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THE THRESHOLD DECISION MAKING EFFECTUATED BY THE ENUMERATING PREFERENCE FUNCTION

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Based on the leximin and leximax preferences, we consider two threshold preference relations on the set X of alternatives, each of which is characterized by an n-dimensional vector ($n \geq 2$) with integer components varying between 1 and $m(m \geq 2)$. We determine explicitly in terms of binomial coefficients the unique utility function for each of the two relations, which in addition maps X onto the natural 'interval' $\{1, 2, \ldots, |\tilde{X}|\}$, where $\tilde{X} = X/I$ is the quotient set of X with respect to the indifference relation I on X induced by the threshold preference. This permits us to evaluate all equivalence classes and indifference classes of the threshold order on X, present an algorithm of ordering the monotone representatives of indifference classes, and restore the indifference class of an alternative via its ordinal number with respect to the threshold preference order.

Keywords: Weak order; surjective utility function; indifference class; ordinal number; ordering algorithm; dual preference.

1. Introduction

In the theory of measurement,[1] one assigns real numbers to things under consideration, which help to understand or interpret them. In this paper, we shall deal with the following situation. Given a (finite) set (of alternatives) X and a preference order P (i.e., asymmetric and negatively transitive binary relation) on X, we would like to

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scale X to real numbers by means of a function, preserving order properties. More precisely, by a (utility) representation of P we mean a real-valued function $\varphi : X \to \mathbb{R}$ such that, given $x, y \in X$, x is preferred to y in the sense that $(x, y) \in P$ if and only if $\varphi(x) > \varphi(y)$. The existence of utility representations with different properties for preference orders was treated in a number of papers.[2–6] In the simplest case when X is finite one may explicitly set $\varphi(x) = |\{y \in X : (x, y) \in P\}|$, where $|A|$ denotes the number of elements in the set $A \subset X$.

In practice it is quite customary that an alternative is characterized by means of $n \geq 2$ grades $x_1, \ldots, x_n$, each of which taking an integer value from 1 ('bad') to $m \geq 2$ ('perfect'). In this way alternatives may be identified with elements of the set $X = \{1, 2, \ldots, m\}^n$ of all n-dimensional vectors x with integer components from 1 to m. Two natural approaches are known[7] to introduce (threshold) preference orders on X based on the leximin and leximax preferences: $x \in X$ is preferred to $y \in X$ in the threshold sense[8] provided $\widehat{x}$ is lexicographically preferred to $\widehat{y}$, where $\widehat{x}$ denotes the vector obtained from x by well ordering its coordinates in ascending (corresponding to the leximin) or descending (corresponding to the leximax) order.

The aim of this paper is to determine explicitly the most 'economic' and 'effective' utility function (called the enumerating preference function) for the threshold preference(s) on X, which, in addition, maps X onto the set $\{1, 2, \ldots, |X/I|\}$ surjectively, where X/I is the quotient set of X with respect to the indifference relation I on X induced by the threshold preference order. This permits us to evaluate all equivalence classes and indifference classes of the threshold order on X and present an algorithm of ordering the monotone representatives of indifference classes. Moreover, since the image of X under the enumerating preference function is 'dense' in the set $\{1, 2, \ldots, |X/I|\}$, we can restore the indifference class of an alternative via its ordinal number in the threshold preference order.

The main results of the paper were announced at the 1st International Conference on Information Technology and Quantitative Management ITQM 2013 (May 16–18, 2013, Suzhou, China).[9]

The paper is organized as follows. Section 2 contains preliminaries on preference (or weak) orders. Main results of the paper are presented in Sec. 3 and their proofs are given in Sec. 4. Sections 5 and 6 are devoted to algorithms of ordering the monotone representatives of alternatives.

2. The Threshold Preference and the EPF

We begin by recalling certain definitions and auxiliary facts needed for our results.

Given a finite set X of cardinality $|X| \geq 2$, elements of which will be called alternatives, let $P \subset X \times X$ be a *weak order* on X (cf. Ref. 10), i.e., P is transitive (if $x, y, z \in X$, $(x, y) \in P$ and $(y, z) \in P$, then $(x, z) \in P$), irreflexive $((x, x) \notin P$ for all $x \in X)$ and negatively transitive (if $x, y, z \in X$, $(x, y) \notin P$ and $(y, z) \notin P$, then $(x, z) \notin P$). It will be convenient to say that P is a (strict) *preference* on X and to interpret the inclusion $(x, y) \in P$ as 'x is preferred to y'. The *indifference* relation

$I = I_P$ on X, induced by P, is given by

$$I = \{(x, y) \in X \times X : (x, y) \notin P \text{ and } (y, x) \notin P\}, \tag{2.1}$$

and it is an *equivalence* relation on X. Given $x, y \in X$, we have: (i) $(x, y) \notin P$ is equivalent to $(y, x) \in P$ or $(y, x) \in I$ (negation of P); (ii) $(x, y) \notin P$ or $(y, x) \notin P$ (completeness of $(X \times X) \backslash P$); (iii) either $(x, y) \in P$, or $(x, y) \in I$, or $(y, x) \in P$ (trichotomy of P).

A typical example of a preference on X is given by a *representable* (more precisely, φ-representable) binary relation $P(\varphi) = \{(x, y) \in X \times X : \varphi(x) > \varphi(y)\}$ with a non-constant function $\varphi : X \to \mathbb{R}$. The function φ (nonuniquely determined, in general) is called a *preference* (or *utility*) *function* for P. Clearly, the indifference relation $I_{P(\varphi)}$ consists of those pairs $(x, y) \in X \times X$, for which $\varphi(x) = \varphi(y)$.

A preference P on X gives rise to the canonical *ranking* of X as follows (cf. Refs. 11 and 12). Given $A \subset X$, let us denote by $\pi(A) = \{x \in A : (y, x) \notin P \text{ for all } y \in A\}$ the set of most preferred alternatives from A. Set $X_1' = \pi(X)$ and, inductively, if $\ell \geq 2$ and nonempty disjoint subsets $X_1', \ldots, X_{\ell-1}'$ of X such that $\bigcup_{k=1}^{\ell-1} X_k' \neq X$ are already determined, then we put $X_\ell' = \pi(X \backslash (\bigcup_{k=1}^{\ell-1} X_k'))$. Since X is finite, there exists a unique positive integer $s = s_P(X)$ (which is equal to the cardinality of the quotient set X/I) such that $X = \bigcup_{\ell=1}^{s} X_\ell'$. Setting $X_\ell = X_{s-\ell+1}'$ for $\ell = 1, 2, \ldots, s$, the collection $\{X_\ell\}_{\ell=1}^{s}$ of pairwise disjoint sets, partitioning X, is said to be the *family of equivalence* (*indifference*) *classes* of the weak order P, and has the following characteristic property: given $x, y \in X$, $(x, y) \in P$ iff (= if and only if) there exist two integers k and ℓ with $1 \leq k < \ell \leq s$ such that $x \in X_\ell$ and $y \in X_k$. Thus, x is preferred to y if x lies in an equivalence class with a greater ordinal number. Also, $(x, y) \in I$ iff $x, y \in X_k$ for some integer $1 \leq k \leq s$.

We define a function $\Phi = \Phi_P : X \to \{1, 2, \ldots, s\}$ as follows: given $x \in X$, there exists a unique integer $1 \leq k \leq s$ such that $x \in X_k$, and so, we set $\Phi(x) = k$. In other words, $X_k = \{x \in X : \Phi(x) = k\}$ and

$$x \in X_{\Phi(x)} = \{y \in X : \Phi(y) = \Phi(x)\} \quad \text{for all } x \in X.$$

The function Φ is well-defined, uniquely determined and surjective preference function for P, which will be called the *enumerating preference function* (EPF, for short). A function $\varphi : X \to \mathbb{R}$ is a preference function for P on X iff $\varphi(x) = f(\Phi(x))$ for all $x \in X$ and some strictly increasing function $f : \{1, 2, \ldots, s\} \to \mathbb{R}$.

In what follows we are going to explicitly determine the EPF for the *threshold preference* (decision making), an important particular case of the leximin, the definitions of which we now recall.

Given two non-negative integers k and l with $k \leq l$, we denote by

$$[k, l] = \{i \in \{0\} \cup \mathbb{N} : k \leq i \leq l\} = \{k, k+1, \ldots, l-1, l\}$$

the (natural) *interval* with the endpoints k and l and 'length' $|[k, l]| = l - k + 1$ expressing the number of elements in $[k, l]$. We also set $[1, 0] = \emptyset$.

A binary relation $\angle_N$ on the set $\mathbb{R}^N$ of all N-dimensional vectors with real components is said to be the *lexicographic order* if, given $u = (u_1, \ldots, u_N)$ and $v = (v_1, \ldots, v_N)$ from $\mathbb{R}^N$, we have: $u \angle_N v$ iff there exists a $p \in [1, N]$ such that $u_i = v_i$ for all $i \in [1, p-1]$ (no condition if $p = 1$, since $[1, 0] = \emptyset$) and $u_p < v_p$. It is well known[10,13] that $\angle_N$ is a linear order on $\mathbb{R}^N$; more precisely, $\angle_N$ is transitive (i.e., $u \angle_N v$ and $v \angle_N w$ imply $u \angle_N w$), the negation of $\angle_N$ is of the form: $\neg(u \angle_N v)$ iff $v \angle_N u$ or $v = u$, and $\angle_N$ is trichotomous (i.e., either $u = v$, or $u \angle_N v$, or $v \angle_N u$).

Given $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$, let us assemble the coordinates of u in ascending order $u_1^* \le u_2^* \le \cdots \le u_N^*$, denote the resulting vector by $u^* = (u_1^*, u_2^*, \ldots, u_N^*)$ and call it the *monotone representative* of u. We say (cf. Refs. 7 and 14) that $u \in \mathbb{R}^N$ is *preferred to* $v \in \mathbb{R}^N$ *in the sense of the leximin* if $v^* \angle_N u^*$. Recall that neither the lexicographic order nor the leximin are representable on $\mathbb{R}^N$.[10]

The set of alternatives X, to be considered throughout the paper, is identified with the Cartesian product $[1, m]^n$ of $n \ge 1$ intervals $[1, m]$ with $m \ge 2$, and so, each alternative $x \in X$ is an n-dimensional vector $x = (x_1, \ldots, x_n)$ with components $x_i \in [1, m]$. Elements of $[1, n]$ may be interpreted as parameters (entities, agents, properties) and elements of $[1, m]$ — as ordered grades or criteria $1 < 2 < \cdots < m - 1 < m$. The vector-grades $x = (x_1, \ldots, x_n)$ (identified with alternatives $x \in X$) may represent expert grades, questionnare data, device readings, tests data, etc.[15] Note that $|X| = |[1, m]^n| = m^n$.

