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Dynamical Systems on 2- and 3-Manifolds

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To Dmitry Victorovich Anosov,
an outstanding mathematician
and a remarkable man

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Contents

Symbols

$\mathbb{N}$	The set of all natural numbers (positive integers)
$\mathbb{Z}$	The set of all integers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$\mathbb{I} = \{x \in \mathbb{R}, 0 \leq x \leq 1\}$	The segment (closed arc) $\mathbb{Z}^n = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n}$, $\mathbb{Z}^0 = \{0\}$
$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$	The n-dimensional Euclidean space, $\mathbb{R}^0 = \{0\}$
$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$	The upper half-space
$O(\underbrace{0, \ldots, 0}_{n})$	The origin of $\mathbb{R}^n$
$\mathbb{C}^n = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n}$	The n-dimensional complex space
$\mathbb{D}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1\}$	The standard n-disk (n-ball), $\mathbb{D}^0 = \{0\}$
$\mathbb{S}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = 1\}$	The standard $(n-1)$-sphere, $\mathbb{S}^{-1} = \emptyset$
$\mathbb{K}^n = \mathbb{S}^{n-1} \times \mathbb{I}$	The standard n-annulus
$\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n}$	The standard n-dimensional torus, $\mathbb{T}^1 = \mathbb{S}^1$
$\mathbb{I}^n = \underbrace{\mathbb{I} \times \cdots \times \mathbb{I}}_{n}$	The standard n-dimensional cube
$\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$	The group of integers modulo $p \in \mathbb{N}$ under operation of addition
$k\mathbb{Z}$	The multiples of $k \in \mathbb{N}$
$\mathrm{cl}X$	The closure of the set X
∂X	The boundary of the set X
$\mathrm{int}X$	The interior of the set X

$\pi_1(X)$	The fundamental group of the connected topological space X
$H_k(X)\ (k \geq 0)$	The k-th homology group of the topological space X
$L(f)$	The Lefschetz number of the continuous mapping $f: X \to X$ of the topological space X
$h(f)$	The topological entropy of the continuous mapping $f: X \to X$ of the topological space X
$\chi(X)$	The Euler characteristic of the topological space X
$C^r(X, Y)\ (r \geq 0)$	The space of the C^r-mappings of the manifold X in the manifold Y, equipped with the C^r-topology
$\mathrm{Diff}^r(X)\ (r \geq 0)$	The space of the C^r-diffeomorphisms of the manifold X equipped with the C^r-topology
TX	The tangent bundle of the manifold X
T_xX	The tangent space at the point $x \in X$
Df	The differential of the mapping f
D_xf	The differential of the mapping f at the point x
$\mathcal{O}_x$	The orbit of the point x
W^s_x	The stable manifold of the point x
W^u_x	The unstable manifold of the point x
Ω_f	The set of the non-wandering points of the diffeomorphism f
Ω_{f^t}	The set of the non-wandering points of the flow f^t
Fix_f	The set of the fixed points of the diffeomorphism f
Fix_{f^t}	The set of the fixed points of the flow f^t
Per_f	The set of the periodic points of the diffeomorphism f
Per_{f^t}	The set of the periodic points of the flow f^t
$\mathrm{per}(x)$	The period of the periodic point x
id	The identity map

Introduction

This book is an introduction to the topological classification of smooth structurally stable cascades on closed orientable 2- and 3-manifolds. First of all, we wish to point out some terminological differences traditional for the Russian school of the dynamical systems. The Russian term "cascade" introduced by D. Anosov means a discrete dynamical system induced by a diffeomorphism on a manifold. The abbreviation "A-diffeomorphisms" means "Axiom A diffeomorphisms". The term "rough system" is slightly different from its English counter part "structurally stable system" but the sets of rough and structural stable systems coincide.

The topological classification is one of the main problems of the theory of dynamical systems. The main idea is to find topological invariants of the decomposition of the manifold into trajectories (topological invariants are understood to be characteristics of the system which are invariant with respect to the topological equivalence or the conjugacy). The results presented in this book are mostly for dynamical systems satisfying Smale's Axiom A. The set of the non-wandering points of such a system is hyperbolic and it coincides with the closure of the set of the periodic points. It is important to note that Smale's Axiom A is the necessary condition of the structural stability (roughness) of a dynamical system.

The topological classification of structurally stable flows (dynamical systems with continuous time) on a bounded part of the plane and on the 2-sphere follows from the results of E. Leontovich and A. Mayer [34, 35] where actually more general class of dynamical systems was considered. The classification was based on the ideas of Poincaré–Bendixson to pick a set of specially chosen trajectories so that their relative position completely defines the qualitative structure of the decomposition of the phase space of the dynamical system into trajectories. E. Leontovich and A. Mayer also exploited the idea of A. Andronov and L. Pontryagin about the structural stability (roughness) of the dynamical system (for details see [3]). M. Peixoto generalized these results [43] and suggested a graph (Peixoto graph) as the complete topological invariant for structurally stable flows on surfaces. Peixoto graph generalizes Leontovich–Mayer invariant which was called a scheme and which was constructed for flows on the plane and on the sphere.

