ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 22, No. 1, 2015, pp. 20–25. © Pleiades Publishing, Ltd., 2015.

Adiabatics Using Phase Space Translations and Small Parameter "Dynamics"

M. Karasev
Department of Applied Mathematics, National Research University "Higher School of Economics", Myasnitskaya str. 20, 101978, Moscow, Russia
E-mail: mkarasev@hse.ru

Received January 23, 2015

Abstract. For slow–fast Hamiltonian systems with one fast degree of freedom, we describe the construction of the complete adiabatic invariant and the complete adiabatic term at once in all asymptotic orders by using the small parameter "dynamics" and parallel translations in the phase space.

DOI 10.1134/S1061920815010045

1. INTRODUCTION

In the well-developed and widely known asymptotic analysis of slow–fast Hamiltonian systems (see, e.g., [1–9]), there are still interesting open questions. In this paper, we deal with some of these questions in the simplest case of 2-dimensional fast fibers with nondegenerate elliptic-type dynamics along them.

The first question: *Is it possible to separate slow and fast dynamics in all asymptotical orders at once* (*not by successive recomputations in each order anew*)?

The second question: *Is it possible to make such a separation geometrically, i.e., by a transformation of the phase space* (*without using any type of operation other than a change of variables*)?

Following the procedure from [10], we obtain positive answer to the first question by considering the small slow–fast asymptotic parameter as a "time" variable and by computing the dynamics in this "time" starting from the "initial data," where the slow evolution is just absent.

Such a parameter "dynamics" automatically generates the all-order expansions for the asymptotic integrals of motion (for the complete adiabatic invariant and for the complete adiabatic term). These objects are computed by a geometric transformation from the values at the freezed slow space point.

In addition to this first transformation, it is possible to introduce a second one, close to the identity, pulling the obtained complete adiabatic term back to the zero-order term by changing the action variable. The composition of these two transformations provides the positive answer to the second question above.

After applying the composed phase space transformation, we obtain the new fast and slow variables in all asymptotic orders at once. In the new slow variables, the Hamiltonian time-dynamics does not depend on the fast dynamics. The corresponding slow trajectories present the guiding center lines around which the "slow" part of the original Hamiltonian trajectories is gyrating. At the first asymptotic order, the gyration is determined by a connection with a Berry-type curvature. The knowledge of the guiding dynamics also allows one to compute the dynamics along the fast fibers, in particular, to determine the Hannay-type angle, in all asymptotic orders at once. This part of our program will be realized in the next paper.

2. INTEGRABILITY VIA FREEZING SLOW VARIABLES AT A FIXED POINT

Let us consider the Hamiltonian $\mathcal{H} = \mathcal{H}(y; x)$ depending on two independent groups of variables:

$$y \in \mathcal{F} \subset \mathbb{R}^{2N}, \qquad x \in \mathcal{D} \subset \mathcal{R}^{2n}, \tag{2.1}$$

with Poisson brackets of two types

$$\{y^\alpha, y^\beta\}_0 = \mathcal{J}^{\alpha\beta}, \qquad \{x^j, x^k\} = \varepsilon J^{jk}, \tag{2.2}$$

This research was supported by the HSE Academic Foundation.

Here $\mathcal{J}$ and J are skew symmetric matrices of the form $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ but of different dimension $2N$ or $2n$.

The parameter ε in (2.2) is assumed to be small $\varepsilon \to 0$; thus the coordinates x^j can be referred to as "slow" ones, and the coordinates y^α are "fast."

The problem is to analyze the Hamiltonian system generated by $\mathcal{H}$ on the phase space $\mathcal{F} \times \mathcal{D}$ with the direct sum of Poisson structures

$$\{\cdot\,,\cdot\}_\varepsilon \stackrel{\text{def}}{=} \{\cdot\,,\cdot\} + \varepsilon\{\cdot\,,\cdot\}. \tag{2.3}$$

Below we deal only with case $N = 1$ for simplicity. A more general situation can be considered as well, following [7, 10].

