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Fairness and efficiency in strategy-proof object allocation mechanisms ☆

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Abstract

I consider the problem of allocating N indivisible objects among N agents according to their preferences when transfers are absent and an outside option may exist. I study the tradeoff between fairness and efficiency in the class of *strategy-proof* mechanisms. The main finding is that for *strategy-proof* mechanisms the following efficiency and fairness criteria are mutually incompatible: (1) *ex-post efficiency* and *envy-freeness*, (2) *ordinal efficiency* and *weak envy-freeness,* and (3) *ordinal efficiency* and *equal division lower bound*. Result 1 is the first impossibility result for this setting that uses *ex-post efficiency*; results 2 and 3 are more practical than similar results in the literature. In addition, for $N = 3$, I give two characterizations of the celebrated random serial dictatorship mechanism: it is the unique *strategy-proof*, *ex-post efficient* mechanism that (4) provides agents that have the same ordinal preferences with assignments not dominated by each other (*weak envy-freeness among equals*), or (5) provides agents that have the same cardinal preferences with assignments of equal expected utility (*symmetry*). These results strengthen the characterization by Bogomolnaia and Moulin (2001); result 5 implies the impossibility result by Zhou (1990).

JEL classification: C78; D71; D78

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Keywords: Probabilistic assignment; Random serial dictatorship; Strategy-proofness; Ex-post efficiency; Weak envy-freeness; Equal division lower bound

1. Introduction

The optimal allocation of goods among individuals is one of the core issues in economics. Normally, researchers analyze this issue using the well-established concepts of markets and auctions, in which individuals receive goods in exchange for transfers. However, in a variety of real-life situations, these transfers are not available for either ethical, institutional or other reasons. Recent literature analyzes numerous examples of such situations. These range from student assignment to primary schools (Abdulkadiroğlu and Sönmez, 2003a) and job placement for graduates (Roth, 1984; Coles et al., 2010), to on-campus housing (Chen and Sönmez, 2002), organ donation (Roth et al., 2004) and distributing military supplies (Kesten and Yazici, 2012).

In this paper I study the simplest version of this class of problems: the object allocation problem,[2] where a set of indivisible objects is allocated to a set of agents solely according to their preferences and such that each agent receives at most one object.[3] The object allocation problem has two stages: first agents report their (ordinal) preferences over objects and then, based on these preferences and using some systematic procedure which we call *a mechanism*, the (probabilistic) assignment is determined. Given the reported preferences and the assignment, we can judge whether the mechanism is efficient (the assignment is not dominated in a certain sense), fair (the agents are treated fairly according to certain criteria), and incentive compatible (agents prefer to report their preferences truthfully). The mutual compatibility of these three types of properties is the focus of this paper.

Since the formal introduction of the object allocation problem by Hylland and Zeckhauser (1979)[4] there has been a search for "nice" mechanisms that would satisfy these major properties: incentive compatibility, efficiency, and fairness. Hylland and Zeckhauser (1979) propose a pseudo-market mechanism that optimally satisfies the latter two properties: the assignment is always *ex-ante efficient* (the assignment is never Pareto dominated) and *envy-free* (each agent prefers her individual assignment to the assignments of others). However, in the pseudo-market mechanism some agents can benefit by misreporting and therefore the mechanism is not *strategy-proof*. Because of this room for profitable manipulation one cannot tell whether the outcome is fair and efficient under the true preferences.

The further search for "nice" mechanisms that are *strategy-proof* gave rise to a series of negative results. Gale (1987) was the first to conjecture that for an object allocation problem with at least three agents, no mechanism can satisfy *ex-ante efficiency*, *strategy-proofness*, and *anonymity*. (*Anonymity* requires that if any two agents exchange the reports, then their assignments are also exchanged.) Later, Zhou (1990) showed a stronger result, where instead of *anonymity* he used *symmetry*. (*Symmetry* requires that any two agents with identical reported cardinal preferences get the same expected utility; it is implied by *anonymity*).

2 The object allocation problem is also known as the assignment problem and the house allocation problem. Occasionally, I will refer to objects as to houses.

3 Each agent might receive an outside option.

4 Hylland and Zeckhauser considered cardinal input, in this paper I mostly focus on ordinal input, but also incorporate few cardinal axioms.

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Subsequently, in their seminal paper Bogomolnaia and Moulin (2001), hereinafter referred to as BM, show a similar but logically independent impossibility result. BM consider agents with strict *ordinal* preferences over objects (as opposed to cardinal preferences in the papers mentioned above). The preferences of each agent therefore create a partial order over the assignments: each agent always prefers one assignment over another if it first-order stochastically dominates the other assignment. An assignment is *ordinally efficient* if it is not stochastically dominated by any other assignment.[5] BM show the following impossibility result: for the case with at least four agents, no mechanism can satisfy *strategy-proofness*, *ordinal efficiency*, and *equal treatment of equals*. (The latter is the ordinal version of *symmetry* that requires that agents with identical ordinal preferences get identical assignments.)[6]

The goal of this paper is to further study the feasibility set of the "nice" mechanisms and the tradeoff between fairness and efficiency. To do that, throughout the paper I consider only *strategy-proof* mechanisms and change the combination of efficiency and fairness criteria.[7] In choosing these criteria I adhere to the ones that I think are most relevant for the real-life applications.

A very standard efficiency criterion is *ex-post efficiency*. By definition, the *ex-post efficient* mechanism exclusively induces the assignments that can be expressed as lotteries over Pareto efficient deterministic assignments (called matchings). In other words, *ex-post efficient* mechanisms are always able to induce a Pareto efficient matching. *Ex-post efficiency* is implied by *ordinal efficiency*, which in turn is implied by *ex-ante efficiency*.[8] Since in the real-life applications we almost exclusively deal with deterministic assignments, *ex-post efficiency* is a reasonable minimum efficiency requirement.

Regarding fairness, throughout the paper I use *envy-freeness* and few weaker concepts related to it. Introduced by Foley (1967), *envy-freeness* "quickly became the dominant argument of justice within microeconomic theory" (Moulin, 2014).[9] The other, weaker concepts that I use share the following requirement with *envy-freeness*: they all require that for a (certain) subset of agents each agent must prefer (in a certain way) her own assignment to the assignments of other agents within this subset. In other words, agents in this subset should not envy each other (in a certain way). Depending on how strict the fairness notion is, the size of this subset varies as does the strictness of the envy among agents in this subset. I introduce these fairness concepts together with the related results below.

There are five main results in this paper: three impossibilities and two characterizations.

The first result states a general impossibility regarding *ex-post efficiency*. I show that when there are at least three agents, there is no *ex-post efficient*, *envy-free*, and *strategy-proof* mech-

5 *Ordinal efficiency* is also often referred to as *sd-efficiency*.

6 For assignments, *equal treatment of equals* is logically independent from *symmetry*, since *equal treatment of equals* applies to a larger set of preferences, but also has stronger implications than *symmetry*. In the same time, if a mechanism satisfies *equal treatment of equals*, then it automatically satisfies *symmetry*.

7 I stick to (strong) *strategy-proofness* as its weaker version is too permissive. For example, *weak strategy-proofness* (requires that nobody can get a strictly stochastically dominant assignment by manipulation) can be combined with the strongest notions of fairness and efficiency. Besides, (strong) *strategy-proofness* has its practical appeal: it is easier to convince agents to report truthfully if truth-telling is a dominant strategy.

8 Each lottery representation of an *ordinally efficient* assignment contains only efficient matchings, but this property alone does not characterize *ordinal efficiency* as shown by Abdulkadiroğlu and Sönmez (2003b). For the case of $N = 3$ *ex-post efficiency* coincides with *ordinal efficiency*; for the case of $N = 2$ all three efficiency criteria are equivalent.

9 An extensive survey of results on *envy-freeness* and adjacent concepts can be found in Arnsperger (1994), as well as Moulin (2014).

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anism. In fact, Lemma 1 shows an even stronger result in which *envy-freeness* and *strategy-proofness* are substituted by a pair of weaker properties.

This result is most relevant for deterministic assignment mechanisms, such as *dictatorship* mechanisms[10] or mechanisms based on the *top trading cycle* algorithm (TTC).[11] These deterministic mechanisms are usually required to be Pareto efficient and *strategy-proof*, but they can be very unfair ex-post.[12] That is why modifications of these mechanisms may involve randomization in order to restore fairness ex-ante. However, as implied by the first result, in these modifications *envy-freeness* can only be achieved at the cost of either *ex-post efficiency* or *strategy-proofness*.

In the rest of the paper, I further study the tradeoff between fairness and efficiency by relaxing the *envy-freeness* requirement and using few weaker fairness criteria instead.

The second result deals with a direct generalization of *envy-freeness*: *weak envy-freeness*. *Weak envy-free* assignment eliminates only inevitable envy: for each agent her assignment is not first-order stochastically dominated by any other agent's assignment.[13] The second result states that for $N \geq 4$, there is no *weak envy-free*, *ordinally efficient*, and *strategy-proof* mechanism. Together with the previous impossibility result, it shows the tradeoff between efficiency and *envy-freeness* for *strategy-proof* mechanisms: when relaxing the fairness criterion from *envy-freeness* to *weak envy-freeness*, the impossibility threshold for efficiency shifts from *ex-post efficiency* to *ordinal efficiency*. This result complements the well-known impossibility in BM, which uses *equal treatment of equals* instead of *weak envy-freeness*.

Next, I focus on an alternative approach to fairness: the so-called "fair share guaranteed". Here, the agents' assignments are compared not to one another's, as in *envy-freeness*, but to the "fair" assignment of equal division such that each agent receives each object with equal probability $\frac{1}{N}$. And if the assignment first-order stochastically dominates equal division then it satisfies *equal division lower bound*.[14]

The third result states that for $N \geq 4$ there is no *strategy-proof* and *ordinally efficient* mechanism that satisfies *equal division lower bound*. This result is important for a large class of mechanisms that satisfy *equal division lower bound* by construction: either originate from equal division, or make the fare share feasible for each agent. For example, the pseudo-market mechanism by Hylland and Zeckhauser (1979), the core from random endowments mechanism by Abdulkadiroğlu and Sönmez (1998), the top trading cycles from equal division mechanism by Kesten (2009). These mechanisms must fail either in the efficiency dimension, or in the incentive dimension. Indeed, the pseudo-market mechanism and the top trading cycles from equal division mechanism satisfy *ordinal efficiency* but are not *strategy-proof*; while the core from random endowments mechanism is *strategy-proof* but not *ordinally efficient*.