Structurally stable (rough) flows on surfaces have only finitely many singularities and finitely many closed orbits, all of which are hyperbolic. They also have no trajectories joining saddle points and no nontrivial recurrent trajectories (i.e., recurrent trajectories other than singularities and closed orbits). Under these conditions the topological classification of such flows is reduced to a combinatorial problem. The absence of nontrivial recurrent trajectories for structurally stable flows on the plane and on the sphere is immediate from the topology of these surfaces but this is not so trivial for structurally stable flows on orientable surfaces of genus $g > 0$. At first it was proved by A. Mayer for structurally stable flows with no singularities on the 2-torus [36][1] and later by M. Peixoto [41, 42] for structurally stable flows on surfaces of any genus (see also [39]). M. Peixoto also proved denseness (in C^1-topology) of the structurally stable flows in the space of flows on surfaces.

In contrast to the case of flows on surfaces, manifolds of dimension more than 2 (more than 1) admit flows (cascades) with homoclinic trajectories and this implies a complicated structure of the set of the trajectories which was first understood by A. Poincaré [44]. Later G. Birkhoff [10] while studying measure preserving maps proved the existence of infinitely many periodic orbits on the annulus in the neighborhood of a homoclinic point. S. Smale in 1961 [47] constructed an example of a structurally stable diffeomorphism of the 2-sphere with infinitely many periodic orbits which is now known as “Smale horseshoe”. This was the key example that showed the difference between structurally stable flows (cascades) on manifolds of dimension more than 2 (more than 1) and structurally stable flows on surfaces. Another important discovery was made by D. Anosov in [4] where he studied geodesic flows on Riemannian manifolds of negative curvature and in [5] where he introduced the most important class of Y-systems (now flows and diffeomorphisms of this class bear his name) and proved the structural stability (roughness) of the systems of this class. Generalizing further S. Smale introduced a special class of Smale’s Axiom A systems, i.e., systems whose hyperbolic non-wandering set coincides with the closure of the set of the periodic points. The non-wandering set of a system of this class decomposes into finitely many closed invariant basic sets and on each of them the system acts transitively. The dynamics of such a system on each nontrivial basic set (neither a periodic orbit nor a fixed point) is in some way similar to the behavior of the Smale horseshoe on its non-wandering set.

At first in 1960 S. Smale [46] speculated that on manifolds of dimension more than 2 the structurally stable flows are exactly the flows that have finitely many singular points and finitely many periodic orbits, all of them hyperbolic, while the invariant manifolds of the periodic orbits intersect transversally (it was analogous to the structurally stable flows on the 2-sphere). But later S. Smale and J. Palis [38, 40] showed that these flows are indeed structurally stable but unlike the structurally stable flows on surfaces they are not the only ones (one can consider a flow that is a

[1]Actually in [36] A. Mayer found the conditions of structural stability for cascades (discrete dynamical systems) on the circle and he also got the topological classification for these cascades.

suspension over the Smale horseshoe diffeomorphism; it is structurally stable but it has the countable set of the periodic orbits). Moreover, S. Smale [48] showed that the structurally stable flows on manifolds of dimension more than 2 are not generic.

Nevertheless, the flows similar to the structurally stable flows on surfaces are very important for applications as well as for the general theory. The latter is due to the fact that the dynamics of these flows is closely connected with the topology of the phase space. In particular S. Smale inequalities [46] similar to Morse inequalities hold for them. Therefore this class of flows (named Morse–Smale flows) was thoroughly studied. Note that though a Morse–Smale flow has only finitely many hyperbolic singular points and finitely many closed orbits the dynamics of such a flow on its wandering set can be quite complicated. For example L. Shilnikov and V. Afraimovich [1] showed that the restriction of a Morse–Smale flow to the closure of the heteroclinic trajectories is topologically conjugated to the suspension over the topological Markov chain. Later (similarly to flows) the discrete dynamical systems with a finite hyperbolic non-wandering set and such that the manifolds of distinct periodic points intersect transversally were called the Morse–Smale systems.

Thus the approaches to the topological classification of the structurally stable dynamical systems on manifolds come roughly under two headings:

I. singling out special classes of Morse–Smale systems for which it is possible to construct a complete topological invariant;
II. construction of complete topological invariants for the restriction of the dynamical system to some neighborhood (support) of the given nontrivial basic set.