Two natural partial orders $\succcurlyeq$ and $\succ$ on $X = [1, m]^n$ are introduced in the usual way: given $x, y \in X$, we write $x \succcurlyeq y$ (or $y \preccurlyeq x$) if $x_i \ge y_i$ for all $i \in [1, n]$, and $x \succ y$ (or $y \prec x$) — if $x \succcurlyeq y$ and $x_{i_0} > y_{i_0}$ for some $i_0 \in [1, n]$.

The monotone representative of an alternative $x \in X$ is of the form

$$x^* = \Big(\underbrace{1, \ldots, 1}_{v_1(x)}, \underbrace{2, \ldots, 2}_{v_2(x)}, \ldots, \underbrace{m-1, \ldots, m-1}_{v_{m-1}(x)}, \underbrace{m, \ldots, m}_{v_m(x)} \Big)$$

$$\underbrace{}_{n}$$

$$= (1^{v_1(x)}, 2^{v_2(x)}, \ldots, (m-1)^{v_{m-1}(x)}, m^{v_m(x)}), \tag{2.2}$$

the number $v_j(x) \equiv v_j^{(n)}(x) = |\{i \in [1, n] : x_i = j\}|$ being the multiplicity of the grade $j \in [1, m]$ in the vector $x = (x_1, \ldots, x_n)$. Clearly, $v_j(x^*) = v_j(x)$. In what follows if the multiplicity of a grade j is zero, i.e., $v_j(x) = 0$, then the expression j^0 will be omitted in (2.2) (e.g., the vector $(1, \ldots, 1)$ from $[1, m]^n$ is simply (1^n)).

Given $A \subset X$, we denote by $A^* = \{x^* : x \in A\}$ the set of all monotone representatives of elements from A.

Also, given $x \in X$ and $j \in [1, m]$, we set

$$V_0(x) = 0 \quad \text{and} \quad V_j(x) \equiv V_j^{(n)}(x) = \sum_{k=1}^j v_k(x) \tag{2.3}$$

and note that $0 \le v_j(x) \le n$, $0 \le V_{j-1}(x) \le V_j(x) \le n$ and

$$\sum_{j=1}^m v_j(x) = V_m(x) = n. \tag{2.4}$$

The following two properties will play an important role below[8]:

$$x, y \in X, x \succ y \quad \text{implies} \quad x^* \succ y^*, \quad \text{and} \tag{2.5}$$

$x^* \succ y^*$ iff $\exists\, k \in [1, m-1]$ such that $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$,
$$v_k(x) < v_k(y) \text{ and } V_p(x) \leq V_p(y) \text{ for all } p \in [k+1, m-1]$$
(with no last condition if $k = m-1$). $\tag{2.6}$

The *threshold preference* $P = P_{m-1}$ on $X = [1, m]^n$ is defined by (see Refs. 16–18, if $m = 3$ and for all $m \geq 2$) $P_{m-1} = \{(x, y) \in X \times X : v(x) \angle_{m-1} v(y)\}$, where, given $x \in X$, $v(x) = (v_1(x), \ldots, v_{m-1}(x)) \in [0, n]^{m-1}$. The decision-making rule $v(x) \angle_{m-1} v$ (y) is said to be the *threshold rule*. More explicitly, if $m = 2$, we have $(x, y) \in P = P_1$ iff $v_1(x) < v_1(y)$, and if $m \geq 3$, we find $(x, y) \in P = P_{m-1}$ iff $v_1(x) < v_1(y)$ or there exists a $k \in [2, m-1]$ such that $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$ and $v_k(x) < v_k(y)$.

Let us show that the threshold preference P_{m-1} is the restriction of the leximin preference on $\mathbb{R}^n$ to the set $X = [1, m]^n$.

Lemma 2.1. *Given $x, y \in X$, we have: $(x, y) \in P_{m-1}$ iff $y^* \angle_n x^*$.*

Proof. *Necessity.* Let $(x, y) \in P_{m-1}$ and $k \in [1, m-1]$ be as in the explicit form of P_{m-1} above. Taking into account (2.3), we set $p = V_k(x) + 1$ and note that $p \leq V_k(y)$, which implies $p \in [1, n]$. Since $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$, then $y_i^* = x_i^*$ for all $i \in [1, p-1]$ and $y_p^* = k < k+1 \leq x_p^*$. It follows that $y^* \angle_n x^*$.

Sufficiency. Now, let $y^* \angle_n x^*$. Then there exists a $p \in [1, n]$ such that $y_i^* = x_i^*$ for all $i \in [1, p-1]$ and $y_p^* < x_p^*$. We set $k = y_p^*$ and note that the inequality $k < x_p^*$ implies $k \in [1, m-1]$. By condition $y_i^* = x_i^*$ for all $i \in [1, p-1]$, we find that $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$. Let us put $q = |\{i \in [1, p-1] : y_i^* = x_i^* = k\}|$. Then $v_k(y) \geq q+1$ and, since $k = y_p^* < x_p^* \leq x_{p+1}^* \leq \cdots \leq x_n^*$, we have $v_k(x) = q$. It follows that $v_k(x) < v_k(y)$, and so, $v(x) \angle_{m-1} v(y)$ implying $(x, y) \in P_{m-1}$. $\square$

Thus, the threshold preference P is a weak order on X. The indifference relation (2.1), induced by P, is given by: $(x, y) \in I$ iff $v_j(x) = v_j(y)$ for all $j \in [1, m]$ iff $v(x) = v(y)$ iff $x^* = y^*$, i.e., vectors x and y can be transformed into each other by certain permutations of their coordinates (anonymity of agents). Denoting by $I_x = \{y \in X : (x, y) \in I\}$ the indifference class of $x \in X$, the family of equivalence classes of the threshold preference P is given by $\{X_\ell\}_{\ell=1}^s = X/I = \{I_x : x \in X\}$ with the value $s = s_P(X)$ equal to[8]

$$s = |X^*| = |[1, m]^{n*}| = C_{n+m-1}^{m-1} = C_{n+m-1}^n = \frac{(n+m-1)!}{n!\,(m-1)!}, \tag{2.7}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ is the usual binomial coefficient if $k \in [0, n]$ (and $0! = 1$). Note also that x^* is a (monotone) representative of the equivalence class I_x for all $x \in X$, and the restriction of P to $X^* \times X^*$, denoted by P^*, is a linear order on X^* (i.e., P^* is transitive, irreflexive, and weakly connected: given $x^*, y^* \in X^*$ with $x^* \neq y^*$, $(x^*, y^*) \in P^*$ or $(y^*, x^*) \in P^*$).

In order to get a better feeling of the threshold preference order, it will be helpful to take a look at the ordering in ascending preference of, say, the set $[1,5]^{3*}$ of monotone representatives of elements from $X = [1,m]^n$ with $m = 5$ and $n = 3$.

Here we have the decomposition $X = [1,5]^3 = \bigcup_{\ell=1}^s X_\ell$ into the family of indifference classes with $s = 35$ (cf. (2.7)). For instance, X_{12} denotes the equivalence class of $x^* = (1,3,5)$, i.e., $X_{12} = I_{x^*}$. At the same time to each vector in Table 1 an ordinal number is assigned, which is given as the lower index at the right of the vector, and this ordinal number is exactly the value of the EPF at the vector, e.g., $\Phi(1,3,5) = 12$. The greater the number is the more preferable is the alternative. Also, it is seen from Table 1 that the EPF exhibits how 'far' from each other are vectors in the threshold ordering: clearly, $(2,2,5) \angle_3 (2,3,3)$, but only the values $\Phi(2,2,5) = 19$ and $\Phi(2,3,3) = 20$ show that the two vectors are 'neighbors'.

Since the EPF is also a preference function, it is desirable to have a characterization of preference functions for the threshold preference in fewer axioms. This has been done in Refs. 16 and 19 for $m = 3$ and extended in Refs. 8, 17 and 18 for the general case when $m \geq 2$ is arbitrary (Theorem A below is of different nature as compared to Refs. 13 and 14):

Theorem A. *Let $P = P_{m-1}$ on $X = [1,m]^n$ and $\varphi : X \to \mathbb{R}$. Then $P = P(\varphi)$ iff, given $x, y \in X$, the function φ satisfies axioms* $(A.1)_2$ *and* $(A.2)_2$ *if $m = 2$, or axioms* $(A.1)_m$, $(A.2)_m$ *and* $(A.3)_m$ *if $m \geq 3$, where*

$(A.1)_m$ *if $v_j(x) = v_j(y)$ for all $j \in [1, m-1]$, then $\varphi(x) = \varphi(y)$ (anonymity);*
$(A.2)_m$ *if $x \succ y$ in X, then $\varphi(x) > \varphi(y)$ (Pareto domination);*
$(A.3)_m$ *given $k \in [3, m]$, the following condition $(A.3.k)_m$ holds: if $v_j(x) = v_j(y)$ for all $j \in [1, m-k]$, $v_{m-k+1}(x) + 1 = v_{m-k+1}(y) \neq n - V_{m-k}(y)$, $V_{m-k+2}(x) = n$ and $V_{m-k+1}(y) + v_m(y) = n$, then $\varphi(x) > \varphi(y)$ (noncompensatory threshold).*

In Table 1, neighbor vectors separated by comma obey axiom $(A.2)_m$, and those separated by semicolon obey axiom $(A.3)_m$.

An explicit preference function for P_{m-1} on $X = [1,m]^n$ was given in Ref. 20:

$$\varphi_0(x) = \psi_0(x) + 1 - \frac{m^n - 1}{m - 1} \quad \text{if } x = (x_1, \ldots, x_n) \in X,$$

where $\psi_0(x) = \sum_{i=1}^n m^{n-i} x_i^*$ is the value in the decimal system $\langle 0, 1, \ldots, 9 \rangle$ of the number $x_1^* x_2^* \ldots x_n^*$ constructed from the vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ from (2.2) and

Table 1.   Example of the threshold preference.

