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MAPPINGS OF BOUNDED VARIATION WITH VALUES IN A METRIC SPACE: GENERALIZATIONS

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1. Introduction

The present paper addresses the theory of mappings $f: I \to X$ of bounded (Φ, σ)-variation (see the definition in Sec. 2) which are defined on a compact interval I of the real line $\mathbb{R}$ and take values in a metric or normed space X. We prove the structural theorem for these mappings (Lemma 4 and Theorem 5) and establish a compactness theorem in the space of mappings of bounded (Φ, σ)-variation (Theorem 6), which in the classical case ($X = \mathbb{R}$, $\Phi(\rho) = \rho$, and $\sigma(t) = t$) reduces to the well-known Helly selection principle ([13], Chap. 8, Sec. 4). We study properties of differentiability in the weak and strong senses for these mappings (Theorem 7) and generalize criteria due to Riesz [14], Medvedev [11] and the author [6] for the case of reflexive Banach space- and metric space-valued mappings (Corollaries 9 and 10). We show that any absolutely continuous mapping $f: I \to X$ from I into a metric space X is a mapping of bounded (Φ, σ)-variation with an appropriately chosen function Φ such that $\Phi(\rho)/\rho \to \infty$ as $\rho \to \infty$ for any continuously differentiable function $\sigma: I \to \mathbb{R}$ such that $\sigma' > 0$ (Corollary 11). We prove an explicit formula for the (Φ, σ)-variation of a smooth mapping (Theorem 12). Finally, we show (Theorem 13) that any set-valued mapping with compact graph from a compact interval of the real line into subsets of a Banach space X that is of bounded (Φ, σ)-variation with respect to the Hausdorff metric admits a regular selection of bounded (Φ, σ)-variation with respect to the original norm in X (this result generalizes the previous results of the author on the existence of regular selections of set-valued mappings of bounded variation [2]–[6]).

The short version of the main results of the present paper was presented at the International Conference Dedicated to the 90th Anniversary of the Birth of L. S. Pontryagin, August, 31–September, 6, Moscow, 1998 ([7]).

2. Definitions

In what follows, we assume that X and Y are metric spaces with respective distance functions $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ that will, for the sake of brevity, be denoted by the same symbol $d(\cdot,\cdot)$. Let $\mathcal{M}$ be the set of all continuous convex functions $\Phi: [0, \infty[\to [0, \infty[$ such that $\Phi(\rho) = 0$ if and only if $\rho = 0$. The set of all functions $\Phi \in \mathcal{M}$ with $\lim_{\rho\to\infty} \Phi(\rho)/\rho = \infty$ will be denoted by $\mathcal{N}$. Suppose that $\sigma: I \to Y$ is a fixed *injective* mapping from the compact interval $I = [a, b] \subset \mathbb{R}$ ($a < b$) into Y (later on, the assumptions on σ will be made more strict—see (4), (8) and (17)).

Given a mapping $f: I \to X$, a partition $T = \{t_i\}_{i=0}^m$ of the interval I (i.e., $a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$), and a function $\Phi \in \mathcal{M}$, we set

$$V_{\Phi,\sigma}[f, T] = \sum_{i=1}^{m} \Phi\Big(\frac{d(f(t_i), f(t_{i-1}))}{d(\sigma(t_i), \sigma(t_{i-1}))}\Big) \cdot d(\sigma(t_i), \sigma(t_{i-1})).$$

The supremum of $V_{\Phi,\sigma}[f, T]$ with respect to all partitions T of the interval I will be denoted by $\mathsf{V}_{\Phi,\sigma}(f, I)$, or simply by $\mathsf{V}_{\Phi,\sigma}(f)$ if I is clear, and will be called the (*total*) Φ-*variation of* f *with respect to* σ, or the

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(Φ, σ)*-variation of f on I.* We denote by

$$BV_{\Phi,\sigma}(I;X) = \{f : I \to X \mid \mathsf{V}_{\Phi,\sigma}(f) < \infty\}$$

the set of all mappings from I into X *of bounded* (Φ, σ)*-variation.*

In the special case where $\Phi(\rho) = \rho$, $Y = \mathbb{R}$, and $\sigma(t) = t$, a mapping $f : I \to X$ of bounded (Φ, σ)-variation will be called a mapping *of bounded variation* (in the classical sense of C. Jordan), its total (Φ, σ)-variation will be written as $\mathsf{V}_1(f, I)$ or $\mathsf{V}_1(f)$, and the set of all these mappings will be denoted by $BV_1(I;X)$.

A mapping $f : I \to X$ is said to be σ*-absolutely continuous* if, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$ and $\sum_{i=1}^n d(\sigma(b_i), \sigma(a_i)) \le \delta(\varepsilon)$, then $\sum_{i=1}^n d(f(b_i), f(a_i)) \le \varepsilon$. We set

$$AC_\sigma(I;X) = \{f : I \to X \mid f \text{ is } \sigma\text{-absolutely continuous}\}.$$

If $Y = \mathbb{R}$ and $\sigma(t) = t$, σ-absolutely continuous mappings will simply be called *absolutely continuous* and the set of all these mappings will be denoted, as usual, by $AC(I;X)$.

A mapping $f : E \subset \mathbb{R} \to X$ is called σ*-Lipschitzian* if the following quantity, which is called the σ*-Lipschitz constant* of f, is finite:

$$\mathrm{Lip}_\sigma(f) = \sup\left\{\frac{d(f(t), f(s))}{d(\sigma(t), \sigma(s))} \;\middle|\; t,\, s \in E,\, t \ne s\right\}.$$

The set of all σ-Lipschitzian mappings from E into X is denoted by

$$C_\sigma^{0,1}(E;X) = \{f : E \to X \mid \mathrm{Lip}_\sigma(f) < \infty\}.$$

In particular, if $Y = \mathbb{R}$ and $\sigma(t) = t$, we call mappings from $C_\sigma^{0,1}(E;X)$ *Lipschitzian* (or *Lipschitz continuous*), and we drop the subscript σ in the notation of $\mathrm{Lip}(f)$—the *Lipschitz constant* of f—and of $C^{0,1}(E;X)$.

In the sequel we are going to make use of *Jensen's inequalities* for convex continuous functions $\Phi \in \mathcal{M}$, which we now recall (e.g., [13], Chap. 10, Sec. 5):

(a) *Jensen's inequality for sums*: if $\{\alpha_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$ are nonnegative numbers and $\sum_{i=1}^n \alpha_i > 0$, then

$$\Phi\left(\frac{\sum_{i=1}^n \alpha_i x_i}{\sum_{i=1}^n \alpha_i}\right) \le \frac{\sum_{i=1}^n \alpha_i \Phi(x_i)}{\sum_{i=1}^n \alpha_i}\ ; \tag{1}$$

(b) *Jensen's integral inequality*: if $\alpha : [a, b] \to \mathbb{R}$ and $x : [a, b] \to \mathbb{R}$ are nonnegative Lebesgue integrable functions and $\int_a^b \alpha(t)\,dt > 0$, then (in the case where all the integrals exist) we have

$$\Phi\left(\frac{\int_a^b \alpha(t)x(t)\,dt}{\int_a^b \alpha(t)\,dt}\right) \le \frac{\int_a^b \alpha(t)\Phi(x(t))\,dt}{\int_a^b \alpha(t)\,dt}\ . \tag{2}$$

3. Relations Between Functional Spaces

We begin with the following general proposition on embeddings of the above function spaces, which is valid under the assumptions given above.

Proposition 1. (a) $C_\sigma^{0,1}(I;X) \subset AC_\sigma(I;X)$.

(b) *If $\sigma \in BV_1(I;Y)$, then $C_\sigma^{0,1}(I;X) \subset BV_{\Phi,\sigma}(I;X) \subset BV_1(I;X)$ for all $\Phi \in \mathcal{M}$ and*

$$\mathsf{V}_1(f, I) \le \mathsf{V}_1(\sigma, I) \cdot \Phi^{-1}\left(\frac{\mathsf{V}_{\Phi,\sigma}(f, I)}{d(\sigma(b), \sigma(a))}\right), \qquad f \in BV_{\Phi,\sigma}(I;X), \tag{3}$$

where $I = [a, b]$ and $\Phi^{-1} : [0, \infty[\ \to [0, \infty[$ is the inverse function of Φ.

(c) *If* $\Phi \in \mathcal{N}$, *then* $BV_{\Phi,\sigma}(I;X) \subset AC_\sigma(I;X)$.

(d) *The inclusion* $AC_\sigma(I;X) \subset BV_1(I;X)$ *holds if* $\sigma : I \to Y$ *is continuous and satisfies the condition:*

$$\mathsf{V}_1(\sigma,[s,t]) = d(\sigma(t),\sigma(s)) \qquad \forall t,\ s \in I,\ s \le t. \tag{4}$$

In particular, the above inclusion holds if $\sigma : I \to \mathbb{R}$ *is continuous and strictly increasing.* (Condition (4) will be discussed below; see Remark 1.)

