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Combinatorics of fronts of Legendrian links and the Arnol'd 4-conjectures

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Abstract. Each convex smooth curve on the plane has at least four points at which the curvature of the curve has local extrema. If the curve is generic, then it has an equidistant curve with at least four cusps. Using the language of contact topology, V. I. Arnol'd formulated conjectures generalizing these classical results to co-oriented fronts on the plane, namely, the four-vertex conjecture and the four-cusp conjecture. In the present paper these conjectures and some related results are proved. Along with a simple generalization of the Sturm–Hurwitz theory, the main ingredient of the proof is a theory of pseudo-involutions which is constructed in the paper. This theory describes the combinatorial structure of fronts on a cylinder. Also discussed is the relationship between the theory of pseudo-involutions and bifurcations of Morse complexes in one-parameter families.

Contents

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Introduction

0.1. Contact manifolds, Legendrian submanifolds, fronts. A *contact form* on a manifold V of dimension $2n+1$ is a 1-form α such that the form $\alpha \wedge (d\alpha)^n$ vanishes nowhere. (All objects in the paper are assumed to be C^∞-smooth unless otherwise stated.) By a *contact manifold* we mean a manifold V^{2n+1} equipped with a field ξ of tangent hyperplanes $\xi_x \subset T_xV$ locally defined as the zero sets of a contact 1-form, that is, $\xi = \ker\alpha$. If the field ξ is co-orientable, then such a form can be defined globally. An n-dimensional submanifold L of a contact manifold (V^{2n+1}, ξ) is said to be *Legendrian* if $T_xL \subset \xi_x$ for any $x \in L$; if $\xi = \ker\alpha$, then this condition is equivalent to the condition $\alpha|_L \equiv 0$.

Let M be a smooth manifold. In this case the 1-jets of functions on M form the manifold $J^1(M) = T^*M \times \mathbb{R}$. The contact structure on $J^1(M)$ is defined by the natural 1-form $\alpha = du - p\,dq$, where u is the coordinate on $\mathbb{R}$ and (p,q) are the canonical Liouville coordinates on T^*M. Another important example of a contact manifold is the spherization ST^*N of the cotangent bundle of a smooth manifold N, that is, the manifold of co-oriented hyperplanes cotangent to N (contact elements). The natural co-oriented contact structure on ST^*N is defined by the following construction: the contact co-oriented tangent hyperplane at a point $X \in ST^*N$ is the pre-image of the co-oriented hyperplane $X \subset TN$ under the action of the differential of the projection $\rho\colon ST^*N \to N$.

Let (V,ξ) be a contact manifold. A smooth fibre bundle $\tau\colon (V,\xi) \to B$ is said to be *Legendrian* if its fibres are Legendrian submanifolds. The projection $\tau(L) \subset B$ of a Legendrian submanifold $L \subset V$ is called the *front* of L. Each point $x \in L$ determines a hyperplane $\tau_*\xi_x \subset T_{\tau(x)}B$ smoothly depending on x that coincides with the tangent hyperplane $T_{\tau(x)}(\tau(L))$ at any smooth point of the front (that is, at the points to which the Legendrian submanifold is projected diffeomorphically). This hyperplane is referred to as a *tangent hyperplane* to the front $\tau(L)$. The natural projections $\rho\colon ST^*N \to N$ and $\sigma\colon J^1(M) \to J^0(M) = M \times \mathbb{R}$ are Legendrian fibre bundles. Hyperplanes tangent to fronts of the projection σ are not vertical (they do not contain the u direction); hyperplanes tangent to fronts of the projection ρ have a natural co-orientation induced by the co-orientation of the contact plane. Every generic Legendrian submanifold of $J^1(M)$ (and, more generally, every Legendrian submanifold occurring in a generic finite-parameter family) can be reconstructed from its front; the same holds for Legendrian submanifolds of ST^*N and their co-oriented fronts in N.

Let us assume that the dimension of the contact manifold is equal to three. Then the front of a generic Legendrian link (a compact Legendrian submanifold) is a curve whose singularities are transverse crossings and semicubical cusps. Cusps correspond to points at which the link is tangent to a fibre of the projection. Every curve in $J^0(M)$, $\dim M = 1$, having singularities of this kind and having no vertical tangents is the front of a Legendrian link in $J^1(M)$; every co-oriented contact curve in N, $\dim N = 2$, with such singularities is the front of a Legendrian link in ST^*N.

One of the main results of the present paper is Theorem 0.1, formulated by Arnol'd [1] as a conjecture.

Theorem 0.1 (the Arnol'd conjecture on four cusps). *Let $\{L_{t\in[0,1]}\}$ be a smooth path in the space of Legendrian knots in $ST^*\mathbb{R}^2$ such that the fronts $\rho(L_0)$ and $\rho(L_1)$*

are convex curves diffeomorphic to circles and having opposite co-orientations. Then there is a point $t_0 \in [0,1]$ such that the Legendrian knot L_{t_0} is tangent to the fibres of the projection ρ at least at four points. If the family $\{L_t\}$ is generic, then there is a point $t_0 \in [0,1]$ such that the front $\rho(L_{t_0})$ has at least four cusps.

0.2. Vertices. In order to formulate another theorem which was also proposed by Arnol'd as a conjecture, we need the notion of *vertex of a Legendrian curve* in $ST^*\mathbb{R}^2$. We give the necessary definitions. For a Legendrian link $L \subset ST^*\mathbb{R}^2$ we define the generalized curvature map $\mathrm{Curv}_L\colon L \to S^1 = \mathbb{R}\mathrm{P}^1$. Let ξ denote the vector bundle over $ST^*\mathbb{R}^2$ formed by all vectors belonging to the contact planes, and consider the projectivization $P\xi$. Let $z \in L$. A line $l_z \subset \xi_z$ tangent to L at z determines an element of the fibre of the bundle $P\xi$ over the point z. Let us construct a trivialization of the vector bundle ξ. This will determine a trivialization of the bundle $P\xi$, that is, a map from $P\xi$ to $\mathbb{R}\mathrm{P}^1$. The map $\mathrm{Curv}_L\colon L \to \mathbb{R}\mathrm{P}^1$ is the composition of the map $z \mapsto l_z$ and the trivializing map.

To construct a trivialization of the bundle ξ, we fix a Euclidean structure and an orientation on $\mathbb{R}^2$. A contact element in $ST^*\mathbb{R}^2$ can be uniquely represented by a unit covector $\cos q\,dx_1 + \sin q\,dx_2 \in T^*_{(x_1,x_2)}\mathbb{R}^2$, where (x_1,x_2) are oriented Euclidean coordinates on $\mathbb{R}^2$ and $q \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Thus, (x_1,x_2,q) are coordinates on $ST^*\mathbb{R}^2$. The natural contact structure on $ST^*\mathbb{R}^2$ is defined by the form $\cos q\,dx_1 + \sin q\,dx_2$. The vector fields $v_0 = \frac{\partial}{\partial q}$, $v_1 = -\sin q\frac{\partial}{\partial x_1} + \cos q\frac{\partial}{\partial x_2}$ are tangent to the contact distribution and thus define a trivialization of it. Let $z \in L$ and let $w = a_0v_0(z) + a_1v_1(z) \in T_zL$ be a non-zero tangent vector. We represent the space $\mathbb{R}\mathrm{P}^1$ in the form $\mathbb{R}\cup\{\infty\}$ and set $\mathrm{Curv}_L(z) = \frac{a_0}{a_1}$ if $a_1 \neq 0$ and $\mathrm{Curv}_L(z) = \infty$ if $a_1 = 0$.

(We note that the vector fields v_0 and v_1 have the following geometric description. The field v_0 is everywhere tangent to the fibres of the bundle $\rho\colon ST^*\mathbb{R}^2 \to \mathbb{R}^2$ and corresponds to the rotation with unit velocity of a contact element at the given point. The field v_1 corresponds to the shift with unit velocity of a contact element along the line (the geodesic) tangent to the element. This description enables one to immediately extend the construction of the map Curv_L to the case of Legendrian links in ST^*M^2, where M^2 is a two-dimensional oriented Riemannian manifold.)

The map Curv_L generalizes the notion of curvature of a planar curve in the following sense.

Proposition 0.2. *If the link L is tangent to a fibre of the projection ρ at a point z, then* $\mathrm{Curv}_L(z) = \infty$; *otherwise the number* $\mathrm{Curv}_L(z)$ *coincides up to sign with the curvature of the front $\rho(L)$ at the point $\rho(z)$.*

Proposition 0.2 is proved in §1. We refer to the critical points of the map Curv_L as *vertices* of L. This definition agrees with the classical definition of vertices of a smooth immersed curve in $\mathbb{R}^2$ if this curve is regarded as the front of a Legendrian link. For a generic Legendrian link L the projections of the vertices belong to the smooth part of the front $\rho(L)$.

We denote by $\mathcal{L}_1$ the connected component of the space of Legendrian knots in $ST^*\mathbb{R}^2$ that contains a knot whose front is a circle in $\mathbb{R}^2$. The following conjecture was proposed by Arnol'd (see [4], p. 97, where this conjecture is presented in a different but equivalent form).

Theorem 0.3 (the Arnol'd conjecture on four vertices). *Every Legendrian knot $L \in \mathcal{L}_1$ has at least four vertices.*

0.3. Bifurcations of fronts. Let $\{L_{t \in [0,1]}\}$ be a generic one-parameter family of immersed Legendrian submanifolds of $ST^*\mathbb{R}^2$. In this case the front $\rho(L_t)$ can have more complicated singularities than cusps and transverse intersections of two branches at only finitely many points $t_1 < \ldots < t_k$. The fronts $\rho(L_t)$ remain diffeomorphic to one another as t ranges within the limits of one of the intervals $[0, t_1[,]t_1, t_2[, \ldots,]t_k, 1]$. When the parameter passes through a value t_i, the fronts of the family $\{L_t\}$ are subjected to a bifurcation in one of the ways shown in Fig. 1 (the direction in which the parameter increases can be opposite to the direction indicated by the arrow; the co-orientations of the branches for the versions shown in Fig. 1 a–c can be arbitrary).

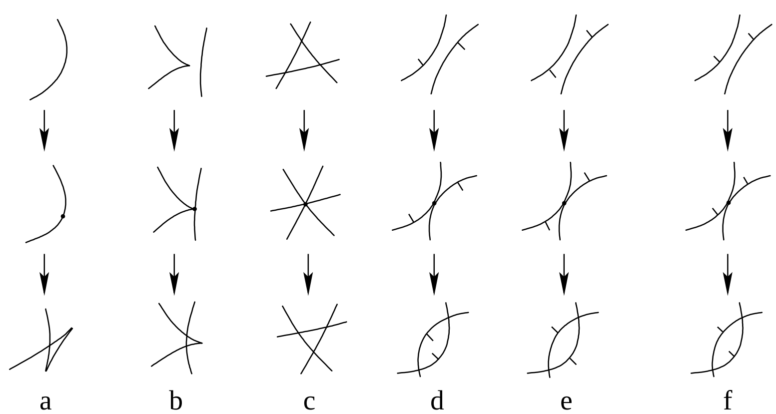


Figure 1

The bifurcation shown in Fig. 1 f (a positive self-tangency) is the only bifurcation for which the Legendrian manifold L_{t_i} has a self-intersection. The assertion of Theorem 0.1 for generic families has the following equivalent formulation: each family of fronts obtained one from another by a chain of bifurcations (shown in Fig. 1 a–f) that begins and ends with convex curves of opposite co-orientations contains a front with four cusps. In studying the relationship between these bifurcations and the combinatorics of fronts, Arnol'd constructed a theory of finite-order invariants for fronts [2]. He also discovered that the assertion of Theorem 0.1 fails if the Legendrian manifolds in the family $\{L_t\}$ can have self-intersections, or, equivalently, if one admits bifurcations of positive self-tangency for the fronts [1]. It turns out that the embedding condition is essential for Theorem 0.3 as well, namely, the Legendrian knot whose front is shown in Fig. 2 has exactly two vertices and can be joined by a path in the set of Legendrian immersions to a Legendrian knot in $\mathcal{L}_1$.

For convex smooth curves in the plane our Theorem 0.3 gives the classical four-vertex theorem (see [7]). For closed embedded planar curves this fact was proved by Mukhopadhyaya [20]. In [23] Sedykh obtained a generalization of the four-vertex

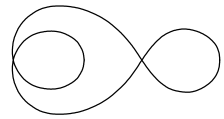


Figure 2

theorem that involves flattening points of a spatial curve lying on the boundary of its convex hull.[1]

Equidistant curves of a convex curve provide an important example of a family of fronts for which the assertion of Theorem 0.1 follows from the classical four-vertex theorem. Studying this example led Arnol'd to the formulation of the above conjecture on four cusps. Let E_0 be an embedded co-oriented curve in the plane. We consider the family $\{\mathrm{E}_{t\in[0,T]}\}$ of equidistant curves, where E_t is obtained from E_0 by shifting each point by the distance t along the positive normal; we co-orient E_t in the direction of this normal.