$(1,1,1)_1$,	$(1,1,2)_2$,	$(1,1,3)_3$,	$(1,1,4)_4$,	$(1,1,5)_5$;		
$(1,2,2)_6$,	$(1,2,3)_7$,	$(1,2,4)_8$,	$(1,2,5)_9$;	$(1,3,3)_{10}$,	$(1,3,4)_{11}$,	$(1,3,5)_{12}$;
$(1,4,4)_{13}$,	$(1,4,5)_{14}$,	$(1,5,5)_{15}$;	$(2,2,2)_{16}$,	$(2,2,3)_{17}$,	$(2,2,4)_{18}$,	$(2,2,5)_{19}$;
$(2,3,3)_{20}$,	$(2,3,4)_{21}$,	$(2,3,5)_{22}$,	$(2,4,4)_{23}$,	$(2,4,5)_{24}$,	$(2,5,5)_{25}$;	
$(3,3,3)_{26}$,	$(3,3,4)_{27}$,	$(3,3,5)_{28}$,	$(3,4,4)_{29}$,	$(3,4,5)_{30}$,	$(3,5,5)_{31}$;	
$(4,4,4)_{32}$,	$(4,4,5)_{33}$,	$(4,5,5)_{34}$,	$(5,5,5)_{35}$.			

considered in the m-ary system $\langle 0, 1, \ldots, m-1 \rangle$. For instance, in the context of Table 1 (with $m = 5$ and $n = 3$) we have $\varphi_0(x_1, x_2, x_3) = 25x_1^* + 5x_2^* + x_3^* - 30$ for $x = (x_1, x_2, x_3) \in [1, 5]^3$, and so, $\varphi_0(3, 5, 5) = 75$ and $\varphi_0(4, 4, 4) = 94$, whereas the values of the EPF Φ at these vectors (see Table 1) are $\Phi(3, 5, 5) = 31$ and $\Phi(4, 4, 4) = 32$ (and so, the vectors are neighbors in the threshold ordering). The values of φ_0 are scattered from 1 (the minimal value of φ_0 attained at $(1^n) = (1, \ldots, 1)$) and m^n (the maximal value of φ_0 attained at $(m^n) = (m, \ldots, m)$), although, as it will be seen from Lemma 2.2 below, φ_0 takes on only s (cf. (2.7)) different values, where s is much smaller than m^n (e.g., for $m = 10$ and $n = 60$ we have $s = 56,672,074,888 < 10^{11} \ll 10^{60}$). Clearly, φ_0 is not the EPF for P_{m-1}.

Lemma 2.2. *If $\varphi : X \to \mathbb{R}$ is a preference function for $P = P_{m-1}$ on $X = [1, m]^n$ and $\varphi(A) = \{\varphi(x) : x \in A\}$ is the image of a subset $A \subset X$ under φ, then*

$$\varphi(A) = \varphi(A^*) \quad and \quad |\varphi(A)| = |\varphi(A^*)| = |A^*|. \tag{2.8}$$

Proof. In fact, given $l \in \varphi(A)$, we have $l = \varphi(x)$ for some $x \in A$, and so, $x^* \in A^*$ and, by axiom $(A.1)_m$ from Theorem A, $\varphi(x^*) = \varphi(x) = l$ implying $l \in \varphi(A^*)$. Conversely, if $l \in \varphi(A^*)$, then $l = \varphi(x)$ for some $x \in A^*$, and so, there exists an $a \in A$ such that $a^* = x$, which, again by virtue of axiom $(A.1)_m$, gives $\varphi(a) = \varphi(a^*) = \varphi(x) = l$ and $l \in \varphi(A)$. This proves the first equality in (2.8). In order to establish the third equality in (2.8), it suffices to verify that φ maps A^* into $\mathbb{R}$ injectively. Given $x^*, y^* \in A^*$ with $x^* \neq y^*$, by virtue of property (iii) of P from the beginning of this section, we have $(x^*, y^*) \in P$ or $(y^*, x^*) \in P$, and so, since φ is a preference function for P, either $\varphi(x^*) > \varphi(y^*)$ or $\varphi(y^*) > \varphi(x^*)$. Thus, φ maps A^* onto the image $\varphi(A^*)$ bijectively, and so, $|\varphi(A^*)| = |A^*|$. $\qquad\square$

3. Main Results

It follows from (2.8) that the number of elements in the image $\Phi(X)$ of the EPF Φ for $P = P_{m-1}$ is equal to $s = |X^*|$ from (2.7), and so, Φ maps X onto the natural interval $[1, |X^*|]$. The first main result of the paper asserts that Φ can be given explicitly in terms of binomial coefficients and quantities (2.3) as follows.

Theorem 3.1. *A function Φ maps $X = [1, m]^n$ onto $[1, |X^*|]$ and is a preference function for $P = P_{m-1}$ on X (i.e., Φ is the EPF for P) iff it is of the form*

$$\Phi(x) = \sum_{j=1}^{m} C_{n - V_j(x) + m - j - 1}^{m-j} \quad for \ all \ x \in X, \tag{3.1}$$

where $C_k^{k+1} = 0$ if $k \in [0, m-1]$, and $C_{-1}^0 = 1$.

It is to be noted that, by virtue of (2.4), the last two terms in (3.1) corresponding to $j = m - 1$ and $j = m$ are equal to $C_{v_m(x)}^1 = v_m(x)$ and $C_{-1}^0 = 1$, respectively.

In particular, the EPF Φ for P_4 on $X = [1,5]^n$ assumes the form:

$$\Phi(x) = \frac{1}{24}(n - v_1(x) + 3)(n - v_1(x) + 2)(n - v_1(x) + 1)(n - v_1(x))$$

$$+ \frac{1}{6}(n - v_1(x) - v_2(x) + 2)(n - v_1(x) - v_2(x) + 1)(n - v_1(x) - v_2(x))$$

$$+ \frac{1}{2}(n - v_1(x) - v_2(x) - v_3(x) + 1)(n - v_1(x) - v_2(x) - v_3(x)) + v_5(x) + 1,$$

where $v_1(x) + v_2(x) + v_3(x) + v_4(x) + v_5(x) = n$ (cf. Table 1 with $n = 3$).

As a corollary of Theorem 3.1, we are able to characterize the family of equivalence classes $\{X_\ell\}_{\ell=1}^s$ of P as well as the family of indifference classes $\{I_x\}_{x \in X}$ in Theorem 3.3 below. For this, we need the following auxiliary result, which is of independent interest and needed in the proof of Theorem 3.1.

Theorem 3.2. *Given two integers $n_0 = n \geq 1$ and $m \geq 2$, an integer ℓ belongs to the interval $[1, C_{n+m-1}^{m-1}]$ iff there exists a unique collection of $m - 2$ integers $n_1, n_2, \ldots, n_{m-2}$ satisfying $0 \leq n_j \leq n_{j-1}$ for all $j \in [1, m-2]$ such that*

$$\ell \in [L+1, L+1+n_{m-2}], \quad \text{where } L = \sum_{j=1}^{m-2} C_{n_j+m-j-1}^{m-j}. \tag{3.2}$$

Theorem 3.3. *Given $\ell \in [1, |X^*|]$, we have:*
(a) $X_\ell = \{x \in X : \Phi(x) = \ell\}$; *in other words, $x \in X_{\Phi(x)}$ and*

$$I_x = X_{\Phi(x)} = \{y \in X : \Phi(y) = \Phi(x)\} \quad \text{for all } x \in X;$$

(b) *given $x \in X$, x lies in X_ℓ iff (in the notation of Theorem 3.2)*

$$v_j(x) = n_{j-1} - n_j \quad \text{for all } j \in [1, m-2], \tag{3.3}$$

$$v_{m-1}(x) = L + 1 + n_{m-2} - \ell \quad \text{and} \quad v_m(x) = \ell - L - 1. \tag{3.4}$$

Note that Theorem 3.3(b) answers the question: given $\ell \in [1, s] = [1, |X^*|]$, what are the vectors $x \in X$ satisfying $x \in X_\ell$? Taking into account Theorem 3.3(a), this can be reformulated as: find all solutions $x \in X$ of the equation $\Phi(x) = \ell$. In other words, in Theorem 3.3(b) the equivalence class X_ℓ of the threshold preference P is restored via its ordinal number ℓ. The number of elements in X_ℓ can be calculated as follows: if the generic vector x from X_ℓ satisfies conditions (3.3) and (3.4), then

$$|X_\ell| = \frac{n!}{v_1(x)! \cdots v_m(x)!} = \frac{n!}{\prod_{j=1}^{m-2}(n_{j-1} - n_j)! \cdot (L+1+n_{m-2} - \ell)! \cdot (\ell - L - 1)!}.$$

4. Proofs of the Main Results

Throughout the proofs we apply the summation over lower indices formula for binomial coefficients (see Ref. 21, formulas (5.9) and (5.10)): if $p, q \geq 0$ are integers,

then

$$\sum_{k=0}^{q} C_{p+k}^{p} = C_{p}^{p} + C_{p+1}^{p} + \cdots + C_{p+q}^{p} = C_{p+q+1}^{p+1} = C_{p+q+1}^{q}. \qquad (4.1)$$

Proof of Theorem 3.2. If there are integer numbers $n_1, n_2, \ldots, n_{m-2}$ satisfying $0 \le n_j \le n_{j-1}$ for all $j \in [1, m-2]$ such that (3.2) holds, then, by virtue of (4.1),

$$1 \le L+1 \le \ell \le L+1+n_{m-2} \le \sum_{j=1}^{m-2} C_{n+m-j-1}^{m-j} + 1 + n$$

$$= \sum_{j=1}^{m} C_{n+m-j-1}^{m-j} = \sum_{j=1}^{m} C_{n-1+m-j}^{n-1} = \sum_{k=0}^{m-1} C_{n-1+k}^{n-1} = C_{n+m-1}^{m-1}. \qquad (4.2)$$