This result can also be observed using the proof of Theorem 2 in Athanassoglou and Sethuraman (2011). The focus in that paper is on an object allocation problem with fractional endowments, and the main axiom is *individual rationality* (the assignment for each agent has to be

[10] In a dictatorship mechanism the assignment is determined by one of the agents, though the acting agent may be constantly changing.

[11] The TTC algorithm is attributed to David Gale, it was introduced in Shapley and Scarf (1974). The mechanisms based on TTC are used in various settings: in school choice, organ donation, and housing problems; for details see Abdulkadiroğlu and Sönmez (2003a), Roth et al. (2004) and Abdulkadiroğlu and Sönmez (2010) correspondingly.

[12] In fact, ex-post fairness is an extremely restrictive property, as shown by Kesten and Yazici (2012).

[13] I discuss the practical relevance of *weak envy-freeness* in Section 4.

[14] An extensive review on comparison to equal division and other notions of fairness for allocation rules can be found in Moulin (2014) and Thomson (2007).

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at least as good as her endowment). Athanassoglou and Sethuraman (2011) show that *ordinal efficiency*, *strategy-proofness* and *individual rationality* are mutually incompatible. Their proof, however, relies on the endowment of equal division, which makes *individual rationality* identical to *equal division lower bound* in the current paper.[15]

Finally, for the case $N = 3$, I present two characterization results for the popular *random serial dictatorship mechanism* (RSD), also known as the random priority mechanism. RSD is *strategy-proof* and *ex-post efficient*, and by adding a fairness notion it can be pinned down. In the first characterization, this fairness notion is *weak envy-freeness among equals*, which requires that for each two agents with identical preferences their assignments do not dominate one another.

This result implies the characterization by BM that uses *equal treatment of equals*. Another implication is that for $N = 3$ RSD can be characterized as the unique mechanism that is *strategy-proof*, *ex-post efficient*, and *weak envy-free*. These results underline the central role of RSD among the *strategy-proof* and *ex-post efficient* mechanisms and fit nicely into the series of other characterization and equivalence results regarding RSD.[16]

In the second characterization the fairness notion is *symmetry*. This characterization implies the impossibility by Zhou (1990), it also implies the characterization by BM (since *symmetry* is implied by *equal treatment of equals*).

Despite the negative results presented in this paper, we can, however, still hope to find a *strategy-proof*, fair, and efficient mechanism in some relevant settings. For large markets in which every object has an increasing number of copies (for example, one can think of seats in one school as copies of a unique seat; the number of seats increases while the number of schools remains the same), Che and Kojima (2010) show that RSD is *asymptotically ordinally efficient*. For the same large market, Kojima and Manea (2010) show that the *ordinally efficient* and *envy-free* probabilistic serial mechanism (first introduced by BM) is *asymptotically strategy-proof*.[17] Liu and Pycia (2013) extend these results by showing that all *regular*, *asymptotically efficient*, *asymptotically strategy-proof* and *symmetric* mechanisms are asymptotically equivalent. In case of cardinal reports, Azevedo and Budish (2013) and He et al. (2015) show that the pseudo-market mechanism is *asymptotically incentive compatible*. Therefore, the impossibility results presented here do not hold asymptotically for large markets.

Table 1 summarizes the main findings of this paper as well as the relevant results of BM.

The paper proceeds as follows: Section 2 introduces the framework, Section 3 presents the impossibility with *ex-post efficiency* (Theorem 1), Section 4 presents the two impossibilities with *ordinal efficiency* (Theorem 2 and Theorem 3), Section 5 presents two characterizations of RSD for $N = 3$ (Proposition 1 and Proposition 2) and corollaries. Section 6 concludes by discussing the implications of the findings and the remaining open questions.

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[15] I am grateful to Acelya Altuntas for observing this connection.

[16] Knuth (1996) and Abdulkadiroğlu and Sönmez (1998) show the equivalence between the symmetrized TTC mechanism and RSD. This result is further generalized in various directions. Pathak and Sethuraman (2011) show the equivalence for symmetrizations within partitions of the object set, and for a similar partitioned symmetrization within the agent set Carroll (2014) shows an invariance with respect to priority structure. Ekici (2015) extends the original equivalence into the existing tenants framework – where some objects are initially owned by some agents. Lee and Sethuraman (2011) show the equivalence for more general hierarchical priority structures. Pycia and Ünver (2015) characterize the set of *group strategy-proof* and Pareto efficient mechanisms, and Bade (2016) shows that symmetrizations of these mechanisms are equivalent to RSD.

[17] Based on the probabilistic serial mechanism Budish et al. (2013) develop fair and efficient mechanisms for various non-standard settings.

Table 1
Summary of results.

		Strategy-proof mechanisms			
		Envy-free	Weak envy-free	Equal division lower bound	Equal treatment of equals
Ex-post efficient	$N=3$	Ø [Theorem 1; BM*]	RSD (unique) [Corollary 3]	RSD (possibly not unique)	RSD (unique) [Corollary 2; BM]
	$N>3$	Ø [Theorem 1]	RSD (possibly not unique)	RSD (possibly not unique)	RSD (possibly not unique)
Ordinally efficient	$N>3$	Ø	Ø [Theorem 2]	Ø [Theorem 3]	Ø [BM]

Notes: The table presents the mutual compatibility of fairness and efficiency within the set of strategy-proof random assignment mechanisms. Ø denotes the empty set, exclamation mark denotes uniqueness, BM stands for Bogomolnaia and Moulin (2001).

* The case of three agents is also mentioned by BM, p. 310, though informally.

2. The model

In this section I introduce the framework: I define the object allocation problem and the axioms.

Let $A = \{a_1, a_2, ..., a_N\}$ be the set of N agents and $H = \{h_1, h_2, ..., h_N\} \cup \{h_0\}$ be the set of N houses and the outside option h_0. Each agent $a \in A$ is endowed with a strict preference relation $\succ_a$ on H with a corresponding weak preference relation $\succcurlyeq_a$. A set of individual preferences of all agents constitutes a **preference profile** $\succ = (\succ_a)_{a \in A}$. Let $\mathcal{R}$ be the set of all possible individual preferences and $\mathcal{R}^N$ be the set of all possible preference profiles. In what follows I assume that the sets A and H are fixed and that the problem is defined by the preference profile $\succ$ only.

A (probabilistic) **assignment** P is a substochastic matrix of size N. (If the outside option h_0 is not available, then P is doubly stochastic.) Each element $P_{a,h}$ in P is a probability with which agent a is assigned house h. Let $\mathcal{P}$ be the set of all possible assignments P. A **matching** μ is a deterministic assignment. Let $\mathcal{M}$ be the set of all possible matchings μ. According to the Birkhoff–von Neumann theorem, any assignment P can be represented as a convex combination of matchings in $\mathcal{M}$ (but this representation is not necessarily unique).

For comparison of different assignments we need the following definitions. A set of houses that agent a weakly prefers to some house h is the **upper contour set of house h at $\succ_a$**: $U(\succ_a, h) = \{h' \in H : h' \succcurlyeq_a h\}$. Given the individual assignment P_a, the overall probability of agent a being assigned some house that is at least as good as house h is her **surplus at h under P_a**: $F(\succ_a, h, P_a) = \sum_{h' \in U(\succ_a, h)} P_{a,h'}$. In other words, the surplus at h is the probability of being assigned some object from the upper contour set of h.

An individual assignment P_a **dominates** another individual assignment P'_a **at** $\succ_a$ (denoted by $P_a \geq_a P'_a$) if it first-order stochastically dominates it.[18] Additionally, P_a **strictly dominates** P'_a (denoted by $P_a >_a P'_a$) if the two assignments are not identical, $P_a \neq P'_a$. Finally, an assignment P **dominates** another assignment P' **at** $\succ$ (denoted by $P \geq P'$) if for each agent a her individual assignment P_a dominates P'_a at $\succ_a$; additionally, P **strictly dominates** P' (denoted by $P > P'$) if the assignments are not identical.

[18] An equivalent condition is that all surpluses of P_a weakly exceed the surpluses of P'_a: for each $h \in H$ $F(\succ_a, h, P_a) \geq F(\succ_a, h, P'_a)$.

Properties of a mechanism From here on we deal with systematic procedures called *mechanisms* that associate each preference profile $\succ \in \mathcal{R}^N$ with some assignment $P \in \mathcal{P}$: $P = \varphi(\succ)$, where φ denotes a mechanism.

Efficiency. A matching is **(Pareto) efficient at a preference profile** if it is not dominated by any other matching at this preference profile. For an assignment efficiency can be defined in three ways: ex-post, ordinal, and ex-ante (the latter I define at the end of this section). An assignment is **ex-post efficient (ExPE) at a preference profile** if it can be represented as a lottery over matchings that are efficient at this preference profile. An assignment is **ordinally efficient (OE) at a preference profile** if it is not stochastically dominated by any other assignments at this preference profile.

Incentive compatibility. A mechanism φ is **strategy-proof (SP)** if at any preference profile no agent can benefit by misreporting her preferences: for each $a \in A$, for each $\succ \in \mathcal{R}^N$ and for each $\succ'_a \in \mathcal{R}$ the following holds: $\varphi(\succ) \geq_a \varphi_a(\succ'_a, \succ_{-a})$.

Now I introduce an auxiliary notion of incentive compatibility which is weaker than *strategy-proofness*; I use this property for the first impossibility result below. This notion restricts the set of (potentially) profitable manipulation strategies for agents. A mechanism is **upper shuffle-proof (USP)** if no agent a can change her surplus at some object h by "shuffling" the objects that are strictly preferred over h (in other words: by misreporting the preferences within the upper contour set of h excluding h itself). Formally, for each $a \in A, h \in H$, and for each $\succ \in \mathcal{R}^N$, $\succ'_a \in \mathcal{R}$ such that $U(\succ_a, h) = U(\succ'_a, h)$, the following holds: $F(\succ_a, h, \varphi_a(\succ)) - \varphi_{ah}(\succ) = F(\succ_a, h, \varphi_a(\succ')) - \varphi_{ah}(\succ')$ (the difference represents the sum of assignment probabilities for houses that are strictly better than h).[19] For example, if $N = 3$ *upper shuffle-proofness* requires that no agent can benefit – in terms of the sum of assignment probabilities for the top two houses – by swapping these two houses. Agents could still possibly benefit: either in some other respect (not in terms of the surplus of the second best object), or from using other strategies (that involve other swaps).

