are no other periodic cutpoints of F.

Let $\widetilde{X}$ be the union of all rays landing at x and $\{x\}$ itself. Some edges of the convex hull X of $\mathrm{Ar}_f(x)$ are contained inside G (otherwise x would not be a cutpoint of F). Apply the map ψ_G which collapses all edges of G to points. It semiconjugates $\sigma_3|_{\mathrm{Bd}(G)}$ to σ_2 so that the restriction of $\mathcal{L}_f$ onto G induces a σ_2-invariant geolamination $\mathcal{L}_f^2$ which contains, by the above, some periodic leaves. By Proposition II.6.10b of [Thu85], the lamination $\mathcal{L}_f^2$ has an invariant gap H of non-zero rotation number or the leaf $H = \overline{\frac{1}{3}\frac{2}{3}}$. Theorem II.5.3 of [Thu85] shows that if H is a gap then either H is a Siegel gap, or it is a gap with countably many vertices, or it is a finite gap. However in the first two cases it follows that the lamination $\mathcal{L}_f^2$ contains an isolated critical leaf. On the other hand, the construction of $\mathcal{L}_f^2$ implies that all non-degenerate leaves of $\mathcal{L}_f^2$ are either (pre)periodic with non-degenerate images, or limits of (pre)periodic, a contradiction. Thus, either $H = \overline{\frac{1}{3}\frac{2}{3}}$, or H is a finite gap of rational rotation number.

Consider the convex hull H_1 of $\psi_G^{-1}(H')$. Then H_1 has either the same number of vertices as H, or twice as many vertices as H (if vertices of H are ψ_G-images of edges of G). We want to prove that there is an f-fixed point associated to H_1 such that external rays of f whose arguments are vertices of H_1 land at that point. Indeed, suppose otherwise. Then by definition of our laminations we may assume that there are σ_2-pullbacks of $\psi_G(X)$ accumulating on each edge of H. Let $\ell = \overline{ab}$ be an edge of H. Then the corresponding σ_3-pullbacks of X will accumulate on the corresponding edge of $\overline{a_1b_1}$ of H_1. The corresponding cuts of F formed by the corresponding pullbacks of $\widetilde{X}$ can be chosen so that their "vertices" (i.e., corresponding pullbacks of x) converge to a point y_ℓ belonging to the impression of $R_f(a_1)$ and the impression of $R_f(b_1)$. Thus, impressions of $R_f(a_1)$ and $R_f(b_1)$ are non-disjoint.

If H_1 and H have the same number of vertices, it follows that the union K of all impressions of angles with arguments which are vertices of H_1 is a continuum. If H_1 has twice as many vertices as H, for every vertex l of H there is an edge $\ell = \overline{uv}$ of H_1 such that $\psi_G(\ell) = l$. By [BOPT13b, Theorem 7.7], the external rays $R_f(u)$, $R_f(v)$ land at the same point. Hence in that case the union K of impressions of angles which are vertices of H_1 is a continuum too. Clearly, K is invariant and separated from impressions of rays with arguments which are not vertices of H_1 (either by the just discussed pullbacks of $\widetilde{X}$, or by the appropriate fibers approaching F). By [BCO11, Lemma 37] then K is a fixed repelling or parabolic point. Since $[f] \in \mathrm{CU}$, K is parabolic. Since H, and hence H_1, are of non-zero rotation number, the multiplier at K is not one. On the other hand, x is a Cremer or Siegel point. Thus, f has at least two periodic points of multiplier not equal to 1, a contradiction with $[f] \in \mathrm{CU}$. This shows that $\mathcal{L}_{\sim_f} = \mathcal{L}_f$. □

Proof of Theorem C. In view of Lemma 5.2, it remains to prove that, for $[f] \in \mathrm{CU}$, the lamination $\mathcal{L}_f = \mathcal{L}_{\sim_f}$ is cubioidal. Let us prove that $\mathcal{L}_f$ has at most one rotational set G, and G is invariant. Suppose that G' is a finite $\sim_f$-class. Then, by Theorem 4.8, it corresponds to a periodic repelling or parabolic cutpoint $y(G) = y$ of $J(f)$. Since $[f] \in \mathrm{CU}$, then, by Definition 1.1(2), the point y is parabolic and by Definition 1.1(3) $y = 0$. Hence $\mathcal{L}_f$ cannot have two finite rotational classes. Now, if G is a Siegel gap of $\sim_f$ then there must exist a Siegel periodic point y of f inside $\varphi_f^{-1} \circ p_{\sim_f}(G)$; thus, $y = 0$. Hence $\mathcal{L}_f$ has at most one rotational set G, and G is invariant.