Assume that the Hamiltonian $\mathcal{H} \in C^\infty(\mathcal{F} \times \mathcal{D})$ is real and, for any fixed x, the energy levels of $H(\cdot; x)$ are connected closed nondegenerate curves in $\mathcal{F}$. Then the integrals

$$\frac{1}{4\pi} \oint y\mathcal{J}\,dy \stackrel{\text{def}}{=} \mathcal{S}_0$$

along these energy levels determine the family of action functions $\mathcal{S}_0(\cdot; x)$ smoothly depending on $x \in \mathcal{D}$. The Hamiltonian is represented via the actions as follows:

$$\mathcal{H}(y, x) = f_0(\mathcal{S}_0(y, x); x), \qquad y \in \mathcal{F}, \quad x \in \mathcal{D}. \tag{2.4}$$

Here $f_0 = f_0(s; x)$ is a smooth function on $\mathcal{A} \times \mathcal{D}$, where $\mathcal{A}$ is an interval in $\mathbb{R}$. We refer to this function as to the *zero adiabatic term*.

The Hamiltonian (2.4) is not integrable with respect to brackets (2.3) (except the case $\varepsilon = 0$). The obstruction to integrability is, of course, the dependence of the action $\mathcal{S}_0(y; x)$ on x-variables.

Let us try to "freeze" this x-dependence in the action. We choose any fixed point $\underline{x} \in \mathcal{F}$, and construct a freezing transformation g_0 on $\mathcal{F} \times \mathcal{D}$ in such a way that

$$g_0 \text{ is canonical along fast fiber with brackets } \{\cdot\,,\cdot\}_0, \tag{2.5}$$

$$\mathcal{S}_0 = g_0^* \underline{\mathcal{S}}_0, \qquad \text{where } \underline{\mathcal{S}}_0(y) \stackrel{\text{def}}{=} \mathcal{S}_0(y; \underline{x}) \text{ is } x\text{-independent}, \tag{2.6}$$

$$g_0^* x^j = x^j \qquad \forall j. \tag{2.7}$$

Then we extend g_0 by an ε-jet[1]of transformations g_ε in such a way that

$$g_\varepsilon \text{ is canonical with respect to } \{\cdot\,,\cdot\}_\varepsilon, \tag{2.8}$$

$$\mathcal{H} = f_\varepsilon(\mathcal{S}_\varepsilon, \mathcal{X}_\varepsilon), \tag{2.9}$$

where

$$\mathcal{S}_\varepsilon \stackrel{\text{def}}{=} g_\varepsilon^* \underline{\mathcal{S}}_0, \qquad \mathcal{X}_\varepsilon^j \stackrel{\text{def}}{=} g_\varepsilon^* x^j \qquad (j = 1, \dots, 2n) \tag{2.10}$$

and f_ε is an ε-jet extending f_0.

It follows from (2.7) that

$$\mathcal{X}_0^j(y; x) = x^j \qquad (j = 1, \dots, 2n). \tag{2.11}$$

Since $\underline{\mathcal{S}}_0$ and x^j are in involution with respect to brackets (2.3) (because of freezing x-dependence in $\underline{\mathcal{S}}_0$), we have the same property for $\mathcal{S}_\varepsilon$ and $\mathcal{X}_\varepsilon^j$, i.e., $\{\mathcal{S}_\varepsilon, \mathcal{X}_\varepsilon^j\}_\varepsilon = 0$. Thus,

$$\{\mathcal{H}, \mathcal{S}_\varepsilon\}_\varepsilon = 0, \qquad \{\mathcal{X}_\varepsilon^j, \mathcal{X}_\varepsilon^k\}_\varepsilon = \varepsilon J^{jk}. \tag{2.12}$$

This means that $\mathcal{S}_\varepsilon$ is the integral of motion for the Hamiltonian system up to $O(\varepsilon^\infty)$. We call $\mathcal{S}_\varepsilon$ the *complete adiabatic invariant*. It follows from (2.10) that $\mathcal{S}_\varepsilon$ is the action in the fast variables $\mathcal{Y}_\varepsilon^\alpha \stackrel{\text{def}}{=} g_\varepsilon^* y^\alpha$, since

$$\mathcal{S}_\varepsilon = \mathcal{S}_0(\mathcal{Y}_\varepsilon, \underline{x}). \tag{2.13}$$

Moreover, equations (2.9) and (2.12) mean that f_ε is the *complete adiabatic term* in the canonical slow variables $\mathcal{X}_\varepsilon^j$.

