A characterization of the egalitarian solution set for ordinally convex NTU games

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1 Introduction

For the class of ordinally convex NTU games with nonlevel characteristic function sets the ESOS set consists of the unique payoff vector belonging to the core [**?**] The axioms are the following: the consistency à la Hart–Mas-Colell and the definition of the egalitarian solution for the class of two-person games. Such system of axioms is generally accepted. However, in the most similar characterizations the solution for two-person games has some axiomatization itself.

The goal of this paper is to give an axiomatic characterization of the egalitarian bargaining solution for bargaining problems whose weak Pareto boundary of the individual rational set coincides with its Pareto boundary. Since two-person bargaining problems coincides with two-person NTU games, we obtain the needed axiomatization of the egalitarian solution for two-person games.

Note that here the term "egalitarian solution" will be used in the sense other than already accepted. In fact, the known bargaining and NTU egalitarian solutions equalize the surplus, not the payoffs themselves. Thus, it seems that such a name fits more to the solution equalizing payoffs of the players up to some conditions.

The paper is organized as follows. We begin with bargaining solutions. In Section2 we define a class of bargaining problems and give a new axiomatic characterization for the lexicographic egalitarian solution (LEG). In Section 3 we consider a class of ordinally convex NTU games and define the LEG solution for this class. Then, by unifying the axiomatization LEG for bargaining problems and the extension of Dutta's theorem [**?**] for convex TU games to the ordinally convex NTU games [**?**] we obtain the main result – the new axiomatization of the LEG solution for the class of ordinally convex NTU games.

2 Lexicographic egalitarian bargaining solution

2.1 Definitions and auxilitary results

Let N be an arbitrary finite set. We denote by $n = |N|$ the number of elements in N, and let $\mathbb{R}^N$ be the n-dimensional Euclidian space. $\mathbb{R}^N_+, \mathbb{R}^N_{++}$ denote the nonnegative and strictly positive orthant of $\mathbb{R}^N$, respectively. Given x,y$\in \mathbb{R}^N$, we write x$\geqq$y, if x-y$\in \mathbb{R}^N_+$, x¿y, if x-$\in \mathbb{R}^N_{++}$, and x$\geq$y, if x$\geqq$y, and x$\neq$y. If x$\in \mathbb{R}^N$, then for $N' \subset N$ by $\mathrm{x}_{N'}$ we denote the projection of x on the space $\mathbb{R}^{N'}$.

A *bargaining problem (BP)* with a finite set N of agents is a pair $B = \langle X, \mathrm{d}\rangle$, where $X \subset \mathbb{R}^N$ is a *bargaining set.* A point $\mathrm{x} \in X$ represents utility levels that can be reached by the agents. The point $\mathrm{d} \in X$ is a *disagreement* or *status quo point.* It represents the utility levels that the players will end up if they do not agree on another point. We will denote by Σ^N_d the class of BP with the disagreement point d, for which

1) X is a upper-bounded and closed subset of $\mathbb{R}^N$;

2) there a point $\mathrm{x} \in X, \mathrm{x} > \mathrm{d}$;

3) X is *d-comprehensive,* i.e. $\mathrm{x} \in X, \mathrm{d} \leq \mathrm{y} \leq \mathrm{x}$ imply $\mathrm{y} \in X$.

Let $\Sigma = \bigcup_{P \subset \mathbb{N}} \Sigma^P$ be the collection of BP with the zero disagreement point. Thus, each class Σ^P is determined by a collection of feasible bargaining sets satisfying conditions 1)–3):

$$\Sigma^P = \{X \subset \mathbb{R}^P_+ \,|\, 0 \in X\}.$$

In the sequel we denote the d-comprehensive hull of X by $ch(X)$.

4) X has a *nonlevel Pareto boundary*, i.e. its weak Pareto boundary coincides with the Pareto boundary. We denote it by ∂X.

A *bargaining solution* F defined on Σ^N_d is a mapping which associates with each BP $\langle X, \mathrm{d}\rangle \in \Sigma^N_d$ a unique point $F(X, \mathrm{d}) \in X$ interpreted as a prediction, or a recommended outcome for that problem.

Give well-known axioms describing properties of bargaining solutions for a fixed population set N.

Pareto-optimality (PO). $F(X, \mathrm{d}) \in \partial X$.
Individual rationality (IR). $F(X, \mathrm{d}) \in IR(X, \mathrm{d}) = \{x \in X \,|\, x \geq d\}$.
Strict individual rationality (SIR). $F(X, \mathrm{d}) \in \{\mathrm{x} \in X \,|\, \mathrm{x} > \mathrm{d}\}$.
Independence of non-individually rational alternatives (INIR). $F(X, \mathrm{d}) = F(IR(X, \mathrm{d}), \mathrm{d})$.
Anonymity (ANO). For each permutation $\pi : N \to N$ $F(\pi X, \pi \mathrm{d}) = \pi F(X, \mathrm{d})$, where $\pi X = \{\mathrm{y} \in X \,|\, \mathrm{y} = \pi \mathrm{x}, \mathrm{x} \in X\}$, $\pi \mathrm{x} = (x_{\pi 1}, x_{\pi 2} \ldots, x_{\pi n})$.
Independence of Irrelevant Alternatives (IIA). If $X' \subset X$ and $F(X, \mathrm{d}) \in X'$, then $F(X', \mathrm{d}) = F(X, \mathrm{d})$.
Continuity (CONT). If $X^m \to_{m \to \infty} X$ in the Hausdorff topology, $\mathrm{d}^m \to_{m \to \infty} \mathrm{d}$, and $\langle X^m, \mathrm{d}\rangle, \in \Sigma^N_{\mathrm{d}^m}, \forall m, \langle X, \mathrm{d}\rangle \in \Sigma^N_d\rangle$, then $F(X^m, \mathrm{d}^m) \to F(X, \mathrm{d})$.
Weak Continuity (WCONT) requires that the property Continuity given above would hold only if $\lim_{m \to \infty} F(X^m, \mathrm{d}^m) \in \partial X$.
Independence of Identical Ordinal Transformations (IORD). For every monotonically increasing function $f : \mathbb{R} \to \mathbb{R}$ $f(F(X, \mathrm{d})) = F(f(X), f(\mathrm{d}))$, where for each $\mathrm{x} \in X$

$f(\mathrm{x}) = (f(x_1), \ldots, f(x_n)),, f(X) = \{\mathrm{y} \in \mathbb{R}^N \,|\, \mathrm{y} = f(\mathrm{x}), \mathrm{x} \in X\}$. Note that if $\langle X, \mathrm{d}\rangle \in \Sigma_d^N$, then $(f(X), f(\mathrm{d})) \in \Sigma_{f(\mathrm{d})}^N$.

