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On the Simple Isotopy Class of a Source–Sink Diffeomorphism on the 3-Sphere

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Abstract—The results obtained in this paper are related to the Palis–Pugh problem on the existence of an arc with finitely or countably many bifurcations which joins two Morse–Smale systems on a closed smooth manifold M^n. Newhouse and Peixoto showed that such an arc joining flows exists for any n and, moreover, it is simple. However, there exist isotopic diffeomorphisms which cannot be joined by a simple arc. For $n = 1$, this is related to the presence of the Poincaré rotation number, and for $n = 2$, to the possible existence of periodic points of different periods and heteroclinic orbits. In this paper, for the dimension $n = 3$, a new obstruction to the existence of a simple arc is revealed, which is related to the wild embedding of all separatrices of saddle points. Necessary and sufficient conditions for a Morse–Smale diffeomorphism on the 3-sphere without heteroclinic intersections to be joined by a simple arc with a "source-sink" diffeomorphism are also found.

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INTRODUCTION

This paper is devoted to solving the Palis–Pugh problem on the existence of an arc with finitely or countably many bifurcations joining two Morse–Smale systems on a closed smooth manifold [1]. In [2], Newhouse and Peixoto proved that any Morse–Smale vector fields are joined by a simple arc. Simplicity means that the entire arc, except finitely many points, consists of Morse–Smale systems, and at the exceptional points, a minimal (in a certain sense) deviation of the vector field from a Morse–Smale system occurs.[1]

The situation with discrete dynamical systems is different. Two orientation-preserving Morse–Smale diffeomorphisms on the circle can be joined by a *simple arc* (see Definition 1 below) if and only if they have the same rotation number. As follows from results of Matsumoto [3] and Blanchard [4], any orientable closed surface admits isotopic Morse–Smale diffeomorphisms which cannot be joined by a simple arc. We say that two isotopic Morse–Smale diffeomorphism belong to the same *simple isotopy class* if they can be joined by a simple arc. According to the paper [4], there exist infinitely many simple isotopy classes of Morse–Smale diffeomorphisms on any orientable surface inside an isotopy class admitting Morse–Smale diffeomorphisms.

The problem of the existence of a simple arc in dimension 3 is complicated by the presence of Morse–Smale diffeomorphisms whose saddle periodic points have separatrices wildly embedded in the underlying manifold. The first "wild" example was constructed by Pixton in [5]. This diffeomorphism belongs to the class (which we called the *Pixton class* in [6]) formed by those three-dimensional Morse–Smale diffeomorphisms whose nonwandering set consists of precisely four points, namely, two sinks, a source, and a saddle (see Fig. 1). According to [7], any Pixton diffeomorphism is joined by a simple arc

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[1] In [2], the notion of a simple arc in the space of vector fields on a given manifold was expounded. In Sec. 1 of this paper, we give a rigorous definition of a simple arc in the space of diffeomorphisms, which is ideologically similar to the corresponding definition for flows.

to a source-sink diffeomorphism. This is caused by the fact that, for any diffeomorphism from the Pixton class, at least one one-dimensional separatrix of its saddle point is tame [8]. By using the connected sum of two 3-spheres on which diffeomorphisms from the Pixton class with wildly embedded separatrices are defined, it is easy to construct a diffeomorphism for which all separatrices of all saddles are wildly embedded (see Fig. 1, in which the 3-balls used to obtain the connected sum are shaded); we prove that such a diffeomorphism is not joined by a simple arc to any source-sink diffeomorphism. The main result of this paper is a criterion for the existence of a simple arc joining a Morse–Smale diffeomorphism without heteroclinic intersections to a source-sink diffeomorphism.

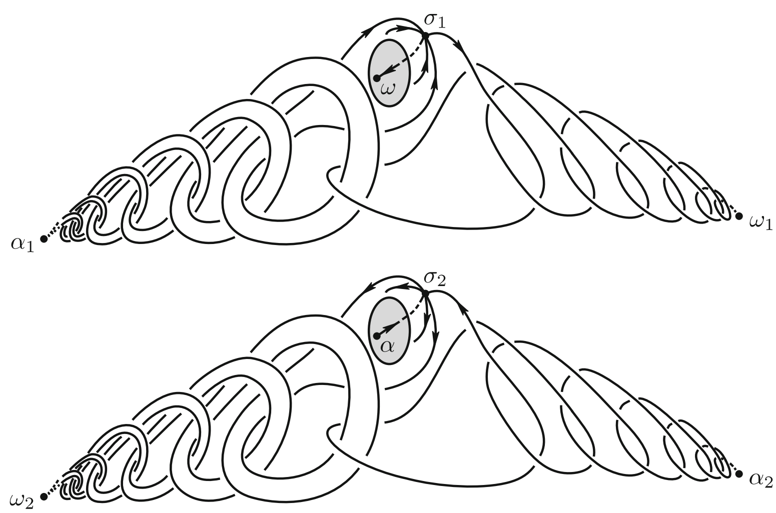

Fig. 1. A connected sum of two Pixton diffeomorphisms.

The key technical point in the solution of the problem stated above is the statement proved in Sec. 3 that any diffeomorphism from the class under consideration different from a source-sink diffeomorphism has a sink or a source periodic point whose domain of attraction or repulsion contains a unique saddle separatrix; moreover, this separatrix is one-dimensional and tame. This fact allows us to apply a nontrivial result of [7] to construct a simple arc from the given diffeomorphism to a Morse–Smale diffeomorphism whose saddle periodic orbits are fewer by one than those of the initial diffeomorphism.[2]

1. STATEMENT OF THE RESULTS

Let $\mathrm{Diff}(M^n)$ be the space of diffeomorphisms on a closed manifold M^n endowed with the C^1-topology. A *smooth arc* in $\mathrm{Diff}(M^n)$ is defined as a smooth map $\xi\colon M^n \times [0,1] \to M^n$ or, equivalently, as a family of diffeomorphisms

$$\{\xi_t \in \mathrm{Diff}(M^n),\ t \in [0,1]\}$$

smoothly depending on t.

Let $KS(M^n)$ be the set of all *Kupka–Smale* diffeomorphisms, i.e., diffeomorphisms whose periodic orbits are hyperbolic and have transversal stable and unstable manifolds. The Kupka–Smale diffeomorphisms with finite nonwandering set form the set $MS(M^n)$ of Morse–Smale diffeomorphisms. For a smooth arc ξ, the set

$$B(\xi) = \{b \in [0,1],\ \xi_b \notin KS(M^n)\}$$

is called the *bifurcation set*. According to [9], for a generic set of arcs (which is the intersection of open dense subsets in the space of smooth arcs), the bifurcation set is countable, and each diffeomorphism ξ_b

[2]In [7], it was proved that any Morse–Smale diffeomorphism without heteroclinic intersections whose nonwandering set consists of four fixed points is joined by a simple arc with a source-sink diffeomorphism by means of a saddle-node bifurcation.

with $b \in B(\xi)$ experiences one of the following bifurcations up to the direction of motion along the arc: a saddle-node bifurcation, a period doubling, a Hopf bifurcation, and a heteroclinic tangency (precise definitions of these bifurcations are given in Sec. 2 below).