Now we describe how to realize this approach by using parallel translations in the phase space and ε-dynamics.

[1]Formal power series in ε with smooth coefficients over the phase space $\mathcal{F} \times \mathcal{D}$ will be referred to as *ε-jets*. All equations below are generally considered in the sense of ε-jets. If these ε-jets happen to be smooth functions in ε near zero, then this means that we are in the integrable situation, see (2.12) below. Of course, this can happen but it is not in a generic case.

3. CONSTRUCTION OF THE FREEZING TRANSFORMATION

The freezing transform (2.5)–(2.7) is constructed as

$$g_0 : (y;x) \to (\mathcal{Y}_0(y;x);x), \tag{3.1}$$

where $\mathcal{Y}_0(\cdot;x)$ is a canonical map with respect to $\{\cdot\,,\cdot\}_0$ for each given x. One can construct $\mathcal{Y}_0$ as the solution of the "evolution" equations

$$D_j\mathcal{Y}_0^\alpha + \{\mathcal{A}_{0j}, \mathcal{Y}_0^\alpha\}_0 = 0, \qquad j = 1,\dots,2n. \tag{3.2}$$

Here we denote $D_j = \partial/\partial x^j$, the slow coordinates x^j play the role of "time," and $\mathcal{A}_{0j}$ play the role of "Hamiltonians."

The Hamiltonian form of (3.2) guarantees that $\mathcal{Y}_0(\cdot;x)$ is the canonical map along fast fibers.

The property (2.6) follows from (3.2) and from the "initial data"

$$\mathcal{Y}_0\big|_{x=\underline{x}} = \mathrm{id} \tag{3.3}$$

if the "Hamiltonians" $\mathcal{A}_{0j}$ obey the equations

$$D_j\mathcal{S}_0 = \{\mathcal{S}_0, \mathcal{A}_{0j}\}_0, \qquad j = 1,\dots,2n. \tag{3.4}$$

Note that the solvability condition for system (3.2) or for system (3.4) is the zero-curvature condition for the connection 1-form $\mathcal{A}_0 = \sum \mathcal{A}_{0j}\,dx^j$, namely,

$$D_j\mathcal{A}_{0k} - D_k\mathcal{A}_{0j} + \{\mathcal{A}_{0j}, \mathcal{A}_{0k}\} = 0, \qquad j,k = 1,\dots,2n. \tag{3.5}$$

Thus, to construct the freezing map g_0, one needs the solution of (3.4), (3.5). This solution can be obtained as the sum

$$\mathcal{A}_0 = \tilde{\mathcal{A}}_0 + a_0(\mathcal{S}_0;x). \tag{3.6}$$

The first summand $\tilde{\mathcal{A}}_0$ obeys (3.4) and has zero average with respect to the $\mathcal{S}_0$-flow. The second summand a_0 in (3.6) is chosen to cancel the curvature generated by $\tilde{\mathcal{A}}_0$.

The precise formula for $\tilde{\mathcal{A}}_0$ is the following one:

$$\tilde{\mathcal{A}}_0(y;x) = \frac{1}{2\pi}\int_0^{2\pi} D\mathcal{S}_0(Y^\tau(y;x);x)(\tau-\pi)\,d\tau, \tag{3.7}$$

where $Y^\tau(\cdot;x)$ is the flow of the action $\mathcal{S}_0(\cdot;x)$ on the fast fiber.