Note that the most bargaining solutions satisfy INIR. This is because of this axiom follows from IR and IIA, and the last axioms are well-used. Thus, if a solution F satisfies INIR, or IR+IIA, then without loss of generality we can consider the feasible sets $IR(X)$ in the definition of a class of BP instead of X.

The last property deals with variable population sets. Let N be an arbitrary finite set. Consider the collection of bargaining problems $\bigcup_{N' \subset N} \bigcup_d \Sigma_d^{N'}$, and let F be a BS for this class

Consistency (CONS) if x= $F(X, \mathrm{d}$, where $X \subset \mathbb{R}^N$, then for each $N' \subset N$ $\mathrm{x}_{N'} = F(X|_{x_{N \setminus N'}}, \mathrm{d}_{N'}$, where $X|_{\mathrm{x}_{N \setminus N'}} \subset \mathbb{R}^{N'}$ is the section of X by $x_i, i \in N \setminus N'$.
Bilateral consistency (BCONS) means the fulfilment of the consistency property only for $|N'| = 2$.

Consistency (bilateral consistency) says that a utility allocation is declares as a fiar co-promise only if it is fair for any subset (two-person set) of agents involved in the bargaining problem.

The *Lexicographic egalitarian solution (LEG)* for a class Σ_d^N prescribies for each BP $\langle X, \mathrm{d}\rangle \in \Sigma_d^N$ the vector $LEG(X, \mathrm{d})$ lexicographically maximal in the set $IR(X, \mathrm{d})$:

$$LEG(X, \mathrm{d}) = \arg \max_{x \in IR(X,d)} \succ_{lexmin} . \tag{1}$$

Let us show that the LEG is well-defined (it is not evident, because we did not supposed convexity of bargaining sets X.)

Lemma 1 *Given a BP $(X, \mathrm{d}) \in \Sigma_d^N$, where X satisfies the additional property 4),*

$$\mathrm{x} = LEG(X, \mathrm{d}) \iff \mathrm{x} = (\underbrace{a_1, \ldots, a_1}_{T_1}, \underbrace{a_2, \ldots, a_2}_{T_2}, \ldots \underbrace{a_m \ldots, a_m}_{T_m}), \tag{2}$$

where

$$a_1 = \max\{\max_{i \in N} d_i, \max\{t \,|\, t\mathrm{e}_N \in X\}\}, \quad \mathrm{e}_N \in \mathbb{R}^N \textit{ is the unit vector;} \tag{3}$$

$$a_j = \max\{\max_{j \in N \setminus R_{j-1}} d_j, \max\{t \,|\, t\mathrm{e}_{N \setminus R_{j-1}} \in X|_{(a_1 \mathrm{e}_{T_1}, a_2 \mathrm{e}_{T_2}, \ldots, a_{j-1} \mathrm{e}_{T_{j-1}})}\}\}, \;\; j = 2, \ldots, m, \tag{4}$$

where $R_l = \bigcup_{i=1}^l T_i, l = 1, \ldots, m$.

Proof.

Note that conditions 1)–3) imply that all maximums in (**??**) and (**??**) are attained.

Prove the inequalities $a_1 \geq a_2 \geq \ldots \geq a_m$. If $m \geq 2$, then by (**??**) and (**??**) the numbers $a_1, \ldots, a_{m-1}$ are equal to some d_j, and they are placed in a decreasing manner. If $a_m = d_j$, then the claim has been proved. Let $a_m = \max\{t \,|\, t\mathrm{e}_{T_m} \in X|_{(a_1 \mathrm{e}_{T_1}, \ldots, a_{m-1} \mathrm{e}_{T_{m-1}}}\}$. This means that $a_m < a_k, k = 1, \ldots, m-1$. In fact, in the contrary, from the inequality $a_m >$

a_{m-1} the relation $(a_1, ...a_2, ...a_{m-1}, a_m) \in X$ by property 4) would imply the existence $a,'$ $a_m > a' > a_{m-1}$ such that $x' \in X$, where

$$x'_i = \begin{cases} x_i, & \text{if } i \notin T_{m-1}, T_m, \\ a', & \text{if } i \in T_{m-1} \cup T_m. \end{cases}$$

However, by the definition of x in the right-hand side of (**??**) such a vector x$'$ cannot belong to X.

Now let x satisfy the right-hand side of (**??**), and y= $LEG(X, \mathrm{d})$. Then y$\succ_{lexmin}$ x. If $m = 1$, then $x_i = x_j = a_1$ for all $i, j \in N$ and the vector x Lorenz (and, hence, lexicographically) dominates all the vectors in X, hence, $y = x$.

Let $m > 1$. Then by Individual Rationality of y (**??**) $y_i \geq a_k = d_i = x_i$ for all $i \in N \setminus T_m, i \in T_k$. Therefore, by Pareto-optimality of x and y the inequality y $\neq$ x implies the existence of $j \in N \setminus T_m$ such that $y_j < a_m = x_j = \min_{i \in N} x_i$, and the relation y $\succ_{lexmin}$ x is wrong. Therefore, $LEG(X, \mathrm{d})$ =x. □

Recall the result of Lensberg which we will use further. Note that the author considered *collective choice problems* and thier solutions. Such problems are defined by bargaining sets placed in the nonnegative orthants of Euclidean spaces. The zero point played a role of a disagreement point, so we could reformulate his result in terms of bargaining problems and bargaining solutions.