Definition 1. An arc ξ is said to be *simple* if the bifurcation set $B(\xi)$ is finite, $\xi_t \in MS(M^n)$ for any $t \in ([0,1] \setminus B(\xi))$, and the bifurcations are of one of the following types:

- saddle-node[3];
- period doubling;
- heteroclinic tangency.

The simplest Morse–Smale diffeomorphism is a source-sink diffeomorphism. The nonwandering set of such a diffeomorphism consists of two points, a source and a sink, and the ambient manifold is homeomorphic to the sphere. In [7], it was proved that all source-sink diffeomorphisms on $\mathbb{S}^3$ belong to the same simple isotopy class, which we denote by I_{NS}. In this paper, we show that this class is not exhausted by source-sink diffeomorphisms and describe all diffeomorphisms in this class which have no *heteroclinic intersections* (that is, no intersections of stable and unstable manifolds of different saddle points).

Let $f \in MS(M^3)$ be a diffeomorphism with a saddle point σ, and let ℓ^u_σ be an unstable *separatrix* of this point (that is, a connected component of the set $W^u_\sigma \setminus \sigma$). A number $\operatorname{per}(\ell^u_\sigma) \in \mathbb{N}$ is called the *period* of the separatrix ℓ^u_σ if $f^{\operatorname{per}(\ell^u_\sigma)}(\ell^u_\sigma) = \ell^u_\sigma$ and $f^m(\ell^u_\sigma) \neq \ell^u_\sigma$ for any positive integer $m < \operatorname{per}(\ell^u_\sigma)$. If the separatrix ℓ^u_σ does not participate in heteroclinic intersections, then $\operatorname{cl}(\ell^u_\sigma) \setminus (\ell^u_\sigma \cup \sigma) = \{\omega\}$, where ω is a sink periodic point (see, e.g., Proposition 2.1.3 in the book [6]). Moreover, if $\dim W^u_\sigma = 1$, then $\operatorname{cl}(\ell^u_\sigma)$ is a topologically embedded[4] arc in M^3. The set $\ell^u_\sigma \cup \sigma$ is a smooth submanifold of M^3. However, the manifold $\operatorname{cl}(\ell^u_\sigma)$ may be wild at the point ω; in this case, the separatrix ℓ^u_σ is said to be *wild*, and otherwise, it is said to be *tame*. The tameness and the wildness of a stable one-dimensional separatrix are defined in a similar way.

Recall that the dynamics of any cascade $f \in MS(M^3)$ can be represented as follows (see, e.g., Chap. 2.2 in the book [6]). Let Ω^q_f, $q = 0, 1, 2, 3$, denote the set of periodic points p for which we have $\dim W^u_p = q$. Then $A_f = W^u_{\Omega^0_f \cup \Omega^1_f}$ is a connected attractor, and $R_f = W^s_{\Omega^3_f \cup \Omega^2_f}$ is a connected repeller with topological dimension at most 1. The sets A_f and R_f do not intersect, and each point from the set $V_f = M^3 \setminus (A_f \cup R_f)$ is wandering and moves from R_f to A_f under the action of f.

We say that A_f and R_f are *separated by a* 2*-sphere* if there exists a smooth 2-sphere $\Sigma_f \subset V_f$ such that A_f and R_f belong to different connected components of $M^3 \setminus \Sigma_f$ (see Fig. 2).

Let $MS_0(M^3)$ denote the class of Morse–Smale diffeomorphisms without heteroclinic intersections on a 3-manifold M^3. The main result of this paper is the following theorem.

Theorem 1. *A diffeomorphism $f \in MS_0(\mathbb{S}^3)$ belongs to the class I_{NS} if and only if the attractor A_f and the repeller R_f are separated by a 2-sphere.*

In Sec. 5, we prove that the diffeomorphism whose phase portrait is described at the end of the introduction (see Fig. 1) is not joined by a simple arc with a source-sink diffeomorphism.

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[3]A saddle-node bifurcation consists in the disappearance of two hyperbolic periodic orbits of the same period. In this paper, we assume that one of these orbits is a node and the other is a saddle.

[4]A C^0 map $g\colon B \to X$ is called a *topological embedding* of a topological manifold B into a manifold X if it is a homeomorphism between B and the subspace $g(B)$ with the topology induced from X. In this case, the image $A = g(B)$ is called a *topologically embedded manifold*. Note that a topologically embedded manifold is not generally a topological submanifold. If A is a submanifold, then it is said to be *tame*, or *tamely embedded*; otherwise, A is said to be *wild*, or *wildly embedded*, and the points at which the conditions in the definition of a topological submanifold are violated are called *points of wildness*.

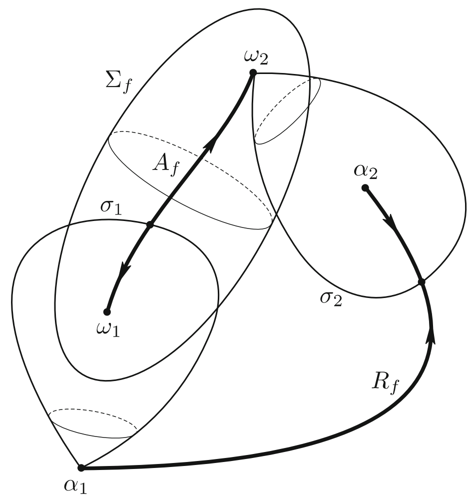

Fig. 2. A diffeomorphism $f \in MS(M^3)$ with attractor A_f and repeller R_f separated by a 2-sphere.

2. BIFURCATIONS ON A TYPICAL ARC

To describe the bifurcation set of a typical arc, we need the following notion.

Let p be a fixed point of a diffeomorphism $f\colon M^n \to M^n$. The differential Df_p induces the decomposition

$$T_pM^n = E^{\mathrm{u}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}}$$

of the tangent space T_pM^n into the direct sum of invariant subspaces. The eigenvalues of the linear maps $Df_p|_{E^{\mathrm{u}}}$, $Df_p|_{E^{\mathrm{c}}}$, and $Df_p|_{E^{\mathrm{s}}}$ are, respectively, inside, on the boundary, and outside the unit disk. In particular, if $\dim E^{\mathrm{c}} = 0$, then the point p is hyperbolic. Otherwise, there exists a smooth invariant submanifold W_p^{c} of M^n which is tangent to E^{c} at p. This submanifold is called a *central manifold* of the nonhyperbolic fixed point p. It is determined not uniquely, but the maps $f|_{W_p^{\mathrm{c}}}$ and $f|_{\widetilde{W}_p^{\mathrm{c}}}$ are topologically conjugate for any central manifolds W_p^{c} and $\widetilde{W}_p^{\mathrm{c}}$. In addition, for the point p, the smooth *stable manifold*

$$W_p^{\mathrm{s}} = \Big\{y \in M^n : \lim_{k\to+\infty} f^k(y) = p\Big\}$$

and the smooth *unstable manifold*

$$W_p^{\mathrm{u}} = \Big\{y \in M^n : \lim_{k\to-\infty} f^k(y) = p\Big\}$$

are defined (see, e.g., [10]). The *central*, *stable*, and *unstable manifolds* of a periodic point of period k are the corresponding manifolds of this point treated as a fixed point of the diffeomorphism f^k.