The connection $\tilde{\mathcal{A}}_0$ has the curvature

$$\tilde{C}_{0jk} \overset{\text{def}}{=} D_j\tilde{\mathcal{A}_{0k}} - D_k\tilde{\mathcal{A}_{0j}} + \{\tilde{\mathcal{A}_{0j}}, \tilde{\mathcal{A}_{0k}}\}_0. \tag{3.8}$$

It follows from $D\mathcal{S}_0 = \{\mathcal{S}_0, \tilde{\mathcal{A}}_0\}$ that this curvature is in involution with the action $\mathcal{S}_0$, and thus, it can be represented as a function in the action

$$\tilde{C}_{0jk} = \tilde{c}_{0jk}(\mathcal{S}_0;x). \tag{3.9}$$

The Bianchi identities for the curvature implies that the 2-form $\frac{1}{2}\sum \tilde{c}_{0jk}dx^j\wedge dx^k$ is closed.

Therefore, to satisfy (3.5), one can choose the 1-form $\sum a_{0j}dx^j$ to be the primitive, i.e.,

$$D_ja_{0k} - D_ka_{0j} = -\tilde{c}_{0jk} \tag{3.10}$$

(of course, assuming that the domain $\mathcal{D}$ is 2-connected).

Thus the final algorithm is the following one:

– to construct the connection (3.7),

– to calculate the curvature (3.8) and find $\tilde{c}_0$ by (3.9),

– to choose the primitive a_0 from (3.10),

– to determine the zero-curvature connection (3.6),

– to solve the parallel translation equation (3.2) with the condition (3.3),

– to define the freezing transform g_0 by (3.1).

4. ε-DYNAMICAL SYSTEM FOR THE CANONICAL TRANSFORMATION G_ε

Now we describe the algorithm for the derivation of the jet g_ε extending the freezing map g_0. First of all, we analyze the condition on g_ε to be canonical with respect to the brackets (2.3).

We shall use the notation $z = (y, x)$ for points from the total phase space $\mathcal{F} \times \mathcal{D}$. The Poisson structure (2.3) on this space is given (in Darboux coordinates) by the matrix

$$\Psi_\varepsilon = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & \varepsilon J \end{pmatrix}, \tag{4.1}$$

i.e., $\{L, M\}_\varepsilon = dL \cdot \Psi_\varepsilon \cdot dM$, where $d \stackrel{\text{def}}{=} (\partial, D)$, $\partial = \partial/\partial y$, $D = \partial/\partial x$.

The transformation

$$g_\varepsilon : z \to \mathcal{Z}_\varepsilon(z), \qquad \mathcal{Z}_\varepsilon = (\mathcal{Y}_\varepsilon, \mathcal{X}_\varepsilon), \tag{4.2}$$

of the phase space can be described by a differential equation in the parameter ε.

Theorem 4.1. *Let the map* (4.2) *be determined by the following ε-dynamical system*:

$$\frac{\partial}{\partial \varepsilon}\mathcal{Z}_\varepsilon = \Psi_\varepsilon d\underline{\mathcal{B}}_\varepsilon(\mathcal{Z}_\varepsilon) + v_\varepsilon(\mathcal{Z}_\varepsilon). \tag{4.3}$$

Here $\underline{\mathcal{B}}_\varepsilon$ is a scalar ε-jet over the phase space $\mathcal{F} \times \mathcal{D}$, and v_ε is determined by

$$v_\varepsilon \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ J\underline{\mathcal{A}}_\varepsilon \end{pmatrix} - \frac{1}{2}\sum_{j,k} \underline{\mathcal{A}}_{\varepsilon j} J^{jk} \Psi_\varepsilon \, d\underline{\mathcal{A}}_{\varepsilon k}, \tag{4.4}$$

where $\underline{\mathcal{A}}_\varepsilon$ is a 1-form ε-jet obeying the ε-dynamical equation