Fairness. An assignment P is **envy-free (EF)** if every agent prefers her assignment to any other agent's assignment: for each $a, a' \in A$ $P_a \geq_a P_{a'}$. An assignment P is **weak envy-free (wEF)** if for each two agents a, a' none of them strictly prefers the assignment of the other: $P_{a'} \not>_a P_a$. **Equal treatment of equals (ETE)** requires that for each $a, a' \in A$ with $\succ_a = \succ_{a'}$ the individual assignments are identical: $P_a = P_{a'}$. A combination of these two properties is **weak envy-freeness among equals**: for each two agents a, a' with identical preferences $\succ_a = \succ_{a'}$ none of them strictly prefers the assignment of the other: $P_{a'} \not>_a P_a$.[20] P satisfies **equal division lower bound (EDLB)** if $P \geq ED$, where ED denotes the equal division assignment.

Next, I introduce two auxiliary notions of fairness. The first notion is a modification of *envy-freeness*: it restricts the set of agents "eligible to envy" to only those agents that have the same upper contour set of some house, and the envy is considered only for this particular house. Formally, an assignment P is **upper envy-free (UEF)** if any two agents with identical upper contour sets of some house h receive equal assignment probabilities of h: for each $a, a' \in A, h \in H$ such that $U(\succ_a, h) = U(\succ_{a'}, h)$ it follows that $P_{ah} = P_{a'h}$. The second notion is a generalization of *equal treatment of equals*. An assignment P satisfies the **strong equal treatment of equals (SETE)** if any two agents with identical preferences from the top house down to some partic-

[19] *Upper shuffle-proofness* is equivalent to *lower invariance* in Mennle and Seuken (2014), which requires that the probabilities of *all* objects ranked below h stay the same, the equivalence can be easily obtained using induction.

[20] *Equal treatment of equals* can be seen as (strong) *envy-freeness* among equals: if two agents have the same preferences, one of them never envies the other if and only if they have identical assignment.

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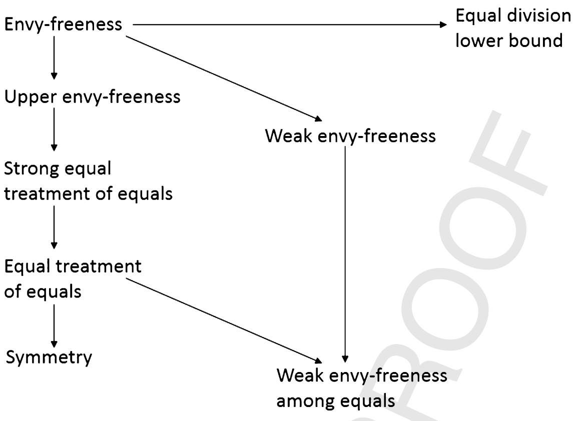

Fig. 1. Logical relations between fairness notions.

ular house receive identical assignments from the top down to that house. Observe that *upper envy-freeness* and *strong equal treatment of equals* differ from the definitions of *envy-freeness* and *equal treatment of equals* in that the set of agents that can be compared is different, namely, is restricted for *envy-freeness* and enlarged for the *equal treatment of equals*.

I also introduce one fairness notion and one efficiency notion for the cardinal framework. Assume that each agent $a \in A$ reports her utility $u_a = \{u_{ah}\}_{h \in H} : u_{ah} \in \mathbb{R}$ for each object $h \in H$. An assignment P is **symmetric** if every two agents a and a' with the same reported utilities $u_a = u_{a'}$ receive equal expected utility: $\sum u_{ah} P_{ah} = \sum u_{a'h} P_{a'h}$. An assignment P is **ex-ante efficient at utility** $U = \{u_a\}_{a \in A}$ if there does not exist any other assignment P' such that for each agent a assignment P provides an (expected) utility that is at least as high as the utility provided by the assignment P': $\sum P_{ah} u_{ah} \geq \sum P'_{ah} u_{ah}$, and, at least for one of the agents, the inequality is strict.

Finally, a mechanism is said to satisfy one of the fairness or efficiency properties introduced above if it always induces assignments with this property.

The efficiency notions can be logically ordered: *ex-post efficiency* is implied by *ordinal efficiency*, which in turn is implied by *ex-ante efficiency*.

The fairness notions can be logically ordered as well, as the following remark shows. Fig. 1 illustrates the remark.

Remark. The following logical relations hold:

1. envy-freeness $\Longrightarrow$ upper envy-freeness $\Longrightarrow$ strong equal treatment of equals $\Longrightarrow$ equal treatment of equals $\Longrightarrow$ symmetry;
2. envy-freeness $\Longrightarrow$ weak envy-freeness $\Longrightarrow$ weak envy-freeness among equals;
3. envy-freeness $\Longrightarrow$ equal division lower bound;
4. weak envy-freeness, equal division lower bound and upper envy-freeness (as well as strong equal treatment of equals and equal treatment of equals) are logically independent;
5. equal treatment of equals $\Longrightarrow$ weak envy-freeness among equals.

The proof is in the appendix.

3. $N \geq 3$ and ex-post efficiency

I begin by studying the tradeoff between the properties of a mechanism when fairness is of a higher concern than efficiency. The following theorem considers the set of *strategy-proof* mechanisms that are moderately efficient (at least *ex-post efficient*) and very fair (*envy-free*, which implies all other fairness criteria). The set of such mechanisms turns out to be empty:

Theorem 1. *For $N \geq 3$ there does not exist a mechanism that is ex-post efficient, strategy-proof, and envy-free.*

This result is a direct corollary to a stronger result of Lemma 1:

Lemma 1. *There does not exist a mechanism that is ex-post efficient, upper shuffle-proof, and upper envy-free.*

Proof. I first prove the claim for $N = 3$ and I do it by contradiction. Suppose there exists a mechanism φ satisfying ExPE, USP an UEF.

For convenience of the proofs I use a novel notation: instead of a preference profile I use a rank table, that is, a matrix $N \times N$ with rows corresponding to agents and columns corresponding to houses. The elements of the table are the ranks of the respective house in the preferences of a respective agent. For instance, for the preference profile $\succ$:

$$\succ: \begin{array}{c||ccc} a_1 & h_1 & h_2 & h_3 \\ a_2 & h_1 & h_3 & h_2 \\ a_3 & h_2 & h_1 & h_3 \end{array},$$

the corresponding rank table $r(\succ)$ is as follows (the superscripts of the elements denote the probabilities in the assignment $\varphi(\succ)$):

$$r(\succ) = \begin{array}{cccc} & (h_1) & (h_2) & (h_3) \\ (a_1:) & 1^{\frac{1}{2}} & 2^{\frac{1}{4}} & 3^{\frac{1}{4}} \\ (a_2:) & 1^{\frac{1}{2}} & 3^{0} & 2^{\frac{1}{2}} \\ (a_3:) & 2^{0} & 1^{\frac{3}{4}} & 3^{\frac{1}{4}} \end{array}.$$

To see that $\varphi(\succ)$ is indeed as shown let us begin with the assignment probabilities of house h_1. Agent a_3 receives zero probability $\varphi_{a_3h_1}(\succ) = 0$ due to ExPE. Agents a_1 and a_2 receive equal probabilities $\varphi_{a_1h_1}(\succ) = \varphi_{a_2h_1}(\succ) = \frac{1}{2}$ since φ satisfies SETE (implied by UEF), otherwise the agent who received less of her top house h_1 might have envied another agent. Next, consider the assignment probabilities of house h_2. Since agent a_2 dislikes house h_2 while agent a_3 prefers this house over others, agent a_2 is never assigned h_2 due to ExPE: $\varphi_{a_2h_2}(\succ) = 0$. Therefore agent a_2 is left with one half of probability of house h_3: $\varphi_{a_2h_3}(\succ) = 1 - \varphi_{a_2h_1}(\succ) = \frac{1}{2}$. Finally, observe that the remaining assignment probability of house h_3 is spread equally between agents a_1 and a_3 due to UEF (their upper contour sets at h_3 are identical). Thus, $\varphi_{a_1h_3}(\succ) = \varphi_{a_3h_3}(\succ) = \frac{1}{4}$ and $\varphi_{a_1h_2}(\succ) = \frac{1}{4}$, $\varphi_{a_3h_2}(\succ) = \frac{3}{4}$.

Next consider another preference profile $\succ'$ that differs from $\succ$ in that agent a_3 copies the report of agent a_1:

$$r(\succ') = \begin{matrix} 1^{\frac{1}{3}} & 2^{\frac{1}{2}} & 3^{\frac{1}{6}} \\ 1^{\frac{1}{3}} & 3^{0} & 2^{\frac{2}{3}} \\ 1^{\frac{1}{3}} & 2^{\frac{1}{2}} & 3^{\frac{1}{6}} \end{matrix}.$$

Agents receive equal probability shares of h_1 due to SETE (implied by UEF): $\varphi_{a_1h_1}(\succ') = \varphi_{a_2h_1}(\succ') = \varphi_{a_2h_1}(\succ') = \frac{1}{3}$. Next, due to ExPE agent a_2 receives zero of h_2 as before: $\varphi_{a_2h_2}(\succ') = 0$. Therefore, $\varphi_{a_2h_3}(\succ') = \frac{2}{3}$ and, again using SETE, $\varphi_{a_1h_2}(\succ') = \varphi_{a_3h_2}(\succ') = \frac{1}{2}$ and $\varphi_{a_1h_3}(\succ') = \varphi_{a_3h_3}(\succ') = \frac{1}{6}$.

Finally, observe that φ cannot satisfy USP since when shifting from $\succ_{a_3}$ to $\succ'_{a_3}$ the agent's a_3 upper contour set at h_3 remains the same ($U(\succ_{a_3}, h_3) = U(\succ'_{a_3}, h_3)$) but the assignment probability has changed. This contradiction completes the proof for $N = 3$.