The family of equidistant curves of a generic curve E_0 admits local bifurcations of the forms shown in Fig. 1 a–e. From the physical point of view, a family of equidistant curves describes a wave propagation in a homogeneous isotropic medium. For more general systems (non-homogeneous, anisotropic, non-autonomous) a bifurcation of positive self-tangency cannot occur, because a wave cannot overtake itself. Thus, the embedding condition for Legendrian submanifolds in Theorem 0.1 is natural.

We return to the family $\{\mathrm{E}_t\}$ of equidistant curves. Local singularities of the curves E_t cover the caustic of the curve E_0, that is, the curve formed by the centers of curvature. The caustic has a semicubical singularity at any center of curvature corresponding to a vertex of the curve E_0, and the family $\{\mathrm{E}_t\}$ of fronts is subjected to the bifurcation shown in Fig. 1 a.

Let E_0 be a generic quadratically convex curve, co-oriented inwards. The radius of curvature defines a smooth function $R\colon \mathrm{E}_0 \to \mathbb{R}$ whose critical points are the vertices of the curve E_0. For a generic value $t \in \mathbb{R}$ the number of cusps in E_t is equal to the number of points in E_0 at which the function R takes the value t. In particular, for each sufficiently large T the front E_T is a smooth convex curve co-oriented outwards, and the family of Legendrian knots corresponding to the family $\{\mathrm{E}_{t\in[0,T]}\}$ of fronts satisfies the assumptions of Theorem 0.1 (up to a change of

[1] *Russian Editor's note*: The authors' results should also be compared with those in the following papers: R. Osserman, "The four-or-more vertex theorem", *Amer. Math. Monthly* **92** (1985), 332–337; T. Bisztriczky, "On the four-vertex theorem for space curves", *J. Geom.* **27** (1986), no. 2, 166–174; G. Cairns and R. W. Sharpe, "On the inversive differential geometry of plane curves", *Enseign. Math.* (2) **36** (1990), nos. 1-2, 175–196; S. I. R. Costa and M. Firer, "Four-or-more-vertex theorems for constant curvature manifolds", in: Real and Complex Singularities (Sao Carlos, 1998), Chapman and Hall/CRC, Boca Raton, FL 2000, pp. 164–172; M. Maeda, "Remarks on the four-vertex theorem", *Sci. Rep. Yokohama Nat. Univ. Sect.* I *Math. Phys. Chem.* (1997), no. 44, 65–72; E. Heil, "A four-vertex theorem for space curves", *Math. Pannon.* **10** (1999), no. 1, 123–132; O. R. Musin, "Chebyshev systems and zeros of a function on a convex curve", *Proc. Steklov Inst. Math.* **221** (1998), 236–246; V. Ovsienko and S. Tabachnikov, "Sturm theory, Ghys theorem on zeroes of the Schwarzian derivative and flattening of Legendrian curves", *Selecta Math.* (N.S.) **2** (1996), no. 2, 297–307.

the parameter t). For this family, the assertion of Theorem 0.1 follows from the classical four-vertex theorem, because each function on the circle having at least two minima takes some regular value at least four times.

0.4. A remarkable contactomorphism. By introducing a Euclidean structure on $\mathbb{R}^2$ one canonically determines a contactomorphism between $ST^*\mathbb{R}^2$ and $J^1(S^1)$, where the circle S^1 is identified with the unit circle in $\mathbb{R}^2$. We describe this map. A contact element in $ST^*\mathbb{R}^2$ is uniquely represented by a unit covector $\cos q\,dx_1 + \sin q\,dx_2 \in T^*_{(x_1,x_2)}\mathbb{R}^2$, where $q \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Thus, the triple (x_1, x_2, q) defines coordinates on $ST^*\mathbb{R}^2$. The natural contact structure on $ST^*\mathbb{R}^2$ is given by the form $\cos q\,dx_1 + \sin q\,dx_2$. The map $ST^*\mathbb{R}^2 \to J^1(S^1)$, $(x_1, x_2, q) \mapsto (p, q, u)$, where $p = -x_1 \sin q + x_2 \cos q$, $u = x_1 \cos q + x_2 \sin q$, is a contactomorphism, because it takes the form $du - p\,dq$ to the form $\cos q\,dx_1 + \sin q\,dx_2$. This contactomorphism is shown in Fig. 3. One can similarly define a contactomorphism between $ST^*\mathbb{R}^{n+1}$ and $J^1(S^n)$ for any n. We identify $ST^*\mathbb{R}^2$ with $J^1(S^1)$. This identification plays an important role in the proof of Theorems 0.1 and 0.3. The main objects of our study are the fronts $\sigma(L) \subset J^0(S^1)$ of Legendrian links $L \subset J^1(S^1)$.

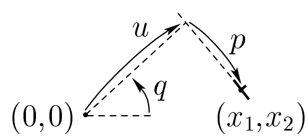


Figure 3

0.5. Decompositions of fronts. The theory of pseudo-involutions of fronts in $J^0(M)$, where $\dim M = 1$, is the combinatorial tool used in the proof of Theorems 0.1 and 0.3. Let us present here some definitions of this theory in the simpler case of $M = \mathbb{R}$ and formulate a result using the language of proper decompositions rather than the language of pseudo-involutions (which is equivalent for $M = \mathbb{R}$).

Let $\Sigma \subset J^0(\mathbb{R})$ be the front of a generic Legendrian link in $J^1(\mathbb{R})$. Along with the restriction on the singularities of the front, we extend the genericity conditions by including the assumption that the q-coordinates of the crossings of this front are different. By a *section* of Σ we mean a subset that is the graph of a continuous function $h\colon [q_1, q_2] \to \mathbb{R}$ such that the points $(q_1, h(q_1))$ and $(q_2, h(q_2))$ are cusps of Σ.

An unordered collection $D = \{\Gamma_1, \dots, \Gamma_N\}$ of sections of Σ is called a *decomposition* of Σ if $\Sigma = \bigcup_{i=1}^N \Gamma_i$ and any two different sections intersect at only finitely many points. A crossing x of Σ is said to be *switching* for the decomposition D if a section in D containing the point x has a break at this point. Each decomposition D is uniquely determined by the set $\mathrm{Sw}(D)$ of its switching points. A decomposition $\{\Gamma_1, \dots, \Gamma_N\}$ of Σ is said to be *proper* if the sections Γ_i can be partitioned into pairs in such a way that the sections forming a pair intersect exactly at their ends. This partition into pairs defines a free involution θ_D on the set D. We note that a front need not have a proper decomposition. For example, a front containing the 'zigzag' fragment (disjoint from the other branches of the front) has no proper decomposition.

Proper decompositions were first introduced by Eliashberg [12]. The present paper was mainly inspired by the constructions in [12].

0.6. Positivity. Suppose that D is a proper decomposition and $x = (q_0, u_0) \in \mathrm{Sw}(D)$. Let Γ and Γ' be sections in D passing through x. The sections Γ, Γ', $\theta_D(\Gamma)$, and $\theta_D(\Gamma')$ are pairwise distinct, because the decomposition D is proper. We consider the line $l_{q_1} = \{q = q_1\}$, where $q_1 \neq q_0$ is close to q_0, and write $z_1 = \Gamma \cap l_{q_1}$, $z_2 = \Gamma' \cap l_{q_1}$, $z_1' = \theta_D(\Gamma) \cap l_{q_1}$, and $z_2' = \theta_D(\Gamma') \cap l_{q_1}$.

A switching point x is said to be *positive* (for the decomposition D) if the following either-or condition holds: either the intervals $[z_1, z_1']$ and $[z_2, z_2']$ are disjoint or one of them is a subset of the other. This definition does not depend on the choice of q_1. A proper decomposition D of the front Σ is said to be *positive* if all the switching points are positive.

Theorem 0.4. *Let L and L' be generic Legendrian links in $J^1(\mathbb{R})$ homotopic in the class of Legendrian links. Then the fronts $\sigma(L)$ and $\sigma(L')$ have the same number of positive decompositions.*

It should be noted that the notion of positive proper decomposition was independently introduced by Fuchs in [16], where the relationship between the existence of a positive decomposition for a front of a Legendrian knot $L \subset J^1(\mathbb{R})$ and the augmentations of the differential graded algebra associated with L was studied (for the definition of differential graded algebra and its augmentations, see [11]).

0.7. Plan of the paper. A proof of Theorem 0.1 for a special case is given in §1. This proof illustrates the ideas used in the full proof, which involve the Hurwitz theorem. In §2 we give the basic definitions of the theory of pseudo-involutions and formulate the main result of this theory, Theorem 2.5 on the extension of positive pseudo-involutions to generic one-parameter families of fronts. Theorem 2.5 is proved in §3, where the desired extension of a pseudo-involution is constructed explicitly. In §4 we study the (non-)uniqueness of the extension of positive pseudo-involutions and the monodromy group arising from the extension along loops. The extension of pseudo-involutions to non-generic fronts is studied in §5. In §6 we prove some results on the combinatorics of decompositions of fronts. In §7 we formulate generalizations of the Hurwitz theorem in which a function on the circle is replaced by the front of a Legendrian submanifold of $J^1(S^1)$. The proofs are given in §8. In §9 we prove the Arnol'd conjectures on cusps and vertices (by using results in the previous sections) and present some generalizations of these conjectures and other results on cusps and vertices of fronts. Critical points of Legendrian knots (and their relationships to the Hurwitz theorem) are discussed in §10. The theory of pseudo-involutions enables one to construct invariants of Legendrian knots. These invariants are described in §11. In §12 we explain the relationship between the theory of generating families for Legendrian manifolds and the theory of pseudo-involutions. In particular, we explain how a generating family of a special type produces a pseudo-involution on the front of the Legendrian submanifold of $J^1(S^1)$ determined by this family.

We express our deep gratitude to our teacher V. I. Arnol'd, who discovered (among many other things) the remarkable relationship between differential geometry and contact topology. We are extremely thankful to him for his support and persistence, without which this paper would not have been completed. We thank F. Aicardi, Ya. M. Eliashberg, E. Ferrand, D. B. Fuchs, V. V. Goryunov, M. L. Kontsevich, M. F. Prokhorova, V. D. Sedykh, and O. Ya. Viro for their stimulating interest in this research and for helpful discussions.

§ 1. Application of the Hurwitz theorem

1.1. Curvature and 1-forms. Let us give a description of the map Curv_L in terms of differential forms. We consider the 1-forms $\omega = dq$ and $\beta = -\sin q\, dx_1 + \cos q\, dx_2$ on $ST^*\mathbb{R}^2$. They satisfy the relations $\langle \omega, v_0\rangle = \langle \beta, v_1\rangle = 1$ and $\langle \omega, v_1\rangle = \langle \beta, v_0\rangle = 0$, where v_0 and v_1 are the vector fields in the definition of Curv_L given in 0.2. Thus, we have $\mathrm{Curv}_L(z) = \langle \omega(z), w\rangle / \langle \beta(z), w\rangle$ for each point $z \in L$ and each non-zero tangent vector $w \in T_z L$.

1.2. Proof of Proposition 0.2. If the link L is tangent to a fibre of ρ, then $\langle \beta(z), w\rangle = 0$, and the assertion is proved. Suppose that L is not tangent to a fibre of ρ at the point z. We choose a local parameterization $t \mapsto (x_1(t), x_2(t), q(t))$, $0 \mapsto z$ of L such that $\dot{x}_1^2 + \dot{x}_2^2 \equiv 1$, that is, $t \mapsto (x_1(t), x_2(t))$ is a natural parameterization of the front $\rho(L)$. The curvature of $\rho(L)$ at the point $\rho(z)$ is equal to $\sqrt{\ddot{x}_1^2(0) + \ddot{x}_2^2(0)}$ by definition. We show that this number coincides with $|\dot{q}(0)|$. It follows from the contact condition $\dot{x}_1 \cos q + \dot{x}_2 \sin q = 0$ and the condition $\dot{x}_1^2 + \dot{x}_2^2 \equiv 1$ that $(\dot{x}_1(t), \dot{x}_2(t)) = \pm(\sin(q(t)), -\cos(q(t)))$ for any t. Hence, $(\ddot{x}_1(t), \ddot{x}_2(t)) = \pm(\cos(q(t)), \sin(q(t)))\dot{q}(t)$, and therefore $\sqrt{\ddot{x}_1^2(0) + \ddot{x}_2^2(0)} = |\dot{q}(0)|$. Since we have the relations $\big\langle \omega(z), (\dot{x}_1(0), \dot{x}_2(0), \dot{q}(0))\big\rangle = \dot{q}(0)$ and $\big\langle \beta(z), (\dot{x}_1(0), \dot{x}_2(0), \dot{q}(0))\big\rangle = -\dot{x}_1(0)\sin(q(0)) + \dot{x}_2(0)\cos(q(0)) = \pm 1$ (because the vectors $(\cos(q(0)), \sin(q(0)))$ and $(\dot{x}_1(0), \dot{x}_2(0))$ are orthogonal), it follows that $\mathrm{Curv}_L(z) = \pm\dot{q}(0)$. □

1.3. Curvature and the Sturm differential operator. Let $L \subset ST^*\mathbb{R}^2 = J^1(S^1)$ be a Legendrian link. We need a description, in terms of the front $\sigma(L)$, of the set of points at which the link L is tangent to the fibres of ρ. Let G_L be the set of non-critical points of the projection $\sigma|_L$. We introduce the map $F_L\colon L \to \mathbb{R}\mathrm{P}^1 = \mathbb{R} \cup \{\infty\}$ and set $F_L(y) = \infty$ for $y \notin G_L$. In a neighbourhood of a point $y \in G_L$ the link L coincides with the 1-graph $j^1 h$ of some function $h\colon U \to \mathbb{R}$, $U \subset S^1$, that is, L is given locally by the equations $u = h(q)$, $p = h'(q)$, where $q \in U$. We consider the Sturm differential operator $D_1 h = d^2h/dq^2 + h$ and write $F_L(y) = D_1 h(q_y)$, where q_y stands for the q-coordinate of the point y.