Conversely, we apply the induction argument on m for each integer $n \ge 1$. If $m = 2$, then $C_{n+m-1}^{m-1} = C_{n+1}^{1} = n+1$, $n_{m-2} = n_0 = n$ and $L = 0$, and so, the assertion in this case is a tautology. If $m = 3$, then $C_{n+m-1}^{m-1} = C_{n+2}^{2}$ and $[1, C_{n+2}^{2}] = \bigcup_{k=0}^{n}[C_{k+1}^{2}+1, C_{k+2}^{2}]$ (disjoint union), and so, given $\ell \in [1, C_{n+2}^{2}]$, there exists a unique number $n_1 \in [0, n]$ such that

$$\ell \in [C_{n_1+1}^{2}+1, C_{n_1+2}^{2}] = [C_{n_1+1}^{2}+1, C_{n_1+1}^{2}+1+n_1],$$

and it remains to note that $L = C_{n_1+1}^{2} = C_{n_1+3-1-1}^{3-1}$. Now, suppose that the necessity part in Theorem 3.2 holds for some $m \ge 3$ and all $n \ge 1$, and assume that $\ell \in [1, C_{n+m}^{m}]$. Noting that $[1, C_{n+m}^{m}] = \bigcup_{k=0}^{n}[C_{k+m-1}^{m}+1, C_{k+m}^{m}]$ (disjoint union), we find a unique $n_1 \in [0, n]$ such that

$$C_{n_1+m-1}^{m}+1 \le \ell \le C_{n_1+m}^{m} = C_{n_1+m-1}^{m} + C_{n_1+m-1}^{m-1},$$

and so, $1 \le \ell' \equiv \ell - C_{n_1+m-1}^{m} \le C_{n_1+m-1}^{m-1}$. Applying the induction hypothesis to the integer ℓ' we obtain a unique collection of $m-2$ non-negative integers $n_1', n_2', \ldots, n_{m-2}'$ satisfying $n_j' \le n_{j-1}'$ for all $j \in [1, m-2]$, where $n_0' = n_1$, such that if $L' = \sum_{j=1}^{m-2} C_{n_j'+m-j-1}^{m-j}$, then

$$L'+1 \le \ell' = \ell - C_{n_1+m-1}^{m} \le L'+1+n_{m-2}'.$$

We set $n_j = n_{j-1}'$ for all $j \in [2, m-1]$. It follows that $0 \le n_j \le n_{j-1}$ for all $j \in [1, (m+1)-2]$, $n_j' = n_{j+1}$ for all $j \in [0, m-2]$,

$$L' = \sum_{j=1}^{m-2} C_{n_{j+1}+m-j-1}^{m-j} = \sum_{j=2}^{m-1} C_{n_j+m-j}^{m+1-j} = \sum_{j=2}^{(m+1)-2} C_{n_j+(m+1)-j-1}^{(m+1)-j}$$

and $L+1 \le \ell \le L+1+n_{(m+1)-2}$, where

$$L = C_{n_1+m-1}^{m} + L' = \sum_{j=1}^{(m+1)-2} C_{n_j+(m+1)-j-1}^{(m+1)-j},$$

and now assertion (3.2) follows with m replaced by $m+1$. $\qquad\square$

Proof of Theorem 3.1. We begin with proving the necessity part. We apply the induction argument on $m \geq 2$ for each integer $n \geq 1$ and divide the proof into several steps for clarity.

Step 1. Suppose $m = 2$. We have: $X = [1, 2]^n$, $(x, y) \in P = P_1$ iff $v_1(x) < v_1(y)$, $v_1(x) + v_2(x) = n$ if $x, y \in X$, $X^* = \{(1^{n-k}, 2^k) : 0 \leq k \leq n\}$ (in the notation (2.2)) and $|X^*| = n + 1$. Let $\Phi : X \overset{\text{onto}}{\longrightarrow} [1, n + 1]$ be a preference function for P on X, and so, axioms $(A.1)_2$ and $(A.2)_2$ from Theorem A are satisfied. Then Φ maps X^* into $[1, n + 1]$ bijectively. Noting that the relation P coincides with $\succ$ on X^* and $(2^n) \succ (1, 2^{n-1}) \succ \cdots \succ (1^{n-1}, 2) \succ (1^n)$ in X, we find from axiom $(A.2)_2$ that $\Phi(2^n) > \Phi(1, 2^{n-1}) > \cdots > \Phi(1^{n-1}, 2) > \Phi(1^n)$. There are $n + 1$ different values in this chain of inequalities and, since the image of X^* under Φ is $[1, n + 1]$, then $\Phi(1^n) = 1$, $\Phi(1^{n-1}, 2) = 2, \ldots, \Phi(1, 2^{n-1}) = n$ and $\Phi(2^n) = n + 1$, and so, $\Phi(1^{n-k}, 2^k) = k + 1$ for all $k \in [0, n]$. It follows that if $x \in X$, then $x^* = (1^{n-k}, 2^k)$ with $k = v_2(x^*) = v_2(x)$, and so, by axiom $(A.1)_2$, we get

$$\Phi(x) = \Phi(x^*) = v_2(x) + 1 \quad \text{for all } x \in X = [1, 2]^n. \tag{4.3}$$

Clearly, the function Φ from (4.3), which is of the form (3.1) with $m = 2$, maps X onto $[1, n + 1] = [1, |X^*|]$ and, by virtue of (2.4) with $m = 2$, satisfies axioms $(A.1)_2$ and $(A.2)_2$, and so, it is a preference function for $P = P_1$ on X.

Thus, Theorem 3.1 is established for $m = 2$ and all integer $n \geq 1$.

Step 2. Now suppose that the necessity part holds for some $m \geq 2$ and all $n \geq 1$, and let us show that it remains valid for $m + 1$ and all $n \geq 1$, as well.

Let $X = [1, m + 1]^n$. The threshold preference $P = P_m$ on X is given for $x, y \in X$ by: $(x, y) \in P$ iff $v_1(x) < v_1(y)$ or there exists a $k \in [2, m]$ such that $v_j(x) = v_j(y)$ for all $j \in [1, k - 1]$ and $v_k(x) < v_k(y)$. Also (cf. (2.4) with m replaced by $m + 1$),

$$v_1(x) + v_2(x) + \cdots + v_m(x) + v_{m+1}(x) = n \quad \text{for all } x \in X \tag{4.4}$$

and, by virtue of (2.7), $|X^*| = |[1, m + 1]^{n*}| = C_{n+m}^m$.

Given $i \in [0, n]$, we set $X(i) = \{x \in X : v_1(x) = v_1^{(n)}(x) = i\}$ and

$$X^*(i) \equiv X(i)^* = X(i) \cap X^* = \{x^* \in X^* : (1^i, 2^{n-i}) \preccurlyeq x^* \preccurlyeq (1^i, (m+1)^{n-i})\}.$$

Clearly, $X(n) = \{(1^n)\}$. Let us fix $i \in [0, n - 1]$ and define a function β_i from $X' \equiv [1, m]^{n-i}$ into $X(i)$ by the rule:

given $x' = (x'_1, \ldots, x'_{n-i}) \in X'$, we set $\beta_i(x') = (1^i, x'_1 + 1, \ldots, x'_{n-i} + 1)$.

Clearly, β_i maps X' into $X(i)$ injectively and X'^* into $X^*(i)$ bijectively, and so, by virtue of (2.7) with n replaced by $n - i$,

$$|X^*(i)| = |X'^*| = |([1, m]^{n-i})^*| = C_{(n-i)+m-1}^{m-1} \quad \text{for all } i \in [0, n]. \tag{4.5}$$

Also, note that

$$v_j(x') \equiv v_j^{(n-i)}(x') = v_{j+1}(\beta_i(x')) \quad \text{for all } j \in [1, m]. \tag{4.6}$$

Now, assume that $\Phi : X \overset{\text{onto}}{\longrightarrow} [1, |X^*|]$ is a preference function for P_m on X.

Step 2a. Let us show that the composed function Φ_i, defined for $x' \in X'$ by $\Phi_i(x') = \Phi(\beta_i(x'))$, is a preference function for $P' = P_{m-1}$ on X'. Let $x', y' \in X'$. First, assume that $m = 2$. The definition of P_1 implies: $(x', y') \in P' = P_1$ iff $v_1(x') < v_1(y')$, and so, by virtue of (4.6), this is equivalent to $v_1(\beta_i(x')) = i = v_1(\beta_i(y'))$ and $v_2(\beta_i(x')) < v_2(\beta_i(y'))$, that is, $(\beta_i(x'), \beta_i(y')) \in P_2 = P_m$. Now, suppose that $m \geq 3$. By the definition of P_{m-1}, we have: $(x', y') \in P'$ iff $v_1(x') < v_1(y')$ or there exists a $k' \in [2, m-1]$ such that $v_j(x') = v_j(y')$ for all $j \in [1, k'-1]$ and $v_{k'}(x') < v_{k'}(y')$, which, by virtue of (4.6), is equivalent to: $v_2(\beta_i(x')) < v_2(\beta_i(y'))$ or there exists a $k' \in [2, m-1]$ such that $v_{j+1}(\beta_i(x')) = v_{j+1}(\beta_i(y'))$ for all $j \in [1, k'-1]$ and $v_{k'+1}(\beta_i(x')) < v_{k'+1}(\beta_i(y'))$. Since $v_1(\beta_i(x')) = i = v_1(\beta_i(y'))$, it follows that $(x', y') \in P'$ iff there exists a $k \in [2, m]$ such that $v_j(\beta_i(x')) = v_j(\beta_i(y'))$ for all $j \in [1, k-1]$ and $v_k(\beta_i(x')) < v_k(\beta_i(y'))$, i.e., $(\beta_i(x'), \beta_i(y')) \in P_m = P$. Thus, given $m \geq 2$ and $x', y' \in X'$, $(x', y') \in P'$ iff $(\beta_i(x'), \beta_i(y')) \in P$, and so, since Φ is a preference function for P on X, we find

$$(x', y') \in P' \quad \text{iff} \quad (\beta_i(x'), \beta_i(y')) \in P \quad \text{iff} \quad \Phi(\beta_i(x')) > \Phi(\beta_i(y'))$$
$$\text{iff} \quad \Phi_i(x') > \Phi_i(y'),$$

which proves that $\Phi_i : X' \to \mathbb{R}$ is a preference function for P' on X'.

Step 2b. Let us prove that

$$\Phi(X^*(i)) = [\Phi(1^i, 2^{n-i}), \Phi(1^i, (m+1)^{n-i})] \quad \text{for all } i \in [0, n]. \tag{4.7}$$

If $x^* \in X^*(i)$, then $(1^i, 2^{n-i}) \preccurlyeq x^* \preccurlyeq (1^i, (m+1)^{n-i})$, and so, by axioms $(A.1)_{m+1}$ and $(A.2)_{m+1}$ for the preference function Φ for P on X (Theorem A), we get $\Phi(1^i, 2^{n-i}) \leq \Phi(x^*) \leq \Phi(1^i, (m+1)^{n-i})$, which establishes the inclusion $\subset$ in (4.7). Conversely, suppose ℓ lies in the set on the right in (4.7). Since, by the assumption, $\Phi(X) = [1, |X^*|]$, and, by Lemma 2.2, $\Phi(X^*) = \Phi(X)$, we find that $\ell \in [1, |X^*|]$, and so, there exists an $x^* \in X^*$ such that $\ell = \Phi(x^*)$. Noting that X^* is the disjoint union of sets $X^*(k)$ over all $k \in [0, n]$, we find a $k \in [0, n]$ such that $x^* \in X^*(k)$. If we show that $k = i$, then $\ell \in \Phi(X^*(i))$, which completes the proof of (4.7). In fact, if $k < i$, then $v_1(x^*) = k < i = v_1(1^i, (m+1)^{n-i})$, and so, by the definition of P, $(x^*, (1^i, (m+1)^{n-i})) \in P$ implying $\ell = \Phi(x^*) > \Phi(1^i, (m+1)^{n-i})$, which is a contradiction. Similarly, if $k > i$, then $v_1(1^i, 2^{n-i}) = i < k = v_1(x^*)$, whence $((1^i, 2^{n-i}), x^*) \in P$, and so, $\Phi(1^i, 2^{n-i}) > \Phi(x^*) = \ell$, which is also a contradiction. Thus, $k = i$.