Lemma (Lensberg 1987).*Let Σ_0 be the class of BP with the zero disagreement point, whose bargaining sets X are convex and satisfy 1)–3). If a bargaining solution F for this class satisfies PO, CONT and CONS, then it satisfies IIA.*

In the next subsection we give a modification of this Lemma (Lemma **??**), where we consider the class $\Sigma = \bigcup_{P \subset \mathcal{N}} \Sigma^P$, and the result is established for the class Σ^2 of two-person bargaining problems.

2.2 Two-person bargaining problems

These are the simplest bargaining problems. However, they are of a big importance, because all the characterizations of BS and the solutions of the NTU games as well with the help of consistency, reduce the initial problem to that with two agents or players. Then the characterizations of BS for two-person problems can help for the same problems and for the characterization of solutions for NTU games with arbitrary finite sets of agents/players.

In this subsection we use the following notation:

Denote the class of bargaining problems with the set of agents P, disagreement point d, and whose bargaining sets satisfy properties 1)–3) and 5) by Σ^P_d, and $\Sigma^P = \bigcup_{\mathrm{d} \in \mathbb{R}^P}$, and for $|P| = 2$ $\Sigma^P = \Sigma^2$.

Let $(X, \mathrm{d}) \in \Sigma^2$. Denote

$$\bar{x}_i = \max_{x=(x_1,x_2) \in X} x_i, \;\; d'_i = \max_{(\bar{x}_j, x_i) \in X} x_i, \; i = 1, 2.$$

$$X_z = \{(x_1, x_2) \in X \,|\, x_1 \geq z\}, \;\; X^z = \{(x_1, x_2) \,|\, x_2 \leq z\}, \;\; X^y_z = \{(x_1, x_2) \in X \,|\, x_1 \leq z, x_2 \leq y\}.$$

Instead of property 4) for two-person bargaining probelms we consider a weaker condition:
5) For two-peson bargaining problems the Pareto boundary of X, ∂X, is a connected set.

The reason of consideration property 5) instead of 4) is the application of the IIA axiom that can lead to bargaining sets with pieces of weak Pareto boundary.

There is a characterization of the LEG for the classes Σ_d^2 with disagreement points with equal coordinates $\mathrm{d} = (d, \ldots, d)$ following from Nielsen's result in [**?**]:

heorem 1 (Nielsen 1983) *The LEG is the unique bargaining solution for the class Σ_0^2 satisfying PO, SIR, IIA and IORD.*

Note that on the class Σ_0^2 the LEG does satisfy SIR. However, for arbitrary $\mathrm{d} > 0$ this axiom does not hold. This is the reason of taking into consideration Axiom WCONT.

Lemma 2 *Let $\Sigma = \bigcup_{P \subset \mathcal{N}} \Sigma^P$ be the class of BP, whose bargaining sets satisfy 1)–3) and 5).If F is a bargaining solution for the class Σ satisfies PO, BCONS, and WCONT, then for the class Σ^2 F satisfies IIA.*

Proof. The proof follows that of Lemma 1 in [**?**]. In fact, this proof does not use convexity of BS. Moreover, it does not depend on the choice of zero as the disagreement point.

Repeat the proof replacing CONT by WCONT taking into account the properties of the bargaining sets. Let F be a BS for Σ satisfying PO, BCONT, and WCONS. $P = \{i, j\}$ and $X, X' \in \Sigma_d^2, X' \subset X, y = F(X, \mathrm{d}) \in X'$. We must show that $y = F(X', \mathrm{d})$ also. Assume first that there is a neighborhood U of y such that

$$X' \cap U = X \cap U. \tag{5}$$

Case 1. The weak Pareto boundary of X coincides with its Pareto boundary.

Let now k be an agent $k \neq \{i, j\}$, and let $Q = \{i, j, k\}$. Define $X^1 = X' \times \{e_k\}$ and for all $\varepsilon \geq 0$ let C_ε be the cone with vertex $(1 + \varepsilon)e_k$ spanned by X^1. Define $T^\varepsilon = \mathrm{ch}(C^\varepsilon) \cap \mathrm{ch}(S \times \{e_k\})$ and $U^1 = U \times \{e_k\}$, and note that for all $\varepsilon \geq 0$, $U^1 \cap X^1 \subset T^\varepsilon$ and $T^\varepsilon \in \Sigma^Q$.

Let $\mathrm{z} = F(T^0)$. We claim that $\mathrm{z} = (\mathrm{y}, 1)$. Note that whatever z is, the projection $T_0|_{z_k}$ of T_0 on $\mathbb{R}^P$ w.r.t. z_k equals X, and by consistency of F $\mathrm{z}_P = \mathrm{y}$. Since $(\mathrm{y}, 1)$ is the unique Pareto optimal point of T_0 whose projection of $\mathbb{R}^P$ is y, we conclude by PO that $\mathrm{z} = (\mathrm{y}, 1)$.

Consider now $\mathrm{z}^\varepsilon = (\mathrm{z}_P^\varepsilon, z_k^\varepsilon) = F(T^\varepsilon) \in \partial T^\varepsilon$. Let $z^\varepsilon \to \xi$ (by compactness of $IR(X)$ we can always choose a convergent subsequence from z^ε, and for simplicity of notation let it be the sequence x^ε itself). Then by CONS and PO of F $\mathrm{z}_P^\varepsilon = F(T^\varepsilon|_{z_k^\varepsilon}) \in \partial T^\varepsilon|_{z_k^\varepsilon}$. Since $T^\varepsilon \to T^0$, $T^\varepsilon|_{z_k^\varepsilon} \to X'$, and $\xi_P = \lim_{\varepsilon \to 0} \mathrm{z}_P^\varepsilon \in \partial X$ by the assumption of Case 1. Therefore, by WCONT $\mathrm{z}_P^\varepsilon \to \mathrm{z}_P = \mathrm{y}$. It is clear that $z_k^\varepsilon \to 1$ implying $\xi = \mathrm{z}$. In fact, in the contrary $z_k^\varepsilon < 1$ for sufficiently small ε, and in this case $\mathrm{z}^\varepsilon \notin \partial T^\varepsilon$.