Lemma 1.1. *The maps F_L and $1/\mathrm{Curv}_L$ coincide (it is assumed that $1/0 = \infty$ and $1/\infty = 0$). A point z is critical for the projection $\rho|_L$ if and only if $z \in G_L$ and $F_L(z) = 0$.*

Proof. If $z \in L$ is a critical point of the projection $\sigma|_L$, then $F_L(z) = \infty$, and $\mathrm{Curv}_L(z) = 0$ because $\beta|_L(z) = dq|_L(z) = 0$. Suppose that $z \in G_L$ and L coincides with $j^1 h$ near z. Then $1/\mathrm{Curv}_L(z) = \langle \beta(z), v\rangle / \langle \omega(z), v\rangle$. The vector $\zeta = \frac{\partial}{\partial q} + h'(q_z)\frac{\partial}{\partial u} + h''(q_z)\frac{\partial}{\partial p}$, where q_z stands for the qth coordinate of z, is tangent to L at z. Since $\beta = -\sin q\, dx_1 + \cos q\, dx_2 = dp + u\, dq$, we see that $1/\mathrm{Curv}_L(z) = \langle \beta(z), \zeta\rangle / \langle \omega(z), \zeta\rangle = h''(q_z) + h(q_z) = F_L(z)$. The second assertion of the lemma follows from the fact that the equality $\mathrm{Curv}_L(z) = \infty$ is equivalent to the condition $\beta|_L(z) = 0$, and $\beta|_L(z) = 0$ if and only if $v_0(z)$ is tangent to L. □

1.4. Hurwitz theorem and smooth fronts. When proving Theorem 0.1, we can assume that the origin lies inside the (quadratically convex) fronts $\rho(L_0)$ and $\rho(L_1)$. In this case $\sigma(L_0)$ ($\sigma(L_1)$) is the graph of a negative (positive) smooth function on S^1. If $\{\rho(L_t)\}$ is a family of equidistant curves, then all the fronts $\sigma(L_t)$ are obtained from the front $\sigma(L_0)$ by shifts along the u-coordinate.

Suppose that the fronts $\sigma(L_t)$ are the graphs of smooth functions H_t. The following proof of Theorem 0.1 in this special case is due to Arnol'd [1]. Let $t_0 \in [0,1]$ be such that $\int_{S^1} H_{t_0}(q)\,dq = 0$. The coefficients of the zeroth and first harmonics in the Fourier series of the function $D_1 H_{t_0}$ vanish. By the Hurwitz theorem [19], the function $D_1 H_{t_0}$ has at least four zeros. Since the zeros of $D_1 H_{t_0}$ correspond to the zeros of $F_{L_{t_0}}$, this proves the assertion of Theorem 0.1 for this family.

1.5. Hurwitz theorem and fronts with singularities. The main idea of the proof of Theorem 0.1 is to extend this argument to arbitrary families $\{L_t\}$ of Legendrian knots. We shall construct a continuous family $\{H_{t\in[0,1]}\}$ of continuous functions on S^1 such that the graph of H_t is a subset of the front $\sigma(L_t)$ for each t. We claim that if $\int_{S^1} H_{t_0}(q)\,dq = 0$, then $F_{L_{t_0}}$ has at least four zeros.

As an illustration, we consider a Legendrian knot L whose front $\Sigma = \sigma(L)$ has the same combinatorial structure as the front shown in Fig. 4. Omitting the details, we claim that if $\int_{S^1} H(q)\,dq = 0$, where H is a continuous function on S^1 whose graph is a subset of Σ, then L is tangent to the fibres of ρ at least at four points.

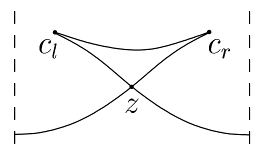


Figure 4

Suppose for simplicity that the function F_L is non-zero at the two pre-images of the crossing point $z = (q_0, u_0)$ of the front Σ. The function $D_1 H$ can be regarded as a distribution on S^1 of the form $C\delta_{q_0} + g$, where δ_{q_0} is the Dirac delta function supported at q_0 and g is a continuous function on $S^1 \setminus \{q_0\}$ having non-zero left and right limits at q_0. It can be seen in Fig. 4 that $C < 0$. The distribution $D_1 H$ vanishes on the trigonometric polynomials of degree ≤ 1. This readily implies (by using a modification of the standard proof of the Hurwitz theorem) that $D_1 H$ changes its sign at least at four points. Suppose that the sign of the distribution $D_1 H$ at the point q_0 is negative (that is, coincides with the sign of C). We say that $D_1 H$ changes sign from the right (from the left) at q_0 if the right (left) limit of g at q_0 is positive.

If the four changes of sign occur away from the point q_0, then the assertion is proved. If changes of sign (one or two) occur at q_0, then we shall seek the missing zeros of the map F_L on the pre-images of the sides $[z, c_l]$ and $[z, c_r]$ of the curvilinear triangle with vertices z, c_r, c_l. We claim that if $D_1 H$ changes sign from the right (left) at q_0, then F_L vanishes at one of the points in the pre-image of $[z, c_l]$ (of $[z, c_r]$). Indeed, the interval $[z, c_l] \subset \Sigma$ is the graph of a function $h\colon [q_l, q_0] \to \mathbb{R}$, where $c_l = (q_l, u_l)$. The function h is smooth everywhere except for the point q_l, and its second derivative h'' tends to $-\infty$ as $q \to q_l$. Therefore, there is a point $q \in [q_l, q_0]$ such that $D_1 h(q) = 0$. The proof for a change of sign from the left at q_0 is similar. This completes the proof.

The proof of Theorem 0.1 for an arbitrary front follows the same scheme. It involves the construction of a certain decomposition of the front into pieces that generalizes the above decomposition of the front Σ into four pieces. The construction of this decomposition is based on the theory of pseudo-involutions.

1.6. Scheme of the proof of Theorem 0.3. We claim that the front $\sigma(L)$ of any Legendrian knot $L \in \mathcal{L}_1$ contains a continuous section $\{u = H_L(q)\}$ of the fibre bundle $J^0(S^1) \to S^1$ that is continuously dependent on L. A shift of the link L (and of the front $\sigma(L)$) along u takes the vertices to vertices. The proof of Theorem 0.3 uses the following trick: by shifting the link L along u, we can achieve the condition $\int_{S^1} H_L(q)dq = 0$. Using the same argument as in the proof of Theorem 0.1, we find four points at which F_L vanishes. The vertices of L are the critical points of the map F_L. We then show (by using an analogue of Rolle's theorem for maps into the circle) that between any two neighbouring zeros of these four zeros there is a vertex of L. This will complete the proof.

§ 2. Pseudo-involutions and the continuation theorem

2.1. Singularities of fronts. Let $M = S^1$, $M = \mathbb{R}$, or $M = I$, where I is a closed interval in $\mathbb{R}$ or S^1. If $M = I$, then by a *Legendrian link* in $J^1(M)$ we mean a compact Legendrian submanifold L with boundary such that $\partial L = \partial J^1(M) \cap L$ (that is, the projection $J^1(I) \to I$ takes the boundary of L to the ends of I), L is transverse to $\partial J^1(I)$ at all points of the boundary, and any two points of ∂L projected onto the same end of I have distinct u-coordinates. We denote by π the projection $J^0(M) \to M$ given by the rule $(q,u) \mapsto q$ and by σ the projection $J^1(M) \to J^0(M)$. Let $\Sigma = \sigma(L) \subset J^0(M)$ be the front of a Legendrian link $L \subset J^1(M)$. We denote by G_Σ the set of non-singular points of Σ, by X_Σ the set of transverse double self-intersections, by C_Σ the set of non-degenerate cusps (points admitting neighbourhoods in which the front is diffeomorphic to the semicubical parabola), and by $Z_\Sigma = \Sigma \setminus (G_\Sigma \cup X_\Sigma \cup C_\Sigma)$ the set of points at which the front has a more complicated singularity. If $M = I$, then we put any point $x \in \Sigma$ with $\pi(x) \in \partial I$ in the set G_Σ.

Let us consider the following two (singular) hypersurfaces in the space $\mathcal{L}$ of all Legendrian links in $J^1(M)$: the hypersurface $\mathcal{S}$ formed by the links L such that $Z_{\sigma(L)}$ is non-empty (that is, the front $\sigma(L)$ has complicated singularities) and the hypersurface $\mathcal{E}$ formed by the links L such that $\sigma(L)$ has two singular points with the same projections on M. The hypersurface $\mathcal{D} = \mathcal{S} \cup \mathcal{E}$ is referred to as the *discriminant*. A Legendrian link $L \subset J^1(M)$ (and its front $\sigma(L)$) is said to be *σ-generic* if $L \in \mathcal{L} \setminus \mathcal{D}$.

2.2. Definition of pseudo-involution. A continuous map $\mathrm{P}\colon G_\Sigma \cup C_\Sigma \to \Sigma$ is called a *pseudo-involution* of a front $\Sigma \subset J^0(M)$ if it satisfies the following four conditions:

(PI1) $\pi|_\Sigma = \pi|_\Sigma \circ \mathrm{P}$;
(PI2) $\mathrm{P}^2(x) = x$ for $\mathrm{P}(x) \in G_\Sigma \cup C_\Sigma$;

(PI3) $\mathrm{P}(x) = x$ if and only if $x \in C_\Sigma$;
(PI4) for any point $x \in X_\Sigma$ there is a neighbourhood $U \subset J^0(M)$ such that $\mathrm{P}(U \cap G_\Sigma)$ is disjoint from U.

2.3. Pseudo-involutions and decompositions. Suppose that $\Sigma \subset J^0(M)$ is the front of a Legendrian link, Λ is a closed interval or a circle, and $\gamma\colon \Lambda \to \Sigma$ is a continuous map such that: (1) locally, the composition of γ and the projection $J^0(M) \to M$ is an embedding; (2) each non-singular point of Σ has at most one γ-pre-image; (3) if Λ is a closed interval, then γ takes each of its ends either to a cusp of Σ or to a point of the boundary $\partial J^0(M) = \pi^{-1}(\partial M)$. A map γ of this kind, regarded up to homeomorphisms of Λ, is called a *section* of the front Σ. If $M = \mathbb{R}$ or $M = I$, then a section γ is uniquely determined by its image $\Gamma = \gamma(\Lambda)$, which we shall also call a *section* (and use both meanings equally). Thus, this definition agrees with that in the Introduction for the case $M = \mathbb{R}$.

Let $\Sigma \subset J^0(M)$ be a σ-generic front. A set D of sections of Σ is called a *decomposition* of Σ if each non-singular point of Σ belongs to the image of exactly one section in D. For $x \in X_\Sigma$ there are sections $\gamma_i\colon \Lambda_i \to \Sigma$ ($i \in \{1,2\}$) in D and points $s_i \in \Lambda_i$ such that $\gamma_i(s_i) = x$ (if $\gamma_1 = \gamma_2$, then $s_1 \neq s_2$). A crossing point x is said to be *switching* for the decomposition D if the images of small neighbourhoods of the points s_1 and s_2 under the maps γ_1 and γ_2 have break points at x, and *non-switching* if the images are smooth. We denote by $\mathrm{Sw}(D)$ the set of switching crossings for D. The map $D \mapsto \mathrm{Sw}(D)$ is a one-to-one correspondence between the set of decompositions of Σ and the set of subsets of X_Σ.

Proposition 2.1. *Each pseudo-involution* P *of a σ-generic front* $\Sigma \subset J^0(M)$ *uniquely determines a decomposition* D_P *of* Σ *and a free involution* $\theta_\mathrm{P}\colon D_\mathrm{P} \to D_\mathrm{P}$ *such that the point* $\mathrm{P}(x)$ *belongs to the image of the section* $\theta_\mathrm{P}(\gamma)$ *for each section* $\gamma\colon \Lambda \to \Sigma$ *in* D_P *and each point* $x \in \gamma(\Lambda) \cap (G_\Sigma \cup C_\Sigma)$.