To define the quasi-transversal intersection of submanifolds, we need the notion of quadratic differential of a map $h\colon A \to B$ at a point $x \in A$, where A and B are smooth manifolds. Recall that the *cokernel* of the first differential $h_x\colon T_xA \to T_{f(x)}B$ is defined as the quotient space

$$\operatorname{Coker} h_x = T_{h(x)}B/h_x(T_xA).$$

In local coordinates $X\colon T_xA \to A$, $Y\colon T_{h(x)}B \to B$, in which

$$X(0) = x, \qquad Y(0) = h(x), \qquad \frac{d}{dt}\Big|_{t=0} X(\zeta t) = \zeta, \qquad \frac{d}{dt}\Big|_{t=0} Y(\zeta t) = \zeta,$$

the map h is written in the form

$$\varphi\colon T_xA \to T_{h(x)}B, \qquad \text{where} \quad \varphi = Y^{-1}hX.$$

The restriction of φ to the kernel $\operatorname{Ker} h_x$ consists of l functions $\varphi_1, \dots, \varphi_l \in \operatorname{Coker} h_x$, each of which depends on k variables $\eta_1, \dots, \eta_k \in \operatorname{Ker} h_x$. The *quadratic differential* of h at the point x is, by definition, the map $h_{xx}\colon \operatorname{Ker} h_x \to \operatorname{Coker} h_x$ written in the local coordinates $\eta_1,\dots,\eta_k$ and $\varphi_1,\dots,\varphi_l$ as

$$(h_{xx}(\eta_1,\dots,\eta_k))_r = \frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial^2\varphi_r}{\partial\zeta_i\,\partial\zeta_j}\eta_i\eta_j, \qquad r = 1,\dots,l$$

(see [11] for details).

Now, suppose that N_1 and N_2 are smooth submanifolds of a manifold M^n, $x \in (N_1 \cap N_2)$, D_1 is a locally normal complement to N_1 at x, and $q\colon M^n \to D_1$ is the natural projection along N_1. We set $g = q|_{N_2}$. The manifolds N_1 and N_2 are said to have *quasi-transversal intersection* at the point x if the space $\operatorname{Coker} g_x$ is homeomorphic to $\mathbb{R}$ and one of the following conditions holds:

(a) $\dim N_1 + \dim N_2 \geq n$ and the quadratic differential g_{xx} is nondegenerate;

(b) $\dim N_1 + \dim N_2 = n - 1$ and $T_xN_1 \cap T_xN_2 = \{0\}$.

For a generic set of arcs ξ, each diffeomorphism ξ_b, $b \in B(\xi)$, experiences one of the bifurcations described below up to the direction of motion along the arc. In the explaining figures, the double arrows schematically show the directions of motion corresponding to exponential contraction and expansion, and the single arrows indicate the directions of motion on a central manifold of a nonhyperbolic point.

We proceed to the description of the possible types of bifurcations.

(1) All periodic orbits of the diffeomorphism ξ_b are hyperbolic except one orbit $\mathcal{O}_p$ of a point p of period k, for which $(Df^k)_p$ has one eigenvalue $\lambda = 1$ and all of the other eigenvalues of $(Df^k)_p$ differ from 1 in absolute value. The stable and unstable manifolds of different periodic orbits of the diffeomorphism ξ_b intersect transversally, and $W^{\mathrm{s}}_p \cap W^{\mathrm{u}}_p = \{p\}$. The passage through ξ_b is accompanied by the merging and subsequent disappearance of hyperbolic periodic points of the same period. Such a bifurcation is called a *saddle-node* bifurcation (see Fig. 3).

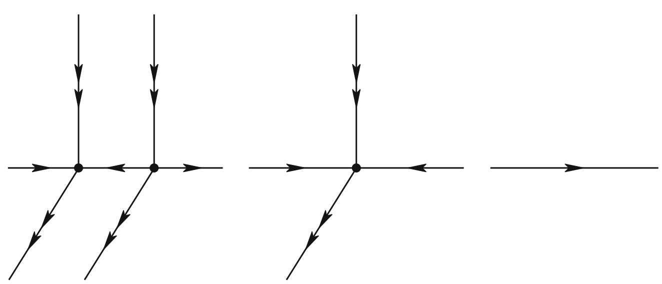

Fig. 3. A saddle-node bifurcation.

(2) All periodic orbits of the diffeomorphism ξ_b are hyperbolic except one orbit $\mathcal{O}_p$ of period k, for which all eigenvalues $(Df^k)_p$ has one eigenvalue $\lambda = -1$ and all of the other eigenvalues of $(Df^k)_p$ are different from 1 in absolute value. The stable and unstable manifolds of different periodic orbits of the diffeomorphism ξ_b intersect transversally, and $W^{\mathrm{s}}_p \cap W^{\mathrm{u}}_p = \{p\}$. Under the passage through ξ_b along the central manifold, the attractor[5] becomes a repeller, and a $2k$-periodic hyperbolic orbit is born. Such a bifurcation is called a *period doubling* (see Fig. 4).

[5] A compact set $A \subset M^n$ is called an *attractor for a diffeomorphism* $f\colon M^n \to M^n$ if A has a neighborhood V for which $f(V) \subset V$ and $A = \bigcap_{n\in\mathbb{N}} f^n(V)$. Such a neighborhood is said to be *trapping*. A set $R \subset M^n$ is called a *repeller* for f if this set is an attractor for f^{-1}.

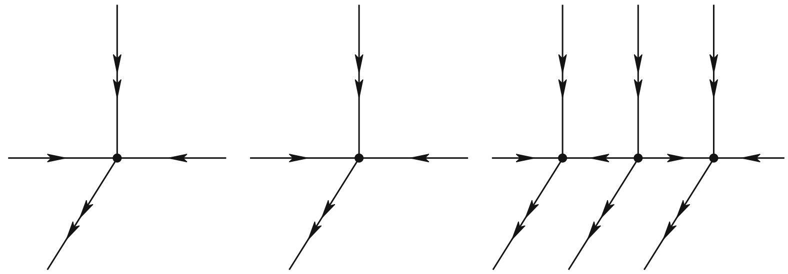

Fig. 4. A period doubling bifurcation.

(3) All periodic orbits of the diffeomorphism ξ_b are hyperbolic, except one orbit $\mathcal{O}_p$ of period k, for which $(Df^k)_p$ has a pair of conjugate eigenvalues λ and $\bar{\lambda}$, where $\lambda = e^{i\theta}$ with $0 < \theta < \pi$, and all of the other eigenvalues of $(Df^k)_p$ differ from 1 in absolute value. The stable and unstable manifolds of different periodic orbits of the diffeomorphism ξ_b intersect transversally, and $W^{\mathrm{s}}_p \cap W^{\mathrm{u}}_p = \{p\}$. Under the passage through ξ_b, the attractor becomes a repeller, near which an invariant circle arises. Such a bifurcation is called a *Hopf* (or *Neimark–Sacker*) *bifurcation* (see Fig. 5).