Proof. (a) For $f \in C^{0,1}_\sigma(I;X)$ and $\varepsilon > 0$, we set $\delta(\varepsilon) = \varepsilon/\max\{1, \mathrm{Lip}_\sigma(f)\} > 0$. If $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$ and $\sum_{i=1}^n d(\sigma(b_i),\sigma(a_i)) \le \delta(\varepsilon)$, then

$$\sum_{i=1}^n d(f(b_i), f(a_i)) \le \mathrm{Lip}_\sigma(f) \cdot \sum_{i=1}^n d(\sigma(b_i),\sigma(a_i)) \le \mathrm{Lip}_\sigma(f)\cdot\delta(\varepsilon) \le \varepsilon.$$

It follows that $f \in AC_\sigma(I;X)$.

(b) 1. For any partition $T = \{t_i\}_{i=0}^m$ of I and any $f \in C^{0,1}_\sigma(I;X)$, we have

$$V_{\Phi,\sigma}[f,T] \le \Phi(\mathrm{Lip}_\sigma(f)) \cdot \sum_{i=1}^m d(\sigma(t_i),\sigma(t_{i-1})) \le \Phi(\mathrm{Lip}_\sigma(f)) \cdot \mathsf{V}_1(\sigma).$$

The first inclusion in (b) thus follows.

2. Let $f : I \to X$ be of bounded (Φ,σ)-variation and $T = \{t_i\}_{i=0}^m$ be a partition of I. Applying Jensen's inequality (1) for sums with

$$\alpha_i = d(\sigma(t_i),\sigma(t_{i-1})), \qquad x_i = \frac{d(f(t_i), f(t_{i-1}))}{d(\sigma(t_i),\sigma(t_{i-1}))}, \qquad i = 1,\ldots,m,$$

we obtain

$$\begin{aligned}
&\Phi\Big(\frac{\sum_{i=1}^m d(f(t_i), f(t_{i-1}))}{\sum_{i=1}^m d(\sigma(t_i),\sigma(t_{i-1}))}\Big)\\
&\le \frac{1}{\sum_{i=1}^m d(\sigma(t_i),\sigma(t_{i-1}))} \cdot \sum_{i=1}^m \Phi\Big(\frac{d(f(t_i), f(t_{i-1}))}{d(\sigma(t_i),\sigma(t_{i-1}))}\Big) d(\sigma(t_i),\sigma(t_{i-1}))\\
&\le \frac{1}{\sum_{i=1}^m d(\sigma(t_i),\sigma(t_{i-1}))} \cdot \mathsf{V}_{\Phi,\sigma}(f,I).
\end{aligned}$$

It follows, by taking the inverse function Φ^{-1}, that

$$\begin{aligned}
V_1[f,T] &= \sum_{i=1}^m d(f(t_i), f(t_{i-1}))\\
&\le \Big\{\textstyle\sum_{i=1}^m d(\sigma(t_i),\sigma(t_{i-1}))\Big\} \cdot \Phi^{-1}\Big(\frac{\mathsf{V}_{\Phi,\sigma}(f,I)}{\sum_{i=1}^m d(\sigma(t_i),\sigma(t_{i-1}))}\Big) \qquad (5)\\
&\le \mathsf{V}_1(\sigma,I) \cdot \Phi^{-1}\Big(\frac{\mathsf{V}_{\Phi,\sigma}(f,I)}{d(\sigma(b),\sigma(a))}\Big),
\end{aligned}$$

and it remains to take the supremum over all partitions T of I.

(c) Let $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$. As in (5), for a mapping $f \in BV_{\Phi,\sigma}(I;X)$, we have

$$\sum_{i=1}^n d(f(b_i), f(a_i)) \le \Big\{\textstyle\sum_{i=1}^n d(\sigma(b_i),\sigma(a_i))\Big\} \cdot \Phi^{-1}\Big(\frac{\mathsf{V}_{\Phi,\sigma}(f,I)}{\sum_{i=1}^n d(\sigma(b_i),\sigma(a_i))}\Big). \tag{6}$$

Setting $v = \mathsf{V}_{\Phi,\sigma}(f,I)$ and taking into account that $\lim_{\rho\to\infty} \Phi(\rho)/\rho = \infty$, we obtain

$$\lim_{t\to+0} t\,\Phi^{-1}(v/t) = v \lim_{\rho\to\infty} \rho/\Phi(\rho) = 0. \tag{7}$$

Hence, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $t\,\Phi^{-1}(v/t) \le \varepsilon$ for $0 < t \le \delta(\varepsilon)$. Then inequality (6) implies the following:

$$\text{if } \sum_{i=1}^{n} d(\sigma(b_i), \sigma(a_i)) \le \delta(\varepsilon), \quad \text{then } \sum_{i=1}^{n} d(f(b_i), f(a_i)) \le \varepsilon,$$

and, therefore, $f \in AC_\sigma(I;X)$.

(d) Let $f \in AC_\sigma(I;X)$. Let $\varepsilon > 0$ be fixed, and let $\delta(\varepsilon) > 0$ be the number from the definition of σ-absolute continuity of f. Since σ is uniformly continuous on I, there exists a partition $T = \{t_i\}_{i=0}^{m}$ of I such that

$$d(\sigma(t_i), \sigma(t_{i-1})) \le \delta(\varepsilon) \qquad \forall\, i = 1, \ldots, m.$$

Now, if $T_i = \{t_{i,j}\}_{j=0}^{m_i}$ is a partition of the closed interval $I_i = [t_{i-1}, t_i]$, then by virtue of (4) and the additivity of $\mathsf{V}_1(\sigma, \cdot\,)$, we have

$$\sum_{j=1}^{m_i} d(\sigma(t_{i,j}), \sigma(t_{i,j-1})) = d(\sigma(t_i), \sigma(t_{i-1})) \le \delta(\varepsilon),$$

so that by the σ-absolute continuity of f, it follows that

$$V_1[f, T_i] = \sum_{j=1}^{m_i} d(f(t_{i,j}), f(t_{i,j-1})) \le \varepsilon.$$

Since the partition T_i of I_i is arbitrary, we have $\mathsf{V}_1(f, I_i) \le \varepsilon$ for all $i = 1, \ldots, m$, and it remains to use the additivity property of $\mathsf{V}_1(f, \cdot\,)$:

$$\mathsf{V}_1(f, I) = \sum_{i=1}^{m} \mathsf{V}_1(f, I_i) \le m\varepsilon.$$

Thus, $f \in BV_1(I;X)$.

Remark 1. Condition (4) does not, in fact, bring any generality as compared to the case where σ is real-valued. By this we mean that if $\sigma : I \to Y$ is injective and satisfies (4), then setting $\sigma_1(t) = \mathsf{V}_1(\sigma, [a,t])$, $t \in I$, we find that $\sigma_1 : I \to \mathbb{R}$ is strictly increasing and bounded and satisfies for $s, t \in I$, $s \le t$, the following relations:

$$\sigma_1(t) - \sigma_1(s) = \mathsf{V}_1(\sigma, [a,t]) - \mathsf{V}_1(\sigma, [a,s]) = \mathsf{V}_1(\sigma, [s,t]) = d(\sigma(t), \sigma(s)).$$

Hence, in the sequel, we will assume that

$$\sigma : I \to \mathbb{R} \textit{ is strictly increasing and bounded.} \tag{8}$$

However, to make sure that condition (4) naturally arises in different contexts, we are going to keep it for a while (until after Theorem 5).

The embeddings in Proposition 1 are depicted in the following diagram:

$$\begin{tikzcd}
C_\sigma^{0,1}(I;X) \arrow[r, "\sigma \in BV_1"] \arrow[dr] & BV_{\Phi,\sigma}(I;X) \arrow[r, "\sigma \in BV_1"] \arrow[d, "\Phi \in \mathcal{N}"] & BV_1(I;X) \\
& AC_\sigma(I;X) \arrow[ur, "\substack{\sigma : I \to \mathbb{R} \\ \text{continuous,} \\ \text{increasing}}"'] &
\end{tikzcd}$$

4. Properties of the (Φ, σ)-Variation

Proposition 2. *Assume that $\Phi \in \mathcal{M}$ and σ satisfies (4) or (8). Then, for any mapping $f : I \to X$, we have*

(a) $V_{\Phi,\sigma}[f,T] \le V_{\Phi,\sigma}[f, T \cup \{t\}]$ *if T is a partition of I and $t \in I \setminus T$;*

(b) $V_{\Phi,\sigma}[f,T_1] \le V_{\Phi,\sigma}[f,T_2]$ *if T_1 and T_2 are partitions of I and $T_1 \subset T_2$;*

(c) $\mathsf{V}_{\Phi,\sigma}(f,T) = V_{\Phi,\sigma}[f,T]$ *for any partition T of I (so that $\mathsf{V}_{\Phi,\sigma}(f,\cdot)$ extends $V_{\Phi,\sigma}[f,\cdot]$ onto all subsets of I);*