Results in these directions led to construction of complete topological invariants for important classes of structurally stable systems with nontrivial basic sets. The bulk of the results for diffeomorphisms on 2- and 3-manifolds is mainly due to the fact that the dynamics of the restriction of a diffeomorphism to its nontrivial basic set in many important cases is determined by the hyperbolic automorphism induced by the restriction of the diffeomorphism to the support of the basic set. Whereas the dynamics of the restriction of the diffeomorphism to its wandering set is defined by a finite graph describing the asymptotic behavior of the wandering points (i.e. the graph contains the information to which basic set the wandering point tends). In addition the graph is equipped with the information on the topology of the embedding of the invariant manifolds of the saddle points into the ambient manifold as well as the information on the structure of the heteroclinic intersections of the invariant manifolds.

The results presented in this book can be summed up as follows.

- The topological classification of the gradient-like diffeomorphisms on 2- and 3-manifolds (see Chaps. 3 and 5)

The constructed topological invariants are the combinations of the classic combinatorial invariants and the new topological invariants introduced in [11, 12, 15, 27] by Ch. Bonatti, V. Grines, V. Medvedev, E. Pecou, O. Pochinka for the description of the topology of the intersection and of the embedding (possibly wild) of the invariant manifolds of the saddle periodic points into the phase space. These invariants are the specially constructed manifolds (the characteristic spaces) containing embedded sets of closed curves, tori, and Klein bottles. For the Morse–Smale diffeomorphisms on 3-manifolds these new invariants led to the study of the bifurcations which occur when the embedding of the invariant manifolds of the saddle periodic points changes its type. In Chap. 4 these bifurcations are described for the so-called Pixton class of the Morse–Smale diffeomorphisms, i.e., such diffeomorphisms whose non-wandering sets consist of exactly four fixed points: two sinks, one saddle, and one source. Our presentation follows [14].

- The construction of smooth global Lyapunov functions for the Morse–Smale diffeomorphisms (see Chap. 7).

C. Conley [18] in 1978 proved that any dynamical system on a closed n-manifold possesses a continuous function which is constant on the so-called chain components and which strictly decreases along the orbits not belonging to the chain recurrent set of the system. Such a function is called a complete or global Lyapunov function and Conley theorem is called the fundamental theorem of dynamical systems. Throughout this book we omit the word “complete” (“global”) for Lyapunov function. If a Lyapunov function is smooth and the set of its critical points coincides with the chain recurrent set, then this function is called the energy function. Very generally smooth flows do admit an energy function (see, e.g. [2], Theorem 6.12), but it is not true even for Morse–Smale diffeomorphisms.

S. Smale was the first to construct energy functions. In 1961 he proved that a gradient-like flows (i.e., Morse–Smale flow without closed orbits) has an energy function which is a Morse function.

K. Meyer [37][2] in 1968 generalized this result and constructed an energy function for any Morse–Smale flow, which actually was a Morse–Bott function. This results prompted M. Shub [45] and F. Takens [49] to put forward a hypothesis that any Morse–Smale diffeomorphism possesses an energy function. D. Pixton proved it to be true for cascades on surfaces but he also constructed an example of a diffeomorphism on the 3-sphere that admits no energy function. The idea was based on the wild embedding of the separatrices of the saddle points into the ambient space. In [21–24] V. Grines, F. Laudenbach, and O. Pochinka showed that the existence of an energy function for a Morse–Smale diffeomorphism $f : M^3 \to M^3$ depends on the type of the embedding of the global attractors and the global repellers which are the closures of the 1-dimensional stable and unstable manifolds of the saddle periodic points respectively.

- Connection between the dynamics of Morse–Smale cascades and the topology of the ambient space (see Chap. 6).

[2]In the paper by K. Meyer [37] there is an inaccuracy noted by F. Laudenbach. The global construction of the energy function in the neighborhood of the closed orbit is not actually given.

There is a nontrivial connection between the periodic data and the behavior of the stable and the unstable manifolds of the saddle periodic points of a Morse–Smale cascade f on the one hand and the topology of the ambient space on the other. Let

$$g_f = \frac{r_f - l_f + 2}{2},$$

where r_f is the number of the saddles and l_f is the number of the node (sink or source) periodic points of the diffeomorphism f. In Sect. 6.1 we present the topological classification of the closed 3-manifolds which admit Morse–Smale diffeomorphisms without heteroclinic curves, i.e., such Morse–Smale diffeomorphisms that the invariant 2-manifolds of their saddle periodic points are disjoint. In this case the ambient manifold is either the 3-sphere (if $g_f = 0$) or the connected sum of g_f copies of $\mathbb{S}^2 \times \mathbb{S}^1$. Our presentation follows [13].