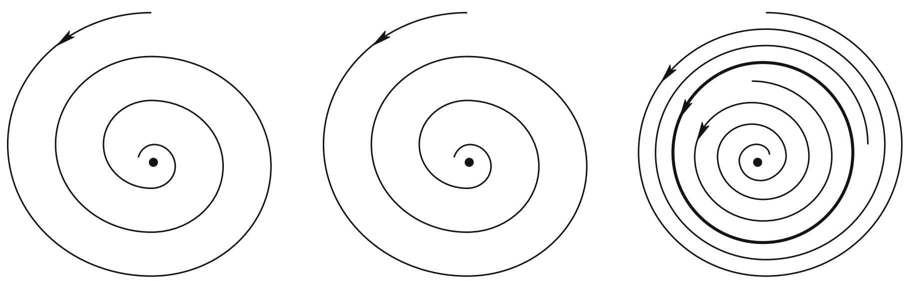

Fig. 5. A Hopf, or Neimark–Sacker, bifurcation.

(4) All periodic orbits of the diffeomorphism ξ_b are hyperbolic, their stable and unstable manifolds have transversal intersection everywhere except on one trajectory, along which the intersection is quasi-transversal. Such a bifurcation is called a *heteroclinic tangency bifurcation* (see Fig. 6).

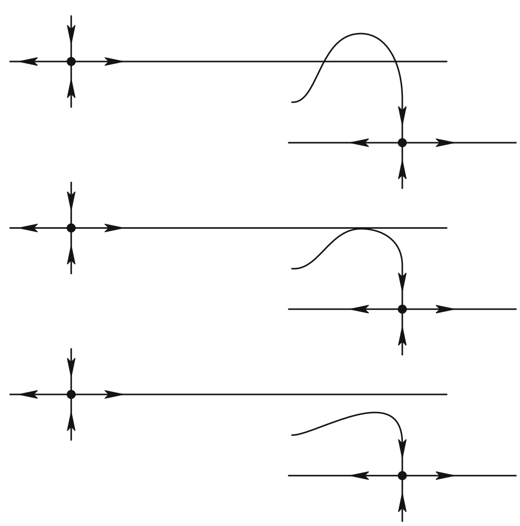

Fig. 6. A heteroclinic tangency bifurcation.

3. RELATIONSHIP BETWEEN TAME SEPARATRICES AND SIMPLE ARCS

3.1. A Tameness Condition for a One-Dimensional Separatrix

We begin this subsection with definitions and facts necessary for understanding what follows; exhaustive information can be found in Chap. 2.1 of the book [6].

Let $f \in MS(M^3)$, and let σ be a saddle point of f such that the unstable *separatrix* ℓ^{u}_σ does not participate in heteroclinic intersections. Then $\mathrm{cl}(\ell^{\mathrm{u}}_\sigma) \setminus (\ell^{\mathrm{u}}_\sigma \cup \sigma) = \{\omega\}$, where ω is a sink periodic point. The type of embedding of the separatrix ℓ^{u}_σ is determined by passing to the orbit space.

We set

$$V_\omega = W^{\mathrm{s}}_{\mathcal{O}_\omega} \setminus \mathcal{O}_\omega \qquad \text{and} \qquad \widehat{V}_\omega = V_\omega / f.$$

Then the natural projection $p_\omega \colon V_\omega \to \widehat{V}_\omega$ is a covering. Since the diffeomorphism $f^{\mathrm{per}(\omega)}|_{W^{\mathrm{s}}_\omega}$ is topologically conjugate to a homothety of $\mathbb{R}^3$, it follows that the manifold $\widehat{V}_\omega$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, and since W^{u}_σ is a smooth submanifold of M^3 and the diffeomorphism $f^{\mathrm{per}(\sigma)}|_{W^{\mathrm{u}}_\sigma}$ is topologically conjugate to a homothety of $\mathbb{R}^{\dim W^{\mathrm{u}}_\sigma}$, it follows that the set $\widehat{\ell}^{\mathrm{u}}_\sigma = p_\omega(\ell^{\mathrm{u}}_\sigma)$ is a *homotopically nontrivial* smooth submanifold of $\widehat{V}_\omega$, i.e., $i_{\widehat{\ell}^{\mathrm{u}}_\sigma *}(\pi_1(\widehat{\ell}^{\mathrm{u}}_\sigma)) \neq 0$, where $i_{\widehat{\ell}^{\mathrm{u}}_\sigma} \colon \widehat{\ell}^{\mathrm{u}}_\sigma \to \widehat{V}_\omega$ is the inclusion map.

In the case $\dim W^{\mathrm{u}}_\sigma = 1$, the manifold $\widehat{\ell}^{\mathrm{u}}_\sigma$ is a *knot* (a homeomorphic image of the circle). The knot $\widehat{\ell}^{\mathrm{u}}_\sigma$ is said to be *trivial* if there exists a homeomorphism $\widehat{\varphi} \colon \widehat{V}_\omega \to \mathbb{S}^2 \times \mathbb{S}^1$ such that

$$\widehat{\varphi}(\widehat{\ell}^{\mathrm{u}}_\sigma) = \{x\} \times \mathbb{S}^1 \qquad \text{for some} \quad x \in \mathbb{S}^2.$$

Statement 1. *If the knot $\widehat{\ell}^{\mathrm{u}}_\sigma$ is trivial in $\widehat{V}_\omega$, then the one-dimensional separatrix ℓ^{u}_σ is tame and has the same period as ω.*

Proof. The first assertion of the statement follows from Theorem 4.2.2 in [6], according to which the separatrix ℓ^{u}_σ is tamely embedded in M^3 if and only if the knot $\widehat{\ell}^{\mathrm{u}}_\sigma$ is trivial in $\widehat{V}_\omega$. To prove the second assertion, note that, according to Proposition 4.1.2 in [6], the knot $\widehat{\ell}^{\mathrm{u}}_\sigma$ is trivial if and only if it has a tubular neighborhood $N(\widehat{\ell}^{\mathrm{u}}_\sigma)$ in $\widehat{V}_\omega$ such that the manifold $\widehat{V}_\omega \setminus N(\widehat{\ell}^{\mathrm{u}}_\sigma)$ is homeomorphic to the solid torus (that is, to $\mathbb{D}^2 \times \mathbb{S}^1$). It follows that, for the trivial knot $\widehat{\ell}^{\mathrm{u}}_\sigma$, the group $i_{\widehat{\ell}^{\mathrm{u}}_\sigma *}(\pi_1(\widehat{\ell}^{\mathrm{u}}_\sigma))$ is isomorphic to $\mathbb{Z}$. The manifold $\widehat{V}_\omega$ is homeomorphic to the quotient space $(W^{\mathrm{s}}_\omega \setminus \omega)/f^{\mathrm{per}(\omega)}$; hence the monodromy theorem implies the existence of an arc $\gamma \subset \ell^{\mathrm{u}}_\sigma$ (going from x to $f^{\mathrm{per}(\omega)}(x)$) which is a lifting of the knot $\widehat{\ell}^{\mathrm{u}}_\sigma$. Thus, $f^{\mathrm{per}(\omega)}(\ell^{\mathrm{u}}_\sigma) = \ell^{\mathrm{u}}_\sigma$. Since $\omega \in \mathrm{cl}(\ell^{\mathrm{u}}_\sigma)$, it follows that $\mathrm{per}(\ell^{\mathrm{u}}_\sigma) \geq \mathrm{per}(\omega)$ and, therefore, the separatrix ℓ^{u}_σ has the same period as the sink ω. □

A similar statement is valid for a stable saddle separatrix in the domain of repulsion of the source α.

