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IS IT SO BAD THAT WE CANNOT RECOGNIZE BLACK SWANS?

УДК 519.86
ББК 65в6
    А32

Analyzing the reasons of financial crises in [1] the author concludes that modern economic models badly describe reality for they are not able to forecast such crises in advance.

We tried to present processes on stock exchange as two random processes one of which happens rather often (regular regime) and the other one – rather rare. Our answer is that if regular processes are correctly recognized with the probability a bit higher than $1/2$, this allows to get positive average gain. We believe that this very phenomenon lies in the basis of unwillingness of people to expect crises permanently and to try recognizing them.

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В [1], анализируя причины финансовых кризисов, автор делает вывод о том, что современные экономические модели плохо описывают реальность, так как не умеют предсказывать такие кризисы. Соответственно, и люди, которых эти модели ничему не учат, не умеют такие кризисы распознавать заранее.

Мы попытались представить процессы, происходящие на бирже, в виде двух случайных процессов, один из которых происходит часто (нормальный режим), а другой – редко (кризис). Ответ, который мы получили, сводится к следующему – если частые, регулярные процессы распознаются правильно даже с вероятностью чуть выше $1/2$, это позволяет почти всегда иметь положительный средний выигрыш. Нам представляется, что именно это лежит в основе нежелания людей все время ожидать кризисы и пытаться распознать их.

1. Introduction

Analyzing the reasons of financial crises in [1] the author concludes that modern economic models badly describe reality for they are not able to forecast such crisis in advance. All extraordinary events, e.g. crises, are named by the author "The Black Swans". Let us give some quotes.

«Before the discovery of Australia, people in the old world were convinced that all swans were white, an unassailable belief as it seemed completely confirmed by empirical evidence. The sighting of the first black swan might have been an interesting surprise for a few ornithologists (and others extremely concerned with the coloring of birds), but that is not where the significance of the story lies. It illustrates a severe limitation to our learning from observations or experience and the fragility of our knowledge. One single observation can invalidate a general statement derived from millennia of confirmatory sightings of millions of white swans. All you need is one single (and, I am told, quite ugly) black bird.

… What we call here a Black Swan (and capitalize it) is an event with the following three attributes. First, it is an *outlier*, as it lies outside the realm of regular expectations, because nothing in the past can convincingly point to its possibility. Second, it carries an extreme impact. Third, in spite of its outlier status, human nature makes us concoct explanations for its occurrence *after* the fact, making it explainable and predictable.

I stop and summarize the triplet: rarity, extreme impact, and retrospective (though not prospective) predictability. A small number of Black Swans explain almost everything in our world, from the success of ideas and religions, to the dynamics of historical events, to elements of our own personal lives. Ever since we left the Pleistocene, some ten millennia ago, the effect of these Black Swans has been increasing. It started accelerating during the industrial revolution, as the world started getting more complicated, while ordinary

events, the ones we study and discuss and try to predict from reading the newspapers, have become increasingly inconsequential.

Just imagine how little your understanding of the world on the eve of the events of 1914 would have helped you guess what was to happen next. (Don't cheat by using the explanations drilled into your cranium by your dull high school teacher). How about the rise of Hitler and the subsequent war? How about the precipitous demise of the Soviet bloc? How about the rise of Islamic fundamentalism? How about the spread of the Internet? How about the market crash of 1987 (and the more unexpected recovery)? Fads, epidemics, fashion, ideas, the emergence of art genres and schools. All follow these Black Swan dynamics. Literally, just about everything of significance around you might qualify.

This combination of low predictability and large impact makes the Black Swan a great puzzle; but that is not yet the core concern of this book. Add to this phenomenon the fact that we tend to act as if it does not exist! I don't mean just you, your cousin Joey, and me, but almost all "social scientists" who, for over a century, have operated under the false belief that their tools could measure uncertainty. For the applications of the sciences of uncertainty to real-world problems has had ridiculous effects; I have been privileged to see it in finance and economics. Go ask your portfolio manager for his definition of "risk," and odds are that he will supply you with a *measure* that *excludes* the possibility of the Black Swan–hence one that has no better predictive value for assessing the total risks than astrology (we will see how they dress up the intellectual fraud with mathematics). This problem is endemic in social matters.

The central idea of this book concerns our blindness with respect to randomness, particularly the large deviations: Why do we, scientists or nonscientists, hotshots or regular Joes, tend to see the pennies instead of the dollars? Why do we keep focusing on the minutiae, not the possible significant large events, in spite of the obvious evidence of their huge influence?» (Prologue, pp. XVII–XIX).

Thus, the author confirms that modern science almost doesn't have tools to predict such unusual events. Moreover, he gives the following rather sharp characterization of modern economic knowledge.

«In orthodox economics, rationality became a straitjacket. Platonified economists ignored the fact that people might prefer to do something other than maximize their economic interests. This led to mathematical techniques such as "maximization," or "optimization," on which Paul Samuelson built much of his work… It involves complicated mathematics and thus raises a barrier to entry by non-mathematically trained scholars. I would not be the first to say that this optimization set back social science by reducing it from the intellectual and reflective discipline that it was becoming to an attempt at an "exact science."» (p. 184).

And, finally,

«… those who started the game of "formal thinking," by manufacturing phony premises in order to generate "rigorous" theories, were Paul Samuelson, Merton's tutor, and, in the United Kingdom, John Hicks. These two wrecked the ideas of John Maynard Keynes, which they tried to formalize (Keynes was interested in uncertainty, and complained about the mind-closing certainties induced by models). Other participants in the formal thinking venture were Kenneth Arrow and Gerard Debreu. All four were Nobeled… All of them can be safely accused of having invented an imaginary world, one that lent itself to their mathematics.» (p. 283).

Not being adherents of any particular points of view[1] in the economic community, we have tried to present the processes occurring on the exchange in the form of two random processes, one of which occurs frequently (normal mode) and the other – rarely (crisis).

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[1] Although we must say that one of the authors of this work – F. Aleskerov – met Professor Paul Samuelson in 1984, and Professor Kenneth Arrow has always had high influence to him. Meetings with Professor Arrow are always the great intellectual feast.

Next, we estimated the average gain with the different probabilities of correct recognition of these processes and used the resulting estimates for the actual processes on the exchange.

Briefly, we've got the following answer: if frequent, regular processes are detected correctly even with a probability slightly higher than $\frac{1}{2}$, it almost always allows to have a positive average gain. This very phenomenon seems underlies the reluctance of people to expect crises all the time and do not try to identify them.

The structure of the paper is as follows. Section 2 proposes the statement of the problem and its solution, Sections 3 and 4 investigate the constraints for nonnegative gain. Section 5 contains a graphical analysis of the results, Section 6 deals with application of the model to real data, and Section 7 concludes. In Appendix 1 and 2 we derive the formulae for the expectation of total gain, in Appendix 3 we consider the estimates of parameters for the various stock market indices, and Appendix 4 provides an alternative evaluation of the model parameters when we use the return of the index instead of its volatility.

2. Problem

The flow of events of two types – type Q (from *quick*) and type R (from *rare*) – enters the device. Each of them is the simplest, i.e. stationary, ordinary and has no aftereffects. The intensity of the flow of events of type Q is equal to λ , the intensity of the flow of events of R is equal to μ, where $\lambda \gg \mu$ (Q-type events are far more frequent than the R-type events).