Again, let G be a finite rotational $\sim_f$-class. Since $y(G) = y$ is a cutpoint of $J(f)$, by Definition 1.1(2), the point y is parabolic. Hence there are parabolic domains attached to y. By Theorem 4.8 they are not collapsed by φ_f. Hence along at least one cycle of edges of G such that the period of the endpoints of these edges is, say, m, there are Fatou gaps of period m attached to G and which do not correspond to one $\sim_f$-class as required in Definition 4.12. By [BOPT13a, Corollary 5.5] this implies that for every periodic leaf ℓ of $\mathcal{L}_f$ whose endpoints are of period t there exists a Fatou gap of $\mathcal{L}_f$ of period t attached to ℓ. It remains to prove that such gaps cannot be contained in convex hulls of $\sim_f$-classes.

By the above the only hypothetical situation which we need to consider is as follows: there is a periodic finite gap or leaf G of $\mathcal{L}_f$ with two cycles of edges on its boundary such that Fatou gaps which *are not* convex hulls of a single $\sim_f$-class are attached to one of these cycles of edges while Fatou gaps which *are* convex hulls of a single $\sim_f$-class are attached to the other cycle of edges. Denote by H a Fatou gap which is one $\sim_f$-class attached to an edge of G; let F be the corresponding CS-fiber. By Lemma 4.10 there is a Cremer or Siegel point $x \in F$. Since $[f] \in \mathrm{CU}$, the point $x = 0$ is fixed and so H is invariant. Clearly, the only way it can happen is when $G = \overline{0\frac{1}{2}}$, a contradiction since if $\overline{0\frac{1}{2}}$ is a leaf of $\mathcal{L}_f$ then from at least one side it has an attached Fatou gap which does not coincide with the convex hull of a $\sim_f$-class as desired (so that $\mathcal{L}_f$ is a CU-lamination). □

We can partially reverse Theorem C. First we prove Lemma 5.3.

Lemma 5.3. *If a cubic polynomial f has no repelling cutpoints then it has a non-repelling fixed point.*

Proof. Consider fixed external rays $R_f(0)$ and $R_f(1/2)$. If they land at the same point w then by the assumptions w is non-repelling as desired. Suppose that the ray $R_f(0)$ lands at z, the ray $R_f(1/2)$ lands at y, and $z \neq y$ are repelling. By [GM93] there exists either an invariant Fatou domain U or a fixed point $x \in J(f) \setminus \{y, z\}$. In the first case f has either an attracting or a Siegel fixed point, and we are done. In the second case there are two possibilities. First, a periodic ray R may land at x. By the assumption about $R_f(0), R_f(\frac{1}{2})$ the ray R is not invariant, hence x is a cutpoint. Since f does not have repelling periodic cutpoints, x is parabolic and we are done. Second, suppose that no periodic ray lands at x. Then x is a Cremer fixed point, and we are done. □

Lemma 5.4. *Suppose that $(\sim_f, \mathcal{L}_f)$ is a cubioidal laminational pair, f has no repelling periodic cutpoints and at most one periodic attracting point. Then $[f] \in$ CU.*

Proof. By Lemma 5.3 we may assume that 0 is an f-fixed point, $|f'(0)| \leqslant 1$, and if there is a fixed non-repelling point with multiplier not equal to 1 then $f'(0) \neq 1$. By Definition 4.12, $\mathcal{L}_{\sim_f} = \mathcal{L}_f$. By Definition 1.1 we need to show that all non-repelling periodic points of f but perhaps 0 have multiplier 1. Assume the contrary: f has a periodic non-repelling point $x \neq 0$, whose multiplier is different from 1.

We need an observation concerning any parabolic point y of f. By [Kiw02] either there is one cycle of rays landing at y, or there are two cycles of rays landing at y. In the first case inside each wedge at y there is a parabolic Fatou domain attached to y. In the second case *a priori* it may happen that there is one cycle of Fatou domains attached to y inside one cycle of wedges at y, and the other cycle of wedges at y contains no Fatou domains attached to y inside them. However since $(\sim_f, \mathcal{L}_f)$ is cubioidal, it follows that if there are two cycles of rays (and hence wedges) at y, then there are two cycles of Fatou domains at y. Now we can consider several cases.