3.2. Characteristic Spaces

Let $f \in MS(M^3)$. Recall that Ω^q_f, $q = 0, 1, 2, 3$, denotes the set of periodic points p for which $\dim W^{\mathrm{u}}_p = q$ and

$$A_f = W^{\mathrm{u}}_{\Omega^0_f \cup \Omega^1_f}, \qquad R_f = W^{\mathrm{s}}_{\Omega^3_f \cup \Omega^2_f}, \qquad V_f = M^3 \setminus (A_f \cup R_f).$$

In [6], the orbit space $\widehat{V}_f = V_f / f$ is referred to as the *characteristic space* of f. Let $p_f \colon V_f \to \widehat{V}_f$ denote the natural projection. As is known (see, e.g., Theorem 1.2 in [12]), the characteristic space is a simple manifold[6].

Statement 2. *For any diffeomorphism $f \in MS(M^3)$, the attractor A_f and the repeller R_f are separated by a 2-sphere if and only if the space $\widehat{V}_f$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$.*

[6]A smooth 3-manifold is said to be *simple* if it is either *irreducible* (that is, any smooth 2-sphere bounds a 3-ball in this manifold) or homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$.

Proof. *Necessity.* The separation of the attractor A_f and the repeller R_f of a diffeomorphism $f \in MS(M^3)$ by a 2-sphere means that there exists a smooth 2-sphere $\Sigma_f \subset V_f$ such that A_f and R_f belong to different connected components of $M^3 \setminus \Sigma_f$. The sphere Σ_f does not bound a 3-ball in V_f; therefore, the manifold V_f is not irreducible. By virtue of Theorem 3.15 in [13], the manifold $\widehat{V}_f$ is not irreducible either. According to Theorem 1.2 in [12], $\widehat{V}_f$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$.

Sufficiency. Suppose that the manifold $\widehat{V}_f$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. Then there is a diffeomorphism $\beta\colon V_f \to \mathbb{S}^2 \times \mathbb{R}$ between V_f and $\mathbb{S}^2 \times \mathbb{R}$. Take a coordinate $r \in \mathbb{R}$ and let $\Sigma_f = \beta^{-1}(\mathbb{S}^2 \times \{r\})$. By construction, the 2-sphere Σ_f separates V_f into two noncompact connected components, while the manifold $M^3 = V_f \cup A_f \cup R_f$ is compact. Since the sets A_f and R_f are connected and disjoint, they must be contained in different connected components of $M^3 \setminus \Sigma_f$. Therefore, the 2-sphere Σ_f is as required. $\square$

Now, take $f \in MS_0(M^3)$. For a saddle point σ of f, let W^2_σ (W^1_σ) denote the two-dimensional (one-dimensional) invariant manifold of σ, and let $\widehat{W}^2_\sigma = p_f(W^2_\sigma)$. Then the set $\widehat{W}^2_\sigma$ is a homotopically nontrivial smooth torus (a homotopically nontrivial Klein bottle) in the manifold $\widehat{V}_f$, provided that the diffeomorphism $f^{\mathrm{per}(\sigma)}$ preserves (reverses) the orientation of W^2_σ (see, e.g., Proposition 2.1.5 in [6]). We set

$$\widehat{W}^2_f = \bigcup_{\sigma \in (\Omega^1_f \cup \Omega^2_f)} \widehat{W}^2_\sigma.$$

Choose a family $\{N(\widehat{W}^2_\sigma),\ \sigma \in (\Omega^1_f \cup \Omega^2_f)\}$ of pairwise disjoint tubular neighborhoods[7] of the surfaces $\widehat{W}^2_\sigma$, $\sigma \in (\Omega^1_f \cup \Omega^2_f)$.

In the case where the manifold $\widehat{V}_f$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, we determine the type of embedding of separatrices by using the following topological facts.

Fact 1. Any homotopically nontrivial smooth torus in the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ bounds a solid torus in this manifold (see, e.g., Proposition 4.1.1 in [6]).

Fact 2. An orientable surface F properly embedded[8] in a manifold X and different from the 2-sphere is incompressible[9] if and only if $\mathrm{Ker}(i_{F*}) = 0$, where $i_F\colon F \to X$ is the inclusion map [14].

Fact 3. If a 3-manifold X is irreducible, then a 2-torus $T \subset X$ not contained in a 3-ball is compressible if and only if it bounds a solid torus in X [14, Exercise 6].

Fact 4. A manifold is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ if and only if it is obtained from two smooth solid tori by attaching their boundaries to each other by means of a diffeomorphism taking meridians[10] to meridians (see, e.g., Proposition 7.1 in [15]).

Remark 1. Let T be a homotopically nontrivial smooth torus in the manifold $\mathbb{S}^2 \times \mathbb{S}^1$. By Fact 3.2, the torus T bounds a solid torus G; we refer to a meridian of G as a *meridian* of the torus T. If the torus T bounds two solid tori, then, according to Fact 3.2, each meridian of one of them is a meridian of the other.

[7] A *tubular neighborhood of a torus* is a manifold diffeomorphic to $\mathbb{T}^2 \times (0,1)$; accordingly, its boundary consists of two tori. A *tubular neighborhood of a Klein bottle* is a locally trivial bundle over the Klein bottle with fiber the interval; its boundary consists of one torus.

[8] A surface F is said to be *properly embedded* in a manifold X if $\partial X \cap F = \partial F$.

[9] A surface F properly embedded in X is said to be *compressible* in X in one of the following two cases:

(1) there exists a noncontractible simple closed curve $c \subset \operatorname{int} F$ and a smoothly embedded 2-disk $D \subset \operatorname{int} X$ for which $D \cap F = \partial D = c$;

(2) there exists a 3-ball $B \subset \operatorname{int} X$ for which $F = \partial B$.

A surface F is said to be *incompressible* in X if it is not compressible in X.

[10] A two-dimensional disk d in a solid torus G is called a *meridian disk* if $\partial G \cap d = \partial d$ and ∂d does not bound a disk in ∂G. The boundary of a meridian disk is called a *meridian*.

Statement 3. *If a diffeomorphism $f \in MS_0(M^3)$ is different from a source-sink diffeomorphism and its characteristic space $\widehat{V}_f$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, then there exists a saddle point σ_* such that at least one connected component of the set $\widehat{V}_f \setminus N(\widehat{W}^2_{\sigma_*})$ is a solid torus disjoint from $\widehat{W}^2_f$.*

Proof. By Fact 3.2, for any saddle point σ, at least one connected component of the set $\widehat{V}_f \setminus N(\widehat{W}^2_\sigma)$ is a solid torus. Since the number of saddle points is finite and the Klein bottle is not embedded in the solid torus[11], it suffices to show that if a torus T is homotopically nontrivial in $\mathbb{S}^2 \times \mathbb{S}^1$ and contained in a solid torus G homotopically nontrivial in $\mathbb{S}^2 \times \mathbb{S}^1$, then T bounds a solid torus in G.