Proof. By dividing M into pieces, one can reduce the proof to the case in which $M = I$ and the front Σ has exactly one singular point (the claim is obvious if all points of Σ are non-singular). Suppose that x is the crossing point of Σ. The non-singular components of Σ are taken by P to sections of Σ. Two of these sections, say Γ_1 and Γ_2, pass through x (otherwise the pseudo-involution P would transpose points close to x, which contradicts the condition (PI4)). In this case the sections of D_P are Γ_1, Γ_2, and the non-singular branches of the front. Clearly, the desired involution θ_P exists and is unique. The case in which the only singular point of the front Σ is a cusp is obvious. $\square$

The elements of the decomposition D_P are called *P-sections* of the front Σ. A point $x \in X_\Sigma$ is called a *switching* crossing for the pseudo-involution P if x is switching for the decomposition D_P, and a *non-switching* crossing for P otherwise. The set of switching crossings of a pseudo-involution P is denoted by $\mathrm{Sw}(\mathrm{P})$.

Let $\Sigma \subset J^0(M)$ be a σ-generic front, where $M = \mathbb{R}$ or $M = I$. A pair (D, θ), where D is a decomposition of Σ and $\theta\colon D \to D$ is a free involution, is said to be *proper* if $\pi(\Gamma) = \pi(\theta(\Gamma))$ and the points of the intersection $\Gamma \cap \theta(\Gamma)$ are cusps of the front Σ for each $\Gamma \in D$.

Proposition 2.2. *If $M = \mathbb{R}$ or $M = I$, then the map* $\mathrm{P} \mapsto (D_{\mathrm{P}}, \theta_{\mathrm{P}})$ *is a one-to-one correspondence between the set of pseudo-involutions of a σ-generic front Σ and the set of proper pairs for Σ.*

Proof. If P is a pseudo-involution of Σ, then the pair $(D_{\mathrm{P}}, \theta_{\mathrm{P}})$ is proper by the construction given in the proof of Proposition 2.1. If (D, θ) is a proper pair, then it follows from the definition of the map $\mathrm{P} \mapsto (D_{\mathrm{P}}, \theta_{\mathrm{P}})$ in Proposition 2.1 that there is exactly one pseudo-involution P such that $(D_{\mathrm{P}}, \theta_{\mathrm{P}}) = (D, \theta)$. $\square$

We note that if $M = \mathbb{R}$, then for each proper decomposition D there is exactly one involution $\theta\colon D \to D$ such that the pair (D, θ) is proper. Thus, Proposition 2.2 implies that the pseudo-involutions of a σ-generic front $\Sigma \subset J^0(\mathbb{R})$ are in one-to-one correspondence with the proper decompositions of the front Σ.

The number of pseudo-involutions is finite, in view of the following assertion.

Proposition 2.3. *Suppose that $\Sigma \subset J^0(M)$ is a σ-generic front. Let $q_0 \in M$ be a point such that the set $\Sigma^{q_0} = \Sigma \cap \pi^{-1}(q_0)$ contains no singular points of the front, and let* P *and* P$'$ *be pseudo-involutions of Σ such that* $\mathrm{Sw}(\mathrm{P}) = \mathrm{Sw}(\mathrm{P}')$ *and both pseudo-involutions act in the same way on Σ^{q_0}. Then* $\mathrm{P} = \mathrm{P}'$.

Proof. It suffices to prove the assertion in the case of $M = I$ (indeed, if the pseudo-involutions are distinct, then so are their restrictions to some σ-generic front $\Sigma^I = \Sigma \cap \pi^{-1}(I) \subset J^0(I)$, where I is a closed interval in M containing q_0). Since $\mathrm{Sw}(\mathrm{P}) = \mathrm{Sw}(\mathrm{P}')$, the decompositions D_{P} and $D_{\mathrm{P}'}$ coincide. It remains to show that $\theta_{\mathrm{P}}(\Gamma) = \theta_{\mathrm{P}'}(\Gamma)$ for each $\Gamma \in D_{\mathrm{P}}$. Suppose that Γ intersects Σ^{q_0} at a point y. Then each of the sections $\theta_{\mathrm{P}}(\Gamma)$ and $\theta_{\mathrm{P}'}(\Gamma)$ contains the point $\mathrm{P}(y) = \mathrm{P}'(y)$, and hence $\theta_{\mathrm{P}}(\Gamma) = \theta_{\mathrm{P}'}(\Gamma)$. If Γ is disjoint from Σ^{q_0}, then Γ contains a cusp. Let Γ_1 be another section in $D_{\mathrm{P}} = D_{\mathrm{P}'}$ containing this cusp. Then each of the sections $\theta_{\mathrm{P}}(\Gamma)$ and $\theta_{\mathrm{P}'}(\Gamma)$ also contains the cusp, and hence $\theta_{\mathrm{P}}(\Gamma) = \theta_{\mathrm{P}'}(\Gamma)$. $\square$

2.4. Positivity. Suppose that P is a pseudo-involution of the front $\Sigma \in J^0(M)$ and $x = (q_0, u_0) \in \mathrm{Sw}(\mathrm{P})$. Let us consider the line $\{q = q_1\} \subset J^0(M)$, where the point $q_1 \neq q_0$ is close to q_0. Let z_1 and z_2 be points in $\Sigma \cap l$ close to x. The crossing point x is said to be *positive* (with respect to P) if the following either-or condition holds: either the intervals $[z_1, \mathrm{P}(z_1)]$ and $[z_2, \mathrm{P}(z_2)]$ are disjoint or one is a subset of the other. A non-positive switching point is said to be *negative*. This definition does not depend on the choice of the point q_1 and agrees with the definition in the Introduction for the case $M = \mathbb{R}$.

A pseudo-involution P is said to be *positive* if all its switching crossings are positive.

2.5. Maslov potential. Let $l\colon [0,1] \to L$ be a generic smooth path with ends in the set G_L of non-singular points of the projection $L \to M$. The *Maslov index* $m(l) \in \mathbb{Z}$ of a generic link L is equal to the number of cusps on this path, counted with regard to signs defined as follows. If the point $\sigma(l(t)) \in \sigma(L)$ goes from the lower to the upper branch as it passes through a given cusp of $\sigma(L)$, then the sign of this cusp is $+1$, and if the point goes from the upper to the lower branch, then the sign is -1. If L is non-generic, then the number $m(l)$ is defined as the Maslov index of a nearby path on a nearby generic link (the result does not depend on the perturbation). The *Maslov number* $m(L)$ of a Legendrian knot L is the absolute value of the Maslov index of a closed path that goes around the knot exactly once. The Maslov number $m(L)$ is always even. Indeed, its parity coincides with that of the number of cusps of $\sigma(L)$ (we assume that L is a generic Legendrian knot). Upon a circuit of L, the pre-images of the right cusps ($\succ$) and those of the left cusps ($\prec$) of the front $\sigma(L)$ alternate. Therefore, there are equally many right cusps and left cusps, and their total number is even. The *Maslov number of a Legendrian link* L is the greatest common divisor of the Maslov numbers for the components of L diffeomorphic to a circle.

Let $m \in \mathbb{Z}$ be a non-negative divisor of the number $m(L)$. A *Maslov potential* on a Legendrian link L is a locally constant function $\mu\colon G_L \to \mathbb{Z}/m\mathbb{Z}$ such that the difference $\mu(z_2) - \mu(z_1)$ is equal to the Maslov index (modulo m) of the path connecting z_1 to z_2 for each pair of points $z_1, z_2 \in G_L$ belonging to the same component of L. All Maslov potentials with values in $\mathbb{Z}/m\mathbb{Z}$ are obtained from one such function by adding a function that is constant on each component of L.

2.6. Maslov pseudo-involution. A crossing point x of the front $\sigma(L)$ is said to be a *Maslov point* with respect to a Maslov potential μ on L if $\mu(z) = \mu(z')$, where z and z' are two distinct points of L projected onto x. A decomposition of a σ-generic front is said to be a *Maslov decomposition* with respect to a Maslov potential μ if all its switching crossings are Maslov points with respect to μ. If L is connected, then for a given m a decomposition of $\sigma(L)$ is either Maslov or non-Maslov, independently of the choice of the potential $\mu\colon G_L \to \mathbb{Z}/m\mathbb{Z}$.

A pseudo-involution P of a σ-generic front $\sigma(L)$ is said to be a *Maslov pseudo-involution* with respect to μ if for each pair y, y' of non-singular points of Σ such that $\mathrm{P}(y) = y'$ and y' is above y (that is, its u-coordinate is greater) the pre-image z of y and the pre-image z' of y' in L satisfy the condition $\mu(z') = \mu(z) + 1$. A section $\gamma\colon \Lambda \to \Sigma$ of a front $\Sigma \subset J^0(M)$ is said to be *long* if either $M = S^1$ and Λ is a circle, or $M = I$ and γ takes both the ends of the interval Λ to the boundary $\partial J^0(M)$. The following assertion can be helpful when verifying whether or not a given pseudo-involution is a Maslov pseudo-involution.

Proposition 2.4. *Suppose that μ is a Maslov potential on a σ-generic link $L \subset J^1(M)$ and P is a pseudo-involution of $\Sigma = \sigma(L)$. Then P is a Maslov pseudo-involution with respect to μ if and only if the decomposition D_{P} is a Maslov decomposition with respect to μ and for each long* P*-section there is a pair y, y' of non-singular points of Σ such that one of them belongs to the image of the section,* $\mathrm{P}(y) = y'$*, y' lies above y, and the pre-image z of y and the pre-image z' of y' in L satisfy the condition $\mu(z') = \mu(z) + 1$.*

Proof. Suppose that P is a Maslov pseudo-involution. We claim that each crossing point $x \in \mathrm{Sw}(\mathrm{P})$ is a Maslov point. Consider a point $z \in L$ such that $\mathrm{P}(\sigma(z)) = x$. Let z' and z'' be points in L belonging to distinct small half-neighbourhoods of z. The points $\mathrm{P}(\sigma(z'))$ and $\mathrm{P}(\sigma(z''))$ belong to different smooth branches of Σ intersecting at z. The Maslov potential μ takes the same value at z' and z'', and hence it takes the same values at the pre-images of $\mathrm{P}(\sigma(z'))$ and $\mathrm{P}(\sigma(z''))$ in L. Then μ takes the same values at the pre-images of x, and x is a Maslov point. Therefore, D_{P} is a Maslov decomposition; the second part of the assertion is satisfied automatically.

We prove the converse assertion. Since D_{P} is a Maslov decomposition, the Maslov potential μ takes the same value on the pre-images in L of all non-singular points of Σ belonging to the image of γ for any P-section γ. Thus, to prove that P is a Maslov pseudo-involution, it suffices to show that for each P-section there is a pair y, y' of non-singular points of Σ such that one of them belongs to the image of this section, $\mathrm{P}(y) = y'$, y' lies above y, and the pre-image z of y and the pre-image z' of y' in L satisfy the condition $\mu(z') = \mu(z) + 1$. By assumption, this condition holds for long sections. The image of each short (non-long) P-section γ contains a cusp. Consider a pair of points $y, y' \in \Sigma$ that are close to this cusp and have the same q-coordinates; one of these points belongs to the image of γ. It follows from the definition of Maslov potential that the above condition holds for the pair y, y'. $\square$

2.7. Families of pseudo-involutions and the continuation theorem. A smooth one-parameter family $\{L_{t\in[a,b]}\}$ of Legendrian links (and the family of their fronts) is said to be *σ-generic* if L_a and L_b are σ-generic and the path $t \mapsto L_t$ transversely intersects the hypersurfaces $\mathcal{S}$ and $\mathcal{E}$ at smooth points of them and is disjoint from $\mathcal{S} \cap \mathcal{E}$.

Let us consider a family $\{\mathrm{P}_{t\in[a,b]}\}$ of pseudo-involutions, where P_t is a pseudo-involution of the front $\sigma(L_t)$. The family $\{\mathrm{P}_t\}$ is said to be *continuous* if the map $(t, x) \mapsto (t, \mathrm{P}_t(x))$, defined on the set

$$\{(t, x) | x \in G_{\sigma(L_t)} \cup C_{\sigma(L_t)}\} \subset [a, b] \times J^0(M),$$

is continuous. A family $\{\mathrm{P}_t\}$ of pseudo-involutions is said to be *positive* if the pseudo-involutions P_t are positive for any t such that L_t is σ-generic.

Let D be a decomposition of a σ-generic front Σ. By the *Euler characteristic* of this decomposition we mean the number

$$\chi(D) = \frac{1}{2}\#(C_\Sigma) - \#(\mathrm{Sw}(D)).$$

By the *Euler characteristic* $\chi(\mathrm{P})$ of a pseudo-involution P we mean the Euler characteristic of the decomposition D_{P}. A family $\{\mathrm{P}_t\}$ is said to be *characteristic* if the Euler characteristics $\chi(\mathrm{P}_t)$ are the same for all values of t such that the link L_t is σ-generic.