(1) Assume that 0 is attracting. Then there is an invariant Fatou domain U containing 0. If x is attracting, Cremer or Siegel then by [Kiw00, Lemma 3.1] there exists a repelling periodic cutpoint, a contradiction. Assume that x is parabolic. Then the fact that the multiplier at x is not 1 implies that x cannot be a boundary point of U. By the

above there are two cases. First, there may be one cycle of rays and one cycle of Fatou domains at x. Clearly, then we can find a point from the orbit of x and a Fatou domain attached to it which can only be separated from 0 by a repelling periodic cutpoint, a contradiction. Second, there may be two cycles of Fatou domains at x. Together with U they will form *three* cycles of Fatou domains of a cubic polynomial f, a contradiction.

(2) Assume that 0 is Cremer or Siegel. If x is attracting, Cremer or Siegel then by [Kiw00, Lemma 3.1] there exists a repelling periodic cutpoint separating 0 and x in $J(f)$, a contradiction. Suppose that x is parabolic. As in (1), the fact that f is cubic implies that there is exactly one cycle of Fatou domains at x. However this implies that there will be one of Fatou domains at one of the points of the orbit of x which can only be separated from 0 by a repelling periodic cutpoint as in [Kiw00, Lemma 3.1], a contradiction.

(3) Assume that 0 is parabolic. By the above there are two subcases here. First, assume that there are two cycles of Fatou domains at 0. Let G be the convex hull of $\mathrm{Ar}_f(0)$. If G is a gap, then each cycle of Fatou domains at 0 consists of at least two domains. If one of them is a cycle of attracting Fatou domains, then we have at least two attracting periodic points of f, a contradiction. If both are cycles of parabolic domains then we cannot have a non-repelling periodic point $x \neq 0$ by the Fatou–Shishikura inequality. Thus, we may assume that G is a leaf. Then having two cycles of Fatou domains at 0 (actually, each cycle in this case consists of just one Fatou domain) means having two cycles of Fatou gaps attached to G which implies that $G = \overline{0\frac{1}{2}}$. If both Fatou domains at 0 are parabolic, we cannot have a non-repelling periodic point $x \neq 0$. Hence one of the Fatou domains at 0 is attracting and the other one is parabolic. However in that case by our choice of f we should have moved the attracting fixed point to 0, a contradiction.

Second, assume that there is one cycle of Fatou domains and one cycle of rays landing at 0. Then it is easy to see (similar to the arguments above) that there must exist a repelling cutpoint separating one of these Fatou domains at 0 from a specifically chosen Fatou domain at one of the points from the orbit of x. In any case, we get a contradiction with the assumption that f has no repelling periodic cutpoints. □

6. Proof of the second part of Theorem B

We need to prove that $\mathcal{LC} \cap \mathrm{CU} = \mathcal{LC} \cap \overline{\mathrm{PHD}}_3^e$ ($\mathcal{LC}$ is the set of classes of polynomials with locally connected Julia sets). By the first part of Theorem B, we have $\overline{\mathrm{PHD}}_3^e \subset \mathrm{CU}$. Hence we have to consider cubic polynomials f such that $[f] \in \mathrm{CU} \setminus \overline{\mathrm{PHD}}_3^e$. By Theorem C, the laminational pair $(\sim_f, \mathcal{L}_f)$ is cubioidal. We may assume that $f \in \mathcal{F}_{nr}$.

6.1. **Main analytic tools.** According to [BOPT13b], there is a well-defined *principal critical point* $\omega_1(f)$ of f that depends holomorphically on f at least in a small neighborhood of f in $\mathcal{F}_{nr}$. If $\lambda = f'(0)$ is a root of unity, then ω_1 is in a parabolic domain attached to 0, in particular, the orbit of $\omega_1(f)$ converges to 0.

Theorem 6.1 ([BOPT13b], Theorem B). *If $f \in \mathcal{F}_{nr}$ and $[f] \notin \overline{\mathrm{PHD}}_3^e$ then there are Jordan domains U^* and V^* such that $f : U^* \to V^*$ is a quadratic-like map hybrid equivalent to $z^2 + c$ with $c \in \overline{\mathrm{PHD}}_2$.*

We will write J^* for the Julia set of the quadratic-like map $f : U^* \to V^*$, and K^* for the filled Julia set of this map. Theorem 6.1 implies that in case $J(f)$ is locally connected we may assume that $\mathcal{L}_{\sim_f}$ has some non-degenerate leaves.

A *stand-alone quadratic invariant gap* U is a quadratic invariant gap U of some lamination considered by itself (without the lamination). We say that U is of *regular critical type* if the major $M = \overline{\theta_1\theta_2}$ of U is critical. If a gap U is of regular critical type, then there exists a unique lamination such that U is its gap. Basically, this lamination is obtained by taking pullbacks of U. This lamination is called the *canonical* lamination of the gap U [BOPT13a].