Step 2c. Given $i \in [0, n-1]$, let us find the EPF $\Psi_i : X' \overset{\text{onto}}{\longrightarrow} [1, |X'^*|]$ for P' on X' (in the notations of Steps 2 and 2a) and apply the induction hypothesis to it.

Since the function Φ_i from Step 2a is a preference function for P' on X', applying (2.8) and (4.5), recalling the definition of Φ_i and that $\beta_i(X'^*) = X^*(i)$ and taking into account (4.7), we get:

$$|X^*(i)| = |X'^*| = |\Phi_i(X'^*)| = |\Phi(\beta_i(X'^*))| = |\Phi(X^*(i))|$$
$$= \Phi(1^i, (m+1)^{n-i}) - \Phi(1^i, 2^{n-i}) + 1. \tag{4.8}$$

Since Φ is a preference function for P on X, then applying (2.8), noting that $(\beta_i(X'))^* = X^*(i)$ and taking into account (4.7) once again and (4.8), we find

$$
\begin{aligned}
\Phi_i(X') = \Phi(\beta_i(X')) &= \Phi((\beta_i(X'))^*) = \Phi(X^*(i)) \\
&= \Phi(1^i, 2^{n-i}) - 1 + [1, \Phi(1^i, (m+1)^{n-i}) - \Phi(1^i, 2^{n-i}) + 1] \\
&= \Phi(1^i, 2^{n-i}) - 1 + [1, |X'^*|].
\end{aligned}
\tag{4.9}
$$

Given $x' \in X'$, we set $\Psi_i(x') = \Phi_i(x') - \Phi(1^i, 2^{n-i}) + 1$. It follows from Step 2a and (4.9) that $\Psi_i : X' \xrightarrow{\text{onto}} [1, |X'^*|]$ is a preference function for P' on X'. Since $X' = [1, m]^{n-i}$ and $P' = P_{m-1}$ is the threshold preference on X', by the induction hypothesis, we get:

$$
\Psi_i(x') = \sum_{j=1}^{m} C^{m-j}_{(n-i)-V_j(x')+m-j-1}, \quad x' \in X',
$$

and so, since, as noticed earlier, the last term in the sum above corresponding to $j = m$ is equal to $C^0_{-1} = 1$, we obtain the following equality:

$$
\Phi_i(x') = \Phi(1^i, 2^{n-i}) + \sum_{j=1}^{m-1} C^{m-j}_{(n-i)-V_j(x')+m-j-1}, \quad x' \in X' = [1, m]^{n-i}.
\tag{4.10}
$$

By virtue of (2.3) and (4.6), we have:

$$
\begin{aligned}
V_j(x') = \sum_{k=1}^{j} v_k(x') &= \sum_{k=1}^{j} v_{k+1}(\beta_i(x')) = \sum_{k=2}^{j+1} v_k(\beta_i(x')) \\
&= V_{j+1}(\beta_i(x')) - v_1(\beta_i(x')),
\end{aligned}
$$

and so, the lower index in the binomial coefficient in (4.10) is equal to

$$
(n-i) - V_j(x') + m - j - 1 = (n-i) + v_1(\beta_i(x')) - V_{j+1}(\beta_i(x')) + m - j - 1.
$$

Taking into account the definition of Φ_i and changing the summation index $j + 1 \mapsto j$ in (4.10), we find that, given $x' \in X'$,

$$
\Phi(\beta_i(x')) = \Phi(1^i, 2^{n-i}) + \sum_{j=2}^{m} C^{(m+1)-j}_{(n-i)+v_1(\beta_i(x'))-V_j(\beta_i(x'))+m-j}.
\tag{4.11}
$$

Given $x \in X(i)$, we have $x^* \in X^*(i) = \beta_i(X'^*)$ and, since β_i maps X'^* into $X^*(i)$ bijectively, there exists a unique $x'^* \in X'^*$ such that $x^* = \beta_i(x'^*)$. Setting $x' = x'^*$ in (4.11) and noting that $v_j(\beta_i(x'^*)) = v_j(x^*) = v_j(x)$ for all $j \in [1, m]$, and so, by axiom $(A.1)_{m+1}$, $\Phi(\beta_i(x'^*)) = \Phi(x^*) = \Phi(x)$, we arrive at the equality

$$
\Phi(x) = \Phi(1^i, 2^{n-i}) + \sum_{j=2}^{m} C^{(m+1)-j}_{(n-i)+v_1(x)-V_j(x)+m-j}, \quad x \in X(i),
\tag{4.12}
$$

where $i \in [0, n-1]$. Note that equality (4.12) holds for $i = n$ as well: in fact, if $i = n$, then $x \in X(i) = X(n)$ iff $v_1(x) = n$ iff $x = (1^n)$, $(1^i, 2^{n-i}) = (1^n, 2^0) = (1^n)$ and $V_j(x) = n$, and so, $C^{(m+1)-j}_{m-j} = 0$ for all $j \in [2, m]$.

It remains to calculate the value $\Phi(1^i, 2^{n-i})$ in (4.12). For this, we need the following equality:

$$\Phi(1^i, 2^{n-i}) = \Phi(1^{i+1}, (m+1)^{n-(i+1)}) + 1 \quad \text{for all } i \in [0, n-1]. \tag{4.13}$$

Step 2d. Proof of (4.13). First, note that $\Phi(1^n) = 1$ and $\Phi((m+1)^n) = |X^*|$. In fact, given $x \in X$, we have $(m+1)^n \succcurlyeq x \succcurlyeq (1^n)$, and so, by axioms $(A.1)_{m+1}$ and $(A.2)_{m+1}$, we have $\Phi((m+1)^n) \geq \Phi(x) \geq \Phi(1^n)$, and since $\Phi : X \xrightarrow{\text{onto}} [1, |X^*|]$, the desired equalities follow.

In order to prove (4.13), let us fix $i \in [0, n-1]$. For the sake of brevity, we set $z^* = (1^i, 2^{n-i})$ and $w^* = (1^{i+1}, (m+1)^{n-(i+1)})$. Note that, since $v_1(z^*) = i < i+1 = v_1(w^*)$, we have $(z^*, w^*) \in P$, and so, $\Phi(z^*) > \Phi(w^*)$. For any $x^* \in \bigcup_{k=0}^{i} X^*(k)$ (disjoint union) we find $\Phi(z^*) \leq \Phi(x^*) \leq \Phi((m+1)^n) = |X^*|$: in fact, if $x^* \in X^*(k)$ with $k \in [0, i-1]$, then $v_1(x^*) = k < i = v_1(z^*)$, and so, $(x^*, z^*) \in P$ implying $\Phi(x^*) > \Phi(z^*)$, and if $x^* \in X^*(i)$, then $x^* \succcurlyeq z^*$, and so, by axiom $(A.2)_{m+1}$, $\Phi(x^*) \geq \Phi(z^*)$. Similarly, if $y^* \in \bigcup_{k=i+1}^{n} X^*(k)$ (disjoint union), then $1 = \Phi(1^n) \leq \Phi(y^*) \leq \Phi(w^*)$: in fact, if $y^* \in X^*(i+1)$, then $y^* \preccurlyeq w^*$, and so, by axiom $(A.2)_{m+1}$, $\Phi(y^*) \leq \Phi(w^*)$, and if $y^* \in X^*(k)$ with $i+1 < k \leq n$, then $v_1(w^*) = i+1 < k = v_1(y^*)$, and so, $(w^*, y^*) \in P$ implying $\Phi(w^*) > \Phi(y^*)$. It follows that

$$\Phi\left(\bigcup_{k=0}^{i} X^*(k)\right) \subset [\Phi(z^*), |X^*|] \quad \text{and} \quad \Phi\left(\bigcup_{k=i+1}^{n} X^*(k)\right) \subset [1, \Phi(w^*)],$$

and, since $X^* = \bigcup_{k=0}^{n} X^*(k) = (\bigcup_{k=0}^{i} X^*(k)) \cup (\bigcup_{k=i+1}^{n} X^*(k))$ (disjoint union), we get

$$[1, |X^*|] = \Phi(X^*) = \Phi\left(\bigcup_{k=0}^{n} X^*(k)\right) \subset [1, \Phi(w^*)] \cup [\Phi(z^*), |X^*|],$$

where $\Phi(w^*) < \Phi(z^*)$. Since the intervals in this inclusion are natural, we get $\Phi(w^*) + 1 = \Phi(z^*)$, and equality (4.13) follows.

Step 2e. In order to establish equality (3.1) for $m+1$ making use of (4.12), let $i \in [0, n]$ and let us calculate the value $\Phi(1^i, 2^{n-i})$. By virtue of (4.1), (4.5), (4.8) and (4.13), we have:

$$\Phi(1^i, 2^{n-i}) = \Phi(1^{i+1}, (m+1)^{n-(i+1)}) - \Phi(1^{i+1}, 2^{n-(i+1)}) + 1 + \Phi(1^{i+1}, 2^{n-(i+1)})$$

$$= |X^*(i+1)| + \Phi(1^{i+1}, 2^{n-(i+1)})$$

$$= |X^*(i+1)| + |X^*(i+2)| + \cdots + |X^*(n)| + \Phi(1^n, 2^0)$$

$$= \sum_{l=i+1}^{n} C_{(n-l)+m-1}^{m-1} + \Phi(1^n) = \sum_{k=0}^{n-i-1} C_{(m-1)+k}^{m-1} + 1$$

$$= C_{(n-i)+m-1}^{m} + 1.$$

It follows from (4.12) that, given $i \in [0, n]$ and $x \in X(i)$,

$$\Phi(x) = C^{(m+1)-1}_{(n-i)+(m+1)-1-1} + \sum_{j=2}^{m} C^{(m+1)-j}_{(n-i)+v_1(x)-V_j(x)+(m+1)-j-1} + 1. \qquad (4.14)$$

Now, given $x \in X$, we find that $x \in X(i)$ with $i = v_1(x)$, and so, applying (4.14) and noting that, by virtue of (4.4),

$$1 = C^0_{-1} = C^{(m+1)-(m+1)}_{(n-v_1(x))+v_1(x)-V_{m+1}(x)+(m+1)-(m+1)-1},$$

we conclude that

$$\Phi(x) = \sum_{j=1}^{m+1} C^{(m+1)-j}_{n-V_j(x)+(m+1)-j-1} \quad \text{for all } x \in X = [1, m+1]^n,$$

as asserted in (3.1) for $m + 1$ in place of m.