By PO there exists $\bar{\varepsilon} > 0$ such that $\mathrm{z}^{\varepsilon} \in U^1$ for all $\varepsilon \in [0, \bar{\varepsilon}]$. But then $\mathrm{z}^{\varepsilon} = \mathrm{z}$ for all $\varepsilon \in (0, \bar{\varepsilon})$ by the fact that $z^{\varepsilon} \to \varepsilon$ in U^1 as $\varepsilon \to 0$, which implies that $F(X') = \mathrm{z}_P = \mathrm{y}$, the desired conclusion.

Case 2. There are points on the weak Pareto boundary of X not belonging to ∂X. Since the Pareto boundary ∂X is connected, these points belong to one or both intervals $A_1 = [d_1, a_1] \times \bar{x}_2$, or $A_2 = \bar{x}_1 \times [d_2, a_2]$ for some a_1, a_2. Then $T^{\varepsilon} \cap \{x_i = \bar{x}_i\} \notin \partial T^{\varepsilon}, i = 1, 2$ and $u \in \partial T^{\varepsilon} \quad \longrightarrow \quad u_i \geq a_i$, $i = 1, 2$. Therefore, $\lim \mathrm{z}^{\varepsilon} \in \partial T^0$ and by WCONT $\lim_{\varepsilon \to 0} \mathrm{z}^{\varepsilon} = \mathrm{z}$. Further the proof coincide with that of Case 1.

To complete the proof, it suffices to observe that if X' does not satisfy condition (**??**) above, then it can be approximated by a sequence of elements from Σ^P that does. WCONT may then be applied once more to conclude that $F(S') = F(S)$ in this case also. □

Lemma 3 *Let $(X, \mathrm{d}) \in \Sigma^2$ be an arbitrary BP with $d_2 > d_1' > d_1$, F be a bargaining solution for Σ^2 satisfying PO, IIA, ORD, and WCONT. Then $F(X, \mathrm{d}) = F(X, \mathrm{d}')$, where $\mathrm{d}' = (d_1', d_2)$.*

Proof. Let $d_n \to d_1', d_n > d_1$ be an arbitrary consequence, $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous increasing functions such that $f_n(x) \to -\infty$ as $x \to -\infty$,

$$f_n(x) = \begin{cases} \text{arbitrary}, & \text{if } x < d_1, \\ d_n + \frac{(x-d_1)(d_1'-d_1)}{d_1'-d_1}, & \text{if } x \in [d_1, d_1'], \\ x, & \text{if } x > d_1'. \end{cases}$$

Then for every $\mathrm{y} = (y_1, y_2) \in \partial IR(X)$ $f_n(\mathrm{y}) = \mathrm{y}$ for all n and by PO and IORD

$$F(f_n(X), f_n(\mathrm{d}) = f_n(F(X, \mathrm{d})) = F(X, \mathrm{d}). \tag{6}$$

It is clear that $f_n(X) = X, f_n(\mathrm{d}) \to \mathrm{d}'$ as $n \to \infty$, and by WCONT and(**??**)

$$F(X, \mathrm{d}) = F(X, \mathrm{d}').$$

□

orollary 1 *Let for a BP (X, d) $d_2' > d_2 \geq \bar{x}_1'$. Then $F(X, \mathrm{d}) = F(X, (d_1, d_2'))$.*

The proof of the Corollary is the same as that of Lemma 3.

The next Theorem characterizing the Lexicographic Egalitarian solution for this class is the extension of the similar result (Roth 1979, Nielsen 1983) to the class Σ_0^2. The unique axiom – WCONT – is neccesary to be added:

heorem 2 *The lexicographic egalitarian solution is the only solution to the class Σ^2 to satisfy PO, IR, ANO, IORD, IIA, and WCONT.*

Proof. The proof is divided for several cases depending on the mutual location of bargaining sets and disagreement points.

Let $(X, \mathrm{d}) \in \Sigma^2$ be an arbitrary BP. By condition 5) the Pareto boundary ∂X is determined by a continuous decreasing function

$$\varphi : [d_1', \bar{x}_1] \to [\bar{x}_2, d_2'] \text{ for some } d_i' \geq d_i, i = 1, 2. \tag{7}$$

By ANO it suffices to consider only the case $d_1 \leq d_2$.

1. $d_1 = d_2 = d$. From the proof of Theorem **??** it follows that if a bargaiing solution F satisfies all the axioms stated in the Theorem except for SIR, than two solution satisfy other axioms: they are LEG and

$$F_1(X, \mathrm{d}) = \begin{cases} (d_1', \varphi(d_1'), & \text{if } x_2 \geq x_1 \text{ for all } x \in \partial X, \\ (d_2, \varphi(d_2)), & \text{if } x_1 \geq x_2 \text{ for all } x \in \partial X, \\ LEG(X, \mathrm{d}) & \text{otherwise.} \end{cases}$$

Let us show that the solution F_1 does not satisfy WCONT. Consider a sequence $X_n \to X$, where in BSs X_n there are points x^n with $x_1 < x_2$ and points y^n with $y_1 > y_2$, and in the BS X $x_2 \geq x_1$ for all $x \in X$ and $LEG(X, \mathrm{d}) = (x, x)$. Then $F_1(X_n, \mathrm{d}) = LEG(X_n, \mathrm{d}) = (x_n, x_n)$ for some $x_n > d$ and $x_n \to x$ as $n \to \infty$. By WCONT we obtain $F_1(X, \mathrm{d}) = LEG(X, \mathrm{d}) \neq F_1(X, \mathrm{d})$.

Before we consider other cases note that by Lemma 3 without loss of generality we may suppose that $d_1 = d_1'$. For simplicity of notation in the sequel we will uppose this equality holds.