In Sect. 6.2 we prove that if a diffeomorphism has no heteroclinic orbits (gradient-like diffeomorphism) and all the frames of the 1-dimensional separatrices of the saddle periodic points are tame, then the ambient manifold admits Heegaard splitting of genus g_f. Our presentation follows [31].

- The topological classification of nontrivial basic sets (i.e., basic sets which are not periodic orbits) of diffeomorphisms on 2-manifolds (see Chap. 9).

The key point in the construction of topological invariants for the basic sets on surfaces is the idea to consider the universal covering of the support of the basic set and study there the asymptotic behavior of the preimages of the invariant manifolds of the points of the basic sets. The universal covering in this case is either the Euclid plane or the Lobachevsky (hyperbolic) plane (or a subset of the Lobachevsky plane). A. Weil was the first to suggest this idea in his report in the First International Topology Conference in Moscow in 1935. D. Anosov applied it in the 1960s to the study of the asymptotic behavior of the covering flow on 2-surfaces distinct from the sphere. The idea was further developed by S. Aranson, V. Grines, E. Zhuzhoma, G. Levitt in the 1970s–1980s and it led to the topological classification for important classes of flows, foliations, and 2-webs with nontrivial recurrent orbits and leaves on surfaces (see the survey [9] and the monograph [6]). Further, in the papers by V. Grines, Kh. Kalay, R. Plykin these methods were applied for the classification of nontrivial basic sets of surface diffeomorphisms (in particular 1-dimensional attractors and repellers). This approach proved to be efficient to show the existence of structurally stable diffeomorphisms in the homotopy classes of the surface diffeomorphisms described in the Nielsen–Thurston theory (see the surveys [7–9, 20] and the monographs [6, 17]).

Notable results in the construction of the algorithmic classification of the 1-dimensional basic sets of A-diffeomorphisms of surfaces were made by A. Zhirov. They are based on the famous example by Plykin of a diffeomorphism of the 2-sphere with one 1-dimensional attractor and four sources. This example at the time greatly helped in understanding of the complex structure of hyperbolic attractors on surfaces. A. Zhirov has recently published a book on this subject [50], thus we do not include these results here.

The results presented in this book provide a base for the classification of 2-dimensional basic sets of A-diffeomorphisms on 3-manifolds. The topological classification of 2-dimensional surface basic sets was given in [26] and $(n-1)$-dimensional orientable expanding (contracting) attractors (repellers) on the n-torus ($n \geq 3$) were classified in [28–30]. As we mentioned before the classification of nontrivial basic sets and Morse–Smale diffeomorphisms gave rise to the topological classification of important classes of structurally stable diffeomorphisms on 2- and 3-manifolds. In his paper [19] V. Grines gave the topological classification of structurally stable cascades on orientable surfaces if the nontrivial basic sets of the cascade are 1-dimensional and the wandering set contains only finitely many heteroclinic orbits. Ch. Bonatti and R. Langevin in their book [16] presented the topological classification of arbitrary structurally stable diffeomorphisms of orientable surfaces. In [28–30] there are the classifications of structurally stable diffeomorphisms on n-manifolds if the non-wandering set of a diffeomorphism contains an orientable expanding attractor or a contracting repeller of co-dimension one. In the recent papers [25, 32, 33] the topological classification was constructed for structurally stable diffeomorphisms of 3-manifolds whose non-wandering sets are 2-dimensional. We omit these results here as the exact wording is fairly complex and the proofs fall outside the scope of this book.

At present there is a number of surveys and books on topics similar to those presented in this book. But the main results on the topological classification of discrete dynamical systems are widely scattered among many papers and surveys. This book tries to present these results systematically. The reader needs be familiar with the basic concepts of the qualitative theory of dynamical systems which are presented in Chap. 1 for convenience. In Chap. 10 we briefly state the necessary definitions and results of algebra, geometry, and topology.

When stating ancillary results at the beginning of each part we sometimes refer to sources which are readily available rather than the ones from which the result originates.

This book tries to present a reasoned exposition of the recent results on the topological classification of A-cascades. We do not try to include all known results but rather focus on the nontrivial effects of the dynamical systems on 2- and 3-manifolds. We present the classical methods and approaches which we consider to be promising for the further research.

The book consists of ten chapters. At the beginning of each chapter we give the necessary definitions and formulate the results. Proofs are presented thereafter with the exact statements of the results given once again for convenience. For the first reading, the reader might omit the proofs but confine oneself to the presented notions and facts.

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Futher Reading

The presentation of most of the material in this book is our own and consists of original or considerably modified proofs of the known results. We try to list all the sources on which the presentation in various parts of the book is based, or that inspired our presentation in other places and many (but not all) of the original sources for the specific results presented in the text. Below we try to list all major monographs and representative textbooks and surveys covering the principal branches of dynamics.

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