The problem of the device is to recognize coming event X. If an event Q occurs and device identified it correctly, then it would promote getting a small reward a, if the error occurred, and the event Q has been recognized as the event R, then the device is 'fined' by an amount b. The probabilities of such outcomes are known and equal p_1 and q_1, respectively. Similarly for the events of the type $R -$ correct identification of coming event R will give the value of c, where $c \gg a$, and incorrect recognizing will give loss $-d$, $d \gg b$. After each coming event received values of 'win' / 'loss' are added to the previous amount (Fig. 1).

How large on average will be the amount of values received for the time t?
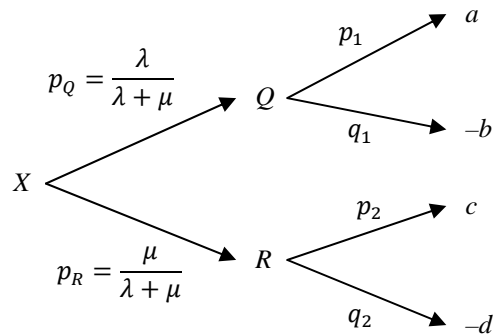
Fig. 1. The general scheme of identification of a random event X

One of the possible implementations of a random process $Z(t)$, equal to the sum of all values of a random variable X received at the time t is given on Fig. 2.
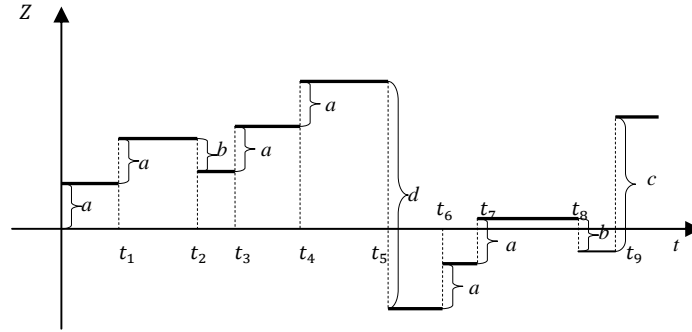
Fig. 2. One of the possible implementations of a random process $\boldsymbol{Z(t)}$

Random value Z of the total sum of the received prizes during the time t is a compound Poisson type variable since the number of terms in the sum $Z = \sum X_i$ is also a random variable and depends on the flow of events received by the device. The formula is derived in Annexes 1 and 2. We give the expression for the expectation of a random variable payoff Z (here W_1 and Y_1 are the random amounts of the gain obtained from the events Q and R, respectively)

$$E(Z) = (\lambda t)E(W_1) + (\mu t)E(Y_1) =$$
$$= \lambda t\big((1 - q_1)a - q_1 b\big) + \mu t\big((1 - q_2)c - q_2 d\big) \qquad (1)$$

3. Analysis of the solution with given q_1 and q_2

Let us require the expectation of Z to be non-negative. The values q_1, q_2 are known and predetermined. Then let us find for which values of other parameters the requirement holds

$$E(Z) = \big((1 - q_1)\lambda a - q_1\lambda b + (1 - q_2)\mu c - q_2\mu d\big) \cdot t \geq 0$$

As $t > 0$, then $E(Z) \geq 0$ holds when

$$\big((1 - q_1)\lambda a - q_1\lambda b + (1 - q_2)\mu c - q_2\mu d\big) \geq 0$$

or

$$\lambda(a - q_1 a - q_1 b) + \mu(c - q_2 c - q_2 d) \geq 0.$$

3.1 We will start with a special case: the gain and loss are the same in both situations, i.e., $b = a, d = c$. Then the condition of nonnegativity of the expectation of the total payoff is $(1 - 2q_1)\lambda a + (1 - 2q_2)\mu c \geq 0$. Dividing by μ and c (both quantities are positive), we obtain

$$(1 - 2q_1)\frac{\lambda}{\mu}\frac{a}{c} \geq (2q_2 - 1).$$

If $q_1 < \frac{1}{2}$, then

$$\frac{\lambda}{\mu}\frac{a}{c} \geq \frac{2q_2 - 1}{1 - 2q_1} \implies \frac{\lambda}{\mu} \geq \frac{2q_2 - 1}{1 - 2q_1} \cdot \frac{c}{a}.$$

If $q_2 > \frac{1}{2}$, then the last inequality implies that the ratio of the intensity of two flows must be greater than the ratio of the gains, if the probability of error in Q-event is less than 0,5, while the probability of making an error and not recognizing the event of type R is higher than 0,5.

If $q_2 \leq \frac{1}{2}$, then the last inequality holds in any case.

If $q_1 > \frac{1}{2}$, then

$$\frac{\lambda}{\mu} \leq \frac{2q_2 - 1}{1 - 2q_1} \cdot \frac{c}{a}. \tag{2}$$

If $q_2 \geq 1/2$, then the inequality is not valid (if we make more mistakes in the occurrence of any event, we will not reach the non-negative values of the expectation of Z in any case).

When $q_2 < 1/2$, we obtain the dependence (2) of the ratio $\frac{\lambda}{\mu}$ on the ratio $\frac{c}{a}$ (R-events should come more often).

If $q_1 = 1/2$, from $\lambda(a - 2q_1 a) + \mu(c - 2q_2 c) \geq 0$ we obtain the inequality $\mu(c - 2q_2 c) \geq 0$, which is valid only for $q_2 \leq 1/2$.

The Table 1 contains the conditions indicating the parameters for the non-negativity of the expectation of the amount of prizes.

Table 1. The conditions indicating the parameters for the non-negativity of the expectation of the gain

		q_2		
		$< 1/2$	$= 1/2$	$> 1/2$
q_1	$< 1/2$	$E(Z) \geq 0$	$E(Z) \geq 0$	for $E(Z) \geq 0$ required $\frac{\lambda}{\mu} \geq \frac{2q_2 - 1}{1 - 2q_1} \cdot \frac{c}{a}$
	$= 1/2$	$E(Z) \geq 0$	$E(Z) = 0$	$E(Z) < 0$
	$> 1/2$	for $E(Z) \geq 0$ required $\frac{\lambda}{\mu} \leq \frac{2q_2 - 1}{1 - 2q_1} \cdot \frac{c}{a}$	$E(Z) < 0$	$E(Z) < 0$

3.2 Consider now more general case $b \neq a, d \neq c$. We find the conditions under which the inequality $\lambda(a - q_1 a - q_1 b) + \mu(c - q_2 c - q_2 d) \geq 0$ holds. Dividing by $\mu > 0$, we obtain

$$(a - q_1 a - q_1 b)\frac{\lambda}{\mu} + (c - q_2 c - q_2 d) \geq 0.$$

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1) If $E(W_1) = a - q_1a - q_1b > 0$, i.e. $(1 - q_1)a > q_1b$ (the expected price of making the right decision for the event Q is higher than the expected price of making the wrong decision), then

$$\frac{\lambda}{\mu} \geq \frac{-(c - q_2c - q_2d)}{(a - q_1a - q_1b)}.$$

If $E(Y_1) = c - q_2c - q_2d \geq 0$, i.e. $(1 - q_2)c \geq q_2d$ (the expected price of making the right decision on the event R is higher than the expected price of making the wrong decision), then taking into account $\lambda > 0, \mu > 0$, we will obtain that $\lambda(a - q_1a - q_1b) + \mu(c - q_2c - q_2d) \geq 0$ always holds.