This completes the proof of the necessity part of Theorem 3.1. Now we turn to the proof of the sufficiency part.

Step 3. First, we prove that the function Φ given by (3.1) is a preference function for $P = P_{m-1}$ on $X = [1, m]^n$ with $m \geq 3$. For this, it suffices to verify that Φ satisfies the three axioms from Theorem A. Let $x, y \in X$.

Axiom $(A.1)_m$. If $v_j(x) = v_j(y)$ for all $j \in [1, m-1]$, then, by virtue of (2.3) and (2.4), $V_j(x) = V_j(y)$ for all $j \in [1, m]$, and so, formula (3.1) implies $\Phi(x) = \Phi(y)$.

Axiom $(A.2)_m$. Suppose that $x \succ y$ in X. Then, by (2.5), $x^* \succ y^*$, and so, condition at the right in (2.6) is satisfied. It follows from (2.3) that $V_j(x) = V_j(y)$ for all $j \in [1, k-1]$, $V_k(x) < V_k(y)$, $V_p(x) \leq V_p(y)$ for all $p \in [k+1, m-1]$ and $V_m(x) = V_m(y) = n$. Therefore,

$$C^{m-j}_{n-V_j(x)+m-j-1} = C^{m-j}_{n-V_j(y)+m-j-1} \quad \text{for all } j \in [1, k-1],$$

$$C^{m-k}_{n-V_k(x)+m-k-1} > C^{m-k}_{n-V_k(y)+m-k-1}, \quad \text{and}$$

$$C^{m-j}_{n-V_j(x)+m-j-1} \geq C^{m-j}_{n-V_j(y)+m-j-1} \quad \text{for all } j \in [k+1, m],$$

and so, summing these (in)equalities over all $j \in [1, m]$ and taking into account equality (3.1), we get $\Phi(x) > \Phi(y)$.

Axiom $(A.3)_m$. Given $k \in [3, m]$, suppose that condition $(A.3.k)_m$ in Theorem A is satisfied. Since $v_j(x) = v_j(y)$ for all $j \in [1, m-k]$, we have:

$$\sum_{j=1}^{m-k} C^{m-j}_{n-V_j(x)+m-j-1} = \sum_{j=1}^{m-k} C^{m-j}_{n-V_j(y)+m-j-1}. \qquad (4.15)$$

Set $\nu = v_{m-k+2}(x)$. Then condition $V_{m-k+2}(x) = n$ implies $V_{m-k+1}(x) = n - \nu$ and $V_j(x) = n$ for all $j \in [m-k+2, m]$, and condition $v_{m-k+1}(x) + 1 = v_{m-k+1}(y)$ implies $V_{m-k+1}(x) + 1 = V_{m-k+1}(y)$, and so, $V_{m-k+1}(y) = n - \nu + 1$ (in particular, it follows that $\nu \in [2, n]$). Finally, condition $V_{m-k+1}(y) + v_m(y) = n$ implies $V_j(y) = V_{m-k+1}(y) = n - \nu + 1$ for all $j \in [m-k+1, m-1]$, $V_m(y) = n$ and $v_m(y) = \nu - 1$.

It follows that

$$\sum_{j=m-k+1}^{m} C_{n-V_j(x)+m-j-1}^{m-j} = C_{n-V_{m-k+1}(x)+m-(m-k+1)-1}^{m-(m-k+1)} + \sum_{j=m-k+2}^{m-1} C_{n-n+m-j-1}^{m-j} + 1$$

$$= C_{\nu+k-2}^{k-1} + 1, \tag{4.16}$$

and

$$\sum_{j=m-k+1}^{m} C_{n-V_j(y)+m-j-1}^{m-j} = \sum_{j=m-k+1}^{m-1} C_{n-(n-\nu+1)+m-j-1}^{m-j} + 1$$

$$= \sum_{j=m-k+1}^{m-1} C_{\nu-2+m-j}^{m-j} + 1 = \sum_{j=m-k+1}^{m} C_{\nu-2+m-j}^{\nu-2}$$

$$= \sum_{j=0}^{k-1} C_{\nu-2+j}^{\nu-2} = C_{\nu-2+k}^{k-1}, \tag{4.17}$$

where equality (4.17) follows from (4.1). Now, (3.1) and (4.15)–(4.17) imply that $\Phi(x) = \Phi(y) + 1 > \Phi(y)$, as asserted.

Step 4. Finally, we show that $\Phi : X \xrightarrow{\text{onto}} [1, |X^*|]$, that is, $\Phi(X) = [1, |X^*|]$.

Given $x \in X$, we have $(1^n) \preccurlyeq x \preccurlyeq (m^n)$, and so, by axioms $(A.1)_m$ and $(A.2)_m$, we find $\Phi(1^n) \leq \Phi(x) \leq \Phi(m^n)$. Since $V_j(1^n) = n$ and $V_j(m^n) = 0$ for all $j \in [1, m-1]$, we get: $\Phi(1^n) = \sum_{j=1}^{m-1} C_{m-j-1}^{m-j} + 1 = 1$ and, by virtue of (2.7) and (4.2), $\Phi(m^n) = \sum_{j=1}^{m} C_{n+m-j-1}^{m-j} = |X^*|$, and so, $\Phi(x)$ is in $[\Phi(1^n), \Phi(m^n)] = [1, |X^*|]$ implying $\Phi(X) \subset [1, |X^*|]$.

In order to prove the reverse inclusion $[1, |X^*|] \subset \Phi(X)$, we let ℓ be in $[1, |X^*|] = [1, C_{n+m-1}^{m-1}]$ and apply Theorem 3.2: there is a unique collection of non-negative integers $n_1, n_2, \ldots, n_{m-2}$ satisfying $n_j \leq n_{j-1}$ for all $j \in [1, m-2]$ such that (3.2) holds. Consider a vector $x \in X = [1, m]^n$ (well) defined by equalities (3.3) and (3.4). Then, given $j \in [1, m-2]$, we have:

$$V_j(x) = \sum_{k=1}^{j} v_k(x) = \sum_{k=1}^{j} (n_{k-1} - n_k) = n_0 - n_j = n - n_j$$

and $n - V_j(x) = n_j$, and so, by virtue of (3.1) and (3.2), we get:

$$\Phi(x) = \sum_{j=1}^{m-2} C_{n-V_j(x)+m-j-1}^{m-j} + v_m(x) + 1$$

$$= \sum_{j=1}^{m-2} C_{n_j+m-j-1}^{m-j} + (\ell - L - 1) + 1 = \ell.$$

It follows that $\ell \in \Phi(X)$, and so, $[1, |X^*|] \subset \Phi(X)$.

This completes the proof of Theorem 3.1. $\qquad\qquad\square$

Proof of Theorem 3.3. (a) By the negation property (i) of P from the beginning of Sec. 2, given $x, y \in X$, we have: $(y, x) \notin P$ iff $(x, y) \in P$ or $v_j(x) = v_j(y)$ for all $j \in [1, m - 1]$. Since, by Theorem 3.1, Φ is a preference function for P on X (cf. also property (ii) and axiom $(A.1)_m$), we get:

$$v_j(x) = v_j(y) \text{ for all } j \in [1, m] \quad \text{iff} \quad \Phi(x) = \Phi(y). \qquad (4.18)$$

It follows that $(y, x) \notin P$ iff $\Phi(x) > \Phi(y)$ or $\Phi(x) = \Phi(y)$, i.e., $\Phi(x) \geq \Phi(y)$. As in (2.7), we set $s = |X^*|$.

By the definition of X_s (Sec. 2), we find

$$\begin{aligned} X_s = X_1' = \pi(X) &= \{x \in X : (y, x) \notin P \text{ for all } y \in X\} \\ &= \{x \in X : \Phi(x) \geq \Phi(y) \text{ for all } y \in X\}. \end{aligned}$$

Let us show that the last set is equal to $\{x \in X : \Phi(x) = s\}$. In fact, let $x \in X$. If $\Phi(x) = s$, then since, by Theorem 3.1, $\Phi(y) \in [1, s]$ for all $y \in X$, we get $\Phi(x) = s \geq \Phi(y)$ for all $y \in X$. Now, if $\Phi(x) \geq \Phi(y)$ for all $y \in X$, then setting $y = (m^n)$ we find $s \geq \Phi(x) \geq \Phi(y) = \Phi(m^n) = s$, and so, $\Phi(x) = s$. Thus, $X_s = \{x \in X : \Phi(x) = s\} = \{(m^n)\}$.

Now, suppose that for some $\ell \in [2, s]$ we have already shown that X_k is equal to $\{x \in X : \Phi(x) = k\}$ for all $k \in [\ell, s]$, and let us show that $X_{\ell-1} = \{x \in X : \Phi(x) = \ell - 1\}$. By the definition,

$$X_{\ell-1} = X_{s-(\ell-1)+1}' = X_{s-\ell+2}' = \pi\left(X \setminus \left(\bigcup_{k=1}^{s-\ell+1} X_k'\right)\right)$$

is the set of all $x \in X \setminus (\bigcup_{k=1}^{s-\ell+1} X_k')$ such that $(y, x) \notin P$ for all $y \in X$ which lie outside of $\bigcup_{k=1}^{s-\ell+1} X_k'$. Since, again by the definition, $X_k = X_{s-k+1}'$ for all $k \in [\ell, s]$ or, equivalently, $X_k' = X_{s-k+1}$ for all $k \in [1, s - \ell + 1]$, by the hypothesis above,

$$\bigcup_{k=1}^{s-\ell+1} X_k' = \bigcup_{k=\ell}^{s} X_k = \{x \in X : \Phi(x) \in [\ell, s]\},$$

and so, Theorem 3.1 implies

$$\begin{aligned} X_{\ell-1} = \{x \in X : &\Phi(x) \in [1, \ell - 1] \text{ and } \Phi(x) \geq \Phi(y) \text{ for all } y \in X \\ &\text{such that } \Phi(y) \in [1, \ell - 1]\}. \end{aligned}$$

We claim that $X_{\ell-1} = \{x \in X : \Phi(x) = \ell - 1\}$; in fact, given $x \in X$, we have: clearly, if $\Phi(x) = \ell - 1$, then $x \in X_{\ell-1}$, and if $x \in X_{\ell-1}$, then, by virtue of Theorem 3.1 and equality $\Phi(X) = [1, s]$, we can choose a $y \in X$ such that $\Phi(y) = \ell - 1$, and so, $\ell - 1 \geq \Phi(x) \geq \Phi(y) = \ell - 1$ implying $\Phi(x) = \ell - 1$. In this way we have proved that $X_\ell = \{x \in X : \Phi(x) = \ell\}$ for all $\ell \in [1, |X^*|]$.