(d) *the quantity $\mathsf{V}_{\Phi,\sigma}(f,I)$ is equal to the supremum of $V_{\Phi,\sigma}[f,T]$ taken over all partitions T of I such that every T contains the same finite subset of points from I.*

Proof. (a) Let $T = \{t_i\}_{i=0}^m$ and $t_{k-1} < t < t_k$ for some $k \in \{1, \ldots, m\}$. Setting

$$U_i = U_i(f) = \Phi\Big(\frac{d(f(t_i), f(t_{i-1}))}{d(\sigma(t_i), \sigma(t_{i-1}))}\Big) d(\sigma(t_i), \sigma(t_{i-1})), \qquad i = 1, \ldots, m, \tag{9}$$

we have

$$V_{\Phi,\sigma}[f,T] = \Big(\sum_{i=1}^{k-1} U_i\Big) + U_k + \Big(\sum_{i=k+1}^{m} U_i\Big), \tag{10}$$

where we set the first or the last sum equal to zero if $k = 1$ or $k = m$, respectively. Applying Jensen's inequality (1) with $\alpha_1 = d(\sigma(t), \sigma(t_{k-1}))$, $\alpha_2 = d(\sigma(t_k), \sigma(t))$, and

$$x_1 = \frac{d(f(t), f(t_{k-1}))}{d(\sigma(t), \sigma(t_{k-1}))}, \qquad x_2 = \frac{d(f(t_k), f(t))}{d(\sigma(t_k), \sigma(t))},$$

and noting that, by (4), $\alpha_1 + \alpha_2 = d(\sigma(t_k), \sigma(t_{k-1}))$, we find that

$$\begin{aligned} U_k &\le \Phi\Big(\frac{d(f(t), f(t_{k-1})) + d(f(t_k), f(t))}{d(\sigma(t), \sigma(t_{k-1})) + d(\sigma(t_k), \sigma(t))}\Big) d(\sigma(t_k), \sigma(t_{k-1})) \\ &\le \Phi\Big(\frac{d(f(t), f(t_{k-1}))}{d(\sigma(t), \sigma(t_{k-1}))}\Big) d(\sigma(t), \sigma(t_{k-1})) + \Phi\Big(\frac{d(f(t_k), f(t))}{d(\sigma(t_k), \sigma(t))}\Big) d(\sigma(t_k), \sigma(t)). \end{aligned} \tag{11}$$

Thus, (10) implies $V_{\Phi,\sigma}[f,T] \le V_{\Phi,\sigma}[f, T \cup \{t\}]$.

(b) follows by induction from (a). Items (c) and (d) are consequences of (b).

Proposition 3. *Let $\Phi \in \mathcal{M}$, and let σ satisfy (4) or (8). For $f : I = [a,b] \to X$, we have*

(a) *if $a \le s \le t \le b$, then $\mathsf{V}_{\Phi,\sigma}(f, [s,t]) \le \mathsf{V}_{\Phi,\sigma}(f, [a,b])$;*

(b) *if $a < t < b$, then $\mathsf{V}_{\Phi,\sigma}(f, [a,b]) = \mathsf{V}_{\Phi,\sigma}(f, [a,t]) + \mathsf{V}_{\Phi,\sigma}(f, [t,b])$;*

(c) *if $f_n : I \to X$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} d(f_n(t), f(t)) = 0$ for all $t \in I$, then*

$$\mathsf{V}_{\Phi,\sigma}(f, I) \le \liminf_{n\to\infty} \mathsf{V}_{\Phi,\sigma}(f_n, I).$$

Proof. By virtue of Proposition 2(b), assertion (a) is obvious.

(b) For a partition T_1 of $[a,t]$ and a partition T_2 of $[t,b]$ we have:

$$V_{\Phi,\sigma}[f,T_1] + V_{\Phi,\sigma}[f,T_2] = V_{\Phi,\sigma}[f, T_1 \cup T_2] \le \mathsf{V}_{\Phi,\sigma}(f, I).$$

Since T_1 and T_2 are arbitrary, it follows that

$$\mathsf{V}_{\Phi,\sigma}(f, [a,t]) + \mathsf{V}_{\Phi,\sigma}(f, [t,b]) \le \mathsf{V}_{\Phi,\sigma}(f, I).$$

To prove the converse inequality, assume that $T = \{t_i\}_{i=0}^m$ is a partition of I and that $t_{k-1} \le t \le t_k$ for some $k \in \{1, \dots, m\}$. According to (10) and (11) we have

$$\begin{aligned} V_{\Phi,\sigma}[f,T] &\le V_{\Phi,\sigma}[f, \{t_i\}_{i=0}^{k-1} \cup \{t\}] + V_{\Phi,\sigma}[f, \{t\} \cup \{t_i\}_{i=k}^m] \\ &\le \mathsf{V}_{\Phi,\sigma}(f, [a,t]) + \mathsf{V}_{\Phi,\sigma}(f, [t,b]). \end{aligned}$$

It remains to take into account the arbitrariness of T.

(c) Fix a partition $T = \{t_i\}_{i=0}^m$ of I. By the definition of $\mathsf{V}_{\Phi,\sigma}(f_n, I)$ we have

$$V_{\Phi,\sigma}[f_n, T] \le \mathsf{V}_{\Phi,\sigma}(f_n, I) \quad \text{for all} \quad n \in \mathbb{N}. \tag{12}$$

Using notation (9), the continuity of $d(\cdot,\cdot)$ and Φ, and also the pointwise convergence of f_n to f, we obtain

$$V_{\Phi,\sigma}[f_n, T] - V_{\Phi,\sigma}[f,T] = \sum_{i=1}^m \{U_i(f_n) - U_i(f)\} \to 0 \quad \text{as} \quad n \to \infty.$$

Taking the limit inferior in both sides of inequality (12) we arrive at the inequality

$$V_{\Phi,\sigma}[f,T] \le \liminf_{n\to\infty} \mathsf{V}_{\Phi,\sigma}(f_n, I)$$

for any partition T of I.

Remark 2. In Proposition 3, condition (4) was actually used only in (a) and (b).

5. A Structural Theorem

The following lemma holds for arbitrary injective mappings $\sigma : I \to Y$. It presents examples of mappings of bounded (Φ, σ)-variation.

Lemma 4. *Assume that $\varphi : I \to \mathbb{R}$, $J = \varphi(I)$ is the image of φ, $g \in C^{0,1}(J; X)$, $\mathrm{Lip}(g) \le 1$, and define $f(t) = g(\varphi(t))$, $t \in I$.*

(a) *If $\varphi \in C_\sigma^{0,1}(I;\mathbb{R})$, then $f \in C_\sigma^{0,1}(I;X)$ and $\mathrm{Lip}_\sigma(f) \le \mathrm{Lip}_\sigma(\varphi)$.*

(b) *If $\Phi \in \mathcal{M}$ and $\varphi \in BV_{\Phi,\sigma}(I;\mathbb{R})$, then $f \in BV_{\Phi,\sigma}(I;X)$ and $\mathsf{V}_{\Phi,\sigma}(f) \le \mathsf{V}_{\Phi,\sigma}(\varphi)$.*

(c) *If $\varphi \in AC_\sigma(I;\mathbb{R})$, then $f \in AC_\sigma(I;X)$ and for any $\varepsilon > 0$ the number $\delta_f(\varepsilon) > 0$ from the definition of the σ-absolute continuity of f can be chosen to be equal to the one from the definition of the σ-absolute continuity of φ (in symbols, $\delta_f(\cdot) = \delta_\varphi(\cdot)$).*

Proof. (a) For all t, $s \in I$, we have

$$\begin{aligned} d(f(t), f(s)) &= d(g(\varphi(t)), g(\varphi(s))) \le \mathrm{Lip}(g)|\varphi(t) - \varphi(s)| \\ &\le \mathrm{Lip}(g)\,\mathrm{Lip}_\sigma(\varphi) d(\sigma(t), \sigma(s)) \le \mathrm{Lip}_\sigma(\varphi) d(\sigma(t), \sigma(s)). \end{aligned}$$

(b) If $T = \{t_i\}_{i=0}^m$ is a partition of I, we obtain

$$V_{\Phi,\sigma}[f,T] \le \sum_{i=1}^m \Phi\Big(\mathrm{Lip}(g) \frac{|\varphi(t_i) - \varphi(t_{i-1})|}{d(\sigma(t_i), \sigma(t_{i-1}))}\Big) d(\sigma(t_i), \sigma(t_{i-1})) \le \mathsf{V}_{\Phi,\sigma}(\varphi).$$

(c) Let $\varepsilon > 0$, $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$, and let $\sum_{i=1}^n d(\sigma(b_i), \sigma(a_i)) \le \delta_\varphi(\varepsilon)$, where $\delta_\varphi(\varepsilon)$ is the number from the definition of the σ-absolute continuity of φ. We have

$$\sum_{i=1}^n d(f(b_i), f(a_i)) \le \mathrm{Lip}(g) \sum_{i=1}^n |\varphi(b_i) - \varphi(a_i)| \le \mathrm{Lip}(g) \cdot \varepsilon \le \varepsilon.$$

Remark 3. In Lemma 4(b), the condition $\mathrm{Lip}(g) \le 1$ is particularly important.