2. $\bar{x}_1 \leq d_2$. In this case the domains of individual utilities $[d_1, \bar{x}_1]$, $[d_2, \bar{x}_2]$ may have at most the unique common point $(\bar{x}_1, \bar{x}_1)$ when $\bar{x}_1 = d_2$ or they do not intersect. Consider an arbitrary increasing function $f_1 : [d_1, \bar{x}_1] \to [d_1, \bar{x}_1]$ having only two fixed points $f_1(d_1) = d_1, f_1(\bar{x}_1) = \bar{x}_1$. Therefore, the function $f_2 = \varphi(f_1(\varphi^{-1})) : [d_2, \bar{x}_2] \to [d_2, \bar{x}_2]$ is increasing and has the fixed points $f_2(d_2) = d_2, f_2(\bar{x}_2) = \bar{x}_2$. Define the increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(x) = \begin{cases} x, & \text{if } x \notin (d_1, \bar{x}_1) \cup (d_2, \bar{x}_2), \\ f_1(x), & \text{if } x \in [d_1, \bar{x}_1], \\ f_2(x), & \text{if } x \in [d_2, \bar{x}_2]. \end{cases}$$

Then $f(X) = X, f(\mathrm{d}) = \mathrm{d}$, and by IORD $F(X, \mathrm{d}) = f(F(X, \mathrm{d}))$. Therefore $F(X, \mathrm{d})$ may only be equal to extreme points of ∂X : $F(X, \mathrm{d}) = (\bar{x}_1, d_2') = LEG(X, \mathrm{d})$ or $F(X, \mathrm{d}) = (d_1, \bar{x}_2)$.

The next step is to show the unique possibility: $F(X, \mathrm{d}) = LEG(X, \mathrm{d})$. Let a BP (X, d) satisfy the conditions of Case 2. $\mathrm{d}_1 = (d_1, d_1), \mathrm{d}_y = (d_1, y)$ for $y \in [d_1, d_2]$.

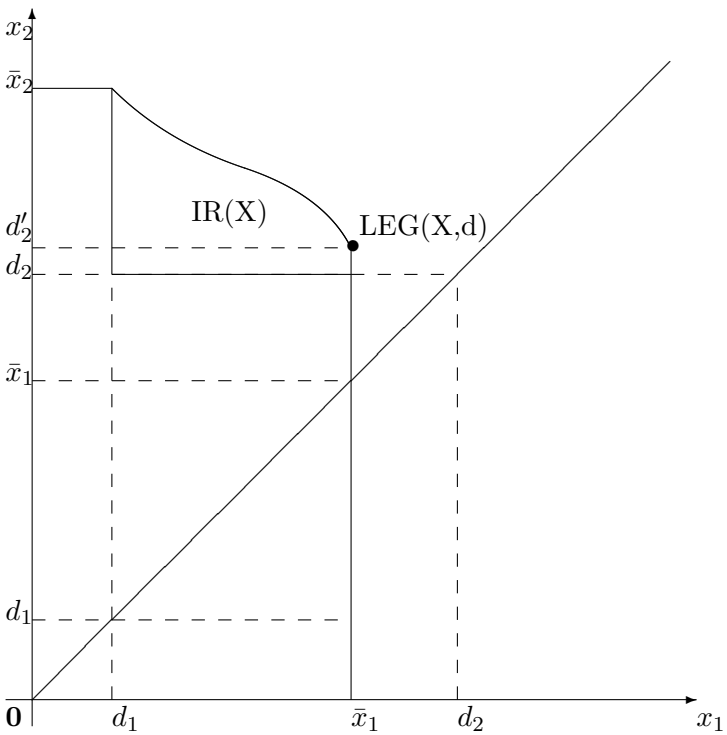

Fig.1

Consider the BP (X, d_1), This BP satisfies Case 1, $F(X, \mathrm{d}_1) = LEG(X, \mathrm{d}_1)$. The BPs (X, d_y) for $y \in [\bar{x}_1, d_2]$ satisfy the conditions of Case 2, so for these y $F(X, \mathrm{d}_y) = LEG(X, \mathrm{d}_y)$ or $F(X, \mathrm{d}_y) = (d_1', \bar{x}_2)$.

Consider now the BPs (X, d_y) for $y \in [d_1, \bar{x}_1]$. By the proof similar to that in Case 2 we obtain that either $F(X, \mathrm{d}_y) = (d_1', \bar{x}_2)$, or $F_1(X, \mathrm{d}_y) \in [y, \bar{x}_1]$. In the last case by IIA

$$F(X, \mathrm{d}_y) = F(X^{\varphi(y)}, \mathrm{d}_y), \tag{8}$$

and by Lemma 3

$$F(X^{\varphi(y)}, \mathrm{d}_y) = F(X, \mathrm{y}) \tag{9}$$

where $\mathrm{y} = (y, y)$ The BP (X, y) satisfies Case 1, and

$$F(X, \mathrm{y}) = LEG(X, \mathrm{y}). \tag{10}$$

Therefore, equalities (**??**)–(**??**) imply

$$F(X, \mathrm{d}_y) = LEG(X, \mathrm{d}_y) = (\bar{x}_1, d_2') = LEG(X, \mathrm{d}) \text{ for } y \in [d_1, \bar{x}_1]. \tag{11}$$

Thus, for $y = \bar{x}_1$ we have equality (**??**), and for $y \in (\bar{x}_1, d_2]$ except for equality (**??**) there may be the possibility $F(X_{d_y}, \mathrm{d}_y) = (d_1, \bar{x}_2)$. However, by WCONT the last equality is impossible, and since for $y = d_2$ we have $\mathrm{d}_y = \mathrm{d}$, we have proved the required uniqueness: $F(X, \mathrm{d}) = LEG(X, \mathrm{d})$.

3. $d_1 < d_2 < \bar{x}_1$.

3a. $\bar{x}_2 \geq \bar{x}_1$.

This case can be divided more on two subcases:

3a$_1$. $a_1 = \varphi^{-1}(\bar{x}_1) \leq d_2$.