Let a and b be generators of the fundamental group of the torus T. Since T is homotopically nontrivial in $\mathbb{S}^2 \times \mathbb{S}^1$, it follows that, up to the interchange of the generators, we have $i_{T*}([a]) \neq 0$ and $i_{T*}([b]) = 0$, where $i_T\colon T \to \mathbb{S}^2 \times \mathbb{S}^1$ is the inclusion map. Let c be a generator of the fundamental group of the solid torus G. Since G is homotopically nontrivial in $\mathbb{S}^2 \times \mathbb{S}^1$ as well, it follows that $i_{G*}([c]) \neq 0$, where $i_G\colon G \to \mathbb{S}^2 \times \mathbb{S}^1$ is the inclusion map; hence $\operatorname{Ker}(i_{G*}) = 0$. Let $j_T\colon T \to G$ denote the inclusion map. Then $i_T = i_G j_T$ and, therefore, $i_{T*} = i_{G*} j_{T*}$. The relations $\operatorname{Ker}(i_{T*}) \neq 0$ and $\operatorname{Ker}(i_{G*}) = 0$ imply $\operatorname{Ker}(j_{T*}) \neq 0$. According to Fact 3.2, the torus T is compressible in G. Since $j_{T*}([a]) \neq 0$, it follows that T is not contained in a 3-ball in G; thus, according to Fact 3.2, T bounds a solid torus in G. □

Statement 4. *Suppose that a diffeomorphism $f \in MS_0(M^3)$ is not a source-sink diffeomorphism, the characteristic space $\widehat{V}_f$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, σ_* is a saddle point satisfying the conditions in Statement 3, and $\dim W^{\mathrm{u}}_{\sigma_*} = 1$ ($\dim W^{\mathrm{s}}_{\sigma_*} = 1$). Then there exists a sink point ω_* (a source point α_*) for which the intersection*

$$\widehat{V}_{\omega_*} \cap p_{\omega_*}(W^{\mathrm{u}}_{\Omega^1_f \cup \Omega^2_f}) \qquad (\widehat{V}_{\alpha_*} \cap p_{\alpha_*}(W^{\mathrm{s}}_{\Omega^1_f \cup \Omega^2_f}))$$

consists of only the trivial node $\widehat{\ell}^{\mathrm{u}}_{\sigma_*}$ ($\widehat{\ell}^{\mathrm{s}}_{\sigma_*}$).

Proof. To be definite, suppose that $\dim W^{\mathrm{u}}_{\sigma_*} = 1$. We set

$$V_0 = \bigcup_{\omega \in \Omega^0_f} V_\omega \qquad \text{and} \qquad \widehat{V}_0 = \bigcup_{\omega \in \Omega^0_f} \widehat{V}_\omega.$$

Then each connected component of the manifold $\widehat{V}_0$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. To better understand the passage from V_f to V_0, note that $V_0 \setminus W^{\mathrm{u}}_{\Omega^1_f} = V_f \setminus W^{\mathrm{s}}_{\Omega^1_f}$. Given a point $\sigma \in \Omega^1_f$, we set

$$N_\sigma = p_f^{-1}(N(\widehat{W}^2_\sigma)) \cup W^{\mathrm{u}}_{\mathcal{O}_\sigma}.$$

By construction, N_σ is an f-invariant neighborhood of the periodic orbit $\mathcal{O}_\sigma$, which contains the set $W^{\mathrm{s}}_{\mathcal{O}_\sigma} \cup W^{\mathrm{u}}_{\mathcal{O}_\sigma}$ (see Fig. 8 and the proof of the existence of such a neighborhood in [6, Theorem 2.1.2]). Let

$$N_{\Omega^1_f} = \bigcup_{\sigma \in \Omega^1_f} N_\sigma.$$

Then $V_0 \setminus N_{\Omega^1_f} = V_f \setminus N_{\Omega^1_f}$.

We set $\widehat{W}^1_\sigma = p_0(W^1_\sigma)$. Note that $\widehat{W}^1_\sigma$ is a pair of knots (a knot) in the manifold $\widehat{V}_0$, provided that the diffeomorphism $f^{\mathrm{per}(\sigma)}$ preserves (reverses) the orientation of W^1_σ (see, e.g., Proposition 2.1.5 in [6]). Let $N(\widehat{W}^1_\sigma) = p_0(N_\sigma)$; then $N(\widehat{W}^1_\sigma)$ is a tubular neighborhood of $\widehat{W}^1_\sigma$. We set

$$\widehat{N}^2_{\Omega^1_f} = p_f(N_{\Omega^1_f}), \qquad \widehat{N}^1_{\Omega^1_f} = p_0(N_{\Omega^1_f}).$$

It follows from $V_0 \setminus N_{\Omega^1_f} = V_f \setminus N_{\Omega^1_f}$ that the manifold $p_0(V_0 \setminus N_{\Omega^1_f})$ is homeomorphic to $p_f(V_f \setminus N_{\Omega^1_f})$. Therefore, $\widehat{V}_0 \setminus \widehat{N}^1_{\Omega^1_f}$ is homeomorphic to $\widehat{V}_f \setminus \widehat{N}^2_{\Omega^1_f}$.

[11] If the Klein bottle were embedded in the solid torus, then it would be embedded in $\mathbb{R}^3$, which is false.

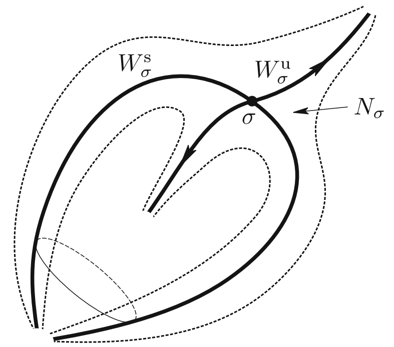

Fig. 7. An f-invariant neighborhood of the saddle point σ.