$$\frac{\partial}{\partial \varepsilon}\underline{\mathcal{A}}_\varepsilon = D\underline{\mathcal{B}}_\varepsilon - \{\underline{\mathcal{A}}_\varepsilon, \underline{\mathcal{B}}_\varepsilon\}_\varepsilon + \frac{1}{2}\sum_{j,k} \underline{\mathcal{A}}_{\varepsilon j} J^{jk} D_k \underline{\mathcal{A}}_\varepsilon, \tag{4.5}$$

and the zero-curvature condition

$$D_j \underline{\mathcal{A}}_{\varepsilon k} - D_k \underline{\mathcal{A}}_{\varepsilon j} - \{\underline{\mathcal{A}}_{\varepsilon j}, \underline{\mathcal{A}}_{\varepsilon k}\}_\varepsilon = 0, \qquad j, k = 1, \dots, 2n. \tag{4.6}$$

Then (4.2) *is canonical, i.e., it preserves the brackets* (2.3).

Moreover, the map (4.2) *satisfies the relation*

$$D \circ g_\varepsilon^* = g_\varepsilon^* \circ \big(D - \mathrm{ad}_\varepsilon(\underline{\mathcal{A}}_\varepsilon)\big), \tag{4.7}$$

where ad_ε is defined by the brackets operation:

$$\{L, M\}_\varepsilon \stackrel{\text{def}}{=} \mathrm{ad}_\varepsilon(L)M.$$

Remark 4.1. Of course, the zero-curvature condition (4.6) is equivalent to the commutativity of components of the vector-operator $D - \mathrm{ad}_\varepsilon(\underline{\mathcal{A}}_\varepsilon)$ in (4.7).

It easily follows from (4.5) that (4.6) holds at ε if it holds at $\varepsilon = 0$.

Relation (4.7) at $\varepsilon = 0$ reads

$$D \circ g_0^* = g_0^* \circ \big(D - \mathrm{ad}_0(\underline{\mathcal{A}}_0)\big). \tag{4.8}$$

If g_0 has the form (3.1), then (4.8) is equivalent to

$$D\mathcal{Y}_0 = \mathcal{J}\partial\underline{\mathcal{A}}_0(\mathcal{Y}_0). \tag{4.9}$$

Thus we need to define $\underline{\mathcal{A}}_0$ by the condition

$$\mathcal{A}_0 = g_0^* \underline{\mathcal{A}}_0 \quad \text{or} \quad \mathcal{A}_0(y; x) = \underline{\mathcal{A}}_0(\mathcal{Y}_0(y; x); x). \tag{4.10}$$

In this case, relation (4.9) coincides with (3.2) and g_0 coincides with the freezing map from Section 3.

The zero-curvature condition (4.6) for $\underline{\mathcal{A}}_0$ at $\varepsilon = 0$ then follows from the zero-curvature condition (3.5).

5. DERIVING AND SOLVING HOMOLOGICAL EQUATION

Note that we still do not know $\underline{\mathcal{B}}_\varepsilon$ in order to determine g^*_ε from (4.3), and we also do not know how to compute the complete adiabatic term f_ε in (2.9). All this is extracted from condition (2.9), as we shall explain now.

Equation (4.3) for g^*_ε can be rewritten as the operator permutation relation

$$\frac{\partial}{\partial\varepsilon} \circ g^*_\varepsilon = g^*_\varepsilon \circ \Big(\frac{\partial}{\partial\varepsilon} - \mathrm{ad}_\varepsilon(\underline{\mathcal{B}}_\varepsilon) + v_\varepsilon \cdot d\Big). \tag{5.1}$$

Condition (2.9) reads

$$\mathcal{H} = g^*_\varepsilon \underline{\mathcal{H}}_\varepsilon, \qquad \underline{\mathcal{H}}_\varepsilon(y;x) \stackrel{\text{def}}{=} f_\varepsilon(\underline{\mathcal{S}}_0(y);x). \tag{5.2}$$