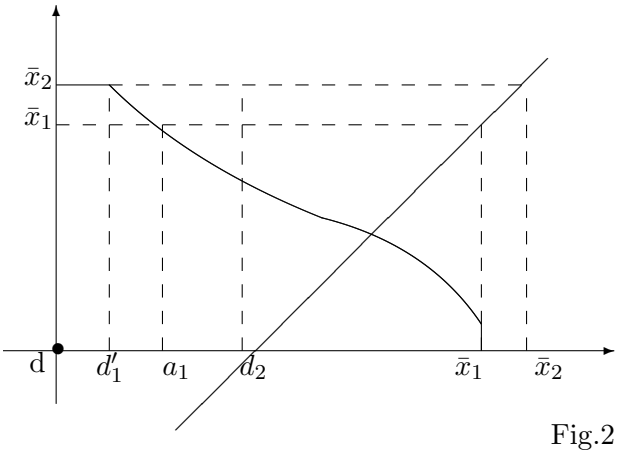

Fig.2

Similar to Case 2 we can consider an arbitrary increasing function f_1 mapping the interval $[d_1, a_1]$ onto itself such than $f_1(d_1) = d_1, f_1(a_1) = a_1$, and $f_2 = \varphi(f_1(\varphi^{-1}))$ maps the interval $[\bar{x}_1, \bar{x}_2]$ onto itself. Then the function

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in [d_1, a_1], \\ f_2(x), & \text{for } x \in [\bar{x}_1, \bar{x}_2], \\ x, & \text{for other } x \end{cases}$$

maps X onto itself, and by IORD we obtain that $F_1(X, \mathrm{d}) \notin (d_1, a_1)$. Therefore, either

$$F(X, \mathrm{d}) = (d_1, \bar{x}_2), \tag{12}$$

or

$$F_1(X, \mathrm{d}) \in [a_1, \bar{x}_1]. \tag{13}$$

If (**??**) holds, then by IIA $F(X, \mathrm{d}) = F(X_{d_1,a_1,}, \mathrm{d})$, where

$$X_{d_1,a_1} = \{(x_1, x_2) \in X \mid d_1 \leq x_1 \leq a_1\}.$$

The BP (X_{a_1}, d) satisfies Case 2, hence,

$$F(X, \mathrm{d}) = F(X_{d,a_1}, \mathrm{d}) = LEG(X_{a_1}, \mathrm{d}),$$

that contradicts (**??**).

Therefore, relation (**??**) takes place. Denote $\mathfrak{d} = (a_1, d_2)$. By IIA we have

$$F(X, \mathrm{d}) = F(X^{\bar{x}_1}, \mathrm{d}'), \tag{14}$$

and by Lemma **??**

$$F(\bar{X}^{\bar{x}_1}, \mathrm{d}) = F(X^{\bar{x}_1}, \mathfrak{d}). \tag{15}$$

Further, either

$$F_1(X^{\bar{x}_1}, \mathfrak{d}) \in [a_1, d_2), \tag{16}$$

or

$$F_1(X^{\bar{x}_1}, \mathfrak{d}) \in [d_2, \bar{x}_1]. \tag{17}$$

In the first case (**??**) by IIA

$$F(X^{\bar{x}_1}, \mathfrak{d}) = F(X_{d_2}, \mathfrak{d}).$$

The BP $(X_{d_2}, \mathfrak{d})$ satisfies Case 2, hence $F(X_{d_2}, \mathfrak{d}) = LEG(X_{d_2}, \mathfrak{d})$, that contradicts (**??**),

Therefore, relation (**??**) holds. Then, similarly to constructions above we obtain that

$$F(X^{\bar{x}_1}, \mathfrak{d}) = F(X^{\bar{x}_1}, \mathrm{d}_2) = LEG(X^{\bar{x}_1}, \mathrm{d}_2) = LEG(X, \mathrm{d}), \tag{18}$$

and equalities (**??**),(**??**, and (**??**), imply

$$F(X, \mathrm{d}) = LEG(X, \mathrm{d}).$$

3a$_2$. $a_1 = \varphi^{-1}(\bar{x}_1) > d_2$.

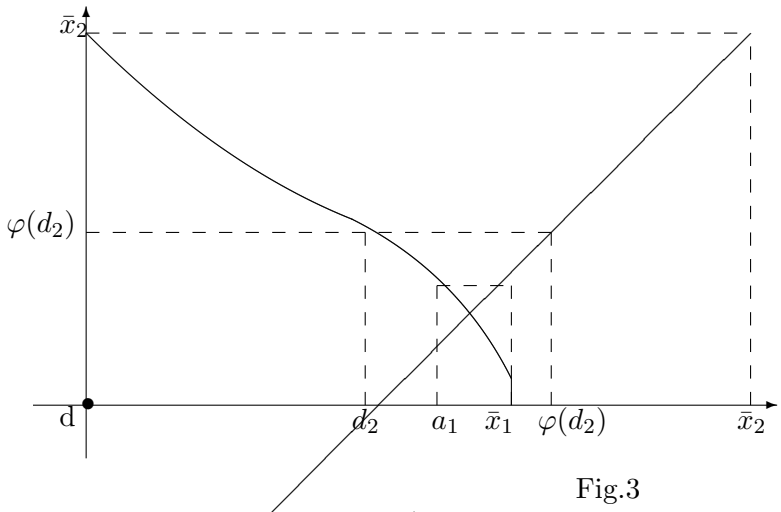

Fig.3

Since $[d_1, d_2] \cap [\bar{x}_1, \bar{x}_2] = \emptyset$, we can, as in Case 2, take an arbitrary increasing function $f_1 : [d_1, d_2] \to [d_1, d_2]$ with fixed ends $f_1(d_1) = d_1$, $f_1(d_2) = d_2$, and then define the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ as follows:

$$f(x) = \begin{cases} x, & \text{if } x \notin (d_1, d_2) \cup (\varphi(d_2), \bar{x}_2'), \\ f_1(x), & \text{if } x \in [d_1, d_2], \\ \varphi(f_1(\varphi^{-1})), & \text{if } x \in [\varphi(d_2), \bar{x}_2']. \end{cases}$$