It turns out that under condition (4) (or equivalently, under condition (8)), mappings f of bounded variation are decomposable as $f = g \circ \varphi$ in the same way as in Lemma 4. More precisely, we have the following:

Theorem 5 (structural theorem). *Let $f \in BV_1(I;X)$. Set $\varphi(t) = \mathsf{V}_1(f,[a,t])$ if $t \in I$, and let $J = \varphi(I)$. Then $\varphi : I \to [0,\infty[$ is a bounded nondecreasing function, and there exists a mapping $g \in C^{0,1}(J;X)$ with $\mathrm{Lip}(g) \le 1$ such that $f(t) = g(\varphi(t))$ for all $t \in I$.*

Moreover, if σ satisfies (4) or (8), then we have

(a) *if $f \in C^{0,1}_\sigma(I;X)$, then $\varphi \in C^{0,1}_\sigma(I;\mathbb{R})$ and $\mathrm{Lip}_\sigma(\varphi) = \mathrm{Lip}_\sigma(f)$*

(b) *if $\Phi \in \mathcal{M}$ and $f \in BV_{\Phi,\sigma}(I;X)$, then $\varphi \in BV_{\Phi,\sigma}(I;\mathbb{R})$ and $\mathsf{V}_{\Phi,\sigma}(\varphi) = \mathsf{V}_{\Phi,\sigma}(f)$*

(c) *if σ is continuous and $f \in AC_\sigma(I;X)$, then $\varphi \in AC_\sigma(I;\mathbb{R})$ and $\delta_\varphi(\cdot) = \delta_f(\cdot)$.*

Proof. The first part of this theorem is proved in [2], Theorem 3.1 and Lemma 3.3. Taking into account the embeddings in Proposition 1, we are going to verify that (a), (b), and (c) hold.

(a) If $t,\ s \in I$, $s \le t$, then for any partition $T = \{t_i\}_{i=0}^m$ of $[s,t]$ we have

$$\begin{aligned} V_1[f,T] &= \sum_{i=1}^m d(f(t_i), f(t_{i-1})) \le \mathrm{Lip}_\sigma(f) \sum_{i=1}^m d(\sigma(t_i),\sigma(t_{i-1})) \\ &\le \mathrm{Lip}_\sigma(f)\cdot \mathsf{V}_1(\sigma,[s,t]), \end{aligned}$$

so that $\mathsf{V}_1(f,[s,t]) \le \mathrm{Lip}_\sigma(f)\cdot\mathsf{V}_1(\sigma,[s,t])$. In view of (4), we obtain

$$\begin{aligned} |\varphi(t)-\varphi(s)| &= \mathsf{V}_1(f,[a,t]) - \mathsf{V}_1(f,[a,s]) = \mathsf{V}_1(f,[s,t]) \\ &\le \mathrm{Lip}_\sigma(f)\cdot \mathsf{V}_1(\sigma,[s,t]) = \mathrm{Lip}_\sigma(f)\cdot d(\sigma(t),\sigma(s)). \end{aligned}$$

It follows that $\mathrm{Lip}_\sigma(\varphi) \le \mathrm{Lip}_\sigma(f)$. The last inequality is, actually, an equality, as can be seen from Lemma 4(a).

(b) Let $T = \{t_i\}_{i=0}^m$ be a partition of I and $I_i = [t_{i-1},t_i]$, $i = 1,\dots,m$. Applying (3), we have

$$|\varphi(t_i)-\varphi(t_{i-1})| = \mathsf{V}_1(f,I_i) \le \mathsf{V}_1(\sigma,I_i)\cdot\Phi^{-1}\Big(\frac{\mathsf{V}_{\Phi,\sigma}(f,I_i)}{d(\sigma(t_i),\sigma(t_{i-1}))}\Big),$$

and hence,

$$\begin{aligned} V_{\Phi,\sigma}[\varphi,T] &= \sum_{i=1}^m \Phi\Big(\frac{|\varphi(t_i)-\varphi(t_{i-1})|}{d(\sigma(t_i),\sigma(t_{i-1}))}\Big) d(\sigma(t_i),\sigma(t_{i-1})) \\ &\le \sum_{i=1}^m \Phi\Big(\frac{\mathsf{V}_1(\sigma,I_i)}{d(\sigma(t_i),\sigma(t_{i-1}))}\cdot\Phi^{-1}\Big(\frac{\mathsf{V}_{\Phi,\sigma}(f,I_i)}{d(\sigma(t_i),\sigma(t_{i-1}))}\Big)\Big) d(\sigma(t_i),\sigma(t_{i-1})). \end{aligned}$$

Condition (4) and Proposition 3(b) then imply

$$V_{\Phi,\sigma}[\varphi,T] \le \sum_{i=1}^m \mathsf{V}_{\Phi,\sigma}(f,I_i) = \mathsf{V}_{\Phi,\sigma}(f,I).$$

It follows that $\mathsf{V}_{\Phi,\sigma}(\varphi,I) \le \mathsf{V}_{\Phi,\sigma}(f,I)$. Now Lemma 4(b) and the relation $f = g\circ\varphi$ with $\mathrm{Lip}(g) \le 1$ yield $\mathsf{V}_{\Phi,\sigma}(\varphi) = \mathsf{V}_{\Phi,\sigma}(f)$.

(c) By Proposition 1(d), the function $I \ni t \mapsto \varphi(t) = \mathsf{V}_1(f,[a,t])$ is well defined. Let $\varepsilon > 0$, $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$, and let $\sum_{i=1}^n d(\sigma(b_i),\sigma(a_i)) \le \delta_f(\varepsilon)$, where $\delta_f(\varepsilon) > 0$ is the number from the definition of the σ-absolute continuity of f. For any $i \in \{1,\dots,n\}$ and any $\alpha_i < \mathsf{V}_1(f,[a_i,b_i])$, there exists a partition $T_i = \{t_{i,j}\}_{j=0}^{m_i}$ of $[a_i,b_i]$ such that $V_1[f,T_i] \ge \alpha_i$. Since

$$\sum_{i=1}^n\sum_{j=1}^{m_i} d(\sigma(t_{i,j}),\sigma(t_{i,j-1})) = \sum_{i=1}^n d(\sigma(b_i),\sigma(a_i)) \le \delta_f(\varepsilon),$$

by virtue of (4) the σ-absolute continuity of f implies

$$\sum_{i=1}^{n} \alpha_i \le \sum_{i=1}^{n} V_1[f, T_i] = \sum_{i=1}^{n} \sum_{j=1}^{m_i} d(f(t_{i,j}), f(t_{i,j-1})) \le \varepsilon.$$

Passing to the limit $\alpha_i \to \mathsf{V}_1(f, [a_i, b_i])$, we obtain

$$\sum_{i=1}^{n} |\varphi(b_i) - \varphi(a_i)| = \sum_{i=1}^{n} \mathsf{V}_1(f, [a_i, b_i]) \le \varepsilon,$$

so that we can set $\delta_\varphi(\varepsilon) = \delta_f(\varepsilon)$.

From now on, we assume that σ satisfies (8). In this case, the main estimate (3) takes the form

$$\mathsf{V}_1(f, [a, b]) \le (\sigma(b) - \sigma(a)) \cdot \Phi^{-1}\Big(\frac{\mathsf{V}_{\Phi,\sigma}(f, [a, b])}{\sigma(b) - \sigma(a)}\Big), \quad f \in BV_{\Phi,\sigma}(I; X). \tag{13}$$

6. A Selection Principle

Theorem 6 (selection principle). *Assume that K is a compact subset of a metric space X, $\Phi \in \mathcal{M}$, σ satisfies (8), and $\mathcal{F}$ is an infinite family of continuous mappings from I into K such that*

$$v := \sup_{f \in \mathcal{F}} \mathsf{V}_{\Phi,\sigma}(f, I) < \infty. \tag{14}$$

Then there exists a sequence of mappings $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ that converges pointwise on I as $n \to \infty$ to a mapping $f \in BV_{\Phi,\sigma}(I; X)$ such that $\mathsf{V}_{\Phi,\sigma}(f, I) \le v$.

If X is a Banach space, then mappings from $\mathcal{F}$ need not be continuous.

If $\Phi \in \mathcal{N}$ and σ is continuous, we can assume that X is a complete metric space, a family $\mathcal{F}$ of mappings from I into X is such that the sets $\{f(t) \mid f \in \mathcal{F}\}$ are precompact in X for all $t \in I$, and (14) holds. Then the convergence of continuous mappings f_n to f is uniform.