For $N > 3$ consider the following preference profile $\succ'' \in \mathcal{R}^N$. Agents with indices higher than 3 prefer a house with a corresponding index to all others: $\forall a_i \in A : i > 3, \forall h \in H : h \neq h_i \Longrightarrow \succ''_{a_i} : h_i \succ''_{a_i} h$. Additionally let the first three agents prefer the first three houses to any other house: $\forall a_i \in A : i, j = 1, 2, 3, \forall h \in H : h \neq h_j \Longrightarrow \succ''_{a_i} : h_j \succ''_{a_i} h$. Their preferences for the first three houses are as in $\succ'$:

$$r(\succ''): \begin{matrix} 1 & 2 & 3 & ... & N-1 & N \\ 1 & 3 & 2 & ... & N-1 & N \\ 1 & 2 & 3 & ... & N-1 & N \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & 1 & N \\ ... & ... & ... & ... & N-1 & 1 \end{matrix}.$$

At $\succ''$ due to ExPE φ assigns objects with indices above 3 to the corresponding agents with certainty: for each $i > 3$ $\varphi_{a_ih_i}(\succ'') = 1$. Assume the opposite: for some $j > 3$ $\varphi_{a_jh_j} < 1$. For φ to be ExPE there must be an efficient matching μ for which $\mu(h_j) = a_k \neq a_j$. Let us construct a matching that dominates μ. Let $ind()$ denote index function such that for each $l \leq N$ $ind(a_l) = ind(h_l) = l$. Consider the chain C of agents coupled with corresponding houses that begins with (a_j, h_j) where the next agent in the chain is the agent assigned the house of the previous couple at μ: $C = (a_j, h_j), (a_k, h_k), (\mu(h_k), h_{ind(\mu(h_k))}), ...$ If at some point in C we face one of the first three agents, then the next agent in the chain by construction must be some agent a_m with the index above 3 that is assigned one of the first three houses (there is at least one house among the first three which is assigned to an "outsider" with an index higher than 3), $a_m : (m > 3) \cap (ind(\mu(a_m)) \leq 3)$.[21] Since N is finite and since each agent or object can appear only once in a matching, such a chain C inevitably arrives at the couple $(a_{ind(\mu(a_j))}, \mu(a_j))$ and constitutes a cycle that includes both a_j and h_j. Observe that all agents in C prefer the coupled houses to the houses assigned by μ. Therefore if they swap these houses according to C they arrive at a matching that dominates μ for all agents in C which contradicts the assumption that μ is efficient and that φ is ExPE.

Finally it is left to be seen that for the preference profile $\succ''$ we can use the same arguments as for the case with only three agents as considered above to show that ExPE, USP, and UEF are mutually incompatible. □

[21] In other words we treat the first three agents and the first three houses as just one block-agent and one block-house as compared to others in order to avoid any exchanges between them. For instance, if at μ agent a_3 owns h_k, then there is some agent a_m that owns one of (a_1, a_2, a_3). After the transformation a_3 gives h_k away in exchange for this object previously owned by a_m.

Independence of axioms. If we drop *ex-post efficiency*, equal division satisfies *strategy-proofness* and *envy-freeness* (and, therefore, *upper shuffle-proofness* and *upper envy-freeness*). If we drop *strategy-proofness*, then the probabilistic serial mechanism satisfies *ex-post efficiency* and *envy-freeness* (and *upper envy-freeness*). Finally, if drop *envy-freeness*, RSD satisfies *strategy-proofness* and *ex-post efficiency*. We can also see where exactly the impossibility frontier lies. RSD is not *upper envy-free* (for instance, for the preference profile $\succ$ in the proof above), but it satisfies *strong equal treatment of equals*.[22]

Lemma 1 can be seen as a generalization of the statement in BM (p. 310) about the incompatibility of *ex-post efficiency*, *strategy-proofness*, and no envy for the case of three agents. Here I show the incompatibility of *ex-post efficiency* and two weaker properties: *upper shuffle-proofness* and *upper envy-freeness* for any number of agents.[23]

In the following section we interchange the fairness and efficiency requirements: we relax the fairness criterion and strengthen the efficiency criterion in order to obtain other impossibility results.

4. $N \geq 4$ agents and ordinal efficiency

In this section we consider the case with at least four agents, and focus on the strong notion of efficiency: *ordinal efficiency*. The next result shows the incompatibility of *ordinal efficiency* and *weak envy-freeness* with *strategy-proofness*.

Theorem 2. *For $N \geq 4$ there does not exist a mechanism that is ordinally-efficient, strategy-proof, and weak envy-free.*

The proof is in the appendix.

Independence of axioms. If we drop *ordinal efficiency* (and require only *ex-post efficiency*), then RSD satisfies other properties. If we drop *weak envy-freeness*, then the serial dictatorship mechanism satisfies other properties. If we drop *strategy-proofness*, then the probabilistic serial mechanism satisfies other properties.

This result complements the impossibility result in BM, where instead of *weak envy-freeness* the authors used *equal treatment of equals*. Both these properties are natural relaxations of *envy-freeness*, but, as I argue below, *weak envy-freeness* is a more practical fairness property than *equal treatment of equals* (although not as handy to use in determining assignments and therefore, perhaps, not as popular in the literature).

Weak envy-freeness is important for several reasons. First, it passes a version of the "veil of ignorance" test – at the moment when agents observe the assignment and the preference profile (but do not observe the cardinal preferences yet). Since one agent's assignment is not dominated

[22] The fact that RSD satisfies *strong equal treatment of equals* can be easily seen from the underlying dictatorship procedure of RSD: the assignment probabilities for every house depend only on the preferences for the corresponding upper contour set.

[23] Perhaps BM did not show this impossibility result for the general case since they had a different focus: "For problems involving four agents and more, the impossibility result is more severe" (p. 310). However, the result they show (the incompatibility of *strategy-proofness*, *ordinal efficiency* and *equal treatment of equals*) is logically independent from Theorem 1 and especially from Lemma 1 since *ordinal efficiency* is stronger than *ex-post efficiency*.

by another assignment, there always exists a set of Bernoulli utilities that are consistent with ordinal preferences for which each agent would not envy any other agent.[24]

Secondly, *weak envy-freeness* becomes even more appealing in real-life applications, as compared to the case with abstract rational agents, if we account for the so-called endowment effect. The endowment effect is one of the most robust findings in experimental economics. In these experiments, subjects value the items that they are endowed with significantly more than the items that they can purchase.[25] Similarly, one can expect that an agent is less likely to envy the assignments of others due to the endowment effect: she values more what she is given. This might require some readjustment of her *cardinal* preferences ex-post – after being assigned a lottery, but this readjustment can be relatively mild in the case of the *weak envy-free* assignment because it would not require any change in *ordinal* preferences. However, if an assignment is not *weak envy-free*, then envy can be eliminated only if the agent also changes her ordinal preferences – since some other agent will have a stochastically dominant lottery.

Next we consider the third impossibility result that also uses a strong notion of efficiency and a weak notion of fairness, but this time fairness is defined by *equal division lower bound*.

Theorem 3. *For $N \geq 4$ there does not exist a mechanism that is ordinally-efficient, strategy-proof, and satisfies equal division lower bound.*

The proof is in the appendix.

Independence of axioms. The same examples apply as for Theorem 2: RSD, the serial dictatorship mechanism, and the probabilistic serial mechanism.

Equal division lower bound is a relevant fairness concept, both from the practical perspective and the theoretical perspective. For practice, *equal division lower bound* appears to be important for two main reasons. First, equal division seems to be the most natural fair assignment and thus a natural benchmark to compare all other assignments to. Secondly, equal division is often used in practice, for instance, whenever the assignment is made in the absence of the reported preferences.[26]

For theory, *equal division lower bound* is essential since it is satisfied by few popular mechanisms, such as RSD. Indeed, in the RSD procedure each agent has an equal chance of being the first in the ordering (and thus receiving her first best house), the second (and thus receiving at least her second best) and so on. Therefore, under the RSD assignments all agents are weakly better off than under the uniform lottery. Hence, an important implication of Theorem 3 is the restriction that it puts on the feasibility set of mechanisms that dominate RSD.

Corollary 1. *For $N \geq 4$, in a setting without an outside option, any ordinally efficient mechanism that dominates RSD is not strategy-proof.*

[24] This, however, cannot always be translated for the case of an entire assignment since different pairwise comparisons might require mutually incompatible utilities. An assignment for which such non-envy utilities exist is called *possible envy-free*, which is stricter than *weak envy-free*. This distinction is not very common in the assignment literature since most of the known *weak envy-free* mechanisms are also *possible envy-free*. Moreover, since my focus is on negative results, I also concentrate on the lighter notion of *weak envy-freeness*. For more detail on *possible envy-freeness* see Aziz et al. (2014).

[25] Horowitz and McConnell (2002) provide an extensive review of experiments on the endowment effect.

[26] This is the case in the process of assigning Japanese teachers to Japanese schools abroad (Nihonjin gakkō). Each successful applicant is sent for two to three years to one of more than 80 schools all over the world regardless of his or her actual preferences.

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The corollary, however, does not restrict the set of mechanisms that dominate RSD without being *ordinally efficient*. Thus, in the set of *strategy-proof* mechanisms there might still be some room for improvement upon RSD.[27]

5. Case for $N = 3$

I begin by characterizing RSD as the unique mechanism that satisfies *strategy-proofness*, *ex-post efficiency*, and *weak envy-freeness among equals* in a problem with three agents.

Proposition 1. *(First characterization of RSD) For $N = 3$ a mechanism is strategy-proof, ex-post efficient, and weak envy-free among equals if and only if it is RSD.*

The proof is in the appendix.

We get two immediate corollaries from the proposition.

Corollary 2. *(BM) For $N = 3$, a mechanism is strategy-proof, ex-post efficient, and satisfies equal treatment of equals if and only if it is RSD.*

The second corollary follows from the fact that RSD satisfies *weak envy-freeness* (shown in BM):

Corollary 3. *For $N = 3$, a mechanism is strategy-proof, ex-post efficient and weak envy-free if and only if it is RSD.*

I now extend the previous proposition by another characterization in which I use a cardinal fairness criterion: *symmetry*. Although in its sense *symmetry* is similar to *weak envy-freeness among equals* (equal agents receive individual assignments that do not dominate one another), these two properties are logically independent: *symmetry* applies to a smaller subset of utility domain, but for this subset of utilities it also has stronger implications.

Proposition 2. *(Second characterization of RSD) For $N = 3$, a mechanism is strategy-proof, ex-post efficient, and symmetric if and only if it is RSD.*

The proof is in the appendix.

Given this result we can obtain the famous impossibility result by Zhou (1990): *ex-ante efficiency*, *strategy-proofness*, and *symmetry* are mutually incompatible.