If $E(Y_1) = c - q_2c - q_2d < 0$, i.e. $c(1 - q_2) < dq_2$, then the inequality

$$\frac{\lambda}{\mu} \geq \frac{-(c - q_2c - q_2d)}{(a - q_1a - q_1b)}$$

is necessary condition for $E(Z) \geq 0$.

2)   If $E(W_1) = a - q_1a - q_1b < 0$,   i.e. $(1 - q_1)a < q_1b$,   then

$$\frac{\lambda}{\mu} \leq \frac{-(c - q_2c - q_2d)}{(a - q_1a - q_1b)}.$$

If $E(Y_1) = c - q_2c - q_2d > 0$, i.e. $(1 - q_2)c > q_2d$, then the previous condition is necessary for $E(Z) \geq 0$.

If $E(Y_1) = c - q_2c - q_2d \leq 0$, i.e. $(1 - q_2)c \leq q_2d$, then $E(Z) \geq 0$ does not hold.

3) If $E(W_1) = a - q_1a - q_1b = 0$, then $E(Z) = (\lambda(a - q_1a - q_1b) + \mu(c - q_2c - q_2d)) \cdot t \geq 0$ holds if $E(Y_1) = c - q_2c - q_2d \geq 0$, otherwise $E(Z) < 0$.

We represent how the solution depends on these parameters in Table 2.

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Table 2. The conditions of non-negativity of the total gain

		$E(W)$		
		< 0	$= 0$	> 0
$E(Y)$	< 0	$E(Z) < 0$	$E(Z) < 0$	for $E(Z) \geq 0$ required $\dfrac{\lambda}{\mu} \geq -\dfrac{E(Y)}{E(W)}$
	$= 0$	$E(Z) < 0$	$E(Z) = 0$	$E(Z) \geq 0$
	> 0	for $E(Z) \geq 0$ required $\dfrac{\lambda}{\mu} \leq -\dfrac{E(Y)}{E(W)}$	$E(Z) \geq 0$	$E(Z) \geq 0$

4. Analysis of the solution with unknown q_1 and q_2

Now consider the case when the parameters q_1 and q_2 are unknown. What conditions should satisfy the values of q_1 and q_2 for the expected value $E(Z)$ to be nonnegative with all other parameters being fixed?

Both q_1 and q_2 are the probabilities of incorrectly recognized events Q and R, so we have to solve the system of inequalities

$$\begin{cases} E(Z) = \left[\lambda\big((1 - q_1)a - q_1 b\big) + \mu\big((1 - q_2)c - q_2 d\big)\right]t \geq 0, \\ \quad 0 \leq q_1 \leq 1, \\ \quad 0 \leq q_2 \leq 1, \end{cases}$$

when the restrictions on the parameters are $a, b, c, d \geq 0, \lambda, \mu, t > 0$, $a + b > 0, c + d > 0$ (the latter two inequalities mean that both a and b, c and d cannot be equal to zero since the cases with the events with zero losses and gains are not interesting)

$$\begin{cases} q_1 \lambda(a + b) + q_2 \mu(c + d) \leq \lambda a + \mu c, \\ \quad 0 \leq q_1 \leq 1, \\ \quad 0 \leq q_2 \leq 1. \end{cases}$$

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The solution is the range of values q_1 and q_2 defined via the system of inequalities

$$\begin{cases} 0 \le q_1 \le \min\left\{1, \dfrac{\lambda a + \mu c}{\lambda(a + b)}\right\}, \\[2mm] \begin{cases} 0 \le q_2 \le 1 \; if \; 0 \le q_1 \le \min\left\{1, \dfrac{\lambda a - \mu d}{\lambda(a + b)}\right\}, \\[2mm] 0 \le q_2 \le -\dfrac{\lambda(a + b)}{\mu(c + d)} q_1 + \dfrac{\lambda a + \mu c}{\mu(c + d)} \; if \; \max\left\{0, \dfrac{\lambda a - \mu d}{\lambda(a + b)}\right\} \le q_1 \le \min\left\{1, \dfrac{\lambda a + \mu c}{\lambda(a + b)}\right\}. \end{cases} \end{cases}$$

Depending on the values of q_1 and q_2 the corresponding areas look as follows (Fig. 3)
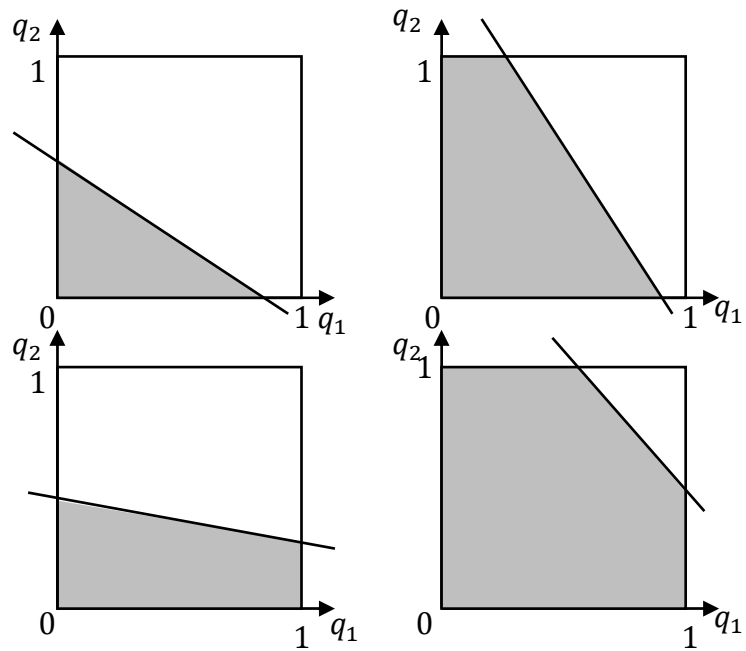

Fig. 3. Examples of possible areas of parameters

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5. Graphical analysis

Let us study how the function $E(Z)$ depends on its arguments and what dependence should exist between arguments for $E(Z)$ to be nonnegative.

1. The dependence $E(Z)$ on t.

Since $E(Z) = \big((1 - q_1)\lambda a - q_1\lambda b + (1 - q_2)\mu c - q_2\mu d\big) \cdot t$, and $t > 0$, then $E(Z)$ for given values of other variables will be a straight line with positive or negative slope depending on the values of the parameters.

Consider the case of equal winnings in each case, i.e. $b = a, d = c$. Let $a = 10, c = 10000, \lambda = 250, \mu = 0.1$. Let us look how the expected payoff looks as a function of time at different values of q_1 and q_2 (probabilities of errors in identification of unknown event).
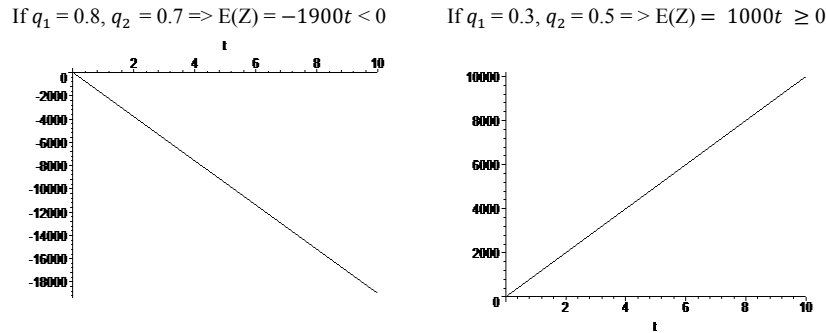
If $q_1 = 0.8$, $q_2 = 0.7 \Rightarrow E(Z) = -1900t < 0$ If $q_1 = 0.3$, $q_2 = 0.5 \Rightarrow E(Z) = 1000t \geq 0$