Then by applying (5.1) to $\underline{\mathcal{H}}_\varepsilon$ and by using the obvious identity $\partial\mathcal{H}/\partial\varepsilon = 0$, one obtains

$$0 = \partial\underline{\mathcal{H}}_\varepsilon/\partial\varepsilon - \{\underline{\mathcal{B}}_\varepsilon, \underline{\mathcal{H}}_\varepsilon\}_\varepsilon + v_\varepsilon \cdot d\underline{\mathcal{H}}_\varepsilon.$$

Taking into account the structure of the Poisson brackets (2.3), we rewrite this equation as follows:

$$\{\underline{\mathcal{H}}_\varepsilon, \underline{\mathcal{B}}_\varepsilon\}_0 = \underline{\mathcal{K}}_\varepsilon - \partial\underline{\mathcal{H}}_\varepsilon/\partial\varepsilon, \tag{5.3}$$

where

$$\underline{\mathcal{K}}_\varepsilon \stackrel{\text{def}}{=} -v_\varepsilon \cdot d\underline{\mathcal{H}}_\varepsilon + \varepsilon\{\underline{\mathcal{B}}_\varepsilon, \underline{\mathcal{H}}_\varepsilon\}. \tag{5.4}$$

Note that $\mathrm{ad}_0(\underline{\mathcal{H}}_\varepsilon) = \omega_\varepsilon \cdot \mathrm{ad}_0(\underline{\mathcal{S}}_\varepsilon)$, where

$$\omega_\varepsilon(y;x) \stackrel{\text{def}}{=} \frac{\partial f_\varepsilon}{\partial s}(\underline{\mathcal{S}}_0(y);x). \tag{5.5}$$

The vector field $\mathrm{ad}_0(\underline{\mathcal{S}}_0)$ on the fast fiber over the fixed point $\underline{x}$ has 2π-periodic trajectories $\underline{Y}^\tau$. Thus the solvability condition for the homological equation (5.3) is

$$\partial\underline{\mathcal{H}}_\varepsilon/\partial\varepsilon = \langle\underline{\mathcal{K}}_\varepsilon\rangle, \tag{5.6}$$

where $\langle\dots\rangle$ is the averaging operation along the fast fiber over $\underline{x}$:

$$\langle M\rangle(y,x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} M(\underline{Y}^\tau(y);x)\,d\tau. \tag{5.7}$$

If (5.6) holds, then (5.3) can be resolved as

$$\underline{\mathcal{B}}_\varepsilon = \frac{1}{\omega_\varepsilon}\underline{\mathcal{K}}^\#_\varepsilon, \tag{5.8}$$

where the # operation is defined as in (3.7)

$$M^\#(y;x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} M(\underline{Y}^\tau(y);x)(\tau-\pi)\,d\tau. \tag{5.9}$$

Note that the function (5.8) has the zero average value: $\langle\underline{\mathcal{B}}_\varepsilon\rangle = 0$, and therefore, equation (5.6) reads

$$\frac{\partial}{\partial\varepsilon}\underline{\mathcal{H}}_\varepsilon + \langle v_\varepsilon \cdot d\underline{\mathcal{H}}_\varepsilon\rangle = 0. \tag{5.10}$$

This ε-dynamical equation is supplemented with the "initial data"

$$\underline{\mathcal{H}}_0(y;x) = f_0(\underline{\mathcal{S}}_0(y);x). \tag{5.11}$$

Equation (5.8) can be rewritten using (5.4) as follows:

$$\underline{\mathcal{B}}_\varepsilon = \frac{\varepsilon}{\omega_\varepsilon}\{\underline{\mathcal{B}}^\#_\varepsilon, \underline{\mathcal{H}}_\varepsilon\} - \frac{1}{\omega_\varepsilon}(v_\varepsilon \cdot d\underline{\mathcal{H}}_\varepsilon)^\#. \tag{5.12}$$

Condition (5.11) generated the "initial" data for $\underline{\mathcal{B}}_\varepsilon$ from (5.12),

$$\underline{\mathcal{B}}_0 = \frac{1}{\omega_0}\sum_{j,k}\big(\underline{\mathcal{A}}_{0j} J^{jk}(D_k - \frac{1}{2}\mathrm{ad}_0(\underline{\mathcal{A}}_{0k}))\underline{\mathcal{H}}_0\big)^\#. \tag{5.13}$$

Equation (5.12) together with (5.13) can be regarded as an evolution system in the ε-variable, since it allows us to compute $\underline{\mathcal{B}}_\varepsilon$ at each next order in ε from knowing it at previous orders.