We say that U is of *periodic type* if its major $M = \overline{\theta_1\theta_2}$ is periodic of some period k. Call such M a *major* (*leaf*) *of periodic type.*

6.2. **The proof of the second part of Theorem B.** By way of contradiction, assume that f is a polynomial with a locally connected Julia set $J(f)$ such that $[f] \in \mathrm{CU} \setminus \overline{\mathrm{PHD}}_3^e$. Recall that there are Jordan domains U^* and V^* such that $f : U^* \to V^*$ is a quadratic-like map with a connected filled Julia set K^*. Define a subset $G' \subset \mathbb{S}^1$ as the set of arguments of all external rays of f landing in K^*; and set G to be the convex hull of G'.

Since $J(f)$ is locally connected, there is an invariant lamination $\sim_f$ and a monotone map $p : \mathbb{C} \to \mathbb{C}$, whose restriction to $\mathbb{S}^1$ semi-conjugates σ_3 with $f|_{J(f)}$, and whose fibers are points, leaves or finite gaps of $\sim_f$.

Lemma 6.2. *Consider a complementary component (a, b) of G' in $\mathbb{S}^1$. Then the rays $R_f(a)$ and $R_f(b)$ land at the same point.*

Proof. Let us first prove that the chord $\overline{ab}$ cannot cross a leaf $\overline{xy}$ of $\sim_f$. Assume the contrary: $\overline{xy} \in \mathcal{L}_{\sim_f}$, where $x \in (a, b)$ and $y \in \mathbb{S}^1 \setminus [a, b]$.

Then the union of $K^* \cup R_f(a) \cup R_f(b)$ separates $R_f(x)$ from $R_f(y)$. Since the landing points of $R_f(x)$ and $R_f(y)$ coincide, this common landing point must belong to K^*, a contradiction with $x \in (a, b)$.

It follows that the chord $\overline{ab}$ is either a leaf of $\sim_f$ or is contained in an infinite gap H of $\sim_f$. Consider the latter case. Then $p(\mathrm{Bd}(H))$ is the boundary of some Fatou component W of f. Consider $X = K^* \cap \mathrm{Bd}(W)$. Clearly, X is connected as if X is disconnected, then so is K^*, a contradiction. On the other hand, no ray with the argument in (a, b) lands in K^*. Hence, X is a closed subarc of $\mathrm{Bd}(W)$ with endpoints $p(a)$ and $p(b)$. By the properties of the locally connected Julia sets for some numbers $n < m$ the union $\bigcup_{i=n}^{m} f^i(X)$ is the boundary of a periodic Fatou domain Q of f. Hence X is a subarc of the appropriate pullback $S \subset J^*$ of $\mathrm{Bd}(Q)$, which is impossible. □

Lemma 6.3. *The set G is a stand-alone quadratic invariant gap.*

Proof. With every complementary component (a, b) of G', we associate the corresponding *G-cut* $\Gamma(a, b) = \Gamma$ consisting of the rays $R_f(a)$, $R_f(b)$, and their common landing point (called the *vertex* of Γ). We claim that the f-image of a G-cut is a G-cut. If v is not critical then $f|_{K^*}$ is a local homeomorphism near v. Hence, if K^* is not locally separated by Γ, then $K^* = f(K^*)$ cannot be locally separated by $f(\Gamma)$. Assume now that v is critical and $f(\Gamma)$ is not a G-cut. Denote by W the wedge of $\mathbb{C}$ with boundary G not containing points of K^*.

Consider two cases. First, let $\sigma_3(a) = \sigma_3(b)$. Then W contains points mapped to K^* and located arbitrarily close to v. This the definition of polynomial-like map. Now, let $\sigma_3(a) \neq \sigma_3(b)$. Since $f(\Gamma)$ is not a G-cut, both components of $f(\Gamma)$ contain points of K^*. Hence, again, W contains points mapped to K^* and located arbitrarily close to v, a contradiction as above. □

Let $M = \overline{ab}$ be the major of G. If M is of regular critical type, then both critical points of f are contained in K^*, a contradiction. Thus M is of periodic type. The point $p(M)$ is a periodic cutpoint of $J(f)$. Since $[f] \in \mathrm{CU}$, this point cannot be repelling. Therefore, $p(M)$ is parabolic of multiplier 1. The Jordan domain U^* (U^*, use in the definition of quadratic like map, intersects the immediate parabolic basin of $p(M)$. Since in an arbitrarily small neighborhood of $p(M)$, there are points of the immediate parabolic basin that stay in this neighborhood, these points must lie in K^*. By the properties of Julia sets this implies that the entire immediate parabolic basin of $p(M)$ is a subset of K^*. It follows that K^* contains both critical points of f, a contradiction.

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