Corollary 4. *(Zhou, 1990) For $N \geq 3$, there does not exist a mechanism that is strategy-proof, ex-ante efficient and symmetric.*

For the case $N = 3$ the proof is just the combination of Proposition 2 and the fact that RSD is not *ex-ante efficient* but only *ex-post efficient*. For the general case $N \geq 3$ we construct a preference profile as it is done in the second part of the proof of the Theorem 1. For this preference profile the problem is effectively reduced to the size of three.

[27] Corollary 1 complements the result in Erdil (2014): in a case with an outside option, any non-wasteful mechanism that dominates RSD is not *strategy-proof*.

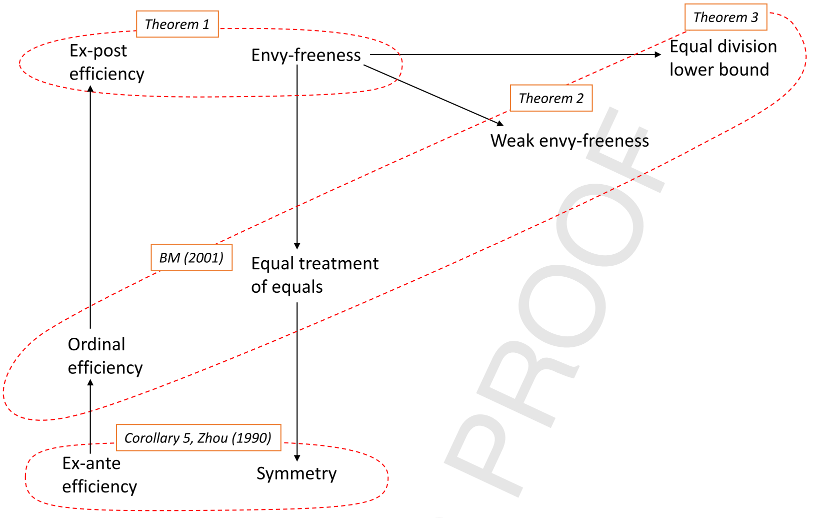

Fig. 2. Tradeoff between fairness and efficiency in an assignment: impossibilities. Notes: Dashed lines denote the mutual incompatibility in the class of strategy-proof mechanisms. The dashed line in the middle corresponds to three impossibilities at once: of ordinal efficiency and one of the contained fairness notions; it also applies to the case $N \geq 4$ whereas two other results apply for $N \geq 3$.

6. Conclusions

This paper considers the standard object allocation problem of assigning N indivisible objects to N agents and shows the impossibility for a *strategy-proof* mechanism to be simultaneously fair and efficient (in three specific ways). Theorem 1 shows the impossibility of combining a weak notion of efficiency – *ex-post efficiency*, with a strong notion of fairness – *envy-freeness*; it is the first known impossibility result in the related literature that involves *ex-post efficiency*. Theorem 2 shows the impossibility for the opposite set of properties: a weak notion of fairness – *weak envy-freeness* and a strong notion of efficiency – *ordinal efficiency*. Finally, Theorem 3 shows a similar impossibility result with a different weak fairness notion: *equal division lower bound* (see Fig. 2).

The paper also shows that for the case of three agents, the trinity of *strategy-proofness*, *ex-post efficiency*, and *weak envy-freeness* for agents with identical preferences uniquely defines the random serial dictatorship mechanism. Alternatively, if we use *symmetry* instead of *weak envy-freeness among equals*, we get the same characterization of the random serial dictatorship mechanism.

The first theorem is, perhaps, of the highest importance for the practical implementation of matching and assignment mechanisms since it deals with the commonly required properties of *strategy-proofness* and *ex-post efficiency*. The other two theorems resemble the impossibility result of Bogomolnaia and Moulin (2001), although, as argued in the current paper, with, perhaps, more relevant notions of fairness.

The results of the paper also fit in the recent literature that supports the central role of RSD among other mechanisms. This literature shows the equivalence of RSD to versions of other mechanisms used in practice. For example, Abdulkadiroğlu and Sönmez (1998) show that RSD is equivalent to the *core from random endowments* mechanism, that initially randomly allocates objects and then proceeds by using the *top trading cycles (TTC)* algorithm in which agents voluntarily exchange the objects that they are endowed with. A recent paper by Bade (2014) generalizes this equivalence result to the set of all symmetrized Pareto optimal, *strategy-proof*, and non-bossy mechanisms. Finally, in specific cases, the equivalence holds for the celebrated *deferred acceptance* mechanism introduced by Gale and Shapley (1962), which is often used for the two-sided matching problems such as the school choice problem (as well as the college admission problem and job placement problem). The mechanism is equivalent to RSD in case schools are initially indifferent between students and the ties are broken randomly for all schools together.

This paper also supports the use of RSD in object allocation problems, when *strategy-proofness* is of high importance. As argued throughout the paper, *strategy-proofness*, *ex-post efficiency* and *weak envy-freeness* are strongly desirable properties for a mechanism used in real-life applications, while *equal division lower bound* might be important when switching from one assignment procedure to another. Not only does RSD possess all four of these properties, but also, as this paper demonstrates, it is impossible to improve on any of the weak properties without violating another: to demand *ordinal efficiency* instead of *ex-post efficiency*, or *envy-freeness* instead of *weak envy-freeness* or *equal division lower bound*.

The central role of RSD as the *strategy-proof* mechanism becomes even more apparent in the problem with three agents. As this paper demonstrates, RSD is the unique *strategy-proof* and *ex-post efficient* mechanism that satisfies some of the weakest (among those presented here) fairness notions: *symmetry* or *weak envy-freeness among equals*. It, however, remains unclear, what combination of properties, together with *strategy-proofness*, characterizes RSD for the general case.[28] The characterization in this paper cannot be directly generalized even for the case of four agents (however, there are also no counter examples found). The reason for this complication is that *weak envy-freeness* (and especially *weak envy-freeness among equals*) is not handy enough as compared to the *equal treatment of equals*. For instance, for two agents with identical preferences *weak envy-freeness* gives precise implications only in case these agents receive identical probabilities for all but two objects. Then the two agents have to have the same assignment for these two objects as well. *Equal treatment of equals*, on the contrary, has implications for the assignment probabilities of all objects. Therefore, I believe, generalizing this characterization would be more difficult than the result that uses *equal treatment of equals*.

Another open question is to what extent one of the three properties can be satisfied should the other two be taken at their extreme. For instance, if *ordinal efficiency* and *envy-freeness* are satisfied, then the probabilistic serial mechanism appears to be the "most" incentive compatible mechanism since it is weakly invariant (limits the set of profitable deviations) and *weakly strategy-proof* (which means that no agent can receive a stochastically dominant assignment by manipulating[29]). Similarly, one could be interested in the "most fair" mechanism that satisfies *strategy-proofness* and *ordinal efficiency* (since the only known serial dictatorship mechanism is very unfair), and in the "most" efficient mechanism that satisfies *strategy-proofness* and *envy-freeness* (again, the only known equal division "mechanism" disregards preferences and is therefore almost always inefficient).

[28] A remarkable characterization of RSD by Pycia and Troyan (2017) uses a stronger notion called *obvious strategy-proofness*, recently introduced by Li (2015).

[29] Probabilistic serial can be characterized by *weak strategy-proofness* for $N = 3$ (BM), but not for $N \geq 5$ as shown by Kesten et al. (2016).

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Appendix A

The proofs below involve *relabeling* the agents and objects in order to show the equivalence between different preference profiles. In general, we are not free to relabel agents or houses without changing the assignment, as that would require the mechanism to be *neutral* toward the "name tags" of the agents and the houses so that the assignment is defined solely by the preference profile. I do not assume this type of neutrality. But if we use the properties of a mechanism (e.g., efficiency, strategy-proofness, fairness) in order to pin down specific values of some assignment probability, then since these properties should also hold for the same (or close enough) preference profile regardless of the name tags, we can find the same probability values for this other preference profile. In other words, all the mechanism's properties that I consider are essentially neutral, i.e., invariant with respect to any relabeling transformation. For instance, an ex-post efficient mechanism remains ex-post efficient regardless of any relabeling, a strategy-proof remains strategy-proof, and so forth. The following Claim expresses this idea more formally:

Claim. If for some mechanism φ and some preference profile $\succ \in \mathcal{R}^N$ one can determine the value of some element in $\varphi_{ah}(\succ), a \in A, h \in H$ using the properties of φ, then this value $\varphi_{ah}(\succ)$ remains the same after any relabeling of agents and houses.

Proof of the Remark in section 2

Proof. *EF* $\Longrightarrow$ *UEF.* We need to show that for each envy-free assignment P it follows that for each $a, a' \in A$ and each $h \in H$ if $U(\succ_a, h) = U(\succ_{a'}, h)$ then $P_{ah} = P_{a'h}$. First observe that $F(\succ_a, h, P_a) = F(\succ_{a'}, h, P_{a'})$ since otherwise one of the two agents might envy another (e.g., if she is almost indifferent between all objects in her upper contour set of h). Then observe that $F(\succ_a, h_a, P_a) = F(\succ_{a'}, h_{a'}, P_{a'})$ where h_a and $h_{a'}$ are the least preferred objects in $U(\succ_a, h)\backslash\{h\}$ and $U(\succ_{a'}, h)\backslash\{h\}$ respectively − for the same reason as earlier. Finally $P_{ah} = F(\succ_a, h, P_a) - F(\succ_a, h_a, P_a)$ and $P_{a'h} = F(\succ_{a'}, h, P_{a'}) - F(\succ_{a'}, h_{a'}, P_{a'})$ and thus $P_{ah} = P_{a'h}$, which completes the proof.

UEF $\Longrightarrow$ *SETE*. Here we need to show that for each UEF assignment P it follows that for each $a, a' \in A$ with identical preferences down to some $h \in H$ the assignment down to this h is the same. More formally, for each $h' \in H$ such that $h' \succ_a h$ and $h' \succ_{a'} h$ it follows that $P_{ah'} = P_{a'h'}$. We prove by induction: consider the top object $h_1 : h_1 \succ_a h'$ for each $h' \in H$ (and $h_1 \succ_{a'} h'$ since the preferences down to h are identical). Using the upper envy-freeness for h_1 (since $U(\succ_a, h_1) = U(\succ_{a'}, h_1)$) we get $P_{ah_1} = P_{a'h_1}$. We then do it for the second top object and so forth until we reach h which would complete the proof.