Let a link L_a be equipped with a Maslov potential μ_a. Then this potential can be uniquely extended to a continuous (in the obvious sense) family $\{\mu_{t\in[a,b]}\}$ of Maslov potentials on links of the family $\{L_{t\in[a,b]}\}$. A family $\{\mathrm{P}_t\}$ of pseudo-involutions is said to be a *Maslov family of pseudo-involutions* with respect to the Maslov potential μ_a (and with respect to the family $\{\mu_t\}$) if P_t is a Maslov pseudo-involution with respect to the potential μ_t for any t such that L_t is σ-generic.

Theorem 2.5 (the continuation theorem). *Suppose that $M = S^1$, $M = \mathbb{R}$, or $M = I$ and that $\{L_{t\in[a,b]}\}$ is a σ-generic family of Legendrian links in $J^1(M)$. Each positive pseudo-involution P_a of the front $\sigma(L_a)$ can be uniquely extended to a continuous positive characteristic family $\{\mathrm{P}_{t\in[a,b]}\}$ of pseudo-involutions.*

If in addition P_a is a Maslov pseudo-involution with respect to a Maslov potential μ_a, then the family $\{\mathrm{P}_t\}$ is a Maslov family with respect to μ_a.

The definitions of continuous, positive, and characteristic families of pseudo-involutions are invariant under a change of direction of the parameter t. This implies the following assertion.

Proposition 2.6. *Under the assumptions of Theorem* 2.5 *the map* $\mathrm{P}_a \mapsto \mathrm{P}_b$ *is a one-to-one correspondence between the sets of positive pseudo-involutions for the fronts* $\sigma(L_a)$ *and* $\sigma(L_b)$, *and this correspondence preserves the Euler characteristics.*

Theorem 0.4 is an immediate corollary to Proposition 2.6.

§ 3. Proof of the continuation theorem

3.1. Fibred diffeomorphisms and pseudo-involutions. Let G_M be the group formed by the diffeomorphisms of $J^0(M)$ fibred over M. A fibred diffeomorphism $g \in \mathrm{G}_M$ defines a one-to-one map of the set of pseudo-involutions of the front $\Sigma \subset J^0(M)$ to the set of pseudo-involutions of the front $g(\Sigma)$ which takes positive pseudo-involutions to positive pseudo-involutions. The following lemma is an immediate corollary to our definitions.

Lemma 3.1. *Let* $\{\Sigma_{t\in[a,b]}\}$ *be a* σ*-generic family of fronts in* $J^0(M)$, *let* $\{g_{t\in[a,b]}\}$ *be a smooth family of elements of the group* G_M, *and let* $g_a = \mathrm{id}$. *In this case* $\{g_t\}$ *defines a one-to-one map from the set of continuous families of pseudo-involutions for* $\{\Sigma_t\}$ *to the set continuous families of pseudo-involutions for* $\{g_t(\Sigma_t)\}$. *This map takes positive families to positive families, characteristic families to characteristic families, and Maslov families with respect to* μ_a *to Maslov families with respect to* μ_a.

If the family $\{L_t\}$ of links in Theorem 2.5 does not intersect the discriminant $\mathcal{D}$, then Lemma 3.1 reduces the assertion of the theorem to the case of a constant family ($L_t \equiv L_a$), for which the assertion obviously holds.

3.2. Typical bifurcations. If $\{L_t\}$ is a σ-generic family of links, $L_c \in \mathcal{D}$, then as t passes through the value c, the fronts $\Sigma_t = \sigma(L_t)$ can be subjected to a bifurcation (up to the action of one-parameter families of fibred diffeomorphisms) as shown in Fig. 5 (the local bifurcations are intersections with $\mathcal{S}$) and Fig. 6 (the non-local bifurcations are intersections with $\mathcal{E}$). Only the components involved in the bifurcation are shown; we note that a bifurcation of type III coincides (up to symmetry) with its inverse III^{-1}, a bifurcation of type XX coincides with XX^{-1}, and the bifurcation of type CC_+ coincides with CC_+^{-1}.

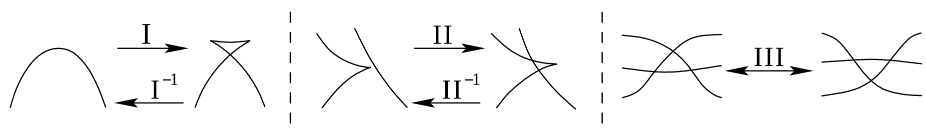


Figure 5

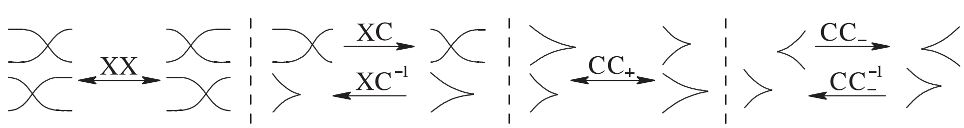


Figure 6

3.3. Simple and semisimple families of fronts. A σ-generic family $\{L_{t\in[a,b]}\}$ of Legendrian links in $J^1(I)$ (and the related family $\{\Sigma_t = \sigma(L_t)\}$ of fronts) is said to be *simple* if the 'boundary' $\partial\Sigma_t = \Sigma_t \cap \partial(J^0(I))$ does not depend on t, the relation $L_t \in \mathcal{D}$ holds for exactly one parameter value $t = c \in\,]a, b[$, and the front Σ_c contains exactly one singular point if the bifurcation is local ($L_c \in \mathcal{S}$) and exactly two singular points with coinciding q-coordinates if the bifurcation is non-local ($L_c \in \mathcal{E}$). A σ-generic family $\{L_{t\in[a,b]}\}$ of Legendrian links in $J^1(M)$ (and the related family $\{\Sigma_t = \sigma(L_t)\}$ of fronts) is said to be *semisimple* with respect to an interval $I \subset M$ if the fronts Σ_t coincide outside $J^0(I)$ for all t and the family $\{\Sigma_t^I = \Sigma_t \cap J^0(I)\}$ of fronts in $J^0(I)$ is simple. By dividing $[a, b]$ into pieces and applying Lemma 3.1, we see that it suffices to prove the assertion of Theorem 2.5 for semisimple families.

Let us show that we can restrict ourselves to the consideration of simple families. Indeed, let a family $\{\Sigma_t\}$ be semisimple with respect to a closed interval I and let P_a be a pseudo-involution of the front Σ_a. In this case every continuous family $\{\mathrm{P}_t^I\}$ of pseudo-involutions for the fronts Σ_t^I such that P_a^I is the restriction of P_a can be uniquely extended to a continuous family $\{\mathrm{P}_t\}$ of pseudo-involutions for the fronts Σ_t, and P_t coincides with P_a outside $J^0(I)$ for any t. Clearly, the family $\{\mathrm{P}_t\}$ is positive (characteristic, a Maslov family) if $\{\mathrm{P}_t^I\}$ is positive (characteristic, a Maslov family, respectively).

Lemma 3.2. *Let $\{\Sigma_{t\in[a,b]}\}$ be a simple family of fronts in $J^0(I)$ and let P_a and P_b be pseudo-involutions of the fronts Σ_a and Σ_b coinciding on $\partial\Sigma_a = \partial\Sigma_b$. Then P_a and P_b can be included into a unique continuous family $\{\mathrm{P}_{t\in[a,b]}\}$ of pseudo-involutions.*

If P_a and P_b are positive, then the family $\{\mathrm{P}_t\}$ is positive. If $\chi(\mathrm{P}_a) = \chi(\mathrm{P}_b)$, then the family $\{\mathrm{P}_t\}$ is characteristic. If $\{\mu_t\}$ is a continuous family of Maslov potentials, P_a is a Maslov pseudo-involution with respect to μ_a, and all switching crossings of the pseudo-involution P_b are Maslov points with respect to μ_b, then $\{\mathrm{P}_t\}$ is a Maslov family.

Proof. By Lemma 3.1, the pseudo-involution P_a can be extended to a unique continuous family $\{\mathrm{P}_t\}$ of pseudo-involutions, where $t \in [a, c[$, and the pseudo-involution P_b can be extended to a unique continuous family $\{\mathrm{P}_t\}$ of pseudo-involutions, where $t \in\,]c, b]$.

We construct the map P_c. Denote by $q_0 \in I$ the projection of the singular points of the front Σ_c and take a point $x \in \Sigma_c$ such that $\pi(x) \neq q_0$. Let us consider a closed interval $I' \subset I \setminus \{q_0\}$ containing an end q' of the interval I and a neighbourhood of $\pi(x)$. There are numbers $a_0 \in [a, c[$, $b_0 \in\,]c, b]$ such that all points of the front $\Sigma_t^{I'}$ are non-singular for $t \in [a_0, b_0]$. Since all pseudo-involutions P_t with $t \in [a_0, b_0] \setminus \{c\}$ act in the same way on $\Sigma_t^{q'} = \pi^{-1}(q') \cap \Sigma_t$, it follows from Lemma 3.1 that there is a unique continuous family $\{\mathrm{P}'_{t\in[a_0,b_0]}\}$ of pseudo-involutions of the fronts $\Sigma_t^{I'}$ such that $\mathrm{P}'_t(y) = \mathrm{P}_t(y)$ for $t \in [a_0, b_0] \setminus \{c\}$ and $y \in \Sigma_t^{I'}$. We set $\mathrm{P}_c(x) = \mathrm{P}'_c(x)$. The map $(t, y) \mapsto (t, \mathrm{P}_t(y))$ is continuous at the point (c, x).

Let x be a non-singular point of Σ_c such that $\pi(x) = q_0$. Since the family $\{\Sigma_t\}$ of fronts is simple, there is a unique continuous family of sections $\Gamma_{t\in[a,b]}$ consisting of non-singular points of Σ_t and such that $x \in \Gamma_c$. The map $(t, y) \mapsto (t, \mathrm{P}_t(y))$, which is already defined for $y \in \Gamma_t$, $t \in [a,b]$, $(t,y) \neq (c,x)$, admits a unique continuous extension to the point (c,x), because the functions $f_t \colon I \to \mathbb{R}$, $t \in [a,b] \setminus \{c\}$, with the sections $\mathrm{P}_t(\Gamma_t)$ as their graphs are uniformly Lipschitzian and converge uniformly outside any given neighbourhood of q_0 as $t \to c$.

If x is a cusp of Σ_c, then we set $\mathrm{P}_c(x) = x$. The continuity of the map $(t,y) \mapsto (t, \mathrm{P}_t(y))$ at (c,x) follows from the fact that every pseudo-involution transposes two branches entering a cusp. This proves the continuity of the family $\{\mathrm{P}_{t\in[a,b]}\}$ and easily implies that P_c is a pseudo-involution of the front Σ_c.

Suppose that P_a is a Maslov pseudo-involution with respect to μ_a, and all switching crossings of P_b are Maslov points with respect to μ_b. For any given long P_b-section there are two points $y, y' \in \partial\Sigma_b$ such that one of them belongs to the image of this section, $\mathrm{P}_b(y) = y'$, and y' is above y. In this case $\mathrm{P}_a(y) = y'$ by the assumption of the lemma. We consider points $z, z' \in \partial L_b$ such that $\sigma(z) = y$ and $\sigma(z') = y'$. Since P_a is a Maslov pseudo-involution, we have $\mu_a(z') = \mu_a(z) + 1$. It follows from the continuity of the family $\{\mu_t\}$ that the restrictions of μ_a and μ_b to $\partial L_b = \partial L_a$ coincide. Thus, $\mu_b(z') = \mu_b(z) + 1$. By Proposition 2.4, P_b is a Maslov pseudo-involution with respect to μ_b. The other assertions of the lemma are now obvious. $\square$

In the remaining part of the proof of the continuation theorem we treat each type of the bifurcations separately.

3.4. Bifurcation I. Suppose that a bifurcation of type I occurs at $t = c$. The crossing x_b of Σ_b is switching for any pseudo-involution of Σ_b. This follows from the fact that a C^1-smooth section of Σ_b connecting the cusps can be mapped by the pseudo-involution P_b only to a section having a break point at x_b. There is exactly one pseudo-involution P_b of Σ_b coinciding with P_a on $\partial\Sigma_b = \partial\Sigma_a$, and $\chi(\mathrm{P}_b) = \chi(\mathrm{P}_a) = 0$. The pseudo-involution P_b is positive, because there are no other points of Σ_b between the two sections of the front Σ_b connecting the cusps. The point x_b is a Maslov point for all Maslov potentials. Applying Lemma 3.2, we complete the proof of the theorem in the case of bifurcations of type I.

3.5. Bifurcation $\mathbf{I^{-1}}$. The crossing of Σ_a is switching for all pseudo-involutions. The subsequent arguments are similar to those used above for a bifurcation of type I.