Fig. 4. Examples of the expected gain as the function of time

2. The dependence $E(Z)$ on a and c

Consider a model with equal payoffs ($b = a, d = c$). Let $\lambda = 250, \mu = 0.1, t = 10$. Let us study how a and c are related. The condition of nonnegativity is $(1 - 2q_1)\lambda a + (1 - 2q_2)\mu c \geq 0$. Let a) $q_1 = 0.3$ and $q_2 = 0.6$; b) $q_1 = 0.2$ and $q_2 = 0.6$; c) $q_1 = 0.1$ and $q_2 = 0.6$; d) $q_1 = 0.05$ and $q_2 = 0.6$. Then, to determine a and c in $E(Z) \geq 0$, we obtain the

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conditions a) $c \geq 5000 \cdot a$; b) $c \geq 8333{,}33 \cdot a$; c) $c \geq 10000 \cdot a$; d) $c \geq 11250 \cdot a$. So we get such area of parameter's values that expected value of total gain will be nonnegative (Fig.5 shows that the less is the error in the identification of events, the more freedom to choose the parameters we have).
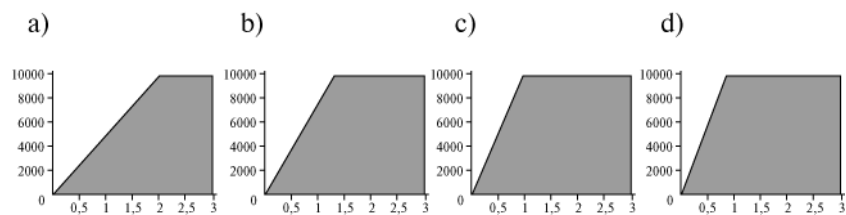

Fig. 5. Area of parameters a and c, which satisfied condition
of nonnegativity of expected total gain

3. The dependence of $E(Z)$ on a and b

Let us consider a model with unequal prize, but with proportional outcomes of two events, i.e. $c = k \cdot a, d = k \cdot b$ (the values of winning and losing for the events of the type R are multiples of similar values of winning and losing for the events of Q). Let $k = 5000, t = 10, \lambda = 200, \mu = 0.1, q_1 = 0.3, q_2 = 0.6$, whereas for the non-negativity of the expectation of the total payoff requires the condition $-3400a + 3600b \leq 0$ or $a \geq 1{,}06 \cdot b$.
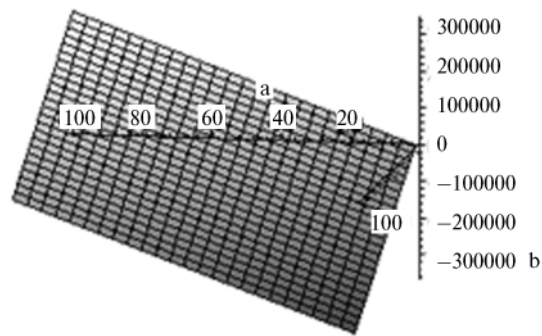
Fig. 6. Expected gain as a function of a and b (the values of a are on the axis OX, the values of b are on the axis OY, the values of expected gain $E(Z)$ are on the axis OZ)

4. The dependence $E(Z)$ on q_1 and q_2

Let us consider the case of equal prizes, i.e. $b = a, d = c$. Let $a = 10, c = 10000, \lambda = 250, \mu = 0.1, t = 10$.

Fig. 7. Expected gain as a function of probabilities (probabilities of mistakes q_1, q_2 are on the axes OX and OY, respectively, the expected gain is on the axis OZ)

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It is necessary to solve the system of inequalities to find out which values of q_1 and q_2 provide the non-negativity of $E(Z)$,
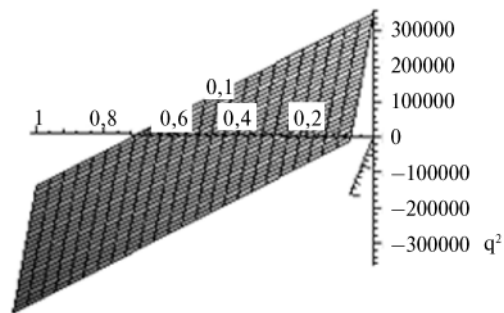
$$\begin{cases} (1 - 2q_1)\lambda a + (1 - 2q_2)\mu c \geq 0, \\ \quad 0 \leq q_1 \leq 1, \\ \quad 0 \leq q_2 \leq 1. \end{cases}$$

In this case $E(Z) \geq 0$ holds when

$$\begin{cases} 0 \leq q_1 \leq \dfrac{7}{10}, \\ \begin{cases} 0 \leq q_2 \leq 1 \ if \ 0 \leq q_1 \leq \dfrac{3}{10} \\ 0 \leq q_2 \leq -\dfrac{5}{2}q_1 + \dfrac{7}{4} \ if \ \dfrac{3}{10} \leq q_1 \leq \dfrac{7}{10}. \end{cases} \end{cases}$$

Graphically, these inequalities separated the region (Fig. 8) on the plane of q_1 и q_2, in which the expected return $E(Z)$ is nonnegative.
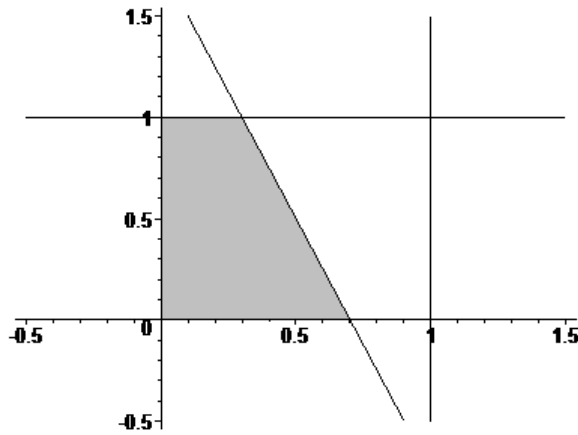

Fig. 8. Area defined by q_1 and q_2 for $E(Z) \geq 0$

6. Application to real data

Let us return to the problem stated above. The device receives the flows of events of two types with different intensity, and depending on which event has received and how the device has recognized it, the previously gained amount will be added to a new one. There are processes that can be described by this model in real life.

Let the device be a stock exchange and events Q and R describe a 'quiet life' and a 'crisis', respectively. According to the model, events Q occur more frequently than R that corresponds to the fact that the crises in our lives are fortunately rare.

The event X can be interpreted as a signal received by a broker about the changes of the economic that helps him to decide is the economy in 'a normal mode' or in a crisis. For example, does the fall of oil prices mean the beginning of the recession in economy or it is a temporary phenomenon and will not change the economy at all.

The values a, b, c, d also have some meaning in such an interpretation. If the event Q occurrs (which means that the economic is stable), and broker correctly recognizes it, then he can get a small income (value a). If the event Q will be taken instead of R, he will loose the amount of $-b$. If the R-event occurred (crisis) and it was not recognized correctly, the broker will lose more (value $-d$). If he could forecast a crisis, he can earn a good deal of money on this – correct identification of the event of type R gives the broker the value c.