Proof. We are going to apply a variant of Helly's selection principle from [2], Theorem 7.1. To this end, we have to verify that the family $\{\mathsf{V}_1(f, I) \mid f \in \mathcal{F}\}$ is bounded. This is a consequence of (13):

$$\mathsf{V}_1(f, I) \le (\sigma(b) - \sigma(a))\Phi^{-1}\Big(\frac{v}{\sigma(b) - \sigma(a)}\Big) \qquad \forall f \in \mathcal{F}.$$

By the Helly selection principle (referred to above), a sequence of mappings $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ converges pointwise on I as $n \to \infty$ to a mapping $f \in BV_1(I; X)$. Actually, $f \in BV_{\Phi,\sigma}(I; X)$, since, by Proposition 3(c), we have

$$\mathsf{V}_{\Phi,\sigma}(f, I) \le \liminf_{n \to \infty} \mathsf{V}_{\Phi,\sigma}(f_n, I) \le v. \tag{15}$$

Assume now that $\Phi \in \mathcal{N}$ and σ is continuous. If t, $s \in I$, $s < t$, by the definition of $\mathsf{V}_{\Phi,\sigma}(f, I)$ and from (13), we have for any $f \in \mathcal{F}$

$$\begin{aligned} d(f(t), f(s)) &\le (\sigma(t) - \sigma(s))\Phi^{-1}\Big(\frac{\mathsf{V}_{\Phi,\sigma}(f, I)}{\sigma(t) - \sigma(s)}\Big) \\ &\le (\sigma(t) - \sigma(s))\Phi^{-1}\Big(\frac{v}{\sigma(t) - \sigma(s)}\Big). \end{aligned} \tag{16}$$

Since $\Phi \in \mathcal{N}$, (7) implies that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) = \delta(\varepsilon, v) > 0$ such that $\rho\Phi^{-1}(v/\rho) \le \varepsilon$ for all $0 < \rho \le \delta(\varepsilon)$. Since σ is continuous, there exists $\delta_1(\varepsilon) > 0$ such that if $0 < t - s \le \delta_1(\varepsilon)$, then $\sigma(t) - \sigma(s) \le \delta(\varepsilon)$. This and (16) yield that $\sup_{f \in \mathcal{F}} d(f(t), f(s)) \le \varepsilon$ for all $0 < t - s \le \delta_1(\varepsilon)$. Hence, we have shown that the family $\mathcal{F}$ is equicontinuous. By Arzelà-Ascoli's theorem, $\mathcal{F}$ is precompact in the space of continuous mappings from I into K equipped with the uniform metric. It follows that there exists a uniformly convergent sequence of mappings $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ whose uniform limit we denote by f. From (15), we conclude that $f \in BV_{\Phi,\sigma}(I;X)$.

If X is a Banach space, then we can apply a refined Helly's selection principle from [3], Theorem 5.1, to obtain what was desired.

Remark 4. If σ is continuous, a theorem similar to Theorem 6 with the uniform convergence assertion holds for a family $\mathcal{F} \subset C_{\sigma}^{0,1}(I;K)$ if $\sup_{f\in\mathcal{F}} \mathrm{Lip}_{\sigma}(f) < \infty$, and for a family $\mathcal{F} \subset AC_{\sigma}(I;K)$, if we assume that $\inf_{f\in\mathcal{F}} \delta_f(\varepsilon) > 0$ for all $\varepsilon > 0$, where $\delta_f(\varepsilon)$ is the number from the definition of σ-absolute continuity of f.

7. Differentiability Properties

If X is a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$), we denote by $C^1(I;X)$ the vector space of all continuously differentiable mappings f whose strong derivative (with respect to the norm in X) evaluated at $t \in I$ is denoted by $f'(t) \in X$. The following abbreviations are commonly used: a.e. = almost everywhere (with respect to the Lebesgue measure on I), a.a. = almost all, etc.

From now on (except for Lemma 8), we will assume that (cf. (8))

$$\sigma \in C^1(I;\mathbb{R}) \quad and \quad \sigma'(t) > 0 \quad for \;\; all \;\; t \in I. \tag{17}$$

Theorem 7. *Let X be a reflexive Banach space with norm $\|\cdot\|$, $I = [a,b]$, $\Phi \in \mathcal{M}$, σ satisfy (17), and let $f \in BV_{\Phi,\sigma}(I;X)$. Then f is a.e. weakly differentiable on I (this is to be made precise in the proof), its weak derivative $t \mapsto f^{\bullet}(t)$ is strongly measurable, and*

$$\int_I \sigma'(t)\Phi\Big(\frac{\|f^{\bullet}(t)\|}{\sigma'(t)}\Big)\,dt \le \mathsf{V}_{\Phi,\sigma}(f,I).$$

If, moreover, $\Phi \in \mathcal{N}$, then $f \in AC(I;X)$ is a.e. strongly differentiable on I, its strong derivative $t \mapsto f'(t)$ is strongly measurable, f can be written in the form

$$f(t) = f(a) + \int_a^t f'(\tau)\,d\tau \qquad for \;\; all \qquad t \in I \tag{18}$$

(with the Bochner integral on the right hand side), and the following equality holds:

$$\mathsf{V}_{\Phi,\sigma}(f,I) = \int_I \sigma'(t)\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\,dt. \tag{19}$$

On the other hand, if $f \in AC(I;X)$ and its strongly measurable strong derivative $t \mapsto f'(t)$, defined a.e. on I, is such that $\int_a^b \sigma'(t)\Phi(\|f'(t)\|/\sigma'(t))\,dt < \infty$, then $f \in BV_{\Phi,\sigma}(I;X)$.

In order to prove Theorem 7, we need a lemma.

Lemma 8. *If X is a normed vector space with the norm $\|\cdot\|$, $I = [a,b]$, $\Phi \in \mathcal{M}$, σ satisfies (8), and $f \in BV_{\Phi,\sigma}(I;X)$, then for any $0 < h < b-a$, we have*

$$\int_a^{b-h} \frac{\sigma(t+h)-\sigma(t)}{h}\,\Phi\Big(\frac{\|f(t+h)-f(t)\|}{\sigma(t+h)-\sigma(t)}\Big)\,dt \le \mathsf{V}_{\Phi,\sigma}(f,[a,b]). \tag{20}$$

Proof. The function $t \mapsto \mathsf{V}_{\Phi,\sigma}(f,[a,t])$ is nondecreasing and bounded on I, so that it is Riemann integrable on I. Fix $0 < h < b-a$. Since $f \in BV_1(I;X)$, it is continuous outside, possibly, a countable subset of I (cf. [2], Theorem 4.1), and hence, the function $[a,b-h] \ni t \mapsto \|f(t+h)-f(t)\|$ has the same continuity properties. Using Proposition 3(b), we have

$$\begin{aligned}\frac{\sigma(t+h)-\sigma(t)}{h}\,\Phi\Big(\frac{\|f(t+h)-f(t)\|}{\sigma(t+h)-\sigma(t)}\Big) &\le \frac{1}{h}\mathsf{V}_{\Phi,\sigma}(f,[t,t+h])\\ &= \frac{1}{h}(\mathsf{V}_{\Phi,\sigma}(f,[a,t+h])-\mathsf{V}_{\Phi,\sigma}(f,[a,t])).\end{aligned}$$

Now it suffices to integrate this inequality with respect to $t \in [a,b-h]$:

$$\begin{aligned}\int_a^{b-h}\frac{\sigma(t+h)-\sigma(t)}{h}\,\Phi\Big(\frac{\|f(t+h)-f(t)\|}{\sigma(t+h)-\sigma(t)}\Big)\,dt &\le \frac{1}{h}\Big(\int_{b-h}^{b}-\int_a^{a+h}\Big)\mathsf{V}_{\Phi,\sigma}(f,[a,t])\,dt\\ &\le \frac{1}{h}\int_{b-h}^{b}\mathsf{V}_{\Phi,\sigma}(f,[a,t])\,dt \le \mathsf{V}_{\Phi,\sigma}(f,[a,b]).\end{aligned}$$

Proof of Theorem 7. 1. Proposition 1(b) yields $f \in BV_1(I;X)$. By Theorem 3.3 from [1], Chap. 1, Sec. 3, the mapping f is a.e. weakly differentiable on I in the sense that there exists a mapping $t \mapsto f^\bullet(t)$ (the weak derivative of f), defined a.e. on I, such that for a.a. $t \in I$ we have

$$\Big(x^*,\frac{f(t+h)-f(t)}{h}-f^\bullet(t)\Big)\to 0 \quad \text{as} \quad \mathbb{R}\ni h\to 0 \qquad \forall x^*\in X^*, \tag{21}$$

where X^* is the strong dual of X and $(\,\cdot\,,\,\cdot\,)$ is the pairing between X^* and X; the weak derivative $f^\bullet$ is strongly measurable and belongs to the Banach space $L^1(I;X)$ of Bochner integrable mappings from I into X. Since $(f(t+h)-f(t))/h$ weakly converges to $f^\bullet(t)$ as $h\to 0$ for a.a. $t\in I$ by (21), it follows that

$$\|f^\bullet(t)\| \le \liminf_{h\to 0}\Big\|\frac{f(t+h)-f(t)}{h}\Big\| \quad \text{for} \quad \text{a.a.} \quad t\in I.$$