3.6. Bifurcation II. Let P_b be a positive pseudo-involution of Σ_b coinciding with P_a on $\partial\Sigma_b = \partial\Sigma_a$. Applying the next lemma to Σ_b, we conclude that the pseudo-involution P_b is unique and $\mathrm{Sw}(\mathrm{P}_b) = \varnothing$.

Lemma 3.3. *If a front Σ has the fragment shown in Fig.* 7 *on the right* (*and this fragment is disjoint from the other parts of the front*), *then the points x and y are non-switching for every positive pseudo-involution* P.

Proof. Suppose that y is switching. In this case the condition (PI4) fails at the point x, because every neighbourhood of x contains points of both P-sections intersecting at the cusp z. Thus, y is non-switching. Suppose that x is switching (and y is non-switching). In this case x is a negative point, because $\mathrm{P}(e_1) = e_3$ and the point $\mathrm{P}(e_2)$ does not belong to the interval $[e_1, e_3]$. $\square$

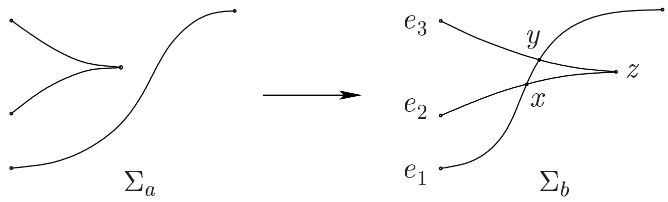


Figure 7

Since $\chi(\mathrm{P}_b) = \chi(\mathrm{P}_a) = 1/2$ and the pseudo-involutions P_a and P_b have no switching crossings, the family of pseudo-involutions with the desired properties is unique by Lemma 3.2.

3.7. Bifurcation $\mathbf{II^{-1}}$. The assertion of the theorem for a bifurcation of type II^{-1} follows as above from Lemma 3.3.

3.8. Bifurcation III. We present the case of bifurcations of type III (see Fig. 8) under four headings III_i, where i stands for the number of switching crossings of the pseudo-involution P_a.

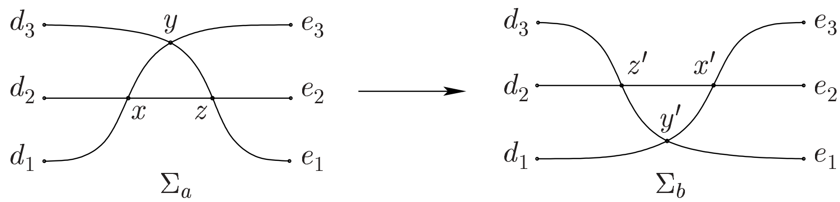


Figure 8

Let us define a one-to-one map $f_a\colon \{d_1, d_2, d_3\} \to \{e_1, e_2, e_3\}$ such that for each $i \in \{1, 2, 3\}$ there is a P_a-section Γ_i connecting d_i to $f_a(d_i)$. Similarly, each decomposition D of the front Σ_b determines a one-to-one map $f_D\colon \{d_1, d_2, d_3\} \to \{e_1, e_2, e_3\}$.

Lemma 3.4. *The map taking a pseudo-involution* P_b *of the front* Σ_b *to a decomposition* D_{P_b} *of* Σ_b *defines a one-to-one correspondence between the set of pseudo-involutions of* Σ_b *coinciding with the pseudo-involution* P_a *on* $\partial\Sigma_a = \partial\Sigma_b$ *and the set of decompositions* D *of* Σ_b *such that* $f_D = f_a$.

The proof of Lemma 3.4 uses the following assertion.

Lemma 3.5. *Each positive pseudo-involution* P_a *transposes no pair of points of the form* d_i, e_i, $i \in \{1, 2, 3\}$.

Proof. Suppose that $\mathrm{P}_a(d_1) = d_2$. Then the condition (PI4) fails at x. If $\mathrm{P}_a(d_2) = d_3$, then x is non-switching (otherwise the condition (PI4) fails at y). Then y is switching (otherwise the condition (PI4) fails at z). Thus, y is a negative switching point, a contradiction. Suppose that $\mathrm{P}_a(d_1) = d_3$. Then x is switching (otherwise the condition (PI4) fails at y). We arrive at a contradiction, because x is negative. The points e_i cannot be permuted for the same reasons. $\square$

Proof of Lemma 3.4. We claim that $f_{D_{\mathrm{P}_b}}(d_i) = f_a(d_i)$ for any $i \in \{1, 2, 3\}$. According to Lemma 3.5, there is a section Γ of Σ_a formed by non-singular points and such that the image $\mathrm{P}_a(\Gamma)$ contains both d_i and $f_a(d_i)$. The section Γ' of Σ_b having the same ends as the section Γ is taken by the pseudo-involution P_b to a P_b-section connecting d_i to $f_a(d_i)$. Thus, the map of Lemma 3.4 is well defined.

Let D be a decomposition of Σ_b. There is exactly one free involution $\theta\colon D \to D$ such that the left ends of the sections $\theta(\Gamma)$ and $\theta_{\mathrm{P}_a}(\Gamma')$ coincide if and only if the left ends of the given sections $\Gamma \in D$ and $\Gamma' \in D_{\mathrm{P}_a}$ coincide. It follows from Lemma 3.5 that the sections $\theta(\Gamma)$ and Γ are disjoint for each $\Gamma \in D$. According to Proposition 2.2, there is exactly one pseudo-involution P_b of Σ_b coinciding with P_a over the left end of I and satisfying $D = D_{P_b}$. If $f_D = f_a$, then these pseudo-involutions coincide over the right end of I as well, and hence the map in Lemma 3.4 is invertible. □

3.9. Bifurcation $\mathbf{III_0}$. It can readily be seen that there is exactly one decomposition D of the front Σ_b such that $f_D = f_a$, namely, the decomposition for which the crossings z', y', and x' (see Fig. 8) are non-switching. By Lemma 3.4, this decomposition determines a pseudo-involution P_b, and we have $\chi(\mathrm{P}_b) = \chi(\mathrm{P}_a) = 0$. Since the pseudo-involution P_b has no switching crossings, it is positive (and it is a Maslov pseudo-involution if P_a is). The assertion of the theorem for the bifurcation in question follows now from Lemma 3.2.

3.10. Bifurcation $\mathbf{III_1}$. Let $\mathrm{Sw}(\mathrm{P}_a) = \{x\}$. A decomposition D of Σ_b such that $f_D = f_a$ is unique, and x' is the only switching crossing of D (it follows from the relation $f_D(d_3) = e_1$ that the points z' and y' are non-switching). By Lemma 3.4, there is a unique pseudo-involution P_b coinciding with P_a on $\partial\Sigma_b = \partial\Sigma_a$; moreover, $\mathrm{Sw}(\mathrm{P}_b) = \{x'\}$ and $\chi(\mathrm{P}_b) = \chi(\mathrm{P}_a) = -1$. Since the point x is positive, it follows from Lemma 3.5 (see also Lemma 3.6 below) that x' is positive. The crossing x' is a Maslov point, because x is a Maslov point. The assertion of the theorem in the case of $\mathrm{Sw}(\mathrm{P}_a) = \{x\}$ follows from Lemma 3.2. The case $\mathrm{Sw}(\mathrm{P}_a) = \{z\}$ is completely similar.

Let $\mathrm{Sw}(\mathrm{P}_a) = \{y\}$. Then there are exactly two decompositions D of Σ_b satisfying the condition $f_D = f_a$. All the crossings z', y', and x' are switching for the first decomposition, in which case $-3 = \chi(\mathrm{P}_b) \neq \chi(\mathrm{P}_a) = -1$. For the second decomposition we have $\mathrm{Sw}(D) = \{y'\}$, and the argument is the same as in the case $\mathrm{Sw}(\mathrm{P}_a) = \{x\}$ above.

3.11. Lemmas on signs. We formulate and prove some auxiliary assertions needed below. Let P be a pseudo-involution of a σ-generic front $\Sigma \subset J^0(M)$ and let d_1 and d_2 be two different non-singular points of the front such that $\pi(d_1) = \pi(d_2)$ and $\mathrm{P}(d_1) \neq d_2$. We introduce the sign $\varepsilon_{\mathrm{P}}(d_1, d_2)$ as follows: let $\varepsilon_{\mathrm{P}}(d_1, d_2) = -1$ if the intersection of the intervals $[d_1, \mathrm{P}(d_1)]$ and $[d_2, \mathrm{P}(d_2)]$ is non-empty and differs from each of them; otherwise let $\varepsilon_{\mathrm{P}}(d_1, d_2) = +1$. Let $x \in \mathrm{Sw}(\mathrm{P})$. We write $\varepsilon_{\mathrm{P}}(x) = +1$ if x is a positive crossing of the pseudo-involution P, and $\varepsilon_{\mathrm{P}}(x) = -1$ otherwise (a crossing of this type is said to be negative). Thus, the sign of a switching crossing coincides with the sign of a pair of non-singular points close to it. The following assertion is obvious.

Lemma 3.6. *Let* P *be a pseudo-involution of a σ-generic front $\Sigma \subset J^0(I)$ and let Γ_1 and Γ_2 be two distinct* P*-sections of Σ not transposed by θ_{P}. Let $d_1, e_1 \in \Gamma_1$ and $d_2, e_2 \in \Gamma_2$ be non-singular points of Σ such that $\pi(d_1) = \pi(d_2)$ and $\pi(e_1) = \pi(e_2)$, and let x be the only crossing point of Σ whose projection on I lies between $\pi(d_1)$ and $\pi(e_1)$. Then $\varepsilon_{\mathrm{P}}(d_1, d_2) = -\varepsilon_{\mathrm{P}}(e_1, e_2)$ if and only if the following two conditions hold*: (1) *the crossing x is non-switching for* P; (2) *two of the sections Γ_1, Γ_2, $\theta_{\mathrm{P}}(\Gamma_1)$, $\theta_{\mathrm{P}}(\Gamma_2)$ meet at x.*

Lemma 3.7. *Suppose that* P *is a pseudo-involution of the front $\Sigma \subset J^0(M)$, $q \in M$, and the set $\Sigma^q = \pi^{-1}(q) \cap \Sigma = \{h_1, \dots, h_{2N}\}$ consists of non-singular points of Σ indexed in ascending order of u-coordinates. If $\varepsilon_{\mathrm{P}}(h_{k-1}, h_k) = \varepsilon_{\mathrm{P}}(h_k, h_{k+1}) = +1$, then $\varepsilon_{\mathrm{P}}(h_{k-1}, h_{k+1}) = +1$.*

Proof. We equip Σ^q with an order relation defined by the rule $h_{k+2} \prec \cdots \prec h_{2N} \prec h_1 \prec \cdots \prec h_{k+1}$. It follows from the condition $\varepsilon_{\mathrm{P}}(h_{k-1}, h_k) = +1$ that $\mathrm{P}(h_{k-1}) \neq h_{k+1}$. Therefore, if $k-1 \le i < j \le k+1$, then $\varepsilon_{\mathrm{P}}(h_i, h_j) = +1$ if and only if $\mathrm{P}(h_j) \prec \mathrm{P}(h_i)$. It follows from the relations $\mathrm{P}(h_k) \prec \mathrm{P}(h_{k-1})$ and $\mathrm{P}(h_{k+1}) \prec \mathrm{P}(h_k)$ that $\mathrm{P}(h_{k+1}) \prec \mathrm{P}(h_{k-1})$. □

3.12. Bifurcation III$_2$. Let $\mathrm{Sw}(\mathrm{P}_a) = \{x, y\}$. The only decomposition D of Σ_b such that $f_D = f_a$ is determined by the rule $\mathrm{Sw}(D) = \{z', x'\}$. By Lemma 3.4, there is a unique pseudo-involution P_b coinciding with P_a on $\partial\Sigma_b = \partial\Sigma_a$, and we have $\mathrm{Sw}(\mathrm{P}_b) = \{z', x'\}$ and $\chi(\mathrm{P}_b) = \chi(\mathrm{P}_a) = -2$.

We prove that the crossings z' and x' of the pseudo-involution P_b corresponding to the decomposition D are positive. The sign $\varepsilon_{\mathrm{P}_b}(z')$ is equal to the sign $\varepsilon_{\mathrm{P}_b}(d_2, d_3) = \varepsilon_{\mathrm{P}_a}(d_2, d_3)$. By Lemma 3.6, $\varepsilon_{\mathrm{P}_a}(d_2, d_3) = \varepsilon_{\mathrm{P}_a}(y) = +1$. Thus, $\varepsilon_{\mathrm{P}_b}(z') = \varepsilon_{\mathrm{P}_a}(y) = +1$. Applying Lemma 3.6 twice, we see that $\varepsilon_{\mathrm{P}_b}(x') = \varepsilon_{\mathrm{P}_b}(d_1, d_3) = \varepsilon_{\mathrm{P}_a}(d_1, d_3)$. The sign $\varepsilon_{\mathrm{P}_a}(d_1, d_2)$ is equal to the sign $\varepsilon_{\mathrm{P}_a}(x) = +1$. It follows from Lemma 3.7 that $\varepsilon_{\mathrm{P}_b}(d_1, d_3) = +1$. Hence, x' is positive. If P_a is a Maslov pseudo-involution, then μ_a takes the same value on the pre-images of all branches shown in Fig. 8, because the crossings x and y are Maslov points. Thus, P_b is a Maslov pseudo-involution. The assertion of the theorem in the case $\mathrm{Sw}(\mathrm{P}_a) = \{x, y\}$ is proved. The case $\mathrm{Sw}(\mathrm{P}_a) = \{y, z\}$ can be treated in the same way.