The passage from $\widehat{V}_f$ to $\widehat{V}_0$ consists in removing $\widehat{N}^2_{\Omega^1_f}$ from $\widehat{V}_f$ and attaching a solid torus to each boundary of the resulting manifold by means of a diffeomorphism taking meridians to meridians. Such a passage is shown in Fig. 8(a) (in Fig. 8(b)). By virtue of Statement 3, the set $\widehat{V}_f \setminus N(\widehat{W}^2_{\sigma_*})$ has a connected component G homeomorphic to the solid torus and disjoint from $\widehat{W}^2_f$; hence, attaching G to a connected component $N(\widehat{\ell}^{\mathrm{u}}_{\sigma_*})$ of $N(\widehat{W}^1_{\sigma_*})$ which is homeomorphic to the solid torus, we obtain a connected component $\widehat{V}_{\omega_*}$ of $\widehat{V}_0$ for which

$$\widehat{V}_{\omega_*} \cap p_{\omega_*}(W^{\mathrm{u}}_{\Omega^1_f \cup \Omega^2_f}) = \widehat{\ell}^{\mathrm{u}}_{\sigma_*}. \quad \square$$

$\square$

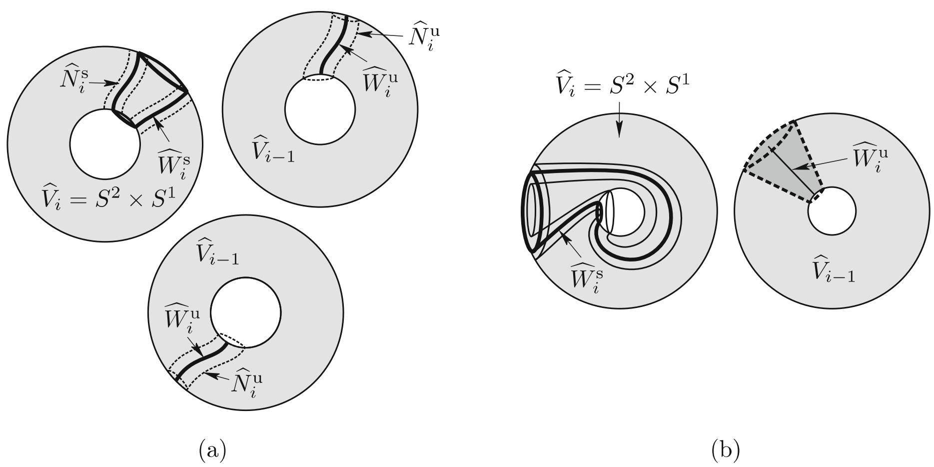

Fig. 8. The passage from the space $\widehat{V}_i$ to the space $\widehat{V}_{i-1}$.

3.3. The Existence of a Simple Arc Decreasing the Number of Periodic Orbits

Statement 5. *Suppose that $f \in MS_0(M^3)$ is not a source-sink diffeomorphism and the characteristic space $\widehat{V}_f$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. Then the nonwandering set of f contains a nodal point (a source or a sink) whose basin (the domain of attraction or repulsion) contains precisely one separatrix of a saddle point; moreover, this separatrix is one-dimensional and tame.*

Proof. The existence of a nodal point with the required properties is proved straightforwardly by successively applying Statements 3, 4, and 1. $\square$

Statement 6. *If the nonwandering set of a diffeomorphism $f \in MS_0(M^3)$ has a nodal point whose basin contains precisely one separatrix of a saddle point and this separatrix is one-dimensional and tame, then there exists a simple arc with a unique bifurcation point which joins f to a diffeomorphism $f' \in MS_0(M^3)$ such that the number of saddle orbits in its nonwandering set is smaller by one than that in the nonwandering set of f.*

Proof. Let σ and ℓ_σ be, respectively, a saddle point and its separatrix satisfying the conditions in the statement. There are two possible cases:

(1) $f^{\mathrm{per}(\sigma)}(\ell) = \ell$;

(2) $f^{\mathrm{per}(\sigma)}(\ell) \neq \ell$.

In case (1), the existence of a simple arc with the required properties is proved by using a saddle-node bifurcation; the method of proof is described in detail in Sec. 4.3.2 of the book [6] (see also [7]) for the case $\mathrm{per}(\sigma) = 1$ and is easily generalized to the case $\mathrm{per}(\sigma) > 1$. In case (2), the construction of the required simple arc uses a period doubling bifurcation; it is described in [16] for $\mathrm{per}(\sigma) = 1$ and is easily generalized to the case $\mathrm{per}(\sigma) > 1$. □

4. A CRITERION FOR A MORSE–SMALE DIFFEOMORPHISM WITHOUT HETEROCLINIC INTERSECTIONS TO BELONG TO THE CLASS I_{NS}

The proof of Theorem 1 is based on the following lemma.

Lemma 1. *If diffeomorphisms $f, f' \in MS_0(M^3)$ are joined by a simple arc, then the spaces $\widehat{V}_f$ and $\widehat{V}_{f'}$ are homeomorphic.*

Proof. Without loss of generality, we can assume that the diffeomorphisms $f, f' \in MS_0(\mathbb{S}^3)$ are joined by a simple arc ξ_t with a unique bifurcation value ξ_b. Then, for $t_1, t_2 < b$ or $t_1, t_2 > b$, the diffeomorphisms ξ_{t_1} and ξ_{t_2} are topologically conjugate and, therefore, the orbit spaces $\widehat{V}_{\xi_{t_1}}$ and $\widehat{V}_{\xi_{t_2}}$ are homeomorphic. Note that it follows from the definition of a simple arc that either $|\Omega^0_{\xi_0}| = |\Omega^0_{\xi_t}|$ or $|\Omega^3_{\xi_0}| = |\Omega^3_{\xi_t}|$ for any $t \in [0,1]$, where $|\cdot|$ denotes the cardinality of a set. To be definite, we suppose that $|\Omega^0_{\xi_0}| = |\Omega^0_{\xi_t}|$ (in the other case, a similar argument is used). Then $|\Omega^1_{\xi_0}| = |\Omega^1_{\xi_t}|$ and $A_{\xi_t} = W^{\mathrm{u}}_{\Omega^0_{\xi_t} \cup \Omega^1_{\xi_t}}$ is an attractor for any $t \in [0,1]$. We set

$$V_{\xi_t} = W^{\mathrm{s}}_{A_{\xi_t} \cap \Omega_{\xi_t}} \setminus A_{\xi_t} \qquad \text{and} \qquad \widehat{V}_{\xi_t} = V_{\xi_t}/\xi_t.$$

We shall prove the existence of an $\varepsilon > 0$ such that the manifolds $\widehat{V}_{\xi_t}$ and $\widehat{V}_{\xi_b}$ are homeomorphic for $b \le t \le b + \varepsilon$. A similar argument proves the existence if an $\widetilde{\varepsilon} > 0$ such that the manifolds $\widehat{V}_{\xi_t}$ and $\widehat{V}_{\xi_b}$ are homeomorphic for $b - \widetilde{\varepsilon} \le t \le b$, which will complete the proof of the lemma.

By using methods of [12], we can construct a smooth trapping neighborhood Q of A_{ξ_b} which is the body bounded by a surface. Choose a tubular neighborhood N of the surface $\xi_b(\partial Q)$ so that $N \cap \partial Q = \varnothing$. We set $S_t = \xi_t(\partial Q)$ for $t \in [b, 1]$. Let us prove the existence of an $\varepsilon > 0$ for which $S_t \subset N$ and the surface S_t separates the boundaries of N for $b \le t \le b + \varepsilon$.

To this end, we set

$$g_t = \xi_t \xi_b^{-1}|_{S_b} \colon S_b \to S_t \qquad \text{for} \quad t \in [b, 1].$$

By Thom's homotopy extension theorem (see, e.g., Theorems 8.1.3 and 8.1.4 in the book [17]), there exists an $\varepsilon > 0$ and a smooth isotopy

$$\{G_t \colon M^3 \to M^3,\ t \in [b, b+\varepsilon]\}$$

satisfying the conditions $G_b = \mathrm{id}$,

$$G_t|_{S_b} = g_t|_{S_b}, \quad G_t|_{M^3 \setminus N} = \mathrm{id}\,|_{M^3 \setminus N} \qquad \text{for any} \quad t \in [b, b+\varepsilon].$$

We have $G_t(N) = N$ and $G_t(S_b) = S_t$, which implies that the surface S_t separates the boundary of the manifold N for $b \le t \le b + \varepsilon$.