Using Fatou's lemma and applying Lemma 8, we obtain

$$\begin{aligned}\int_a^b \sigma'(t)\Phi\Big(\frac{\|f^\bullet(t)\|}{\sigma'(t)}\Big)\,dt &\le \liminf_{h\to 0}\int_a^{b-h}\frac{\sigma(t+h)-\sigma(t)}{h}\,\Phi\Big(\frac{\|f(t+h)-f(t)\|}{\sigma(t+h)-\sigma(t)}\Big)\,dt\\ &\le \mathsf{V}_{\Phi,\sigma}(f,[a,b]).\end{aligned}\tag{22}$$

2. Assume that $\Phi \in \mathcal{N}$. Then $f \in AC_\sigma(I;X)$ by Proposition 1(c), so that, since (17) holds, $f \in AC(I;X)$. According to Theorem 3.4 from [1], Chap. 1, Sec. 3, the mapping f is a.e. strongly differentiable on I (with the strong derivative f' equal a.e. to the weak derivative $f^\bullet$), it can be written in the form (18), and (22) takes place. Now, using (18) and Jensen's integral inequality (2), we obtain the converse inequality for (22): if $T=\{t_i\}_{i=0}^m$ is a partition of I, then

$$\begin{aligned}V_{\Phi,\sigma}[f,T] &= \sum_{i=1}^m \Phi\Big(\frac{\|f(t_i)-f(t_{i-1})\|}{\sigma(t_i)-\sigma(t_{i-1})}\Big)\cdot(\sigma(t_i)-\sigma(t_{i-1}))\\ &\le \sum_{i=1}^m \Phi\Big(\frac{\int_{t_{i-1}}^{t_i}\sigma'(t)\{\|f'(t)\|/\sigma'(t)\}\,dt}{\int_{t_{i-1}}^{t_i}\sigma'(t)\,dt}\Big)\cdot\int_{t_{i-1}}^{t_i}\sigma'(t)\,dt\\ &\overset{(2)}{\le} \sum_{i=1}^m\int_{t_{i-1}}^{t_i}\sigma'(t)\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\,dt = \int_a^b\sigma'(t)\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\,dt.\end{aligned}$$

3. If the last condition of the theorem is fulfilled, then calculations done at the end of step 2 prove that f is of bounded (Φ,σ)-variation.

Remark 5. Without the reflexivity assumption on X one can find Lipschitz continuous mappings $f \in C^{0,1}(I;X)$ that have no point of (weak or strong) differentiability on the interval $]a,b[$, the interior of I (cf. [10] or [4], Sec. 5).

The following corollary is a generalization of the criteria due to Riesz [14] ($X = \mathbb{R}$, $\Phi(\rho) = \rho^q$, $q > 1$, $\sigma(t) = t$), Medvedev [11] ($X = \mathbb{R}$, $\Phi \in \mathcal{N}$, $\sigma(t) = t$), and the author [5] (X a reflexive Banach space, $\Phi \in \mathcal{N}$, $\sigma(t) = t$):

Corollary 9. *If X is a reflexive Banach space, $\Phi \in \mathcal{N}$, and σ satisfies (17), then*

$$f \in BV_{\Phi,\sigma}(I;X) \iff f \in AC(I;X) \text{ and } \int_I \sigma'(t)\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\,dt < \infty.$$

In view of Theorem 5(b), Corollary 9 can be generalized for arbitrary metric spaces X as follows:

Corollary 10. *Let X be a metric space, $\Phi \in \mathcal{N}$, σ satisfy (17), $f \in BV_1(I;X)$, and $\varphi(t) = V_1(f,[a,t])$, $t \in I$. Then*

$$f \in BV_{\Phi,\sigma}(I;X) \iff \varphi \in AC(I;\mathbb{R}) \text{ and } \int_I \sigma'(t)\Phi\Big(\frac{|\varphi'(t)|}{\sigma'(t)}\Big)\,dt < \infty.$$

Corollary 11. *If X is a metric space and σ satisfies (17), then*

$$AC_\sigma(I;X) = \bigcup_{\Phi \in \mathcal{N}} BV_{\Phi,\sigma}(I;X).$$

Proof. The inclusion $\supset$ was obtained in Proposition 1(c). Let us show that for any $f \in AC_\sigma(I;X)$, there exists a function $\Phi \in \mathcal{N}$ depending on f such that $f \in BV_{\Phi,\sigma}(I;X)$. If $\varphi(t) = V_1(f,[a,t])$, $t \in I$, then $\varphi \in AC_\sigma(I;\mathbb{R})$ by Theorem 5(c), and since $\sigma \in C^1(I;\mathbb{R})$, we have $\varphi \in AC(I;\mathbb{R})$. Therefore, the derivative $\varphi' \in L^1(I;\mathbb{R})$. By Corollary 10, it suffices to prove that $\int_I \sigma'(t)\Phi(|\varphi'(t)|/\sigma'(t))\,dt < \infty$. To this end, consider the sets $J_n = \{t \in I \mid (n-1)\sigma'(t) < |\varphi'(t)| < n\sigma'(t)\}$, $n \in \mathbb{N}$. The sets J_n are pairwise disjoint, $\bigcup_{n=1}^\infty J_n = [a,b]$, and

$$\sum_{n=1}^\infty n \int_{J_n} \sigma'(t)\,dt \le \int_a^b |\varphi'(t)|\,dt + (\sigma(b) - \sigma(a)) < \infty.$$

Let $\{\rho_n\}_{n=1}^\infty$ be an increasing sequence of real numbers such that $\rho_1 \ge 1$, $\lim_{n\to\infty} \rho_n = \infty$, and

$$\sum_{n=1}^\infty \rho_n n \int_{J_n} \sigma'(t)\,dt < \infty. \tag{23}$$

Setting

$$\tilde{\Phi}(\tau) = \begin{cases} \tau & \text{if } 0 \le \tau < 1, \\ \rho_n & \text{if } n \le \tau < n+1,\ n \in \mathbb{N}, \end{cases} \qquad 0 \le \tau < \infty,$$

and $\Phi(\rho) = \int_0^\rho \tilde{\Phi}(\tau)\,d\tau$, $\rho \ge 0$, we find that $\Phi \in \mathcal{N}$ (and moreover, $\lim\limits_{\rho\to 0} \Phi(\rho)/\rho = 0$). Since $\Phi(n) = \int_0^n \tilde{\Phi}(\tau)\,d\tau \le \rho_n \cdot n$, we have by (23) that

$$\int_a^b \sigma'(t)\Phi\Big(\frac{|\varphi'(t)|}{\sigma'(t)}\Big)\,dt = \sum_{n=1}^\infty \int_{J_n} \sigma'(t)\Phi\Big(\frac{|\varphi'(t)|}{\sigma'(t)}\Big)\,dt$$
$$\le \sum_{n=1}^\infty \Phi(n) \int_{J_n} \sigma'(t)\,dt \le \sum_{n=1}^\infty \rho_n n \int_{J_n} \sigma'(t)\,dt < \infty,$$

which was to be proved.

8. (Φ, σ)-Variation of a Smooth Mapping

Theorem 12. *Assume that X is a (not necessarily complete) normed vector space with the norm $\|\cdot\|$, $\Phi \in \mathcal{M}$, and σ satisfies (17). Then, for any $f \in C^1(I; X)$, formula (19) holds.*

Proof. 1. To begin with, assume that $\Phi \in \mathcal{N}$. By Theorem 5(b), we know that $\mathsf{V}_{\Phi,\sigma}(f) = \mathsf{V}_{\Phi,\sigma}(\varphi)$, where $\varphi(t) = \mathsf{V}_1(f, [a,t])$, $t \in I$. From [2], Theorem 8.7(b), it follows that $\varphi(t) = \int_a^t \|f'(\tau)\|\, d\tau$ for all $t \in I$. Since $\mathbb{R}$ is a reflexive Banach space, formula (19) yields

$$\mathsf{V}_{\Phi,\sigma}(f) = \mathsf{V}_{\Phi,\sigma}(\varphi) = \int_I \sigma'(t)\Phi\Big(\frac{|\varphi'(t)|}{\sigma'(t)}\Big)\, dt = \int_I \sigma'(t)\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\, dt.$$

However, the general case, $\Phi \in \mathcal{M}$, ought to be considered separately.

2. If X is a Banach space, then the calculations in step 2 of the proof of Theorem 7 imply that

$$\mathsf{V}_{\Phi,\sigma}(f, I) \le \int_I \sigma'(t)\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\, dt. \tag{24}$$

Here the completeness of X was used for the existence of the X-valued integral $\int_{t_{i-1}}^{t_i} f'(t)\, dt$. If X is not complete, we embed X into its completion and note that the norms of elements of X evaluated in X and in the completion of X are the same. This proves that (24) is also valid without the completeness of X.