SETE $\Longrightarrow$ *ETE.* Obvious from the definition of SETE.

ETE $\Longrightarrow$ *wEFE.* Obvious from the definition of wEFE.

EF $\Longrightarrow$ *wEF.* Obvious from the definition of wEF.

wEF $\Longrightarrow$ *wEFE.* Obvious from the definition of wEFE.

EF $\Longrightarrow$ *EDLB.* Consider some agent $a \in A$ and her top object $h_1 \in H$. Since the assignment P is envy-free there is no agent a' with $P_{a'h_1} > P_{ah_1}$ (otherwise a could possibly envy a'). Therefore agent a gets at least her fair share of object h_1 of $\frac{1}{N}$. Next, consider the two top objects $\{h_1, h_2\}$ of agent a. Similarly, there is no agent a' with the total probability $(P_{a'h_1} + P_{a'h_2})$ higher than the total probability of agent a for the same two objects (otherwise a would envy a' once she is indifferent between h_1 and h_2 and does not care as much about the rest). Therefore the total probability $(P_{ah_1} + P_{ah_2})$ is at least as high as the fair share $\frac{2}{N}$. We use the same logic for the other objects and find that agent a is weakly better off under P than under the equal division.

ETE $\Longrightarrow$ symmetry. Obvious.

Independence of weak properties. Finally, it is left to show the mutual independence of the weak notions of fairness which is fairly easy to do by a contour example for each two notions. Indeed, these examples are easy to come up with since all the notions have a different nature: UEF, SETE, ETE, and symmetry can be applied to those preference profiles in which for some agents the preferences are (partially) identical; these properties require the corresponding assignment probabilities to be equal. On the other hand, wEF and EDLB apply to all preference profiles and do not require equalities. Comparing wEF and EDLB, wEF compares assignments between different agents, while EDLB compares them to the fair division. □

Proof of Theorem 2

Proof. I prove by contradiction: assume that there exists a mechanism φ that is OE, SP, and wEF.

First observe that it is enough to prove the claim for the problem where $N = 4$. For the case of more agents, consider the preference profiles similar to the type used in the proof of Theorem 1, namely, where the first four agents prefer the first four houses over all other houses, and other agents prefer the corresponding house of their own index to any other house. Due to OE, all agents with indices higher than 4 receive the corresponding houses with certainty and the object allocation problem is reduced to the size of four.

We begin with the following preference profile:

$$r(\succ^1) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2^0 & 1^{\frac{2}{3}} & 3^{\frac{1}{3}} & 4^0 \\ 3 & 2 & 4^0 & 1^1 \end{array}.$$

Due to OE, and using Corollary 3 we find that $\varphi(\succ^1) = RSD(\succ^1)$. Indeed, agent a_4 is assigned house h_4 with certainty and we can repeat the same arguments used in the proof of Proposition 1 to determine the assignment $\varphi(\succ^1)$.

Consider now two different preference profiles $\succ^2$ and $\succ'^2$:

$$r(\succ^2) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \\ 2 & 1 & 4^0 & 3 \end{array}, \quad r(\succ'^2) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \end{array}.$$

Since φ is OE at $\succ^2$ and $\succ'^2$, at least two agents – but not necessarily all four agents – receive zero probability of their worst houses (it is exactly for this reason that we need to consider two profiles and not just one). W.l.o.g. assume that these are agents a_3, a_4 for $\succ^2$ and a_2, a_4 for $\succ'^2$ (otherwise we can relabel the houses): $\varphi_{a_3h_3}(\succ^2) = \varphi_{a_4h_3}(\succ^2) = 0$ and $\varphi_{a_2h_3}(\succ'^2) = \varphi_{a_4h_3}(\succ'^2) = 0$. We proceed with $\succ^2$ and for the profile $\succ'^2$ the argumentation line would be identical.

$$r(\succ^3) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2^0 & 1^{\frac{2}{3}} & 4^{\frac{1}{3}} & 3^0 \\ 3^0 & 2^0 & 4^0 & 1^1 \end{array}, \quad r(\succ^4) = \begin{array}{cccc} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{array}.$$

Now consider a preference profile $\succ^3$ that can be obtained from $\succ^2$ by changing the preferences of agent a_4 or from $\succ^1$ by changing the preferences of agent a_3.

On the one hand in the assignment of agent a_4 $\varphi_{a_4h_1}(\succ^3) = \varphi_{a_4h_2}(\succ^3) = 0$ due to ExPE of φ and $\varphi_{a_4h_3}(\succ^3) = 0$ due to SP (otherwise agent a_4 might deviate to preference profile $\succ^2$). Therefore $\varphi_{a_4h_4}(\succ^3) = 1$ and $\varphi_{a_3h_4}(\succ^3) = 0$. On the other hand in the assignment of agent a_3 due to SP $\varphi_{a_3h_1}(\succ^3) = 0$ and $\varphi_{a_3h_2}(\succ^3) = \frac{2}{3}$ as it was in the preference profile $\succ^1$.

Next consider the preference profile $\succ^4$ obtained from $\succ^3$ with agents a_3 and a_4 having now identical preferences.

Observe first that $\varphi_{a_3h_3}(\succ^4) = \frac{1}{3}$ remains the same as in $\succ^3$ due to SP. Secondly, due to ExPE $\varphi_{a_1h_4}(\succ^4) = \varphi_{a_2h_4}(\succ^4) = 0$ and $\varphi_{a_3h_1}(\succ^4) = \varphi_{a_4h_1}(\succ^4) = 0$. Thirdly, agent a_4 has to have the same assignment as agent a_3 since their preferences are identical and we could therefore follow the same procedure where a_3 and a_4 are swapped (namely pick a_3 in $\succ^2$ and construct a profile analogous to $\succ^1$). Therefore $\varphi_{a_3h_4}(\succ^4) = \varphi_{a_4h_4}(\succ^4) = \frac{1}{2}$ and $\varphi_{a_3h_2}(\succ^4) = \varphi_{a_4h_2}(\succ^4) = \frac{1}{6}$.

Now we will change the preferences of agents a_3 and a_4 sequentially so that they look symmetric to the preferences of a_1 and a_2. Consider the preference profile $\succ^5$ in which agent a_4 swaps her third and fourth best houses as compared to $\succ^4$.

$$r(\succ^5) = \begin{array}{cccc} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{array}, \quad r(\succ^6) = \begin{array}{cccc} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{array}, \quad r(\succ'^6) = \begin{array}{cccc} 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{array}.$$

Observe that $\varphi_{a_4h_4}(\succ^5) = \frac{1}{2}$ and $\varphi_{a_4h_2}(\succ^5) = \frac{1}{6}$ due to SP and also that $\varphi_{a_1h_4}(\succ^5) = \varphi_{a_2h_4}(\succ^5) = 0$ and $\varphi_{a_3h_1}(\succ^5) = \varphi_{a_4h_1}(\succ^5) = 0$ due to OE. Therefore $\varphi_{a_3h_4}(\succ^5) = \varphi_{a_4h_4}(\succ^5) = \frac{1}{2}$ and using wEF for a_3 and a_4 we then get that $\varphi_{a_3h_2}(\succ^4) = \varphi_{a_4h_2}(\succ^4) = \frac{1}{6}$.

Now we do the same swap with houses h_1 and h_3 in the preferences of agent a_3 and get the preference profile $\succ^6$. We calculate her assignment using the same argument as above.[30]

This result is derived from the fact that $\varphi_{a_3h_3}(\succ^2) = \varphi_{a_4h_3}(\succ^2) = 0$. But if we use the same procedure for $\succ'^2$ instead of $\succ^2$ then we get the assignment for a profile $\succ'^6$.

The preference profile $\succ'^6$ is effectively identical to $\succ^6$ if we relabel houses h_1 and h_4 and agents a_1 and a_4. Due to the Claim at the beginning of this section we can conclude that agent a_2 at $\succ^6$ has to have the same assignment as at $\succ'^6$: $\varphi_{a_2h_1}(\succ^6) = \varphi_{a_2h_4}(\succ'^6) = \frac{1}{2}$, $\varphi_{a_2h_2}(\succ^6) = \varphi_{a_2h_2}(\succ'^6) = \frac{1}{6}$ and $\varphi_{a_2h_3}(\succ^6) = \varphi_{a_2h_3}(\succ'^6) = \frac{1}{3}$. Then the full assignment at $\succ^6$ is as follows:

$$r(\succ^6) = \begin{array}{cccc} 1^{\frac{1}{2}} & 2^{\frac{1}{2}} & 3^0 & 4^0 \\ 1^{\frac{1}{2}} & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{array}.$$

Finally, agent a_2 strongly envies agent a_1, which is a contradiction. □

[30] If in $\succ^6$ we relabel houses h_1 and h_4 and then swap agents a_1, a_2 and, on the other hand, a_3, a_4, then we get the same preference profile $\succ^6$. However, we would not be able to draw any conclusion regarding the assignment for agents a_1 and a_2 at $\succ^6$ (agents a_3, a_4 after relabeling) since we did not determine the specific values and cannot use the logic of the Claim. For this reason we need a parallel procedure that begins with $\succ'^2$ and ends with $\succ'^6$.

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Proof of Theorem 3

Proof. The proof is by contradiction. Assume that φ satisfies OE, SP and EDLB.