Then by IORD $f(F(X, \mathrm{d})) = F(f(X), \mathrm{d})$, implying $F_1(X, \mathrm{d}) \notin (d_1, d_2)$. Therefore, either

$$F(X, \mathrm{d}) = (d_1, \bar{x}_2), \tag{19}$$

or

$$F_1(X, \mathrm{d}) \geq d_2. \tag{20}$$

Let (**??**) hold. Consider an arbitrary continuous decreasing function $\phi : [d_1, d_2] \to [d_2, \bar{x}_2], \phi(d_1) = \bar{x}_2, \phi(d_2) = d_2$ such that for all such that for all $x \in [d_1, d_2]$ $\phi(x) \leq \varphi(x)$. Then by IIA

$$F(X, \mathrm{d}) = F(X^{\phi}, \mathrm{d}), \tag{21}$$

where

$$X^{\phi} = \{(x_{,}x_2) \in X^{\bar{x}_2} \mid x_1 \in [d_1, d_2], x_2 \leq \phi(x_1)\}.$$

The BP (X^{ϕ}, d) satisfies Case 2, hence,

$$F(X^{\phi}, \mathrm{d}) = LEG(X^{\phi}, \mathrm{d}), \tag{22}$$

and the last equality contradicts (**??**) and (**??**).

Let now (**??**) hold. By IIA

$$F(X, \mathrm{d}) = F(X^{\varphi(d_2)}, \mathrm{d}), \tag{23}$$

and by Corollary to Lemma 3

$$F(X^{\varphi(d_2)}, \mathrm{d}) = F(X, \mathrm{d}_2). \tag{24}$$

The BP (X, d_2) satisfies Case 2, hence (**??**),(**??**) imply

$$F(X, \mathrm{d}) = F(X, \mathrm{d}_2) = LEG(X, \mathrm{d}_2) = LEG(X, \mathrm{d}).$$

3b.$\bar{x}_1 > \bar{x}_2$.

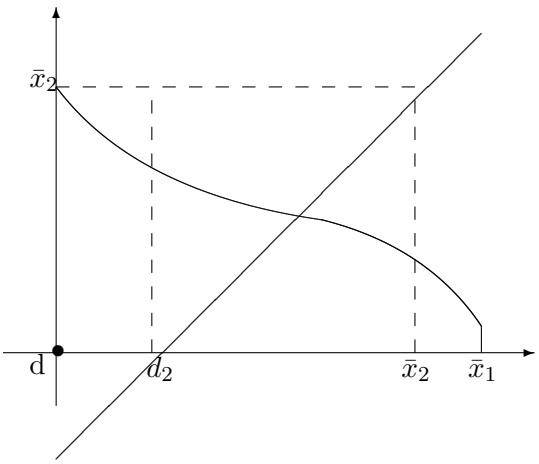

Fig.4

Since $[\bar{x}_2, \bar{x}_1] \cap [d_2, \bar{x}_2] = x_2$, similar to Case 3a, by considering arbitrary increasing functions $f_1 : [\bar{x}_2, \bar{x}_1] \to [\bar{x}_2, \bar{x}_1]$ with fixed ends and the functions $f_2 = \varphi(f_1(\varphi^{-1}))$, we obtain that $F_1(X, \mathrm{d}) \notin (\bar{x}_2, \bar{x}_1)$. Thus, by IIA either

$$F(X, \mathrm{d}) = (\bar{x}_1, \varphi(\bar{x}_1)), \tag{25}$$

or

$$F(X, \mathrm{d}) = F(X_{\bar{x}_2}, \mathbf{d}). \tag{26}$$

Let (**??**) hold. Then by IIA

$$F(X, \mathrm{d}) = F(X_{\bar{x}_2}, \mathrm{d}) = F(X^{\varphi(\bar{x}_2)}, \mathrm{d}). \tag{27}$$

By Lemma 3

$$F(X^{\varphi(\bar{x}_2)}, \mathbf{d}) = F(X^{\varphi(\bar{x}_2)}, \mathrm{d}_2). \tag{28}$$

The BP $(X^{\varphi(\bar{x}_2)}, \mathrm{d}_2)$ satisfies Case 1, and we obtain

$$F(X^{\varphi(\bar{x}_2)}, \mathrm{d}_2) = LEG(X^{\varphi(\bar{x}_2)}, \mathrm{d}_2), \tag{29}$$

that inconsistent with (**??**).

Therefore, equality (**??**) takes place. The BP $(X_{\bar{x}_2}, \mathrm{d})$ satisfies Case 3a implying

$$F(X_{\bar{x}_2}, \mathrm{d}) = LEG(X_{\bar{x}_2}, \mathrm{d}) = LEG(X, \mathrm{d}) = F(X, \mathrm{d}).$$

□

orollary 2 *Theorem 1 holds, if property 5) of the class Σ^2 is replaced by property 4).*

3 Egalitarian solution for ordinally convex NTU games

In [**?**] the egalitarian solution a class of ordinally convex NTU games was axiomatized as a value which is consistent à la Hart–Mas-Colell and is the solution of constrained egalitarianism for two-person superadditive games. The last class of games coincides with two-person bargaining porblems considered in the previous section. Thus, by unifying the results of Section 3? and Theorems 3 and 4 in [**?**] a characterization of egalitarian solution for ordinally convex NTU games will be given in this Section without using the definition of two-person games' solution as an axiom.

A *non-transferable utility game (NTU-game)* is a pair $\langle N, V\rangle$, where N is a finite set of players, $V : 2^N \setminus \emptyset \to \bigcup_{S\subset N} \mathbb{R}^S$ is a mapping, called the *characteristic function*, that associates with each coalition $S \subset N$ a set $V(S) \subset \mathbb{R}^S$ of feasible payoff vectors for S. Standard assumptions about the values of the characteristic functions V are the following: for each coalition $S \subset N$ the set $V(S)$ is
– a nonempty strict subset of $\mathbb{R}^S$,
– closed and comprehensive.

Solutions for NTU games are defined by the same way as for TU fgames. Almost all solutions for TU games have analogs in some classes of NTU games. Among them one egalitarian-type solution – monotonic solution (cf. Kalai and Samet 1989) – is the direct extension of the Shapley value in the form of Harsanyi's dividends to NTU games. In this Section we give and extension of the egalitarian solution of Dutta for convex TU games to a class of ordinally convex NTU games.