Now we estimate the parameters of the model, specially the intensity of flows of these events. We will use time series of returns of the stock index – the series is stationary as in small samples (about 10–20 points), and for a long period (several years). Periods corresponding to the only regular or the only crises days have been selected to check whether time series are stationary; the Dickey-Fuller Unit root test and analysis of autocorrelation and partial autocorrelation functions were used to control of stationary property.

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Parameters λ and μ are indicators of the economy, so we took stock index S&P 500 for their evaluation. S&P500 includes 500 selected stocks of the USA having the largest capitalization. The list is owned and formed by Standard & Poor's [3].

The time interval has been taken for over 10 years – from August 1999 until December 2009. We took the mean of the value of opening and closing as the index value for the day (Fig. 9).

Fig. 9. Stock index S&P500
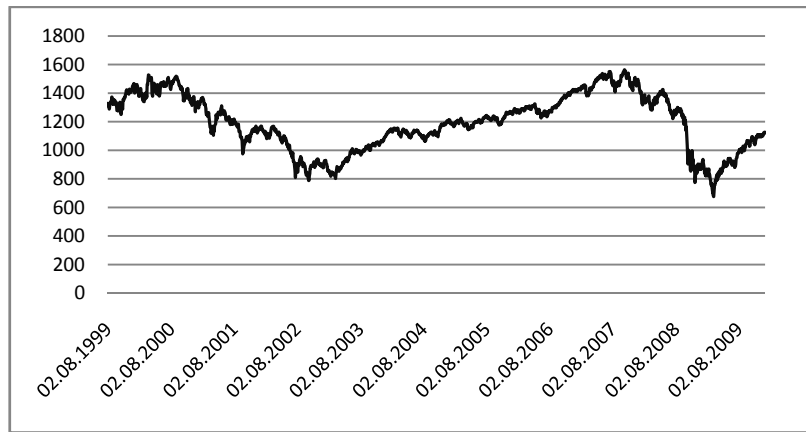
We evaluated the volatility of the index with a sliding interval of 20 days to find when the problems occur in the economy and the recession begins. We took the previous 20 index values S_i for each day and calculate the standard deviation of the sample mean $\bar{S}$ for this sample

$$\sigma_j = \sqrt{\frac{1}{19} \sum_{i=j-20}^{j} (S_i - \bar{S})^2}, j = 21, 22, \ldots$$

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We accept that if the value of volatility is greater than 6%, this means the occurrence of an event of R-type, i.e. a crisis. Figure 10 shows that high values of volatility correspond to the drop of the index. An alternative method of finding the 'crisis days' is given in Appendix 4.
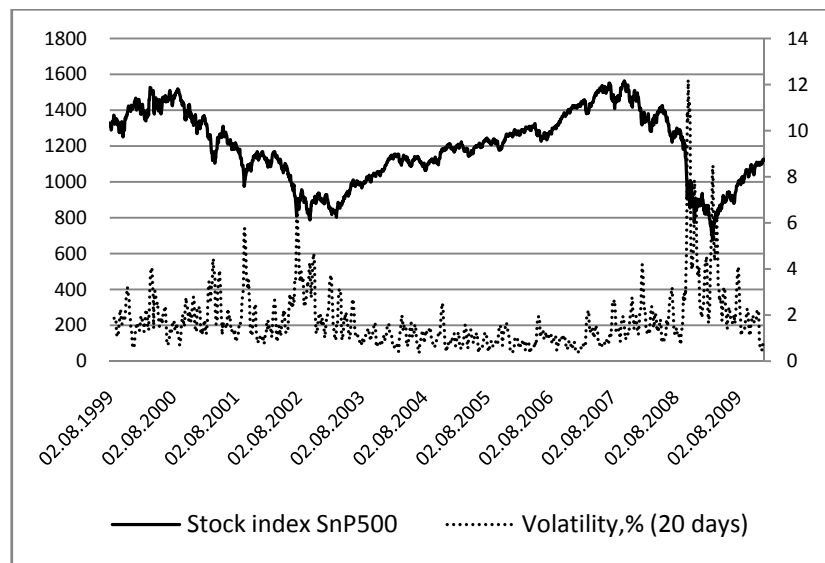
Fig. 10. Stock index S&P500 and its volatility

The threshold 6% was exceeded 42 times of all 2645 (that is 1.6%) for these data (from August 1999 till December 2009). Taking 250 days as an average number of working days per year, we estimate $\mu = 4$ and $\lambda = 246$ for the year.

Thus, our average estimate for the year is 246 "normal" days and 4 days characterized by high volatility and the fall of the index. The corresponding estimates are equal to 249 and 1 for the 10% threshold, respectively.

We can find the value of separate "gain" / "loss" now. We estimated a and b at points corresponding to the event Q (which is determined by the value of the volatility of the index − it should be less than the threshold value for the

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event Q). If the index goes up at this moment, it means that the event a was realized, and if it goes down – then $-b$ is observed. The same approach was used for the events of the type R (that we define as the excess of volatility over the threshold value), i.e. if the index has increased in compare with the previous value, the change of the index is c; if the index has fallen, the change is $-d$. We calculated the average in the obtained samples and took them as an estimate of the values of "gain" / "loss".

Such estimates are $a = 7, -b = -7.5, c = 23.5, -d = -26$ in case of 6% threshold value. These values for the threshold value of 10% are shown in Table 3.

Table 3. Estimates for S&P500 index

Stock index S&P500						
August 1999 – December 2009 ($n = 2664$ observations)						
Threshold value for volatility	λ	μ	a	$-b$	c	$-d$
6%	246	4	7	–8	24	–26
10%	249	1	7	–8	35	–29

One can see which should be the probability of error in the identification process for the expected gain of broker to be non-negative under these values of parameters. For the 6% threshold the corresponding region in the q_1-q_2 plane is shown on Fig. 11.

As it is seen, the probability of an error in recognition of the frequent events has a greater influence on the expected gain, which is not surprising for such values of other parameters. In fact, it is enough to identify Q-events in half of the cases to ensure a positive outcome of the game.
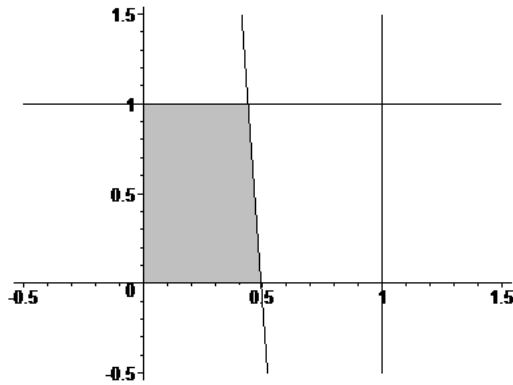
Fig. 11. Area of q_1, q_2, where expected gain is nonnegative

7. Conclusion

Defending economics we can say also that in [5] both crises of 2000 and 2005 were forecast, however, this book did not attract much attention of the scientific community.

In [4] it is pointing out that "…despite its title, Taleb's book mostly is about how statistical models, especially in finance, should pay more attention to low probability gray swans. It would be much more interesting – though much more challenging – to discuss truly aberrant black swans events to which no probabilities are attached because the model we use does not even contemplate their possibility."

Instead of analyzing such probabilities, we showed using very simple model that with a small reward for the correct (with probability slightly higher ½) recognition of the frequent events (and if crisis events are detected with very low probability) the average player's gain will be positive. In other words, players do not need to play more sophisticated games, trying to identify crises events in advance.