The converse inequality will immediately follow from (20) if we show that

$$\lim_{h\to+0} \int_a^{b-h} \frac{\sigma(t+h)-\sigma(t)}{h}\, \Phi\Big(\frac{\|f(t+h)-f(t)\|}{\sigma(t+h)-\sigma(t)}\Big)\, dt = \int_a^b \sigma'(t)\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\, dt.$$

We set $\Delta_h f(t) = (f(t+h) - f(t))/h$ and $\Delta_h\sigma(t) = (\sigma(t+h) - \sigma(t))/h$. We have

$$\begin{aligned} &\Big|\Delta_h\sigma(t)\cdot\Phi\Big(\frac{\Delta_h f(t)}{\Delta_h\sigma(t)}\Big) - \sigma'(t)\cdot\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\Big| \\ &\le |\Delta_h\sigma(t)|\cdot\Big|\Phi\Big(\frac{\Delta_h f(t)}{\Delta_h\sigma(t)}\Big) - \Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\Big| + |\Delta_h\sigma(t) - \sigma'(t)|\cdot\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big). \end{aligned}$$

Hence,

$$\begin{aligned} &\Big|\int_a^{b-h} \Delta_h\sigma(t)\cdot\Phi\Big(\frac{\Delta_h f(t)}{\Delta_h\sigma(t)}\Big)\, dt - \int_a^b \sigma'(t)\cdot\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\, dt\Big| \\ &\qquad\le \int_a^{b-h} |\Delta_h\sigma(t)|\cdot\Big|\Phi\Big(\frac{\Delta_h f(t)}{\Delta_h\sigma(t)}\Big) - \Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\Big|\, dt \\ &\qquad\qquad + \int_a^{b-h} |\Delta_h\sigma(t) - \sigma'(t)|\cdot\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\, dt + \int_{b-h}^b \sigma'(t)\Phi\Big(\frac{\|f'(t)\|}{\sigma'(t)}\Big)\, dt. \end{aligned}$$

The three integrals on the right-hand side tend to zero as $h \to +0$.

9. Regular Selections of Set-Valued Mappings

Now we turn to the existence of regular selections of set-valued mappings. First, a few definitions are in order.

If A and B are nonempty subsets of a metric space (X, d), the *excess of A over B* is defined by

$$\mathrm{e}(A, B) = \sup_{x \in A} \mathrm{dist}(x, B), \qquad \text{where} \quad \mathrm{dist}(x, B) = \inf_{y \in B} d(x, y),$$

and the *Hausdorff distance between A and B* is defined by

$$D(A, B) = \max\{\mathrm{e}(A, B), \mathrm{e}(B, A)\}.$$

The mapping D is a metric (called the *Hausdorff metric*) on the set of all nonempty closed bounded (and, in particular, compact) subsets of X.

Given $I = [a, b]$, a *set-valued mapping from I into X* is a mapping $F : I \to 2^X$, where 2^X is the class of all subsets of X, such that $F(t) \subset X$ for all $t \in I$. The set $\mathrm{Gr}(F) = \{(t, x) \in I \times X \mid x \in F(t)\}$ is called the *graph of F* and the set $\mathrm{R}(F) = \bigcup_{t \in I} F(t)$ is called the *range of F*.

If a set-valued mapping $F : I \to \dot{2}^X = 2^X \setminus \{\varnothing\}$ has closed bounded or compact images $F(t)$ for all $t \in I$, then, using the Hausdorff metric D, we can introduce the notions of set-valued mappings *of bounded (Φ, σ)-variation*, *σ-absolutely continuous* set-valued mappings, and *σ-Lipschitz* set-valued mappings ($\Phi \in \mathcal{M}$ and σ satisfies (17)) in a similar manner as was previously done for metric-space valued mappings. The respective classes of set-valued mappings will be denoted by $BV_{\Phi,\sigma}(I; \dot{2}^X)$, $AC_\sigma(I; \dot{2}^X)$, and $C^{0,1}_\sigma(I; \dot{2}^X) = C^{0,1}(I; \dot{2}^X)$. The total (Φ, σ)-variation of $F : I \to \dot{2}^X$ will still be denoted by $\mathsf{V}_{\Phi,\sigma}(F, I)$ and the *Lipschitz constant* of F by $\mathrm{Lip}(F)$. In view of Corollary 11, σ-absolutely continuous mappings are of no interest any more, and hence, we do not consider them in the sequel.

By a *regular selection* of a set-valued mapping $F : I \to \dot{2}^X$ we mean a (single-valued) mapping $f : I \to X$ such that $f(t) \in F(t)$ for all $t \in I$. Moreover, the mapping f should have the same "regularity" properties (relative to the variation) as the initial set-valued mapping F—this is made precise in the following theorem:

Theorem 13 (existence of regular selections). *Assume that X is a Banach space with the norm $\|\cdot\|$, $\Phi \in \mathcal{M}$, and σ satisfies (17). If the graph $\mathrm{Gr}(F)$ of the set-valued mapping $F \in BV_{\Phi,\sigma}(I; \dot{2}^X)$ is compact (and hence, the images $F(t)$ are compact subsets of X for all $t \in I$), then, for any $t_0 \in I$ and $x_0 \in F(t_0)$, there exists a mapping $f \in BV_{\Phi,\sigma}(I; X)$, a regular selection of F such that $f(t) \in F(t)$ at all points $t \in I$ where F is continuous (the set of these points is at most countable), $f(t_0) = x_0$, $\mathsf{V}_{\Phi,\sigma}(f, I) \le \mathsf{V}_{\Phi,\sigma}(F, I)$ and $\mathsf{V}_1(f, I) \le \mathsf{V}_1(F, I)$.*

Moreover, if F is continuous or $\Phi \in \mathcal{N}$, then the selection f is continuous as well and $f(t) \in F(t)$ for all $t \in I$.

Proof. 1. For each $n \in \mathbb{N}$, let $T_n = \{t_i^n\}_{i=0}^n$ be a partition of the closed interval $I = [a, b]$ (i.e., $a = t_0^n < t_1^n < \ldots < t_{n-1}^n < t_n^n = b$) with the properties

(i) $t_0 \in T_n$, i.e., $t_0 = t_{k(n)}^n$ for some $k(n) \in \{0, 1, \ldots, n\}$;

(ii) if $\lambda(T_n) = \max_{1 \le i \le n}(t_i^n - t_{i-1}^n)$, then $\lim_{n \to \infty} \lambda(T_n) = 0$.

First we define elements $x_i^n \in F(t_i^n)$, $n \in \mathbb{N}$, $i = 0, 1, \ldots, n$, inductively as follows. To begin with, assume that $n \in \mathbb{N}$ and $a < t_0 < b$.

(a) Put $x_{k(n)}^n = x_0$.

(b) If $i \in \{1, \ldots, k(n)\}$ and if $x_i^n \in F(t_i^n)$ is already chosen, pick an element $x_{i-1}^n \in F(t_{i-1}^n)$ such that $\|x_i^n - x_{i-1}^n\| = \mathrm{dist}(x_i^n, F(t_{i-1}^n))$.

(c) If $i \in \{k(n) + 1, \ldots, n\}$ and if $x_{i-1}^n \in F(t_{i-1}^n)$ is already chosen, pick an element $x_i^n \in F(t_i^n)$ such that $\|x_{i-1}^n - x_i^n\| = \mathrm{dist}(x_{i-1}^n, F(t_i^n))$.

Now, if $t_0 = a$, so that $k(n) = 0$, then we use only (a) and (c) to define x_i^n, and if $t_0 = b$, so that $k(n) = n$, then we define x_i^n according to (a) and (b).

We define a sequence of mappings $f_n : I \to X$, $n \in \mathbb{N}$, as follows:

$$f_n(t) = x_{i-1}^n + \frac{\sigma(t) - \sigma(t_{i-1}^n)}{\sigma(t_i^n) - \sigma(t_{i-1}^n)}(x_i^n - x_{i-1}^n), \quad t \in [t_{i-1}^n, t_i^n], \quad i = 1, \dots, n. \tag{25}$$

Note that $f_n(t_i^n) = x_i^n$, $f_n(t_{i-1}^n) = x_{i-1}^n$, and, in particular, $f_n(t_0) = x_0$ for all $n \in \mathbb{N}$. Note also that from (b) and (c) and the definition of D, we have

$$\|x_i^n - x_{i-1}^n\| \le D(F(t_i^n), F(t_{i-1}^n)), \qquad n \in \mathbb{N}, \quad i = 1, \dots, n. \tag{26}$$