Thus by solving (5.10)–(5.11) and (5.12)–(5.13), we reconstruct $\underline{\mathcal{B}}_\varepsilon$ in (4.3) (and so, g_ε) together with $\underline{\mathcal{H}}_\varepsilon$ (and so, f_ε) in (5.2), (2.9), (2.10). This finishes the construction of the complete adiabatic invariant and the complete adiabatic term described in Section 2.

6. CONCLUSION

Let us summarize our final algorithm. In order to compute in all ε-orders the complete adiabatic term f_ε, we consider the ε-dynamical system consisting of three evolution equations and their "initial" data:

$$
\begin{aligned}
&\text{— (4.5) for } \underline{\mathcal{A}}_\varepsilon, \quad \text{(4.10) for } \underline{\mathcal{A}}_0,\\
&\text{— (5.12) for } \underline{\mathcal{B}}_\varepsilon, \quad \text{(5.13) for } \underline{\mathcal{B}}_0,\\
&\text{— (5.10) for } \underline{\mathcal{H}}_\varepsilon \quad \text{or} \quad f_\varepsilon, \text{(5.11) for } \underline{\mathcal{H}}_0.
\end{aligned}
$$

In order to compute the formal integral of motion (the complete adiabatic invariant) $\mathcal{S}_\varepsilon$ in all ε-orders, we consider the evolution equation and the "initial data"

$$
\text{— (4.3) for } \mathcal{Z}_\varepsilon = (\mathcal{Y}_\varepsilon; \mathcal{X}_\varepsilon), \qquad \text{(2.11) for } \mathcal{X}_0, \quad \text{(3.2), (3.3) for } \mathcal{Y}_0
$$

and then use formula (2.13) for $\mathcal{S}_\varepsilon$.

REFERENCES

1. V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Nauka, Moscow, 1989).
2. A. I. Neishtadt, "The Separation of Motions in Systems with Rapidly Rotating Phase," J. Appl. Math. Mech. **48** (2), 133–139 (1984).
3. A. I. Neishtadt, "Averaging Method and Adiabatic Invariants," in Hamiltonian Dynamical Systems and Applications, Ed. by W. Craig (Springer Verlag, 2008), pp. 53–66.
4. D. V. Treschev, "The Continuous Averaging Method in the Problem of Separation of Fast and Slow Motions," Regul. Chaotic Dyn. **2** (3/4), 9–20 (1997).
5. M. V. Berry, "Classical Adiabatic Angles and Quantal Adiabatic Phase," J. Phys. A: Math. Gen. **18**, 15–27 (1985).
6. J. H. Hannay, "Angle Variable Holonomy in Adiabatic Excursion of an Integrable Hamiltonian," J. Phys. A: Math. Gen. **18**, 221–230 (1985).
7. R. Montgomery, "The Connection Whose Holonomy Is the Classical Adiabatic Angles of Hannay and Berry and Its Generalization to the Non-Integrable Case," Comm. Math. Phys. **120**, 269–294 (1988).
8. J. E. Marsden, R. Montgomery, and T. Ratiu, "Reduction, Symmetry and Phases in Mechanics," Mem. Amer. Math. Soc. **88** (436), 1–110 (1990).
9. M. Avendano-Camacho, J. A. Vallejo, and Yu. Vorobiev, "Higher Order Corrections to Adiabatic Invariants of Generalized Slow-Fast Hamiltonian Systems," J. Math. Phys. **54**, 082704 (2013).
10. M. V. Karasev, "Adiabatic Approximation via Hodograph Translation and Zero-Curvature Equations," Russ. J. Math. Phys. **21** (2), 197–218 (2014).