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This conclusion resembles the logic of precautionary behavior, that prescripts to play the game with almost reliable small wins. However, these conclusions are obtained with the same (equal to zero) endowments of players. We have not taken into account neither the training nor the risk-taking behavior with a large initial endowments. This will be done in separate papers.

Appendix 1. The formula for the expected value

Let us consider the flows separately. If only events of the first type Q come to the device, then the total value of the received payoff at the time t is equal to

$$Z_Q = \sum_{i=1}^{N_Q} W_i,$$

where all W_i (random variable of payoff when the event Q occurs) are distributed equally with the distribution shown in Table 4.

Table 4

W_1	a	$-b$
P_W	p_1	q_1

$$F_1(x) = \begin{cases} 0, x < -b, \\ q_1, -b \leq x < a, \\ 1, x \geq a, \end{cases}$$

and N_Q is the number of Q-events occurred during the time t, it is distributed according to the Poisson distribution (the law of rare events) with parameter λt (because the flow of Q-type events is the simplest) [2].

This sum of a Poisson number N_Q terms, where N_Q and W_i are independent, is named a compound Poisson random variable. Its distribution is given by a pair $P(\lambda t; F_1(x))$, and the distribution function can be obtained by applying the formula of total probability with the hypotheses $\{N_Q = m\}$

$$F(x) = \sum_{m=0}^{\infty} P\{W_1 + \cdots + W_m \leq x\} \cdot P\{N_Q = m\} = \sum_{m=0}^{\infty} F_1^{(m)}(x) \cdot \frac{(\lambda t)^m}{m!} e^{-\lambda t},$$

where $P\{N_A = m\} = P_m(t) = \frac{(\lambda t)^m}{m!} e^{-\lambda t}$, $F_1^{(m)}(x)$ – m-fold convolution $F_1(x)$.

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Then the expected value of a random variable Z_Q will be equal to

$$EZ_Q = \sum_{j=0}^{\infty} E(Z|N_Q = j) \cdot P\{N_Q = j\} = \sum_{j=0}^{\infty} E(j \cdot W_1) \cdot P\{N_Q = j\} =$$

$$= E(W_1) \cdot \sum_{j=0}^{\infty} jP\{N_Q = j\} = E(W_1) \cdot E(N_Q)$$

$$= (p_1 a - q_1 b) \cdot (\lambda t).$$

The flow of events of type R can be considered similarly. The gain from the second type of events in time t is equal

$$Z_R = \sum_{i=1}^{N_R} Y_i,$$

where all Y_i are distributed similarly (Table 5).

Table 5

Y_1	c	$-d$
P_Y	p_2	q_2

$$F_2(x) = \begin{cases} 0, x < -d, \\ q_2, -d \leq x < c, \\ 1, x \geq c, \end{cases}$$

and N_R is distributed by the Poisson law with parameter μt. Then the expected value of a random variable Z_R is equal

$$EZ_R = E(Y_1) \cdot E(N_R) = (p_2 c - q_2 d) \cdot (\mu t).$$

An important property of a compound Poisson distribution includes that the distribution of the sum of independent variables having the same distribution belongs to the same class again.

As $Z_Q \sim P(\lambda t; F_1(x))$ and $Z_R \sim P(\mu t; F_2(x))$, then $Z = Z_Q + Z_R \sim P((\lambda + \mu)t; G(x))$, where

$$G(x) = \frac{\lambda}{\lambda + \mu} F_1(x) + \frac{\mu}{\lambda + \mu} F_2(x).$$

Considering the given inequalities $\lambda \gg \mu$, $c \gg a$, $d \gg b$, we obtain

$$G(x) = \begin{cases} 0, x < -d, \\ \dfrac{\mu q_2}{\lambda + \mu}, -d \leq x < -b, \\ \dfrac{\lambda q_1 + \mu q_2}{\lambda + \mu}, -b \leq x < a, \\ \dfrac{\lambda + \mu q_2}{\lambda + \mu}, a \leq x < c, \\ 1, x \geq c. \end{cases}$$

The expected value of the random variable $Z = Z_Q + Z_R$ is

$$E(Z) = (\lambda t)E(W_1) + (\mu t)E(Y_1) =$$
$$= \lambda t\big((1 - q_1)a - q_1 b\big) + \mu t\big((1 - q_2)c - q_2 d\big) \ (1)$$

Appendix 2. The formula for the expected value of Z by considering the values of the total winnings

We can get the expected gain for t years without considering the events of each type separately, but examining the value of the total winnings.

Because of our device does not know what type of event has been received at this moment, the value derived from the recognition of the event is a random variable with a discrete set of values (Table 6)

Table 6. The law of distribution of the single winning

X_i	$-d$	$-b$	a	c
P	p_d	p_b	p_a	p_c

where $p_d + p_b + p_a + p_c = 1$.

Since the events Q and R are the simplest flows (they are stationary, ordinary, and have no aftereffects), then the superposition of these flows will also be the simplest with intensity $\lambda + \mu$. Consequently, the probability that unknown event X will be an event of type Q, is $p_Q = \frac{\lambda}{\lambda+\mu}$, and the probability that the event X is an R-event is $p_R = \frac{\mu}{\lambda+\mu}$. So, p_d (the probability that the random variable X takes the value $-d$) is equal to the probability that the occurred event is the R-type and the device has not recognized it, i.e. $p_d = p_R \cdot q_2$.

We can find other probabilities similarly. Then the distribution law of a payoff X is as follows (Table 7).

Table 7. The law of distribution of the single winning

X_i	$-d$	$-b$	a	c
P	$\dfrac{\mu \cdot q_2}{\lambda + \mu}$	$\dfrac{\lambda \cdot q_1}{\lambda + \mu}$	$\dfrac{\lambda \cdot p_1}{\lambda + \mu}$	$\dfrac{\mu \cdot p_2}{\lambda + \mu}$

$$
F_Z(x) = \begin{cases}
0, x < -d, \\
\dfrac{\mu \cdot q_2}{\lambda + \mu}, -d \le x < -b, \\
\dfrac{\mu \cdot q_2}{\lambda + \mu} + \dfrac{\lambda \cdot q_1}{\lambda + \mu}, -b \le x < a, \\
\dfrac{\mu \cdot q_2}{\lambda + \mu} + \dfrac{\lambda \cdot q_1}{\lambda + \mu} + \dfrac{\lambda \cdot p_1}{\lambda + \mu}, a \le x < c, \\
1, x \ge c.
\end{cases}
$$

The total value of received payoffs for time t equals

$$
Z = \sum_{i=1}^{N_Z} X_i,
$$

where all X_i have the same distribution by the law of distribution (Table 7), and N_Z is the number of events occurred during the time t, it is distributed according to the Poisson distribution with parameter $(\lambda + \mu)t$ (because the flow of events is the simplest flow with the intensity $\lambda + \mu$).

This sum of a Poisson number N_Z terms, where N_Z and X_i are independent, is called a compound Poisson random variable. Its distribution is given by a pair of $P(\lambda t; F_Z(x))$, and the explicit form of the distribution function can be obtained by applying the formula of total probability with hypotheses $\{N_Z = m\}$

$$F(x) = \sum_{m=0}^{\infty} P\{X_1 + \cdots + X_m \le x\} \cdot P\{N_Z = m\} =$$

$$= \sum_{m=0}^{\infty} F_Z^{(m)}(x) \cdot \frac{((\lambda + \mu)t)^m}{m!} e^{-\lambda t},$$

where $P\{N_Z = m\} = P_m(t) = \frac{((\lambda+\mu)t)^m}{m!} e^{-\lambda t}$, $F_Z^{(m)}(x)$ – m-fold convolution of $F_Z(x)$.