Let us consider the case in which $\mathrm{Sw}(\mathrm{P}_a) = \{x, z\}$. There are exactly two decompositions D_1 and D_2 of Σ_b such that $f_{D_1} = f_{D_2} = f_a$; moreover, $\mathrm{Sw}(D_1) = \{z', y'\}$ and $\mathrm{Sw}(D_2) = \{y', x'\}$. We claim that exactly one of the two pseudo-involutions $\mathrm{P}_{\langle i \rangle}$ corresponding to the decompositions D_i is positive. Since $\varepsilon_{\mathrm{P}_a}(x) = +1$, it follows that $\varepsilon_{\mathrm{P}_a}(d_1, d_2) = +1$. By Lemma 3.6, the relation $\varepsilon_{\mathrm{P}_a}(z) = +1$ implies that $\varepsilon_{\mathrm{P}_a}(d_1, d_3) = +1$. Since $\varepsilon_{\mathrm{P}_a}(d_1, d_2) = +1$, it follows from Lemma 3.6 that $\varepsilon_{\mathrm{P}_{\langle 1 \rangle}}(y') = +1$. Since $\varepsilon_{\mathrm{P}_a}(d_1, d_3) = +1$, it follows from Lemma 3.6 that $\varepsilon_{\mathrm{P}_{\langle 2 \rangle}}(y') = +1$. Moreover, $\varepsilon_{\mathrm{P}_{\langle 1 \rangle}}(z') = \varepsilon_{\mathrm{P}_a}(d_2, d_3)$ and we have $\varepsilon_{\mathrm{P}_{\langle 2 \rangle}}(x') = -\varepsilon_{\mathrm{P}_a}(d_2, d_3)$ by Lemma 3.6. Thus, exactly one of the pseudo-involutions $\mathrm{P}_{\langle 1 \rangle}$, and $\mathrm{P}_{\langle 2 \rangle}$ is positive: $\mathrm{P}_{\langle 1 \rangle}$ for $\varepsilon_{\mathrm{P}_a}(d_2, d_3) = +1$ and $\mathrm{P}_{\langle 2 \rangle}$ for $\varepsilon_{\mathrm{P}_a}(d_2, d_3) = -1$. We have the equality $\chi(\mathrm{P}_{\langle 1 \rangle}) = \chi(\mathrm{P}_{\langle 2 \rangle}) = \chi(\mathrm{P}_a) = -2$. The proof of the Maslov property is the same as in the case $\mathrm{Sw}(\mathrm{P}_a) = \{x, y\}$.

3.13. Bifurcation III$_3$. In this case $\mathrm{Sw}(\mathrm{P}_a) = \{x, y, z\}$. There are exactly two decompositions D of Σ_b such that $f_D = f_a$. There is a case in which $\mathrm{Sw}(\mathrm{P}_b) = \{y'\}$ and $\chi(\mathrm{P}_b) \neq \chi(\mathrm{P}_a)$. In the other case $\mathrm{Sw}(\mathrm{P}_b) = \{z', y', x'\}$. Applying Lemma 3.6, we can readily see that the switching points z', y', and x' are positive if the switching points x, y, and z are positive. The rest of the proof is the same as in the case III_2.

3.14. Bifurcations XC, $\mathbf{CC_{\pm}}$, $\mathbf{XC^{-1}}$, $\mathbf{CC_{-}^{-1}}$. The assertion of the theorem for these bifurcations is obvious.

3.15. Bifurcation XX. We break up the case of this bifurcation (see Fig. 9) into three sub-cases XX_i, where the index i stands for the number of switching crossings of P_a. Extending the pseudo-involution by continuity, we obtain the following assertion.

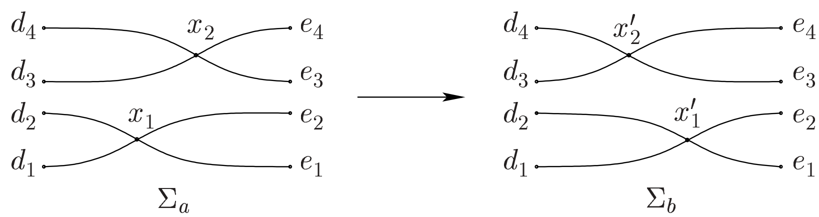


Figure 9

Lemma 3.8. *In the case of a bifurcation of type* XX, *for each pseudo-involution* P_a *there is a continuous family* $\{\mathrm{P}^{\mathrm{S}}_{t\in[a,b]}\}$ *of pseudo-involutions such that the switching crossings depend smoothly on the parameter* t *and* $\mathrm{P}^{\mathrm{S}}_a = \mathrm{P}_a$. *The family is a Maslov family if* P_a *is a Maslov pseudo-involution.*

3.16. Bifurcation $\mathbf{XX_0}$. It follows from the condition $\chi(\mathrm{P}_b) = \chi(\mathrm{P}_a)$ that P_b has no switching crossings. The existence of the desired family of pseudo-involutions follows from Lemma 3.8, and the uniqueness of the family follows from Proposition 2.3 and Lemma 3.2.

3.17. Bifurcation $\mathbf{XX_1}$. We denote by Γ_i the P_a-section of Σ_a containing the point d_i, $i \in \{1,2,3,4\}$. We note that $\theta(\Gamma_1) \neq \Gamma_2$, where $\theta = \theta_{\mathrm{P}_a}$.

Let $\mathrm{Sw}(\mathrm{P}_a) = \{x_1\}$. Suppose that $\{\theta(\Gamma_1), \theta(\Gamma_2)\} \neq \{\Gamma_3, \Gamma_4\}$. Let P' be a pseudo-involution of Σ_b coinciding with P_a on $\partial\Sigma_b = \partial\Sigma_a$. We claim that $\mathrm{Sw}(\mathrm{P}') = \{x_1'\}$. Let $x_1' \notin \mathrm{Sw}(\mathrm{P}')$. There is an index $i \in \{1,2\}$ such that $\theta(\Gamma_i) = \Gamma_5 \notin \{\Gamma_3, \Gamma_4\}$. In this case $\mathrm{P}_a(d_i) = d_5$ and $\mathrm{P}_a(e_i) = e_5$, where d_5 and e_5 are the ends of the P_a-section Γ_5. Consider the P'-section Γ_5' with ends d_5 and e_5 and the P'-sections Γ_i' with ends d_i and e_{3-i}. It follows from the relation $\mathrm{P}_a(d_5) = d_i \in \Gamma_i'$ that $\theta_{\mathrm{P}'}(\Gamma_5') = \Gamma_i'$. On the other hand, $\mathrm{P}_a(e_5) = e_i \notin \Gamma_i'$, a contradiction. Suppose that $\mathrm{Sw}(\mathrm{P}') = \{x_1', x_2'\}$. There is an $i \in \{3,4\}$ such that $\theta(\Gamma_i) = \Gamma_5 \notin \{\Gamma_1, \Gamma_2\}$. Consider the P'-section Γ_5' with ends d_5 and e_5 and the P'-sections Γ_i' with ends d_i and e_i. It follows from the relation $\mathrm{P}_a(d_5) = d_i \in \Gamma_i'$ that $\theta_{\mathrm{P}'}(\Gamma_5') = \Gamma_i'$. On the other hand, $\mathrm{P}_a(e_5) = e_{7-i} \notin \Gamma_i'$, a contradiction.

It follows now from Lemma 3.8 that if $\{\theta(\Gamma_1), \theta(\Gamma_2)\} \neq \{\Gamma_3, \Gamma_4\}$, then there is a unique pseudo-involution P_b coinciding with P_a on $\partial\Sigma_b = \partial\Sigma_a$; moreover, $\mathrm{Sw}(\mathrm{P}_b) = \{x_1'\}$. The pseudo-involution P_b is positive, because $\varepsilon_{\mathrm{P}_b}(x_1') = \varepsilon_{\mathrm{P}_b}(e_1, e_2) = \varepsilon_{\mathrm{P}_a}(x_1) = +1$ by Lemma 3.6. The assertion of the theorem follows in this case from Lemma 3.8 and Lemma 3.2.

Suppose that $\{\theta(\Gamma_1), \theta(\Gamma_2)\} = \{\Gamma_3, \Gamma_4\}$. Then it is easy to see that there are exactly two pseudo-involutions P and P′ of Σ_b coinciding with P_a on $\partial\Sigma_a = \partial\Sigma_b$; moreover, $\mathrm{Sw}(\mathrm{P}) = \{x_1'\}$ and $\mathrm{Sw}(\mathrm{P}') = \{x_2'\}$. We claim that P′ is positive and P is not. Since x_1 is positive for P_a, we see that $\mathrm{P}(d_1) = \mathrm{P}'(d_1) = \mathrm{P}_a(d_1) = d_4$, $\mathrm{P}(d_2) = \mathrm{P}'(d_2) = \mathrm{P}_a(d_2) = d_3$, $\mathrm{P}(e_1) = e_3$, and $\mathrm{P}(e_2) = e_4$. Thus, $\varepsilon_{\mathrm{P}}(x_1') = \varepsilon_{\mathrm{P}}(e_1, e_2) = -1$ and $\varepsilon_{\mathrm{P}'}(x_2') = \varepsilon_{\mathrm{P}}(d_3, d_4) = +1$.

If P_a is a Maslov pseudo-involution with respect to μ_a, then $\mu_a(d_4) = \mu_a(d_1)+1$, $\mu_a(d_3) = \mu_a(d_2) + 1$, and $\mu_a(d_1) = \mu_a(d_2)$. Therefore, $\mu_a(d_3) = \mu_a(d_4)$, and hence $\mu_b(d_3) = \mu_b(d_4)$, and P′ is a Maslov pseudo-involution. Applying Lemma 3.2, we complete the proof for any bifurcation of type XX_1 (the case $\mathrm{Sw}(\mathrm{P}_a) = \{x_2\}$ can be reduced to the case $\mathrm{Sw}(\mathrm{P}_a) = \{x_1\}$ by symmetry).

3.18. Bifurcation $\mathbf{XX_2}$. The condition $\chi(\mathrm{P}_b) = \chi(\mathrm{P}_a)$ implies that both the crossings of the front Σ_b are switching points. A continuous family $\{\mathrm{P}_t\}$ of pseudo-involutions with two switching crossings exists by Lemma 3.8 and is unique by Proposition 2.3 and Lemma 3.2. It remains to show that P_b is positive. It follows from $\varepsilon_{\mathrm{P}_a}(x_1) = \varepsilon_{\mathrm{P}_a}(x_2) = +1$ and from Lemma 3.6 that $\varepsilon_{\mathrm{P}_a}(e_1, e_2) = \varepsilon_{\mathrm{P}_a}(d_3, d_4) = +1$. The pseudo-involution P_b is positive, because $\varepsilon_{\mathrm{P}_b}(x_1') = \varepsilon_{\mathrm{P}_b}(e_3, e_4) = \varepsilon_{\mathrm{P}_a}(e_3, e_4) = +1$ and $\varepsilon_{\mathrm{P}_b}(x_2') = \varepsilon_{\mathrm{P}_b}(d_1, d_2) = \varepsilon_{\mathrm{P}_a}(d_1, d_2) = +1$. This completes the proof of Theorem 2.5.

§ 4. Non-uniqueness of a continuous extension and the monodromy of pseudo-involutions

4.1. Non-characteristic continuations. It turns out that if one omits the assumption that the family $\{\mathrm{P}_t\}$ is characteristic under the conditions of Theorem 2.5, then the uniqueness of a continuation can fail; however, as before there is an explicit description for all possible extensions of a pseudo-involution P_a to a continuous positive (not necessarily characteristic) family $\{\mathrm{P}_{t\in[a,b]}\}$ of pseudo-involutions. There are at most two continuations of this kind if the σ-generic family $\{L_t\}$ intersects the discriminant $\mathcal{D}$ exactly once. A description of the continuations can be extracted from the proof of Theorem 2.5; here we present the description in a more explicit form. We assume that the family $\{L_t\}$ is simple.

4.2. Unique continuations. If the bifurcation has one of the forms I, II, XC, and $\mathrm{CC}_\pm$ (or one of their inverses; see 3.1), then the corresponding continuous positive family $\{\mathrm{P}_t\}$ is unique. The same holds if the bifurcation is of type III_0, III_2 (see 3.9 and 3.12), or XX_1 (see 3.17).