As before, it is sufficient to prove the claim for the case $N = 4$. These preference profiles have the following rank tables, we consider them sequentially:

$$r(\succ^1) = \begin{array}{cccc} 1^{\frac14} & 2^{\frac12} & 3^0 & 4^{\frac14} \\ 1^{\frac14} & 2^{\frac12} & 3^0 & 4^{\frac14} \\ 1^{\frac14} & 3^0 & 2^{\frac12} & 4^{\frac14} \\ 1^{\frac14} & 3^0 & 2^{\frac12} & 4^{\frac14} \end{array}, \quad r(\succ^2) = \begin{array}{cccc} 1^{\frac14} & 2^{\frac12} & 4^0 & 3^{\frac14} \\ 1^{\frac14} & 2^{\frac12} & 3^0 & 4^{\frac14} \\ 1^{\frac14} & 3^0 & 2^{\frac12} & 4^{\frac14} \\ 1^{\frac14} & 3^0 & 2^{\frac12} & 4^{\frac14} \end{array}, \quad r(\succ^3) = \begin{array}{cccc} 1^{\frac14} & 2^{\frac12} & 4^0 & 3^{\frac14} \\ 1^{\frac14} & 2^{\frac12} & 4^0 & 3^{\frac14} \\ 1^{\frac14} & 3^0 & 2^{\frac12} & 4^{\frac14} \\ 1^{\frac14} & 3^0 & 2^{\frac12} & 4^{\frac14} \end{array},$$

$$r(\succ^4) = \begin{array}{cccc} 1^{\frac14} & 2^{\frac14} & 4^0 & 3^{\frac12} \\ 1^{\frac14} & 2^{\frac14} & 4^0 & 3^{\frac12} \\ 1^{\frac14} & 2^{\frac14} & 3^{\frac12} & 4^0 \\ 1^{\frac14} & 2^{\frac14} & 3^{\frac12} & 4^0 \end{array}, \quad r(\succ^5) = \begin{array}{cccc} 1^{\frac14} & 2 & 4^0 & 3 \\ 1^{\frac14} & 2 & 4^0 & 3 \\ 1^{\frac14} & 2^{\frac14} & 3^{\frac14} & 4^{\frac14} \\ 1^{\frac14} & 3^0 & 2^{\frac34} & 4^0 \end{array}, \quad r(\succ^6) = \begin{array}{cccc} 1^{\frac14} & 2^{\frac14} & 4 & 3 \\ 1^{\frac14} & 2^{\frac14} & 4 & 3 \\ 1^{\frac14} & 2^{\frac14} & 4 & 3 \\ 1^{\frac14} & 3^{\frac14} & 2 & 4 \end{array}.$$

First, consider the preference profile $\succ^1$. Due to EDLB each agent has a right to receive at least $\frac{1}{4}$ of her most preferred house h_1 and at most $\frac{1}{4}$ of her least preferred house h_4. Then, due to ordinal efficiency, either $\varphi_{a_1h_3}(\succ^1) = \varphi_{a_2h_3}(\succ^1) = 0$ or $\varphi_{a_3h_2}(\succ^1) = \varphi_{a_4h_2}(\succ^1) = 0$. Since the assignment of h_1 and h_4 is determined, it turns out that both conditions hold.

Consider now a profile $\succ^2$ that is derived from the previous profile using the swap of houses h_3 and h_4 in the preferences of agent a_1. The assignment $\varphi(\succ^2)$ is the same as before for the following reasons. First, the assignment of house h_1 is symmetric due to EDLB. Second, $\varphi_{a_1h_2}(\succ^2) = \frac{1}{2}$ because of SP (otherwise agent a_1 might deviate from/to $\succ^1$). Third, $\varphi_{a_1h_3}(\succ^2) = 0$ due to OE. As a result, we find the remaining element $\varphi_{a_1h_4}(\succ^2) = \frac{1}{4}$. Therefore, the assignment of house h_4 is again symmetric due to EDLB. Finally, using OE we find the assignment of houses h_2 and h_3: $\varphi_{a_1h_3}(\succ^2) = \varphi_{a_2h_3}(\succ^2) = 0$ and $\varphi_{a_3h_2}(\succ^2) = \varphi_{a_4h_2}(\succ^2) = 0$ (again: only one of these conditions has to be satisfied due to OE, but in fact both of them hold because of the previous findings).

Next, consider the preference profile $\succ^3$ derived using the same swap of houses h_3 and h_4 but this time for agent a_2. It turns out that the assignment is again the same. First, $\varphi_{a_1h_3}(\succ^2) = \varphi_{a_2h_3}(\succ^2) = 0$ due to ExPE. Second, both $\varphi_{a_1h_2}(\succ^3)$ and $\varphi_{a_2h_2}(\succ^3)$ are equal to $\frac{1}{2}$ because of SP (otherwise one of the two agents a_1, a_2 would have switched from/to preference profile $\succ^2$). The rest of the assignment can be found using EDLB as before.

Next, we consider a different preference profile $\succ^4$ in which the agents have opposite tastes regarding the other pair of houses: h_3 and h_4 (and not h_2 and h_3 as before). The assignment $\varphi(\succ^4)$ can be determined using the same argumentation line as in the case of $\succ^1$.

Finally, we consider the preference profile $\succ^5$, which can be derived from the profile $\succ^4$ using a swap of houses h_2, h_3 in the preferences of agent a_4, *and at the same time* from the profile $\succ^3$ using the swap of houses h_2, h_3 in the preferences of agent a_3. The assignment $\varphi(\succ^5)$ can be determined using the following arguments. First, since φ is SP, the elements $\varphi_{a_3h_4}(\succ^5)$ and $\varphi_{a_4h_4}(\succ^5)$ must correspond to the elements of $\varphi(\succ^3)$ and $\varphi(\succ^4)$ respectively: $\varphi_{a_3h_4}(\succ^5) = \frac{1}{4}$ and $\varphi_{a_4h_4}(\succ^5) = 0$. Second, due to OE, and since $\varphi_{a_3h_4}(\succ^5) = \frac{1}{4} > 0$, agents a_1, a_2 get zero of h_3: $\varphi_{a_1h_3}(\succ^5) = \varphi_{a_2h_3}(\succ^5) = 0$. Third, due to OE $\varphi_{a_4h_2}(\succ^5) = 0$. Fourth, the assignment of house h_1 is identical due to EDLB. Therefore $\varphi_{a_4h_3}(\succ^5) = \frac{3}{4}$ and then $\varphi_{a_3h_3}(\succ^5) = \frac{1}{4}$ and $\varphi_{a_3h_2}(\succ^5) = \frac{1}{4}$.

The fact that $\varphi_{a_3h_2}(\succ^5)$ equals $\frac{1}{4}$ and is therefore different from $\varphi_{a_1h_2}(\succ^5)$ or $\varphi_{a_2h_2}(\succ^5)$ (since their sum has to be equal to one) contradicts SP. Indeed, consider the profile $\succ^6$ which is different from $\succ^5$ in that agent a_3 swaps her preferences for houses h_3 and h_4 and thus becomes identical to agents a_1 and a_2. Since each of the agents a_1, a_2, a_3 could swap their least preferred houses h_3, h_4 in order to deviate from/to $\succ^5$, due to SP we conclude that $\varphi_{a_1h_2}(\succ^6) = \varphi_{a_2h_2}(\succ^6) = \varphi_{a_3h_2}(\succ^6) = \frac{1}{4}$ and therefore $\varphi_{a_4h_2}(\succ^5) = \frac{1}{4}$ which contradicts the OE. □

Proof of Proposition 1 (First characterization of RSD)

I now use the Claim in order to restrict the attention to only six types of preference profiles (since all other preference profiles are equivalent to one of these) and pin down all the assignment probabilities.

Proof. We know that RSD is strategy-proof, ex-post efficient, and satisfies *weak envy-freeness among equals*. In order to show the other direction, we first consider the case in which all agents prefer each house over the outside option. We sequentially check all possible the preference profiles.

Let φ satisfy SP, ExPE and wEFE.

For $N = 3$ there are the following six types of preference profiles (any other preference profile can be represented as one of these after the relabeling of agents and houses as discussed in the Claim above):

$$\text{type 1 (2 profiles): } \begin{cases} h_1 \succ_{a_1} h_3 \succ_{a_1} h_2 \\ h_1 \succ_{a_2} h_3 \succ_{a_2} h_2 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases}, \quad \text{type 4 (1 profile): } \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_1 \succ_{a_3} h_2 \succ_{a_3} h_3 \end{cases},$$

$$\text{type 2 (2 profiles): } \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_3 \succ_{a_2} h_2 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases}, \quad \text{type 5 (2 profiles): } \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases},$$

$$\text{type 3 (1 profile): } \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_1 \succ_{a_3} h_3 \succ_{a_3} h_2 \end{cases}, \quad \text{type 6 (8 profiles): } \begin{cases} h_1 \succ_{a_1} (h_2, h_3) \\ h_2 \succ_{a_2} (h_1, h_3) \\ h_3 \succ_{a_3} (h_2, h_3) \end{cases}.$$

We begin with the profile of type 1. Due to ExPE we get $\varphi_{a_3h_2} = 1$. Therefore agents a_1 and a_2 receive equal expected shares of the remaining houses $\varphi_{a_1h_1} = \varphi_{a_2h_1} = \varphi_{a_1h_2} = \varphi_{a_2h_2} = \frac{1}{2}$, otherwise one of them weakly envies another which contradicts wEFE.

In type 2, due to SP, agent a_2 receives the same expected share of house h_1 as before in type 1: $\varphi_{a_2h_1} = \frac{1}{2}$. Using ExPE we get $\varphi_{a_2h_2} = \varphi_{a_3h_1} = 0$ and thus $\varphi_{a_1h_1} = \varphi_{a_2h_3} = \frac{1}{2}$. Suppose also $\varphi_{a_1h_3} = x \in [0, \frac{1}{2}]$. Then the remaining probabilities are as follows: $\varphi_{a_1h_2} = \varphi_{a_3h_3} = \frac{1}{2} - x$ and $\varphi_{a_3h_2} = \frac{1}{2} + x$.

Next, consider the preference profile of type 3. Since both agents a_1 and a_2 can transform this profile to one of type 2 considered above by switching their top objects, due to SP we get: $\varphi_{a_1h_3} = \varphi_{a_2h_3} = \frac{1}{2} - x$. (Here we implicitly used the Claim above.) Using wEFE for these two agents and the fact that $\varphi_{a_3h_2} = 0$ due to ExPE, we get $\varphi_{a_1h_2} = \varphi_{a_2h_2} = \frac{1}{2}$ and $\varphi_{a_1h_1} = \varphi_{a_2h_1} = x$. Consequently, the remaining expected share of house h_1 goes to agent a_3: $\varphi_{a_3h_1} = 1 - 2x$.

Finally, consider the symmetric preference profile of type 4. Each agent can swap her second and third choices and transform the preference profile to that of type 3. Due to SP, their expected shares of the top house h_1 are all equal: $\varphi_{a_1h_1} = \varphi_{a_2h_1} = \varphi_{a_3h_1} = 1 - 2x$. Therefore $x = \frac{1}{3}$ and the assignments of types 1–4 are identical to RSD assignments.[31]

Now that we have determined the unknown x it is easy to show that the assignments for the remaining profiles are also equal to RSD. This concludes the proof for the case where each object is preferred over the outside option.