Let

$$K_{\xi_t} = Q \setminus \operatorname{int} \xi_t(Q).$$

Then K_{ξ_t} is a fundamental domain[12] of the action of the diffeomorphism ξ_t on V_{ξ_t}. The orbit space $\widehat{V}_{\xi_t}$ is homeomorphic to the topological space obtained from K_{ξ_t} by identifying its boundaries by a means of a diffeomorphism ξ_t (see, e.g., Statement 10.2.22 in [6]). Let us show that there exists a homeomorphism $h_t\colon K_{\xi_b} \to K_{\xi_t}$ between K_{ξ_t} and K_{ξ_b} which satisfies the conditions

$$h_t|_{\partial Q} = \mathrm{id}\,|_{\partial Q} \qquad \text{and} \qquad h_t|_{\xi_b(\partial Q)} = g_t;$$

this will complete the proof of the lemma.

We set

$$R = K_{\xi_b} \setminus N \qquad \text{and} \qquad P_t = \operatorname{cl}(K_{\xi_t} \setminus R).$$

Note that $K_{\xi_t} = R \cup P_t$. We also set $S = R \cap P_t$ and $S_t = \xi_t(\partial Q)$. By construction, S and S_t are diffeomorphic surfaces. Moreover, since S_t separates the boundaries of N for $b \le t \le b + \varepsilon$, it follows that P_t is diffeomorphic to the manifold $S \times [0,1]$ (see, e.g., Corollary 3.2 in [18] or Theorem 3.3 in [19]). Moreover, we can construct a family of diffeomorphisms $\nu_t\colon P_t \to S \times [0,1]$ with the property $\nu_t(s) = \{s\} \times [0,1]$ for $s \in S$ so that this family is continuous in $t \in [b, b+\varepsilon]$. We set

$$\mu_t = \nu_t g_t \nu_b^{-1}|_{S \times \{1\}}\colon S \times \{1\} \to S \times \{1\}.$$

By construction, the map μ_t is isotopic to the identity map, which implies the existence of a diffeomorphism

$$q_t\colon S \times [0,1] \to S \times [0,1]$$

coinciding with the identity map on $S \times \{0\}$ and with μ_t on $S \times \{1\}$. The map h_t coinciding with the identity map on R and with the diffeomorphism $\nu_t^{-1} q_t \nu_b$ on P_b is as required. □

Proof of Theorem 1. Let us prove Theorem 1, that is, show that a diffeomorphism $f \in MS_0(\mathbb{S}^3)$ belongs to the class I_{NS} if and only if the attractor A_f and the repeller R_f are separated by a 2-sphere.

Necessity. Suppose that a diffeomorphism $f \in MS_0(\mathbb{S}^3)$ belongs to the class I_{NS}. Since the characteristic space $\widehat{V}_g$ of a source-sink diffeomorphism $g\colon \mathbb{S}^3 \to \mathbb{S}^3$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ (see, e.g., Theorem 2.2.1 in [6]), it follows by Lemma 1 that the characteristic space $\widehat{V}_f$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. By Statement 2, the attractor A_f and the repeller R_f are separated by a 2-sphere.

Sufficiency. Suppose that the attractor A_f and the repeller R_f of a diffeomorphism $f \in MS_0(\mathbb{S}^3)$ are separated by a 2-sphere. Then, by Statement 5, the nonwandering set of f contains a saddle point whose one-dimensional separatrices l_1 and l_2 are contained in the basins of nodal points (sinks or sources) a_1 and a_2, respectively; moreover, $a_1 \ne a_2$, and at least one of the separatrices l_1 and l_2 is tame and has the same period as the corresponding node. According to Statement 6, there exists a simple arc with one bifurcation of saddle-node or period doubling type which joins the diffeomorphism f to some diffeomorphism $f' \in MS_0(\mathbb{S}^3)$ for which the number of saddle orbits is smaller by one than that for f. By Lemma 1, the characteristic space $\widehat{V}_{f'}$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, and by Statement 2, the attractor $A_{f'}$ and the repeller $R_{f'}$ of the diffeomorphism f' are separated by a 2-sphere. Continuing, we construct the required arc. □

[12] A *fundamental domain of the action* of a map g on X is defined as a closed set $D_g \subset X$ for which there exists a set $\widetilde{D}_g$ with the following properties:

(1) $\operatorname{cl}(\widetilde{D}_G) = D_G$;

(2) $g^k(\widetilde{D}_G) \cap \widetilde{D}_G = \varnothing$ for all $k \in (\mathbb{Z} \setminus \{0\})$;

(3) $\bigcup_{k \in \mathbb{Z}} g^k(\widetilde{D}_g) = X$.

5. AN EXAMPLE OF A DIFFEOMORPHISM $f \in MS_0(\mathbb{S}^3)$ NOT BELONGING TO THE CLASS I_{NS}

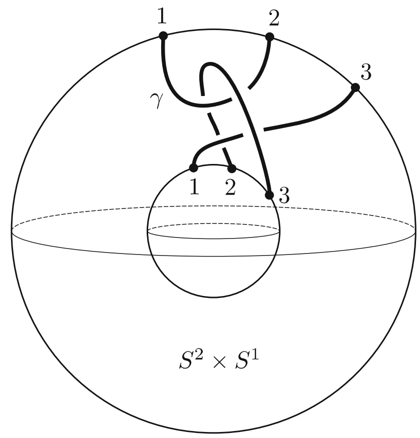

Fig. 9. A realization of the diffeomorphism f.

In fact, such a diffeomorphism f is the connected sum of two Pixton diffeomorphisms, as shown in Fig. 1. By virtue of Lemma 1 and Statement 2, to prove that such a diffeomorphism is not joined by a simple arc with a source-sink diffeomorphism, it suffices to show that its characteristic space $\widehat{V}_f$ is not homeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$. To this end, note that f can be realized by using methods of [20] from an abstract scheme $S = (\widehat{V}, T^{\mathrm{s}}, T^{\mathrm{u}})$ with the following structure. Consider a knot γ on the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ (see Fig. 9, in which a development of γ is shown). We choose a tubular neighborhood $V(\gamma)$ of γ and attach two copies of the manifold $\mathbb{S}^2 \times \mathbb{S}^1 \setminus \operatorname{int} V(\gamma)$ to each other along the boundary tori by the identity map. The resulting manifold is $\widehat{V}$, and the boundaries of a tubular neighborhood of the locus of attachment are two-dimensional tori T^{s} and T^{u}. By construction, none of these tori bounds a solid torus in $\widehat{V}$, which means, according to Fact 3.2, that the manifold $\widehat{V}$ is not homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. Since $\widehat{V}$ is homeomorphic $\widehat{V}_f$, the required assertion follows.

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