Now, given $x \in X$, there is an $\ell \in [1, s]$ such that $x \in X_\ell = \{x \in X : \Phi(x) = \ell\}$, and so, $x \in X_{\Phi(x)}$, i.e., $X_{\Phi(x)}$ is the indifference class of x, which establishes the equality $X_{\Phi(x)} = I_x$.

(b) It was shown in Step 4 of the proof of Theorem 3.1 that if $x \in X$ satisfies (3.3) and (3.4), then $\Phi(x) = \ell$, and so, by item (a), $x \in X_\ell$.

Now, suppose that $x \in X_\ell$, so that $\Phi(x) = \ell$. By Theorem 3.2, there exists a unique collection of $m - 2$ non-negative integers $n_1, n_2, \ldots, n_{m-2}$ satisfying $n_j \leq n_{j-1}$ for all $j \in [1, m - 2]$ such that the inclusion in (3.2) holds. Consider a vector $x' \in X$ having the properties (3.3) and (3.4). Then $\Phi(x') = \ell$, and so, $\Phi(x) = \Phi(x')$. Taking into account (4.18), we find that $v_j(x) = v_j(x')$ for all $j \in [1, m]$, and so, x satisfies conditions (3.3) and (3.4) as well. $\qquad\square$

5. The Algorithmic Order on X^*

Recall that, given $x, y \in X$, we have: $(x, y) \in P$ iff $(x^*, y^*) \in P^*$, where P^* is the restriction of the relation P to $X^* \times X^*$, and that P^* is a linear order on X^*. Moreover, $I_x = I_{x^*}$ for all $x \in X$. It follows that if we are interested in more properties of the relation P on X, then it suffices to study them for P^* on X^*. Recall also that the restriction of the function Φ from (3.1) to X^* is a bijection between X^* and $[1, |X^*|]$, so that the pairs (X^*, P^*) and $([1, |X^*|], >)$ are order isomorphic in the sense that, given $x^*, y^* \in X^*$, $(x^*, y^*) \in P$ iff $\Phi(x^*) > \Phi(y^*)$.

Let $x \in X = [1, m]^n$. Since $X = \bigcup_{\ell=1}^s X_\ell$ (disjoint union) with $s = |X^*|$, there exists a unique $\ell \in [1, |X^*|]$ such that $x \in X_\ell$. By Theorem 3.2, the number ℓ determines uniquely a collection of $m - 2$ non-negative integers $n_1, n_2, \ldots, n_{m-2}$ with appropriate properties, so that, in particular, equalities (3.3) and (3.4) hold. Setting $n_{m-1} = v_m(x) = \ell - L - 1$ and $n_m = 0$, we find that $v_{m-1}(x) = n_{m-2} - n_{m-1}$ and $v_m(x) = n_{m-1}$, and so, $0 \leq n_{m-1} \leq n_{m-2}$. Thus, we have shown that, given $x \in X$, there exists a unique collection of m integers $n_1, n_2, \ldots, n_{m-1}$ and $n_m = 0$ satisfying $0 \leq n_j \leq n_{j-1}$ for all $j \in [1, m]$ such that

$$v_j(x) = n_{j-1} - n_j \quad \text{for all } j \in [1, m - 1] \quad \text{and} \quad v_m(x) = n_{m-1}. \qquad (5.1)$$

Moreover, Theorems 3.2, 3.3(a) and definitions of n_{m-1} and n_m imply

$$
\begin{aligned}
\Phi(x) = \ell &= L + n_{m-1} + 1 \\
&= \sum_{j=1}^{m-2} C_{n_j+m-j-1}^{m-j} + C_{n_{m-1}+m-(m-1)-1}^{m-(m-1)} + C_{n_m+m-m-1}^{m-m} \\
&= \sum_{j=1}^{m} C_{n_j+m-j-1}^{m-j}. \qquad (5.2)
\end{aligned}
$$

On the other hand, due to the uniqueness of collection $\{n_j\}_{j=0}^m$, it is clear that, given $x \in X$, we have:

$$n_j \equiv n_j(x) = n - V_j(x) \quad \text{for all } j \in [1, m] \qquad (5.3)$$

and, in particular, numbers (5.3) satisfy conditions (5.1), and so, the monotone representative x^* of x is of the form:

$$x^*(\tilde{n}) = (1^{n-n_1}, 2^{n_1-n_2}, 3^{n_2-n_3}, \ldots, (m-1)^{n_{m-2}-n_{m-1}}, m^{n_{m-1}}), \qquad (5.4)$$

where $\tilde{n} = (n_1, n_2, \ldots, n_{m-1})$, $n_0 = n$ and $n_j \in [0, n_{j-1}]$ for all $j \in [1, m-1]$. Denote by $\tilde{N}$ the set of all such vectors $\tilde{n}$. In this way we have shown that the set $\tilde{N}$ is bijective to X^* via the map (5.4) (cf. also (5.3)). Moreover, $\tilde{N}$ and X^* are order isomorphic in the following sense: given $\tilde{n} = (n_1, n_2, \ldots, n_{m-1})$, $\tilde{n}' = (n_1', n_2', \ldots, n_{m-1}') \in \tilde{N}$, we have: $(x^*(\tilde{n}), x^*(\tilde{n}')) \in P^*$ iff $n_1' < n_1$ or there exists a $k \in [2, m-1]$ such that $n_j' = n_j$ for all $j \in [1, k-1]$ and $n_k' < n_k$. In fact, in order to see this, it suffices to note only that $v_j(x^*(\tilde{n})) = n_{j-1} - n_j$ and $v_j(x^*(\tilde{n}')) = n_{j-1}' - n_j'$ for all $j \in [1, m-1]$ and $n_0 = n_0' = n$.

Thus, the linear order on $\tilde{N}$, exposed in the previous paragraph, defines the *algorithmic order on* X^* via (5.4) corresponding to the more greater P-preferability, which can be described by the following rule: write out one by one a string of vectors $x^*(\tilde{n})$ of the form (5.4) in such a way that n_1 assumes successively the values $0, 1, \ldots, n$, and if n_1 is fixed, then the number n_2 assumes successively the values $0, 1, \ldots, n_1$, and if n_1 and n_2 are fixed in the ranges $0 \leq n_1 \leq n$ and $0 \leq n_2 \leq n_1$, then the number n_3 assumes successively the values $0, 1, \ldots, n_2$, and so on, and finally, if $n_1, n_2, \ldots, n_{m-2}$ are fixed in their respective ranges $(0 \leq n_1 \leq n, \ 0 \leq n_2 \leq n_1, \ldots, 0 \leq n_{m-2} \leq n_{m-3})$, then the number n_{m-1} assumes successively the values $0, 1, \ldots, n_{m-2}$. According to the algorithmic order on X^*, to each $x^* \in X^*$ there corresponds a unique natural number, which is the ordinal number of x^* and, if x^* is of the form (5.4) for some collection $\tilde{n} = (n_1, n_2, \ldots, n_{m-1}) \in \tilde{N}$, then this ordinal number of x^* is given by formula (5.2). Table 1 is the illustration.

6. The Dual Threshold Preference

The threshold preference $P = P_{m-1}$ from Sec. 2 can be applied to rank the set of alternatives $X = [1, m]^n$ if their utmost perfection is of main concern (e.g., Ref. 15). However, if one is interested in at least one good feature of alternatives, then the dual threshold preference (see Ref. 8, Sec. 5) should be employed. The aim of this section is to obtain the (dual) EPF for the dual threshold preference.

We begin by recalling several definitions and known facts.

Making use of the lexicographic order, the *dual threshold preference* $\overline{P} = \overline{P}_{m-1}$ on $X = [1, m]^n$ is defined by[8]

$$\overline{P}_{m-1} = \{(x, y) \in X \times X : \overline{v}(y) \angle_{m-1} \overline{v}(x)\},$$

where, given $x \in X$, $\overline{v}(x) = (v_m(x), v_{m-1}(x), \ldots, v_2(x)) \in [0, n]^{m-1}$ and, as usual, $v_j(x)$ is the multiplicity of grade $j \in [1, m]$ in the vector $x = (x_1, x_2, \ldots, x_n)$. More explicitly, if $m = 2$, then $(x, y) \in \overline{P}_1$ iff $v_2(y) < v_2(x)$, and if $m \geq 3$, then we find $(x, y) \in \overline{P}_{m-1}$ iff $v_m(y) < v_m(x)$ or there exists a $k \in [2, m-1]$ such that $v_j(x) = v_j(y)$ for all $j \in [k+1, m]$ and $v_k(y) < v_k(x)$.

Let us show that the dual threshold preference $\overline{P} = \overline{P}_{m-1}$ is the restriction of the leximax preference on $\mathbb{R}^n$ to $X = [1, m]^n$. Recall that $x \in \mathbb{R}^n$ *is preferred to* $y \in \mathbb{R}^n$ *in the sense of the leximax* if $y_* \angle_n x_*$, where $x_* = (x_{*1}, x_{*2}, \ldots, x_{*n}) \in \mathbb{R}^n$ is the *dual monotone representative* of x, whose coordinates $x_{*i} \in \{x_1, x_2, \ldots, x_n\}$, $i \in [1, n]$, are

assembled in descending order $x_{*1} \geq x_{*2} \geq \cdots \geq x_{*n}$. Note that the dual monotone representative of $x \in X$ is given by

$$x_* = (m^{v_m(x)}, (m-1)^{v_{m-1}(x)}, \ldots, 2^{v_2(x)}, 1^{v_1(x)}). \tag{6.1}$$

Lemma 6.1. *Given* $x, y \in X$, *we have:* $(x, y) \in \overline{P}_{m-1}$ *iff* $y_* \angle_n x_*$.

