Strong Nash equilibrium in Cournot and Bertrand oligopolies*

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Abstract

This paper investigates the existence of strong Nash equilibria (SNE) in Cournot and Bertrand oligopoly models. Given the concavity and continuity of payoffs, I derive the necessary and sufficient conditions for the existence of SNE in these non–cooperative games.

Keywords Non–cooperative game, Strong Nash equilibrium, Coalition, Cournot, Bertrand.

1 Introduction

The paper analyzes the well–studied question of coalition behavior of firms in the industries whereby they avoid competing with one another. As it is well known, the main incentive for firms to form coalitions is the market power generated by decreasing concentration. That is why, for instance, monopolistic competition (Neary, 2007), as well as the proposition for the continuum of traders (Shitovitz, 1973), can not explain the phenomenon of merger.

The most interesting results were obtained in the framework of oligopolies in various productions by comparison the Nash equilibrium profits of firms before and after the merger. Thus, in (Farrell & Shapiro, 1990) the authors obtain conditions of merger profitability and its consequences for rivals. In (Perry & Porter, 1985) authors discuss incentives for horizontal merger. It is shown that participating in cartel or merger is often inefficient for profit maximizing firms, rivals or consumers. In the paper (Salant et al., 1983) authors evaluate losses from merger with the use of comparative–static implication.

Therefore, we can see that in specific oligopoly games, there are opportunities for joint deviations that are mutually beneficial for a subset of players. Nash equilibrium

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does not account for this by definition, and, as a result, it appears to be unstable in that sense. Alternately, the idea of SNE ensures a more restrictive stability. A strong equilibrium is defined as a strategic profile for which no subset of players has a joint deviation that strictly benefits all of them while all other players are expected to maintain their equilibrium strategies. In other words, the existence of SNE is a sufficient condition for firms not to merge. The definition implies as well that an SNE is a Nash equilibrium and is also weakly Pareto efficient among the Nash equilibria since the deviating coalition can be a single player or the whole set of players.

In spite of intuitive validity of SNE approach to merger expediency, there is a lack of literature on this issue. In the paper (Bernheim & Whinston, 1987) the author asserts that SNE does not exist in the linear demand Cournot oligopoly. The intuition here is that SNE can be considered as weakly Pareto efficient refinement of Nash equilibrium. In the paper (Nessah & Tian, 2014) authors provide necessary and sufficient conditions for general economic non–cooperative games and suggest an algorithm for computing SNE. Moreover, they construct a specific example of Cournot oligopoly with linear–quadratic concave demand function which possesses SNE. This paper specifies their findings to the cases of quantity and price competition static oligopoly models.

The remainder of the paper is structured as follows. Sections 2 and 3 set the Cournot and Bertrand models with general demand and technology functions and provide with the necessary and sufficient conditions for SNE existence, respectively. Section 4 concludes.

2 SNE in Cournot model

Consider an oligopoly with firms $I = \{1, \ldots, n\}$ producing the homogeneous products and compete between each other by quantities produced $q = (q_1, \ldots, q_n)$, $q_i > 0 \; \forall i \in I$.

Assumption 1. *The payoff functions $u_i = \pi_i(q)$ are concave*[1] *and continuous $\forall i \in I$.*

Assumption 2. *Nash equilibria $\hat{q}$:*

$$\hat{q} = \arg\max_{q_i} \left(P(q)q_i - C_i(q_i)\right) \quad \forall i \in I; \tag{1}$$

exists in the game $G = (q_i, \pi_i(q))_{i \in I}$;

Assumption 3. $q_i > 0 \; \forall i \in I$.[2]

Proposition 1 (The necessary and sufficient condition for SNE in Cournot game)**.** *Suppose Assumptions 1–3 are satisfied. Then, the Nash equilibrium $\hat{q}$ is SNE in the game $G = (q_i, \pi_i)_{i \in I}$ if and only if $\forall i \; \frac{\partial P(q)}{\partial q_i}(\hat{q}) = 0$.*

[1]Thus, both demand and cost functions are of class $\mathbf{C}^2$. Oligopoly games with demand function of other types need additional studies and also may possess SNE. The example of SNE in salient point can be found in (Nessah & Tian, 2014).

[2]I assume that all firms in the industry produce a positive amount of product. Thus, unlike (Nessah & Tian, 2014) I consider the game G which is not compact. Notice that in the other extreme case of the game G Nash equilibrium $q : q_i \geq 0 \; \forall i \in I$, $\exists! j : q_j > 0$ is obviously strong, and it is a monopoly, in fact.