Then the expected value of Z is equal to

$$E[Z] = \sum_{j=0}^{\infty} E(Z|N_Z = j) \cdot P\{N_Z = j\} = \sum_{j=0}^{\infty} E(j \cdot X_1) \cdot P\{N_Z = j\} =$$

$$= E(X_1) \cdot \sum_{j=0}^{\infty} jP\{N_Z = j\} = E(X_1) \cdot E(N_Z) =$$

$$= \left(-q_2 \frac{d\mu}{\lambda + \mu} - q_1 \frac{b\lambda}{\lambda + \mu} + p_1 \frac{a\lambda}{\lambda + \mu} + p_2 \frac{c\mu}{\lambda + \mu}\right)$$

$$\cdot ((\lambda + \mu)t) == (-q_2 d\mu - q_1 b\lambda + p_1 a\lambda + p_2 c\mu) \cdot t =$$

$$= (\lambda(p_1 a - q_1 b) + \mu(p_2 c - q_2 d)) \cdot t,$$

which is the same as (1).

Appendix 3. The evaluation of the indicators
for the other stock indices

Let us calculate the same indicators for the other indices – the French CAC 40, the German DAX, the Japanese Nikkei 225 and the Hong Kong's Hang Seng, and compare the results.

1. The CAC 40 is a benchmark French stock market index. The index represents a capitalization-weighted measure of the 40 most significant values among the 100 highest market caps on the Paris Bourse (now Euronext Paris). It is one of the main national indices of the pan-European stock exchange group Euronext alongside Brussels' BEL20, Lisbon's PSI-20 and Amsterdam's AEX.

Fig. 12 presents the graphs of the index and the volatility of its returns, Table 8 presents calculated coefficients for our model.
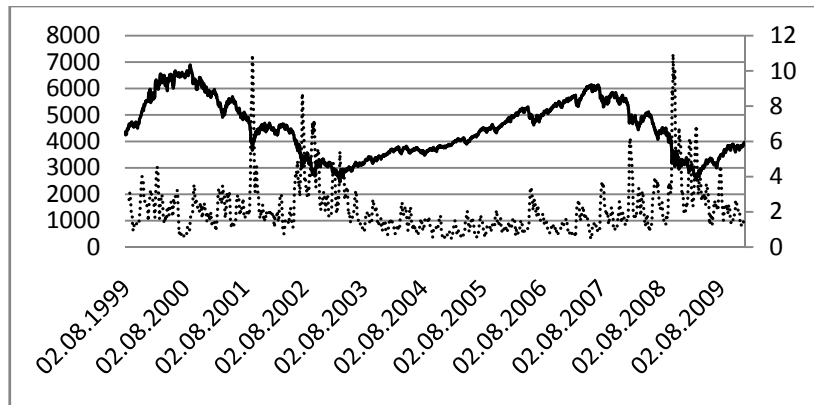

Fig. 12. Stock index CAC 40 and its volatility

Table 8. Estimates for index CAC 40

Index CAC 40						
august 1999 – december 2009 ($n = 2661$ observations)						
The threshold for volatility	λ	μ	a	$-b$	c	$-d$
6%	243	7	35	−39	96	−85
10%	249	1	37	−40	133	−170

2. The DAX (Deutscher Aktien IndeX, formerly Deutscher Aktien-Index (German stock index)) is a blue chip stock market index consisting of the 30 major German companies trading on the Frankfurt Stock Exchange. According to Deutsche Boerse, the operator of Xetra, DAX measures the performance of the Prime Standard's 30 largest German companies in terms of order book volume and market capitalization.

Fig. 13 shows the index and the volatility, Table 9 shows the estimates of the model for the index DAX.
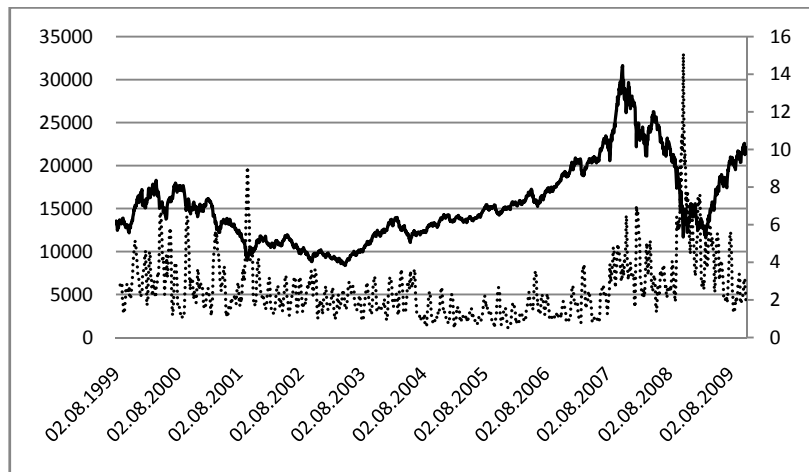

Fig. 13. Index DAX and its volatility

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Table 9. Estimates for the DAX

Index DAX august 1999 – december 2009 ($n = 2651$ observations)						
The threshold for the volatility	λ	μ	a	$-b$	c	$-d$
6%	239	11	39	–45	87	–102
10%	249	1	40	–46	106	–158

3. Nikkei 225 is a stock market index for the Tokyo Stock Exchange (TSE). It has been calculating daily by the Nihon Keizai Shimbun (Nikkei) newspaper since 1950. It is a price-weighted average (the unit is yen), and the components are reviewed once a year. Currently, the Nikkei is the most widely quoted average of Japanese equities, similar to the Dow Jones Industrial Average.

Fig. 14 shows the value of the index and its volatility, Table 10 shows the evaluation of model parameters for a given stock index.
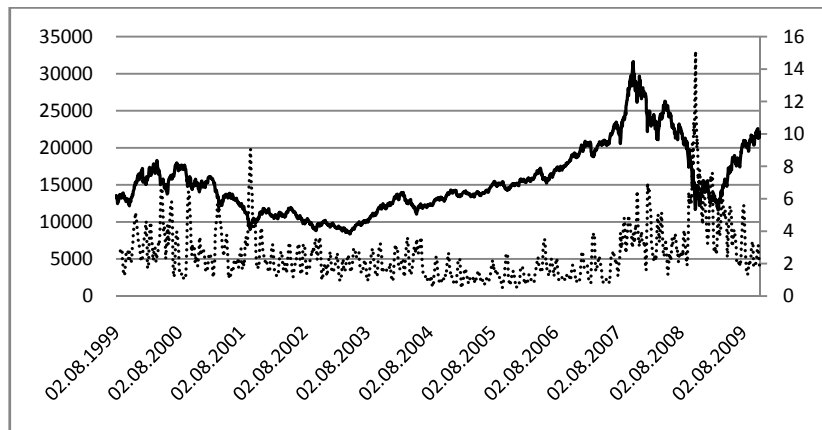

Fig. 14. Index Nikkei 225 and its volatility

Table 10. Model estimates for Nikkei 225

Stock index Nikkei 225 august 1999 – december 2009 ($n = 2557$ observations)						
The threshold value for volatility	λ	μ	a	$-b$	c	$-d$
6%	245	5	102	−112	232	−299
10%	249	1	103	−114	351	−458