Proof. Let $x, y \in X$. We set $r(j) = m - j + 1$ for $j \in [1, m]$, so that $r(r(j)) = j$, and define $\mathbf{r}(x) \in X$ for $x = (x_1, x_2, \ldots, x_n) \in X$ by

$$\mathbf{r}(x) = (r(x_1), r(x_2), \ldots, r(x_n)) = (m - x_1 + 1, m - x_2 + 1, \ldots, m - x_n + 1). \tag{6.2}$$

Clearly, $\mathbf{r}(\mathbf{r}(x)) = x$. It was shown in Ref. 8 that $v_j(\mathbf{r}(x)) = v_{r(j)}(x)$ for all $j \in [1, m]$, $\overline{v}(x) = v(\mathbf{r}(x))$, and $(x, y) \in \overline{P}$ iff $(\mathbf{r}(y), \mathbf{r}(x)) \in P$. Thus, taking into account Lemma 2.1, we find

$$(x, y) \in \overline{P} \quad \text{iff} \quad (\mathbf{r}(x))^* \angle_n (\mathbf{r}(y))^*. \tag{6.3}$$

Next, for $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ we have $x_1^* \leq x_2^* \leq \ldots \leq x_n^*$ and, since $r(1) = m$, $r(2) = m - 1$, ..., $r(m) = 1$, equality (2.2) implies

$$\mathbf{r}(x^*) = (r(x_1^*), r(x_2^*), \ldots, r(x_{n-1}^*), r(x_n^*))$$
$$= (m^{v_1(x)}, (m-1)^{v_2(x)}, \ldots, 2^{v_{m-1}(x)}, 1^{v_m(x)}).$$

Replacing x by $\mathbf{r}(x)$ in this equality, we get, by virtue of (6.1),

$$\mathbf{r}((\mathbf{r}(x))^*) = (m^{v_1(\mathbf{r}(x))}, (m-1)^{v_2(\mathbf{r}(x))}, \ldots, 2^{v_{m-1}(\mathbf{r}(x))}, 1^{v_m(\mathbf{r}(x))})$$
$$= (m^{v_{r(1)}(x)}, (m-1)^{v_{r(2)}(x)}, \ldots, 2^{v_{r(m-1)}(x)}, 1^{v_{r(m)}(x)})$$
$$= (m^{v_m(x)}, (m-1)^{v_{m-1}(x)}, \ldots, 2^{v_2(x)}, 1^{v_1(x)}) = x_*,$$

and so, $(\mathbf{r}(x))^* = \mathbf{r}(x_*)$. Replacing x by $\mathbf{r}(x)$ in the last equality, we find $x^* = \mathbf{r}((\mathbf{r}(x))_*)$ or $\mathbf{r}(x^*) = (\mathbf{r}(x))_*$. Now, it follows from (6.3) that $(x, y) \in \overline{P}$ iff $\mathbf{r}(x_*) \angle_n \mathbf{r}(y_*)$ iff $y_* \angle_n x_*$. $\square$

It is clear that the indifference relation (2.1), induced by the weak order $\overline{P}$ on X, coincides with that induced by the threshold preference P.

As an example, Table 2 (cf. also Table 1) shows the ordering in ascending $\overline{P}$-preference of the set $([1, 5]^3)^*$ of monotone representatives of elements from $X = [1, m]^n$ with $m = 5$ and $n = 3$:

Table 2.   Example of the dual threshold preference.

$(1,1,1)_1$,	$(1,1,2)_2$,	$(1,2,2)_3$,	$(2,2,2)_4$;			
$(1,1,3)_5$,	$(1,2,3)_6$,	$(2,2,3)_7$;	$(1,3,3)_8$,	$(2,3,3)_9$,	$(3,3,3)_{10}$;	
$(1,1,4)_{11}$,	$(1,2,4)_{12}$,	$(2,2,4)_{13}$;	$(1,3,4)_{14}$,	$(2,3,4)_{15}$,	$(3,3,4)_{16}$;	
$(1,4,4)_{17}$,	$(2,4,4)_{18}$,	$(3,4,4)_{19}$,	$(4,4,4)_{20}$;	$(1,5,5)_{21}$,	$(1,2,5)_{22}$,	$(2,2,5)_{23}$;
$(1,3,5)_{24}$,	$(2,3,5)_{25}$,	$(3,3,5)_{26}$;	$(1,4,5)_{27}$,	$(2,4,5)_{28}$,	$(3,4,5)_{29}$,	$(4,4,5)_{30}$;
$(1,5,5)_{31}$,	$(2,5,5)_{32}$,	$(3,5,5)_{33}$,	$(4,5,5)_{34}$,	$(5,5,5)_{35}$.		

It was proved in Ref. 8 that a function $\varphi : X \to \mathbb{R}$ is a preference function for P on X (i.e., $P = P(\varphi)$) iff the function $\overline{\varphi} : X \to \mathbb{R}$, defined by $\overline{\varphi}(x) = -\varphi(\mathbf{r}(x))$ for all $x \in X$ (with $\mathbf{r}(x)$ from (6.2)), is a preference function for $\overline{P}$ on X, that is, $\overline{P} = P(\overline{\varphi})$. Taking into account Theorem 3.1, we shall look for the dual EPF for $\overline{P}$ on X in the form $\overline{\Phi}(x) = c - \Phi(\mathbf{r}(x))$, $x \in X$, where c is an appropriate constant to be found below. Given $j \in [1, m]$, equality (2.3) implies

$$V_j(\mathbf{r}(x)) = \sum_{k=1}^{j} v_k(\mathbf{r}(x)) = \sum_{k=1}^{j} v_{r(k)}(x) = \sum_{k=1}^{j} v_{m-k+1}(x) = \sum_{i=m-j+1}^{m} v_i(x),$$

and so, $n - V_j(\mathbf{r}(x)) = V_{m-j}(x)$, and equality (3.1) gives

$$\overline{\Phi}(x) = c - \Phi(\mathbf{r}(x)) = c - \sum_{j=1}^{m} C_{V_{m-j}(x)+m-j-1}^{m-j} = c - \sum_{i=0}^{m-1} C_{V_i(x)+i-1}^{i}.$$

If we want to have the property of $\overline{\Phi}$ that $\overline{\Phi}$ maps X onto $[1, |X^*|]$, then we should have $\overline{\Phi}(1^n) = 1$. Since $V_i(1^n) = n$ for all $i \in [1, m-1]$, then, by virtue of (2.3) and (4.1), we get:

$$1 = \overline{\Phi}(1^n) = c - C_{-1}^{0} - \sum_{i=1}^{m-1} C_{n+i-1}^{i} = c - 1 - \sum_{i=0}^{m-1} C_{(n-1)+i}^{i} + C_{n-1}^{0}$$

$$= c - \sum_{i=0}^{m-1} C_{(n-1)+i}^{n-1} = c - C_{(n-1)+(m-1)+1}^{m-1} = c - C_{n+m-1}^{m-1},$$

and so, according to (2.7), $c = 1 + C_{n+m-1}^{m-1} = 1 + |X^*|$. Taking into account that $C_{-1}^{0} = 1$, we conclude that

$$\overline{\Phi}(x) = C_{n+m-1}^{m-1} - \sum_{i=1}^{m-1} C_{V_i(x)+i-1}^{i} \quad \text{for all } x \in X. \tag{6.4}$$

Note that $V_i(m^n) = 0$ for all $i \in [1, m-1]$, and so, $\overline{\Phi}(m^n) = |X^*|$. Thus, as a corollary of Theorem 3.1, we get the following

Theorem 6.1. *A function $\overline{\Phi}$ maps $X = [1, m]^n$ onto $[1, |X^*|]$ and is a preference function for $\overline{P} = \overline{P}_{m-1}$ on X (i.e., $\overline{\Phi}$ is the EPF for $\overline{P}$) iff it is of the form (6.4).*

In order to present the dual algorithmic order on X^* corresponding to the weak order $\overline{P}$, following (5.3) we set $n_i = n - V_i(x)$ for all $x \in X$ and $i \in [0, m]$. It follows that $n_0 = n$, $n_m = 0$ and $0 \le n_i \le n_{i-1}$ and $v_i(x) = n_{i-1} - n_i$ for all $i \in [1, m]$. Therefore, the monotone representative x^* of $x \in X$ is of the form (5.4) where $\tilde{n} = (n_{m-1}, n_{m-2}, \ldots, n_2, n_1)$ is such that $n_i \in [0, n_{i-1}]$ for all $i \in [1, m-1]$. If $\tilde{n}' = (n'_{m-1}, n'_{m-2}, \ldots, n'_2, n'_1)$ is such that $n'_i \in [0, n'_{i-1}]$ for all $i \in [1, m-1]$, then we have: $(x^*(\tilde{n}), x^*(\tilde{n}')) \in \overline{P}$ iff $n'_{m-1} < n_{m-1}$ or there exists a number $k \in [1, m-2]$ such that $n'_i = n_i$ for all $i \in [k+1, m-1]$ and $n'_k < n_k$. It follows that the *dual algorithmic order on X^** via (5.4), corresponding to the more greater $\overline{P}$-preferability, can be described by the following rule: write out one by one a string of vectors $x^*(\tilde{n})$ of the

form (5.4) in such a way that n_{m-1} assumes successively the values $0, 1, \ldots, n$, and if n_{m-1} is fixed, then the number n_{m-2} assumes successively the values $n_{m-1}, n_{m-1}+1, \ldots, n$, and if n_{m-1} and n_{m-2} are fixed in the ranges $0 \leq n_{m-1} \leq n$ and $n_{m-1} \leq n_{m-2} \leq n$, then the number n_{m-3} assumes successively the values $n_{m-2}, n_{m-2}+1, \ldots, n$, and so on, and finally, if $n_{m-1}, n_{m-2}, \ldots, n_2$ are fixed and such that $n_i \leq n_{i-1} \leq n$ for all $i \in [3, m-1]$, then the number n_1 assumes successively the values $n_2, n_2+1, \ldots, n$. According to the dual algorithmic order on X^*, to each $x^* \in X^*$ there corresponds a unique natural number, which is the ordinal number of x^* and, if x^* is of the form (5.4) for some collection $\tilde{n} = (n_{m-1}, n_{m-2}, \ldots, n_2, n_1)$ as above, then, by virtue of (6.4), this ordinal number of x^* is equal to

$$\overline{\Phi}(x^*) = C_{n+m-1}^{m-1} - \sum_{i=1}^{m-1} C_{n-n_i+i-1}^{i}.$$

Table 2 illustrates the dual algorithmic order on $X^* = ([1,5]^3)^*$.

Acknowledgments

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