Proof. The strategy profile $\hat{q}$ is the solution of the next system of equations:

$$P(q) + \frac{\partial P(q)}{\partial q_i} q_i = C_i'(q_i), \quad \forall i. \tag{2}$$

Sufficiency. Profit function for each coalition $S \subseteq I$ is

$$\pi_S = P(q) \sum_{i \in S} q_i - \sum_{i \in S} C_i(q_i). \tag{3}$$

The function π_S reaches its maximum for each coalition $S \subseteq I$ due to the assumption 1 by Weierstrass theorem. For each coalition $S \subseteq I$ define the q^S as the strategy profile that maximizes its common profit while firms–outsiders play Nash equilibrium strategies, i.e. q^S is the solution of the system with n equations:

$$\begin{cases} P(q) + \frac{\partial P(q)}{\partial q_i} \sum_{j \in S} q_j = C_i'(q_i), \quad \forall i \in S \\ \\ q_i^S = \hat{q}_i, \quad \forall i \notin S \end{cases} \tag{4}$$

Suppose further that $\frac{\partial P(q)}{\partial q_i}(q^S) = 0$. Then, the solution of (4) $q^S : P(q^S) = C_i'(q_i^S)$ for any S is the same with (2), and therefore, $\forall S\ q^S = \hat{q}$ is SNE in the game G.

Necessity. Now suppose that $\hat{q}$ is SNE in the game G. Then $\hat{q}$ is weakly Pareto efficient:

$$\hat{q} = \arg\max_q \left(P(q) \sum_{i \in I} q_i - \sum_{i \in I} C_i(q_i) \right). \tag{5}$$

Thus, due to the concavity of payoffs functions the next system of equation holds[3] for $\hat{q}$:

$$\begin{cases} P(q) = & C_i'(q_i) - \frac{\partial P(q)}{\partial q_i} \sum_{j \in I} q_j, \\ \\ P(q) = & C_i'(q_i) - \frac{\partial P(q)}{\partial q_i} q_i. \end{cases} \tag{6}$$

Due to the Assumption 3 the system (6) is solvable if $n = 1$ or $\frac{\partial P(q)}{\partial q_i}(\hat{q}) = 0$.

Example 1. *Consider an oligopoly with inverse demand function* $P(q) = 100 - (\sum_{i \in I} q_i - 4)^3$ *(Fig. 1). Assume that* $C_i(q_i) = \frac{25}{2} \theta_i q_i^2$*. Then, the profit of grand coalition is* $\pi_I = P(q) \sum_{i \in I} q_i - \frac{25}{2} \sum_{i \in I} \theta_i q_i^2$ *(Fig. 2). SNE in the game is the vector* $q = (\frac{4}{\theta_1}, \ldots, \frac{4}{\theta_n})$*, and the parameters of the cost functions* θ_i *define the share of the firm* i *in the total output. The profits of each firm and the profit of grand coalition for* $n = 3$ *and* $\theta_1 = 2$, $\theta_2 = 3$, $\theta_3 = 6$ *are illustrated on the Fig. 3.*

It is worth noting that points of inflexion of demand function are not obliged to be Nash equilibria.

Example 2. *Consider an oligopoly with inverse demand function* $P(q) = 15 - \sum_{i \in I} q_i + \cos(\sum_{i \in I} q_i)$ *(Fig. 4). Assume that* $\sum_{i \in I} C_i(q_i) = 15 \cos(\sum_{i \in I} q_i) - \frac{1}{2}(\sum_{i \in I} q_i)^2 + 2 \sin(\sum_{i \in I} q_i)$*. The profit of grand coalition is illustrated on the Fig. 2. The game*

[3] There is no cost efficiency due to limitations of payoffs function of non–cooperative games.