4. The Hang Seng Index is a freefloat-adjusted market capitalization-weighted stock market index in Hong Kong. It is used to record and monitor daily changes of the largest companies of the Hong Kong stock market and it is the main indicator of the overall market performance in Hong Kong.
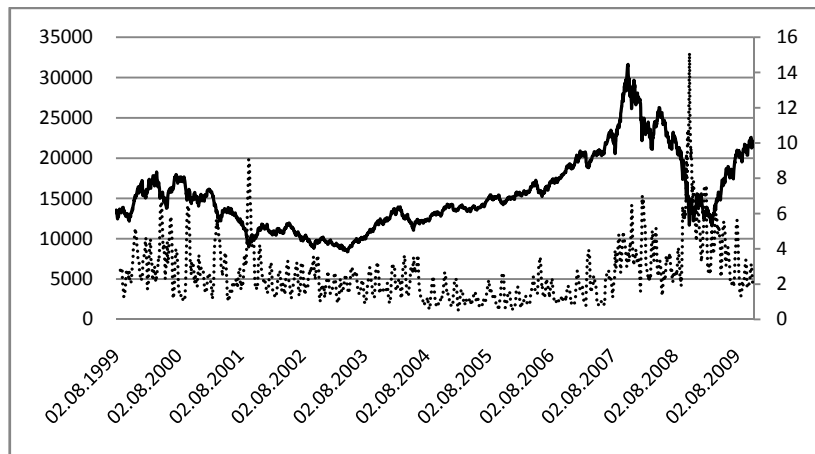

Fig. 15. Index Hang Seng

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Table 11. Model estimates for the index Hang Seng

Index Hang Seng august 1999 – december 2009 ($n = 2557$ points)						
The threshold value for volatility	λ	μ	a	$-b$	c	$-d$
6%	241	9	148	–148	390	–481
10%	249	1	155	–159	481	–777

Taking the threshold value as 6%, we compared the estimates obtained for each of the indices.

Table 12. Model estimates for all indices

Estimates for all indices with the threshold 6%						
Index	λ	μ	a	$-b$	c	$-d$
S&P 500	246	4	7	–8	24	–26
CAC 40	243	7	35	–39	96	–85
DAX	239	11	39	–45	87	–102
Nikkei 225	245	5	102	–112	232	–299
Hang Seng	241	9	148	–148	390	–481

A large discrepancy in the estimates of the parameters is observed due to the fact that the changes were taken for estimation of absolute return indices and all values are measured in percentage points. We obtain almost identical values after counting the figures for the relative changes in each index (Table 13).

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Table 13. Relative model estimates

Relative estimates for all indices with the threshold 6%				
Index	a	$-b$	c	$-d$
S&P 500	0,6%	–0,6%	2,8%	–2,9%
CAC 40	0,8%	–0,8%	3,0%	–2,5%
DAX	0,8%	–0,9%	2,1%	–2,5%
Nikkei 225	0,8%	–0,9%	2,6%	–3,2%
Hang Seng	0,9%	–0,9%	2,6%	–3,0%

We can see that the relative parameter estimates for the indices are approximately equal – index varies with the scope of 0,8–0,9% in the quiet period and its amplitude increases approximately 4 times in time of crisis.

It should be noted that these data should reflect the change in the index in percentages compared to the previous day, rather than racing the index for one day. Daily changes may be more significant, for example, in April 17, 2000 the difference between the highest and the lowest index value of Japanese Nikkei 225 amounted to almost 100 points (approximately 10%). However, April 16 drop was only 3.8% in comparison with the Nikkei the previous day.

What expected gain will have a broker in our model? If we use the data of S&P 500, choose the horizon in 1 year, and the error probability for events Q and and R being 0,3 and 0,8, respectively, the expected gain will be 52%. So the broker can increase its wealth by 52% a year in average.

Appendix 4. Alternative evaluations of the parameters of the model

Of course, volatility is not the only indicator of market behavior and a signal of crisis. The question of the calculation period naturally rises in estimation of the volatility. If we shorten the period, errors in the evaluation of volatility will grow because it will get a higher variability. If we make the period longer, it will reduce errors but will delay the information observed about the market. Perhaps it is also the case in our calculations.

We carry out the same calculations as in Section 6 for estimation of the model parameters, using the index returns instead of the index volatility

$$r_i = \frac{S_i - S_{i-1}}{S_{i-1}}, i = 2, \dots 2665.$$

The fall of the index immediately reflected in its return (Fig. 16), or more precisely on its amplitude. The estimates of all parameters are given in Table 14.
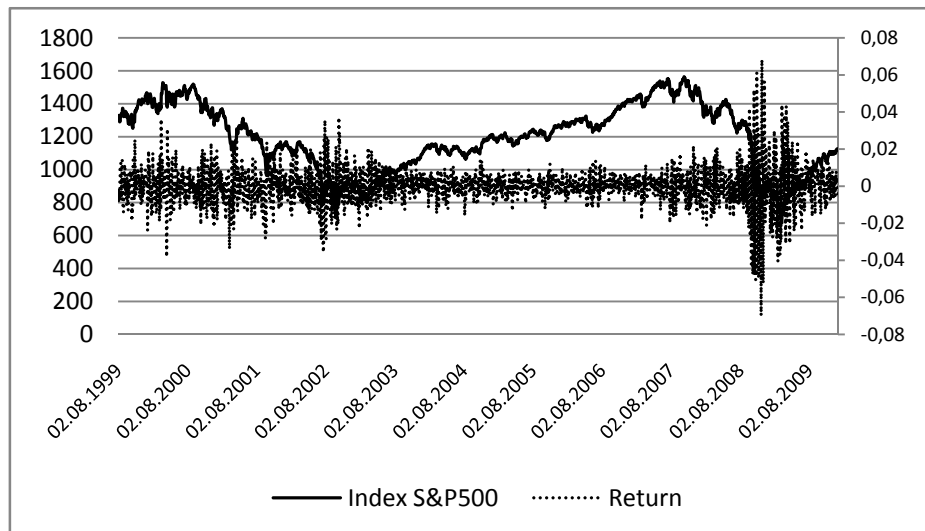

Fig. 16. Index S&P 500 and its return

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Table 14. Estimates of parameters

Index S&P500 august 1999–december 2009 ($n = 2664$ observations)						
The threshold value	λ	μ	a	$-b$	c	$-d$
3%	246	4	7	–7	40	–39
4%	248	2	7	–7	45	–50
5%	249	1	7	–8	–47	–51

So, the return can also serve as a 'yardstick' of the index state and give even more adequate assessment for our model.

In addition, the use of average ratings for such a long period of time inevitably oversimplifies the calculations. It is interesting to look at the distribution of crisis events by years (Table 15). The lack of numbers means the absence of volatility 'jumps' more than 6% in a specified period, i.e. 'quiet life' and the absence of shocks in the market.

Table 15. Estimates of model parameters with 6% threshold by years

	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
a	8	11	8	8	5	5	4	5	7	11	8
b	–9	–11	–9	–8	–5	–4	–4	–4	–7	–12	–9
c				17						30	14
d				–25						–30	–15

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**Так ли уж плохо, что мы не умеем
распознавать черных лебедей?**

(*на английском языке*)

Публикуется в авторской редакции

Notes