Assume now that some agents prefer the outside option over some houses. Let there be a preference profile $\succ'$ for which φ does not coincide with RSD for some agent a and some house h: $\varphi_{ah}(\succ') \neq RSD_{ah}(\succ')$. Let us refer to such preference profile $\succ'$ as *disturbing*, and such agent i as *disturbed in h at* $\succ'$.

W.l.o.g., we can focus on a preference profile $\succ$ such that for each agent a who is disturbed at $\succ$, the reported preference $\succ_i$ is not truncated: for each $h \neq h_0$ $h \succ_a h_0$. This is true since we can arrive to such preference profile $\succ$ starting from any other disturbing preference profile $\succ'$ by extending the truncated report for one of the agents that are disturbed at $\succ'$. Due to strategy-proofness of φ the assignment of agent a should not differ (except for the objects that extend the report).

W.l.o.g., we can focus on a preference profile $\succ$ such that there are two agents a, a' that are disturbed at $\succ$, while the third agent a'' is not disturbed and has truncated preferences. This is true since for each disturbed agent a due to sub-stochasticity of $\varphi(\succ)$ and ExPE, there must be at least one other disturbed agent $a' \neq a$. The third agent a'' is not disturbed at $\succ$ and has truncated preferences since otherwise $\succ$ belongs to one of the six types listed above for which we showed the result already. The disturbed agents are disturbed at least in two objects since their reports are not truncated and thus the assignment probabilities sum up to one.

Preference profile $\succ$ can be one of the six types above, up to one or two houses truncated for one of the agent. (If all three houses are truncated, then the problem is reduced to size 2 and the result is obvious.) We go over these types one by one and agent by agent and arrive to a contradiction.

Consider type 1. If the agent with a truncated report $a'' = a_3$, and other two agents are disturbed, then their assignments mutually violate wEFE. Agent a_3 cannot be disturbed as he gets h_2 with probability 1 due to ExPE.

Consider type 2. If the agent a with truncated report $a'' = a_1$, then a_2 is disturbed in h_1 (since $\varphi_{a_2h_2} = 0$ due to ExPE). If $\varphi_{a_2h_1}(\succ) < \frac{1}{2} = RSD_{a_2h_1}(\succ)$, then a_2 can mimic a_1 to get $\frac{1}{2}$ of h_1 due to wEFE. In case $\varphi_{a_2h_1}(\succ) < \frac{1}{2}$, he could mimic in the opposite way, therefore a_2 is not disturbed. The same in case the agent with a truncated report $a'' = a_3$.

If the agent with a truncated report $a'' = a_2$, then a_1 is disturbed in h_3 (and h_2), and not in h_1 since agent a_3 gets zero probability of h_1 due to ExPE. Assume that $h_1 \succ_{a_3} h_3$ (in the opposite case, we consider a profile such that $h_1 \succ_{a_3} h_3$ in which agent a_3 is also disturbed in h_2). Then agent a_1 can change the preference profile $\succ$ to the one of type 1 by swapping h_1 and h_2 (up to relabeling agents and objects), which leads to a contradiction.

Consider type 3. If the agent with a truncated report $a'' = a_1$, then a_2 and a_3 are disturbed in h_1 and h_3 (and not in h_2 since a_3 gets zero of h_2 due to ExPE). Then agent a_2 can change

[31] Using *upper shuffle-proofness* instead of *strategy-proofness* would not work when moving from the type 1 profile to the type 2 and also from the type 3 to the type 4. In fact, there I use *weak invariance* (Hashimoto et al., 2014) – a "part" of *strategy-proofness* complementary to *upper shuffle-proofness*, that requires the assignment probabilities to be fixed regardless of any changes in the lower contour set. Therefore, for $N = 3$ RSD can also be characterized as an *ex-post efficient*, *weak envy-free among equals*, *upper shuffle-proof*, and a *weakly invariant* mechanism.

the preference profile $\succ$ to the on of type 2 by swapping his two top objects h_1 and h_2, but a preference profile of type 2 cannot be disturbing. Similarly in case $a'' = a_2$.

If the agent with a truncated report $a'' = a_3$, then, if agents a_1 and a_2 are disturbed in h_3, we do as in the previous step. If agents a_1 and a_2 are not disturbed in h_3, then they violate wEFE.

Consider type 4. If any agent is disturbed in h_1, he can change the preference profile $\succ$ to the one of type 3 by swapping her two bottom objects. Alternatively, if no agent is disturbed in h_1, then the two disturbed agents violate wEFE.

Consider type 5. If the agent with a truncated report $a'' = a_1$, then a_2 and a_3 are disturbed in h_2 and h_3, and not in h_1 since agent a_3 gets zero probability of h_1 due to ExPE. Assume that $h_1 \succ_{a_3} h_3$ (in the opposite case, we can also consider a profile $\succ'$ such that $h_1 \succ'_{a_3} h_3$ in which agent a_3 is also disturbed in h_2). Then agent a_3 can change the preference profile $\succ$ to the one of type 4 by swapping h_1 and h_2, which leads to a contradiction. Same if $a'' = a_2$.

If the agent with a truncated report $a'' = a_3$, and if agents a_1 and a_2 are disturbed in h_1, then each of them can change the preference profile $\succ$ to the one of type 2. Alternatively, if no agent is disturbed in h_1, then a_1 and a_2 violate wEFE.

Consider type 6. Finally, φ coincides with RSD due to ExPE. □

Proof of Proposition 2 (Second characterization of RSD)

Proof. We know that RSD is strategy-proof, ex-post efficient and satisfies symmetry. I prove the other direction by checking sequentially all the preference profiles listed above. Let φ satisfy SP, ExPE and symmetry.

First, consider the case where each agent prefers each house over the outside option. For $N = 3$ there are the same six types of preference profiles as in the proof of Proposition 1.

Type 1. Since φ is ExPE we get $\varphi_{a_3 h_2} = 1$. If $u_1 = u_2$ then due to symmetry $\varphi_1(u) = \varphi_2(u)$, otherwise the agents cannot possibly get equal utilities. If, on the contrary, $u_1 \neq u_2$ then one of the agents mimics the other to get a more preferred equal split of objects h_1 and h_3 unless $\varphi_1(u) = \varphi_2(u)$ already. This mimic deviation would violate SP, therefore $\varphi_1(u) = \varphi_2(u) = RSD_1(u) = RSD_2(u)$, where the latter denotes the assignment induced by RSD.

Type 2. Due to the previous result and due to SP, $\varphi_{a_2 h_1} = \frac{1}{2}$. Using ExPE we get $\varphi_{a_2 h_2} = \varphi_{a_3 h_1} = 0$ and thus $\varphi_{a_1 h_1} = \varphi_{a_2 h_3} = \frac{1}{2}$. Suppose also $\varphi_{a_1 h_3} = x \in [0, \frac{1}{2}]$. Then the remaining probabilities are as follows: $\varphi_{a_1 h_2} = \varphi_{a_3 h_3} = \frac{1}{2} - x$ and $\varphi_{a_3 h_2} = \frac{1}{2} + x$. Observe that x cannot effectively depend on the reported utilities u_1, u_3 (given that the ordinal preferences are the same), otherwise φ is not SP.

Type 3. Due to the results for type 2 and due to SP, $\varphi_{a_1 h_3} = \varphi_{a_2 h_3} = \frac{1}{2} - x$. If $u_1 = u_2$ then due to symmetry $\varphi_1(u) = \varphi_2(u)$, otherwise the agents cannot possibly get equal utilities. If $u_1 \neq u_2$ then due to SP $\varphi_1(u) = \varphi_2(u)$, otherwise one of the two agents can profitably mimic another. Since $\varphi_{a_3 h_2} = 0$ due to ExPE, we get $\varphi_{a_1 h_2} = \varphi_{a_2 h_2} = \frac{1}{2}$ and $\varphi_{a_1 h_1} = \varphi_{a_2 h_1} = x$. Consequently, the remaining expected share of house h_1 goes to agent a_3: $\varphi_{a_3 h_1} = 1 - 2x$.

Type 4. Finally, consider the symmetric preference profile of type 4. Each agent can swap her second and third choices and transform the preference profile to that of type 3. Due to SP their expected shares of the top house h_1 are all equal: $\varphi_{a_1 h_1} = \varphi_{a_2 h_1} = \varphi_{a_3 h_1} = 1 - 2x$. Therefore $x = \frac{1}{3}$ and the assignments of types 2 and 3 are identical to RSD assignments.

Continue with type 4. The top object h_1 is equally split, hence it is left to check how object h_2 is assigned. Assume for a contradiction that for some u consistent with type 4 ordinal preferences h_2 is not equally split.

First let us check that this is not possible if two agents have identical utilities and the third differs: w.l.o.g. $u_1 = u_2 \neq u_3$ and due to symmetry and our assumption $\varphi_{a_1h_2}(u) = \varphi_{a_2h_2}(u) \neq \varphi_{a_3h_2}(u)$. If now agent a_3 mimics the other two agents, he gets an equal share of $1/3$ due to the symmetry. But this share cannot differ from $\varphi_{a_3h_2}(u)$, otherwise a_3 could profitably switch in one of the directions. Therefore in this case all objects are equally split.

Next consider the case when all three utilities differ and, w.l.o.g. let agents a_1 and a_2 get different shares of h_2 such that $\varphi_{a_1h_2}(u) \neq 1/3$. Consider a different utility profile $u' : (u'_{-1} = u_{-1}) \cap (u'_1 = u_2)$ which is also consistent with type 4 ordinal preferences. In $\varphi(u')$ agents a_1 and a_2 get the same shares of h_2 $\varphi_{a_1h_2}(u') = \varphi_{a_2h_2}(u') = 1/3$ due to the previous observation. Therefore a_1 can profitably switch between u and u' in one of the directions, which violates SP.

Type 5. Due to SP, agents a_1 and a_2 get the same shares of h_1 as in type 2 profile, and agent a_3 gets none of h_1 due to ExPE. As before, due to symmetry and SP, agents a_1 and a_2 get equal shares of the other two objects as well. The other probabilities corresponding to type 5 profile can be easily determined, they also coincide with those of RSD.

Type 6. Here φ coincides with RSD due to ExPE.

The proof for the case with an outside option is identical to the corresponding part of the proof of Proposition 1, with the only difference that instead wEFE we use the combination of symmetry and SP, similar to the way it is done above. □

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