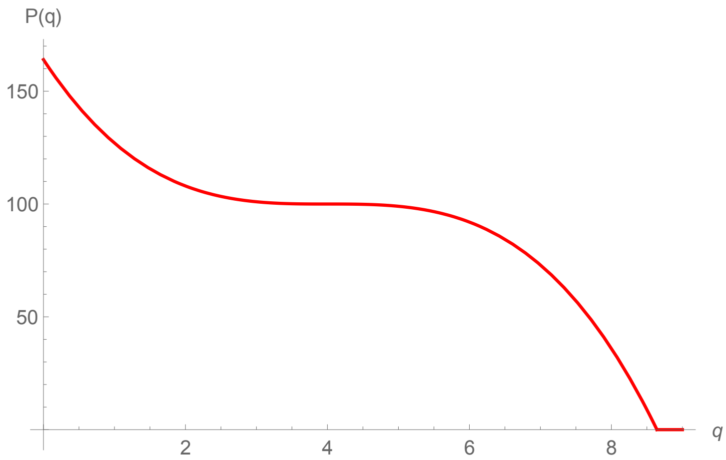

Figure 1: Inverse demand function $P(q) = 100 - (\sum_{i \in I} q_i - 4)^3$

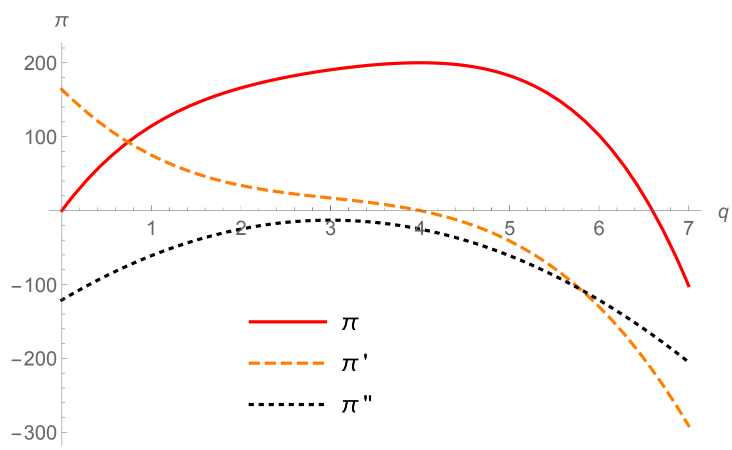

Figure 2: The profit of the coalition of all firms

possesses two flex points of demand function[4] $\hat{q}_1 : \sum_{i \in I} q_i = 3\pi/2$ *and* $\hat{q}_2 : \sum_{i \in I} q_i = 7\pi/2$*, i.e. the condition of the Proposition 1 holds for both* $\hat{q}_1$ *and* $\hat{q}_2$*. However, only* $\hat{q}_2$ *is Nash equilibrium and, therefore, SNE.*

The next corollary holds for the quantity competition oligopoly with linear demand function.

Corollary 1. *Suppose that in the game* $G = (q_i, \pi_i(q))_{i \in I}$ *the production technologies of firms* $C_i(q_i) \geq 0$, $i \in I$, $C_i(\cdot) \in C^2$*; the linear demand function* $P(Q) = a - bQ$*, where* $Q = \sum_{i=1}^{n} q_i$*, and* $a, b > 0$*. Then there are no SNE in the game* G*.*

[4] Solely for the purposes of this example π refers to the generally accepted notation of the ratio of the circumference to its diameter and not to the profit of firms.

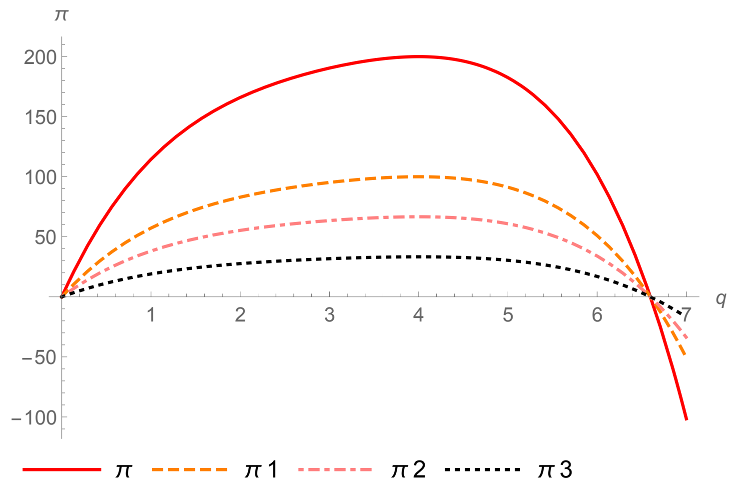

Figure 3: Profits of each firm and profit of grand coalition for the market of size 3