Give some notations: By comprehensiveness of $V(S)$ the boundary of $V(S)$ is a weakly Pareto optimal subset of $V(S)$. Other properties of the sets $V(S)$ may be supposed depending on the subject into consideration. For $S' \subset S \subset N$ denote by $V(S)|_{y_{S'}}$ the section of the set $V(S)$ by the hyperplanes $x_i = y_i, i \in S'$:

$$V(S)|_{y_{S'}} = \{x \in \mathbb{R}^{|S|-|S'|} \mid (x, y_{S'}) \in V(S)\}.$$

The game $\langle N, V\rangle$ is *superadditive (subadditive)*, if for all coalitions $S, T \subset N, S \cap T = \emptyset$ it holds

$$V(S) \cup V(T) \subset (\supset) V(S \cup T).$$

Let $\langle N, V\rangle$ be an arbitrary NTU game. For each coalition $S \subset N$ we denote by $\hat{V}(S) \subset \mathbb{R}^N$ the cylinder over $V(S)$:

$$\hat{V}(S) = \{(x_S, x^S) \in \mathbb{R}^N \mid x_S \in V(S)\}.$$

The NTU game $\langle N, V\rangle$ is *ordinally convex*, if for all $S, T \subset N$

$$\hat{V}(S) \cap \hat{V}(T) \subset \hat{V}(S \cup T) \cup \hat{V}(S \cap T).$$

The *core* of $\langle N, V\rangle$ is the set

$$C(N, V) = V(N) \setminus \bigcup_{S \subset N} \operatorname{int} \hat{V}(S).$$

It is known (cf. Vilkov 1974, Greenberg 1985), that ordinally convex games have non-empty cores.

Let $\mathcal{G}_N^c$ be the class of ordinally convex NTU games with the player set N.

For each finite N we will consider the subclass $\mathcal{G}_N^{c1} \subset \mathcal{G}_N^c$ of NTU games satisfying the following conditions: for each game $\langle N, V\rangle \in \mathcal{G}_N^{c1}$

1^0 The boundaries $\partial V(S)$ of $V(S)$ are Pareto optimal for all $S \subseteqq N$, i.e. if $x, y \in \partial V(S), x \geq y$, then $x = y$. ! , PO

2^0 For each $S \subset N$ the diagonal of $\mathbb{R}^S$ intersects $V(S)$ at a unique point;

3^0 For all $T \subset S \subset N$, $x_T \in \mathbb{R}^T$ the sections

$$V(S)|_{x_T} = \{y \in \mathbb{R}^{|S|-|T|} \mid (x_T, y) \in V(S)\}$$

are not empty.

Property 1^0 is the non-levelness of the boundary of $V(S)$: it demands that these sets have no flat pieces in each coordinate; property 2^0 is a weak property of upper-boundedness: it can be replaced by the property of abence in each $V(S)$ sequences of vectors, whose all coordinates increase unlimitedly.

Note that we suppose neither convexity of the sets $V(S)$, nor their smoothness.

Let us define the lexicographic egalitarian solution (LEG) for the class $\mathcal{G}_N^{c1}$.

Given a game $\langle N, V\rangle \in \mathcal{G}_N^{c1}$, we put $x \in LEG(N,V)$, iff $x \in \partial V(N)$ and the vector x can be represented as

$$x = (\underbrace{a_1, \dots, a_1}_{T_1}, \underbrace{a_2, \dots, a_2}_{T_2}, \dots \underbrace{a_m \dots, a_m}_{T_m}), \tag{30}$$

where

$$a_1 = \max_{S \subset N} \max\{t \,|\, te_S \in V(S)\}, \quad e_S \in \mathbb{R}^S \text{ is the unit vector;} \tag{31}$$

$$a_j = \max_{S \subset N \setminus R_{j-1}} \max\{t \,|\, te_S \in V(R_{j-1} \cup S)|_{(a_1 e_{T_1}, a_2 e_{T_2}, \dots, a_{j-1} e_{T_{j-1}})}\}, \;\; j = 2, \dots, m, \tag{32}$$

where $R_l = \bigcup_{i=1}^{l} T_i, l = 1, \dots, m$.

In [**?**] it was proved that the egalitarian solution for each game $\langle N, V\rangle \in \mathcal{G}_N^{c1}$ is determined uniquely, and the vector $x = LEG(N,V) \in C(N,V)$, and for two-person games it coincides with the lexicographic egalitarian bargaining solution.

The axiomatic characterization of the lexicographic egalitarian solution for the classes $\bigcup_{N' \subset N} \mathcal{G}_{N'}^{c1}$ for each finite N also is similar to that for convex TU games [**?**]:

heorem 3 (Yanovskaya 2005) *The lexicographic egalitarian solution for the class* $\bigcup_{N' \subset N} \mathcal{G}_{N'}^{c1}$ *is the unique solution which coincides with the lexicographic egalitarian solution on the subclass of two-person games and is consistent à la Hart–Mas-Colell.*

Note that the bargaining problems being a particular cases of games from the class $\mathcal{G}_N^{c1}$, are contained in the class Σ^N, considered in subsection 2.2. Therefore, for them Lemma 2 holds, Corollary **??**, Theorem **??**, and Lemma 2 imply th following characterization of the egalitarian solution:

heorem 4 *For each finite N the lexicographic egalitarian solution for the class* $\mathcal{G}^{c1}(N) = \bigcup_{N' \subset N} \mathcal{G}_{N'}^{c1}$ *is the unique value which satisfies PO, IR, ANO, IORD, WCONT, and is consistent à la Hart–Mas-Colell.*

Proof. Let F be an arbitrary value, satisfying the conditions of the Theorem. In view of Theorem **??** it suffices to show that $F = LEG$ for all two-person games from $\mathcal{G}_N^{c1}, |N| = 2$. Lemma **??** implies that F satisfies IIA, and now the claim follows from Theorem **??**. □

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