All mappings $f_n : I \to X$ are continuous, and the restriction of f_n to every closed interval $[t_{i-1}^n, t_i^n]$ is continuously differentiable (see (17)). Taking into account that

$$f_n'(t) = \frac{\sigma'(t)}{\sigma(t_i^n) - \sigma(t_{i-1}^n)}(x_i^n - x_{i-1}^n) \qquad \text{if} \qquad t_{i-1}^n \le t \le t_i^n,$$

and applying Proposition 3(b), Theorem 12, and inequality (26), we find that

$$\begin{aligned}
\mathsf{V}_{\Phi,\sigma}(f_n, I) &= \sum_{i=1}^n \mathsf{V}_{\Phi,\sigma}(f_n, [t_{i-1}^n, t_i^n]) = \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \sigma'(t)\Phi\Big(\frac{\|f_n'(t)\|}{\sigma'(t)}\Big)\,dt \\
&= \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \sigma'(t)\Phi\Big(\frac{\|x_i^n - x_{i-1}^n\|}{\sigma(t_i^n) - \sigma(t_{i-1}^n)}\Big)\,dt \\
&= \sum_{i=1}^n \Phi\Big(\frac{\|x_i^n - x_{i-1}^n\|}{\sigma(t_i^n) - \sigma(t_{i-1}^n)}\Big)(\sigma(t_i^n) - \sigma(t_{i-1}^n)) \\
&\le \sum_{i=1}^n \Phi\Big(\frac{D(F(t_i^n), F(t_{i-1}^n))}{\sigma(t_i^n) - \sigma(t_{i-1}^n)}\Big)(\sigma(t_i^n) - \sigma(t_{i-1}^n)) \\
&= V_{\Phi,\sigma}[F, T_n] \le \mathsf{V}_{\Phi,\sigma}(F, I) < \infty \qquad \forall n \in \mathbb{N}.
\end{aligned} \tag{27}$$

By Proposition 1(b), the mapping F is of bounded Jordan variation, and hence, the calculations above with $\Phi(\rho) = \rho$ and $\sigma(t) = t$ also provide the following estimate:

$$\mathsf{V}_1(f_n, I) \le \mathsf{V}_1(F, I) \quad \text{for all} \quad n \in \mathbb{N}. \tag{28}$$

2. Assume that $\Phi \in \mathcal{N}$. Let us show that the sequence $\{f_n(t)\}_{n=1}^\infty$ is precompact in X for all $t \in I$. To this end, fix $t \in I$. For any $n \in \mathbb{N}$, there exists a number $i(n) \in \{1, \dots, n\}$ depending also on t such that $t_{i(n)-1}^n \le t \le t_{i(n)}^n$. Condition (ii) above implies that the sequences $t_{i(n)-1}^n$ and $t_{i(n)}^n$ tend to t as $n \to \infty$. From (25), (26), and the (absolute) continuity of F, we have

$$\begin{aligned}
\|f_n(t) - x_{i(n)}^n\| &= \frac{\sigma(t_{i(n)}^n) - \sigma(t)}{\sigma(t_{i(n)}^n) - \sigma(t_{i(n)-1}^n)}\|x_{i(n)}^n - x_{i(n)-1}^n\| \\
&\le D(F(t_{i(n)}^n), F(t_{i(n)-1}^n)) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{29}$$

Since the graph of F is compact and $(t_{i(n)}^n, x_{i(n)}^n) \in \mathrm{Gr}(F)$, there exists a subsequence of $\{(t_{i(n)}^n, x_{i(n)}^n)\}_{n=1}^\infty$ (which will be denoted by the same symbol as the sequence itself) that converges to a point $(\tau, x) \in \mathrm{Gr}(F)$ as $n \to \infty$. From $\lim_{n\to\infty} t_{i(n)}^n = t$, it follows that $\tau = t$, so that $x \in F(t)$. At the same time, $\lim_{n\to\infty} x_{i(n)}^n = x$

in X. Relation (29) now implies that $\lim_{n\to\infty} f_n(t) = x$ in X, where $x \in F(t)$; this proves the precompactness of the sequence $\{f_n(t)\}_{n=1}^{\infty}$.

Now we can apply a version of the selection principle (Theorem 6) with $\Phi \in \mathcal{N}$: there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ (which will be denoted by the same symbol as well) which, uniformly on I, converges to a mapping $f \in BV_{\Phi,\sigma}(I;X)$. Clearly, $f(t_0) = x_0$, and Proposition 3(c), (27), and (28) yield

$$\mathsf{V}_{\Phi,\sigma}(f,I) \le \mathsf{V}_{\Phi,\sigma}(F,I) \qquad \text{and} \qquad \mathsf{V}_1(f,I) \le \mathsf{V}_1(F,I). \tag{30}$$

It remains to show that $f(t) \in F(t)$ for all $t \in I$. For a fixed t, from the argument on the precompactness above we have that $\exists\, x \in F(t)$ such that $\lim_{n\to\infty} f_n(t) = x$ in X. From the definition of f, we find that $\lim_{n\to\infty} f_n(t) = f(t)$ in X, so that $f(t) = x \in F(t)$. Therefore, we are through with the case $\Phi \in \mathcal{N}$.

3. Assume that $\Phi \in \mathcal{M}$ and $F \in BV_{\Phi,\sigma}(I;\dot{2}^X)$ is continuous. We are going to apply the following result due to Mordukhovich [12], Supplement, Theorem 1.8: if, under the conditions of Theorem 13, the set-valued mapping $G = F \in C^{0,1}(I;\dot{2}^X)$, then $\exists\, g \in C^{0,1}(I;X)$ such that $g(t) \in G(t)$ for all $t \in I$, $g(t_0) = x_0$, and $\mathrm{Lip}(g) \le \mathrm{Lip}(G)$.

According to Theorem 5(b), F can be decomposed as $F = G \circ \varphi$, where the continuous function $\varphi(t) = \mathsf{V}_1(F,[a,t])$, $t \in I$, belongs to $BV_{\Phi,\sigma}(I;\mathbb{R})$, the set-valued mapping $G : J = \varphi(I) \to \dot{2}^X$ belongs to $C^{0,1}(J;\dot{2}^X)$, $\mathrm{Lip}(G) \le 1$, and $\mathsf{V}_{\Phi,\sigma}(\varphi,I) = \mathsf{V}_{\Phi,\sigma}(F,I)$. Since the graph of F is compact, the graph of G is compact as well. Noting that $x_0 \in F(t_0) = G(\tau_0)$, where $\tau_0 = \varphi(t_0)$, we can apply the result cited above: $\exists\, g \in C^{0,1}(J;X)$ such that $g(\tau_0) = x_0$, $g(\tau) \in G(\tau)$ for all $\tau \in J$ and $\mathrm{Lip}(g) \le \mathrm{Lip}(G) \le 1$. Now, set $f = g \circ \varphi$. Lemma 4(b) gives that $f \in BV_{\Phi,\sigma}(I;X)$ is continuous and

$$\mathsf{V}_{\Phi,\sigma}(f,I) \le \mathsf{V}_{\Phi,\sigma}(\varphi,I) = \mathsf{V}_{\Phi,\sigma}(F,I).$$

Similarly, $\mathsf{V}_1(f,I) \le \mathsf{V}_1(F,I)$. Finally, we have $f(t_0) = g(\varphi(t_0)) = g(\tau_0) = x_0$ and $f(t) = g(\varphi(t)) \in G(\varphi(t)) = F(t)$ for all $t \in I$.

4. Consider the general case where $\Phi \in \mathcal{M}$ and $F \in BV_{\Phi,\sigma}(I;\dot{2}^X)$. We start by arguing as in step 1 down to inequality (28). It is seen from (25) that all the images $f_n(I)$ are contained in the closed convex hull $\overline{\mathrm{co}}\,\mathrm{R}(F)$ of the range $\mathrm{R}(F)$, and since the graph $\mathrm{Gr}(F)$ is compact in $I \times X$, we have that $\mathrm{R}(F)$ is compact in X, and hence, by Lemma 6.2 in [4], the set $\overline{\mathrm{co}}\,\mathrm{R}(F)$ is compact in X as well. Applying the selection principle (Theorem 6) to the sequence $\mathcal{F} = \{f_n\}_{n=1}^{\infty}$, we find a subsequence of $\{f_n\}_{n=1}^{\infty}$ (denoted by the same symbol) which converges pointwise on I to a mapping $f \in BV_{\Phi,\sigma}(I;X)$. Clearly, $f(t_0) = x_0$, and inequalities (30) hold. If $t \in I$ is a point of continuity of F, then $f(t) \in F(t)$: in fact, by (29), from the precompactness argument of step 2 we have $\lim_{n\to\infty} f_n(t) = x \in F(t)$, and $\lim_{n\to\infty} f_n(t) = f(t)$ holds by the construction. We conclude that $f(t) = x \in F(t)$. This ends the proof.

10. Concluding Remarks

Theorem 13 extends the existence results for selections of non-convex valued set-valued mappings of bounded Jordan variation presented in [8, 9] and [15] in the context of a finite-dimensional space X (obtaining only continuous selections) and in [2, 3] ($\Phi(\rho) = \rho$, $\sigma(t) = t$), [4] ($\Phi(\rho) = \rho^q$, $q > 1$, $\sigma(t) = t$), [5] ($\Phi \in \mathcal{N}$, $\sigma(t) = t$), and [6] ($\Phi \in \mathcal{M}$, $\sigma(t) = t$), where X is a general Banach space. Theorem 13 can be generalized to the case where I is a bounded or unbounded, open or half-open interval of the real line $\mathbb{R}$.

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