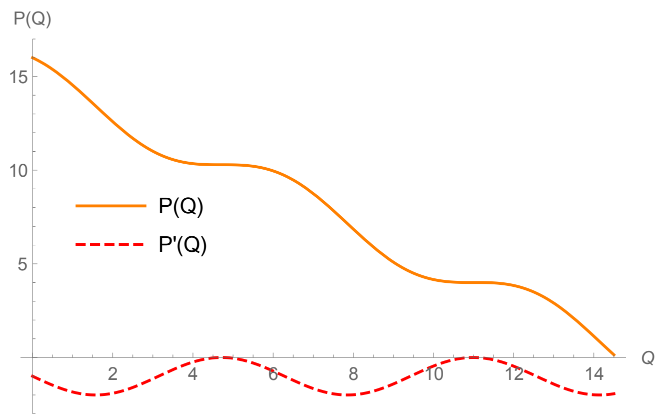

Figure 4: Inverse demand function $P(q) = 15 - \sum_{i \in I} q_i + \cos(\sum_{i \in I} q_i)$

Thus, the firms always have incentives to cooperate in the Cournot models with linear demand function.

3 SNE in Bertrand model

Proposition 2. *Bertrand oligopoly model with homogeneous product does not possess SNE.*

This is true because Nash equilibrium is never weakly Pareto efficient due to the well–known "Bertrand paradox". Examine further the Bertrand model with differentiated products.

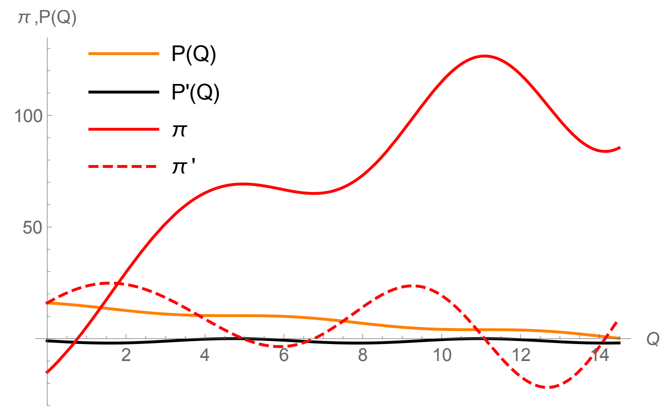

Figure 5: Inverse demand function and the profit of the coalition of all firms

Consider an oligopoly with firms $I = \{1, \dots, n\}$ producing the differentiated products and compete between each other by prices $p = (p_1, \dots, p_n)$, $p_i > 0\ \forall i \in I$. Suppose that each firms i produce just one type of product i.

Assumption 4. *The payoff functions $u_i = \pi_i(p)$ are concave and continuous $\forall i \in I$.*

Assumption 5. $q_i > 0\ \forall i \in I$.

Proposition 3 (The necessary and sufficient conditions for SNE in the game $G = (p_i, \pi_i)_{i \in I}$)**.** *Suppose Assumptions 4 and 5 are satisfied. Then, the game $G = (p_i, \pi_i)_{i \in I}$ does not possess SNE.*

Proof. Suppose $\hat{p}$ is SNE in the game $G = (p_i, \pi_i)_{i \in I}$. Then, by definition it is weakly Pareto efficient:

$$\hat{p} = \arg\max_p \left(\sum_{i \in I} q_i(p) p_i - \sum_{i \in I} C_i(q_i(p)) \right) \tag{7}$$

and it is a Nash equilibrium:

$$\hat{p} = \arg\max_p \left(q_i(p) p_i - C_i(q_i(p)) \right). \tag{8}$$

Consequently, due to the Assumption 4 the following system of equation holds:

$$\begin{cases} q_i(p) = & C_i'(q_i) \frac{\partial q_i(p)}{\partial p_i} - \sum_{j \in I} \frac{\partial q_j(p)}{\partial p_i} p_j, \quad \forall i \in I \\ q_i(p) = & C_i'(q_i) \frac{\partial q_i(p)}{\partial p_i} - \frac{\partial q_i(p)}{\partial p_i} p_i, \quad \forall i \in I \end{cases} \tag{9}$$

The Assumption 5 garantees the only solution of system (9) if $n = 1$.

In other words, firms always have incentives to cooperate in Bertrand model when Assumptions 4 and 5 are satisfied.

4 Conclusion

The paper provides some existence results on strong Nash equilibria in price and quantity static oligopoly games. The condition for SNE existence on demand function for the Cournot model is to have flex point which is Nash equilibrium profile. The condition is specific and, therefore, it is rare for oligopoly to possess SNE. Further, given the concavity and continuity of payoff functions, Bertrand oligopoly never possesses SNE.

In other words, firms have incentives to merge in the most cases. Moreover, for the oligopoly with linear–quadratic utility it is always true. The intuition for this conclusion is that SNE is a refinement of Nash equilibrium which is weakly Pareto efficient.

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