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On Functions Whose All Critical Points Are Contained in a Ball*

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Abstract. In the present note, we answer the following question posed by Arnold. Consider a function with finitely many critical points on a compact connected manifold without boundary. Suppose that a ball embedded in the manifold contains all critical points of the function. Is it possible to reconstruct the manifold by a restriction of the function to the ball? It turns out that one can reconstruct only the Euler characteristic of the manifold.

Key words: Morse function, gradient-like vector field.

This note answers the following question posed by Arnold ([1, Problem 1993-45]). Consider a function with finitely many critical points on a connected closed manifold. An appropriate diffeomorphism takes this function to a function whose all critical points are contained in some (small) open ball. We consider the restriction of the latter function to the ball. The question is, for what pairs (M_1^n, M_2^n) of connected compact manifolds without boundary there exist functions that coincide on a ball containing all of their critical points. The answer is given by the following theorem.

Theorem. *Such functions exist if and only if M_1^n and M_2^n have the same Euler characteristic $\chi(M_1^n) = \chi(M_2^n)$.*

The assertion is obvious for $n = 1$. The necessity of the assumption in the theorem is also obvious, for the total index of the gradient vector field of the restriction of such a function to the ball is equal to the Euler characteristic of the corresponding manifold and depends only on the behavior of the function near the boundary of the ball.

Proof of the theorem for $n > 2$. A Morse function on a manifold will be called an *indexable Morse function* if (1) its value at an arbitrary critical point is equal to the index of that critical point; (2) the function has exactly one local maximum and exactly one local minimum. The proof can be derived from the following assertions.

Assertion 1. *Suppose that the Euler characteristics of M_1^n and M_2^n are the same. Then there exist indexable Morse functions $f_i\colon M_i^n \to \mathbb{R}$, $i = 1, 2$, such that the multiplicities of their critical values coincide.*

Let $f_i\colon M_i^n \to \mathbb{R}$, $i = 1, 2$, be the indexable Morse functions in Assertion 1.

Assertion 2. *There exist embeddings $\varphi_i\colon D^n \to M_i^n$ of the n-dimensional ball in M_i^n such that the restrictions of the functions f_i to the respective images of this ball are the same*: $\varphi_1^* f_1 = \varphi_2^* f_2$.

Proof of Assertion 1. We recall the definition of a gradient-like vector field. Consider an indexable Morse function f on a manifold M. For each critical point, we choose a coordinate patch in which the function has the form

$$f(x_1, \dots, x_n) = i - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

A vector field v is called a *gradient-like vector field* of f if

(1) $(L_v f)(x) > 0$ for any noncritical point $x \in M$;

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(2) in each of the chosen coordinate patches, v has the form

$$v(x_1,\dots,x_n) = -\sum_{j=1}^{i} x_j \frac{\partial}{\partial x_j} + \sum_{j=i+1}^{n} x_j \frac{\partial}{\partial x_j}$$

in some neighborhood of the origin.

Lemma 1. *The following assertion holds for a generic gradient-like vector field v in the space of all gradient-like vector fields of a given indexable Morse function: for each critical point of the function, there exists a phase curve of v joining the point of minimum of the function with this point.*

Proof. For a generic field v, the set of phase curves issuing from a point of index p and terminating at a point of index $q > p$ is a smooth $(q-p-1)$-dimensional submanifold of the $(q-1)$-dimensional sphere formed by all phase curves entering the point of index q. Consequently, for each critical point of index $i > 0$ we can choose one phase curve issuing from the point of index 0 and terminating at that point. □

Consider some indexable Morse function f on a connected closed manifold M^n. Take a positive integer $i < n-1$. If we learn how to change the function in such a way that the multiplicities of the critical values i and $i+1$ increase by 1 and the other multiplicities remain unchanged, then we shall be able to equalize the multiplicities of all respective critical values for the pair f_1, f_2. Indeed, let us equalize the multiplicities of the critical value 1 for these functions by adding critical points of indices 1 and 2; then we equalize the multiplicities of the critical value 2 by adding critical points of indices 2 and 3, etc. At the $(n-2)$nd step, we obtain the desired functions, since the Euler characteristics of M_1^n and M_2^n are the same.

Consider the product $D^{n-1} \times I$, where I is the interval $[1/2, n-1/2]$ and D^{n-1} is the $n-1$-dimensional ball. Suppose that a function g on $D^{n-1} \times I$ coincides with the projection on I. There exists a function g_1 on $D^{n-1} \times I$ such that

(1) g_1 coincides with g in a neighborhood of the boundary;

(2) g_1 has exactly two critical points of indices i and $i+1$, and the critical values at these operators are i and $i+1$, respectively.

The construction of g_1 goes as follows. One creates two critical points of indices i and $i+1$ with values close to $i+1/2$. Then one changes the values using the technique in [2, Theorem 4.1]. To prove Assertion 1, it remains to show that M^n contains a set diffeomorphic to $D^{n-1} \times I$ such that the restriction of f to this set coincides with its projection on I. By Lemma 1, there exists a phase curve of some gradient-like vector field of f joining the point of minimum of f with the point of maximum. The desired set is a part of a neighborhood of this curve. □

Proof of Assertion 2. We join each critical point of an indexable Morse function with its point of minimum by a phase curve of a specially chosen gradient-like vector field. The desired ball on which the indexable Morse function is standard is a neighborhood of the union of these phase curves.

Let k_i be the number of critical points of index i of an indexable Morse function f. Consider the coordinate neighborhood of the point of minimum chosen in the definition of a gradient-like vector field. The phase curves of the gradient-like vector field issuing from this point are represented by rays in this coordinate neighborhood. To each phase curve of this sort, we assign its direction given by a unit vector. We choose an ordered finite sequence $A_i \subset \mathbb{R}^n$, $i = 1, \dots, n$, of disjoint sets of vectors of unit length, the cardinality of A_i being equal to k_i.

Lemma 2. *For the function f, there exists a gradient-like vector field v such that each phase curve issuing from the point of minimum in a direction belonging to A_i terminates at a critical point of index i and all these critical points are distinct.*

Proof. Using Lemma 1, we choose a field v. For each critical point, we choose a phase curve joining it with the point of minimum. Isotopies of the sphere $S^{n-1} = \{f = 1/2\}$ act transitively on finite ordered subsets of given cardinality, since $n-1 > 1$. Using Thom's isotopy lemma [2], we modify the gradient-like vector field v on the set $\{x \mid 1/4 < f(x) < 3/4\}$ so as to satisfy the assumptions of the lemma. □

We denote the union of the phase curves constructed in Lemma 2 by $K(f)$. There exists an orientable neighborhood of $K(f)$.

Consider the coordinate space $\mathbb{R}^n$. Let K be the union of segments joining the origin with points of the set $\bigcup A_i$. There exists a smooth function φ defined in some neighborhood U of K and a gradient-like vector field u of φ in U such that the following conditions hold:

(1) 0 is a critical point, and φ is equal to the sum of squares of the coordinates in a neighborhood of 0;

(2) in a neighborhood of each point of A_i, there is a given affine change of coordinates reducing the function to the form $i - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2$;

(3) all other points of U are nonsingular for φ;

(4) u is a gradient-like vector field for this choice of coordinates, and the segments forming K are its phase curves.

There exists a diffeomorphism π of some neighborhood $V \subset U$ onto a neighborhood of $K(f)$ such that $\pi^* f = \varphi|_V$. Let us describe the construction of this diffeomorphism. We choose diffeomorphisms of sufficiently small neighborhoods of the critical points of φ into coordinate neighborhoods of the critical points of f such that these diffeomorphisms are affine, define the same orientation of $K(f)$, and take f to φ. We extend the mapping defined by these diffeomorphisms in the small neighborhoods to the segments forming K in such a way that these segments are taken to the phase curves chosen in Lemma 2 and f is taken to φ. The mapping thus constructed can readily be extended to neighborhoods of the segments. (Orientation is the only obstruction.)

Thus, for two given indexable Morse functions with coinciding multiplicities of respective critical points we have found diffeomorphisms of neighborhoods V_1 and V_2 of the set K into the corresponding manifold such that the function induced by these diffeomorphisms is standard. Let us restrict these diffeomorphisms to a star-shaped neighborhood of K in $V_1 \cap V_2$. An open star-shaped set is diffeomorphic to a ball. $\square$

Proof of the theorem for $n = 2$. Let us prove the assertion for the Klein bottle K^2 and the torus T^2. We represent the Klein bottle and the torus as bundles (with fiber the circle) over the circle: $p_1 \colon T^2 \to S^1$ and $p_2 \colon K^2 \to S^1$. On the base of the bundle, we take some function f all of whose critical points lie in some connected interval $I \subset S^1$. The preimages $p_1^{-1}(I)$ and $p_2^{-1}(I)$ of this interval are diffeomorphic. The functions $p_i^* f$ coincide on the sets $p_i^{-1}(I)$, and their all critical points lie in these sets. We perturb these functions in the same way on the sets $p_i^{-1}(I)$ so as to obtain Morse functions. For the resulting functions, one can choose the desired balls to lie in $p_i^{-1}(I)$.

Of two-dimensional manifolds, only the connected sum of the torus with the sphere with g handles and the connected sum of the Klein bottle with the same sphere ($g = 0, 1, \dots$) have coinciding Euler characteristics. We take the functions constructed above on K^2 and T^2, which coincide in a disk D^2 containing all critical points of these functions. Let us construct the connected sum by cutting away the ball D^2. We extend the functions in question to the attached spheres with g handles in the same way so as to obtain Morse functions. The resulting functions coincide on the (connected) set that has been attached, and their all critical points lie in that set. It remains to choose appropriate balls in that set. $\square$

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