4.3. Non-unique continuations. If the bifurcation is of type III_3, then there are always exactly two continuous positive families $\{\mathrm{P}_t\}$: one is characteristic, and for the other the number of switching crossings is less by 2. In the notation of Fig. 8, the switching crossings x, y, and z are transformed into the switching crossing y'.

Suppose that the bifurcation is of type III_1. If either x or z is the switching crossing of P_a, then a continuation is unique. Let y be the switching crossing. Suppose that there is a pseudo-involution P_a' that coincides with P_a on $\partial\Sigma_a$ and has the points x, y, and z as switching crossings. Then the pseudo-involution

P_a admits two continuations. One is indicated in Theorem 2.5 and the other can be constructed from the family $\{\mathrm{P}'_t\}$ extending P'_a by Theorem 2.5. If such a pseudo-involution P'_a does not exist, then a continuation is unique.

Suppose that the bifurcation is of type XX_0 or XX_2. A continuation of P_a is non-unique if and only if $\mathrm{P}_a(d_1) = d_4$ and $\mathrm{P}_a(d_2) = d_3$ (see Fig. 9). Then there are exactly two continuations: for one of them the crossings x'_1 and x'_2 are switching and for the other both the crossings are non-switching.

4.4. Characteristic continuations and monodromy. Let $\{L_{t\in[0,1]}\}$ be a σ-generic path (a σ-generic family) in the space of Legendrian links in $J^1(M)$. We deform the path $\{L_t\}$ into another σ-generic path $\{L'_t\}$ in the class of paths in $\mathcal{L}$ having fixed ends. These paths define one-to-one maps Φ and Φ' of the set of positive pseudo-involutions of the front $\sigma(L_0)$ to the set of positive pseudo-involutions of the front $\sigma(L_1)$. It turns out that the maps Φ and Φ' need not coincide.

One can readily see that if the deformation involves only σ-generic paths, then $\Phi = \Phi'$. Moreover, one can show (using the uniqueness of the continuation of a pseudo-involution along a path) that $\Phi = \Phi'$ if each Legendrian link involved in the deformation is either a σ-generic link or a non-singular point of the discriminant $\mathcal{D}$. Therefore, the non-uniqueness of the continuation is determined by the operations of going around codimension-two strata (in $\mathcal{L}$) of the discriminant $\mathcal{D}$. The proof of Theorem 2.5 gives us an explicit description of a continuation of a pseudo-involution. Using this description, we can describe one-to-one maps (monodromy transformations) of the set of positive pseudo-involutions determined by the operations of going around diverse codimension-two strata of the discriminant. A complete description is tedious, and we state only selected results.

There are only two strata for which the monodromy transformation can be non-trivial. We describe these strata and list some properties of the monodromy around them. We introduce the stratum $\mathcal{D}_{2,3}$ consisting of the Legendrian links L such that the front $\sigma(L)$ has a triple point x and a double point x' with the condition $\pi(x) = \pi(x')$, and is σ-generic in all other respects. Let $\{L_{t\in[0,1]}\}$ be a small σ-generic loop going around the stratum $\mathcal{D}_{2,3}$ near L exactly once. One can show that the square of the monodromy map corresponding to a circuit around this stratum is the identity transformation. The positive pseudo-involutions which are not preserved by this monodromy map are contained in the set of pseudo-involutions P of $\sigma(L_0)$ such that exactly two points of the set $\mathrm{Sw}(\mathrm{P})$ are close to the points x and x'.

By definition, the stratum $\mathcal{D}_4$ consists of the Legendrian links L such that the front $\sigma(L)$ has a quadruple point and is σ-generic in other respects. Let $\{L_{t\in[0,1]}\}$ be a small σ-generic loop going around the stratum $\mathcal{D}_4$ near L exactly once. We denote by Φ the monodromy transformation of the set of positive pseudo-involutions of $\sigma(L_0)$ and by A_i the set of positive pseudo-involutions P of $\sigma(L_0)$ such that exactly i points of the set $\mathrm{Sw}(\mathrm{P})$ are close to the quadruple point of $\sigma(L)$. The map Φ preserves each of the sets A_i. One can show that the restriction of Φ to A_i is the identity transformation for $i \notin \{3,4\}$ and both the cubed restriction of Φ to A_4 and the squared restriction of Φ to A_3 are the identity transformations.

We illustrate these phenomena by the following examples.

4.5. First example. We consider the Legendrian knot $L \in \mathcal{D}_{2,3}$ whose front is shown in Fig. 10 a. Suppose that a loop $\{L_{t\in[0,1]}\}$ goes around the stratum $\mathcal{D}_{2,3}$ in a neighbourhood of L exactly once and the front $\sigma(L_0)$ is as shown in Fig. 10 b. We assume for simplicity that the family $\{L_t\}$ intersects the stratum $\mathcal{D}_3$ of fronts with a single triple point and the stratum $\mathcal{D}_{2,2}$ of fronts with two double points with the same q-coordinates a minimal number of times, namely, at 2 and 6 points, respectively. The front $\sigma(L_0)$ admits exactly three positive pseudo-involutions P^0, P^1, and P^2. They are determined by the conditions $\mathrm{Sw}(\mathrm{P}^0) = \{x_2, x_4\}$, $\mathrm{Sw}(\mathrm{P}^1) = \{x_3, x_4\}$, and $\mathrm{Sw}(\mathrm{P}^2) = \{x_1, x_2, x_3, x_4\}$. Using the explicit description of the continuation of the pseudo-involutions $\mathrm{P}_0 \in \{\mathrm{P}^0, \mathrm{P}^1, \mathrm{P}^2\}$ given in the proof of Theorem 2.5, one can show that the monodromy around the loop $\{L_t\}$ transposes the pseudo-involutions P^0 and P^1.

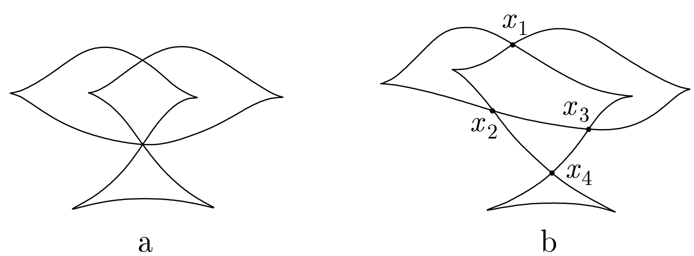


Figure 10

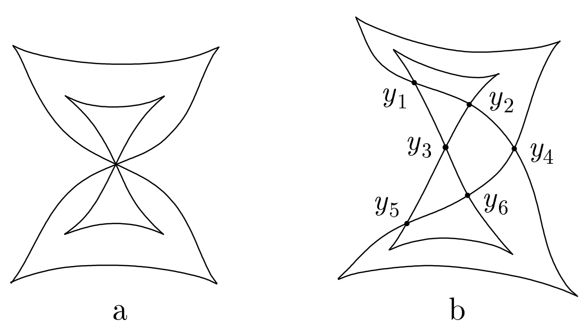


Figure 11

4.6. Second example. We consider the Legendrian link $L \in \mathcal{D}_4$ whose front is shown in Fig. 11 a. Suppose that $\{L_t\}$ is a loop going around the stratum $\mathcal{D}_4$ in a neighbourhood of L exactly once and the front $\sigma(L_0)$ is as shown in Fig. 11 b. We assume for simplicity that the family $\{L_t\}$ intersects the strata $\mathcal{D}_3$ and $\mathcal{D}_{2,2}$ at a minimal number of points, namely, 8 and 6 points, respectively. The front $\sigma(L_0)$ admits exactly five positive pseudo-involutions P^0, P^1, P^2, P^3, P^4. They are determined by the conditions $\mathrm{Sw}(\mathrm{P}^0) = \{y_3, y_4\}$, $\mathrm{Sw}(\mathrm{P}^1) = \{y_3, y_4, y_5, y_6\}$, $\mathrm{Sw}(\mathrm{P}^2) = \{y_1, y_2, y_3, y_4\}$, $\mathrm{Sw}(\mathrm{P}^3) = \{y_1, y_2, y_5, y_6\}$, and $\mathrm{Sw}(\mathrm{P}^4) = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. Using the explicit description of the continuation for $\mathrm{P}_0 \in \{\mathrm{P}^1, \mathrm{P}^2, \mathrm{P}^3\}$ given in the proof of Theorem 2.5, one can show that going around

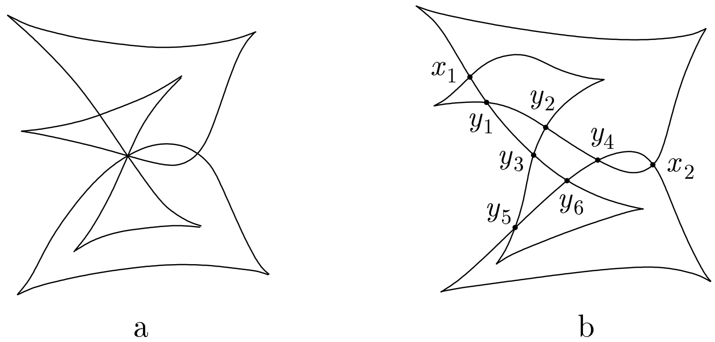


Figure 12

the loop $\{L_t\}$ permutes these three pseudo-involutions cyclically (and the pseudo-involutions P^0 and P^4 are taken to themselves, because the number of switching crossings is preserved by the monodromy map due to the characteristic property).

4.7. Third example. We consider the Legendrian link $L \in \mathcal{D}_4$ whose front is shown in Fig. 12 a. Suppose that $\{L_t\}$ is a loop going around the stratum $\mathcal{D}_4$ in a neighbourhood of L exactly once and the front $\sigma(L_0)$ is as shown in Fig. 12 b. The front $\sigma(L_0)$ admits exactly five positive pseudo-involutions P^0, P^1, P^2, P^3, P^4, determined by $\mathrm{Sw}(\mathrm{P}^0) = \{y_2, y_3\}$, $\mathrm{Sw}(\mathrm{P}^1) = \{x_2, y_2, y_5, y_6\}$, $\mathrm{Sw}(\mathrm{P}^2) = \{x_2, y_2, y_3, y_4\}$, $\mathrm{Sw}(\mathrm{P}^3) = \{y_2, y_3, y_5, y_6\}$, and $\mathrm{Sw}(\mathrm{P}^4) = \{x_2, y_2, y_3, y_4, y_5, y_6\}$. Using the explicit description of the continuation for $\mathrm{P}_0 \in \{\mathrm{P}^1, \mathrm{P}^2\}$ given in the proof of Theorem 2.5, one can show that these two pseudo-involutions are transposed under a circuit around the loop $\{L_t\}$ (and the pseudo-involutions P^0, P^3, and P^4 are preserved by this operation).

We note that the link L belongs to the connected component of the space of Legendrian links containing the link in the previous example. This readily implies that every permutation of the pseudo-involutions P^1, P^2, P^3 of the front $\sigma(L_0)$ can be realized by a monodromy transformation along some σ-generic loop.

§ 5. Extension of pseudo-involutions to the discriminant

5.1. Let k be an integer. We consider the set $\mathcal{L}_k^\sigma$ of all σ-generic Legendrian links in $J^1(M)$ whose fronts admit exactly one positive pseudo-involution, with Euler characteristic equal to k. Let $\mathcal{L}'_k$ be the closure of $\mathcal{L}_k^\sigma$ in the space $\mathcal{L}$ of Legendrian links. It follows from Proposition 2.6 that $\mathcal{L}'_k$ is a union of some connected components of $\mathcal{L}$.

Let us consider the map $\mathcal{P}_0$ taking a Legendrian link $L \in \mathcal{L}_k^\sigma$ to the positive pseudo-involution $\mathcal{P}_0(L)$ of $\sigma(L)$ such that $\chi(\mathcal{P}_0(L)) = k$. Our objective is to extend the map $\mathcal{P}_0$ to a continuous map $\mathcal{P}$ defined on $\mathcal{L}'_k$ and taking a Legendrian link to a pseudo-involution of the front of the link. The continuity condition here is understood as follows. Let us consider the set $A_{\mathcal{P}} = \{(x, L) | x \in G_{\sigma(L)} \cup C_{\sigma(L)}\} \subset J^0(M) \times \mathcal{L}'_k$ and the map $\Phi_{\mathcal{P}} \colon A_{\mathcal{P}} \to J^0(M)$ given by the rule $\Phi_{\mathcal{P}}(x, L) = \mathcal{P}(L)(x)$. We equip the set $A_{\mathcal{P}}$ with the topology induced by the product of the natural topology on $J^0(M)$ and the C^∞-topology on $\mathcal{L}'_k$. The map $\mathcal{P}$ is said to be *continuous* if $\Phi_{\mathcal{P}}$ is continuous.

Theorem 5.1. *The map $\mathcal{P}_0$ can be uniquely extended to a continuous map $\mathcal{P}$ taking each link $L \in \mathcal{L}'_k$ to a pseudo-involution of $\sigma(L)$.*

5.2. Proof of Theorem 5.1. If x is a cusp of the front $\Sigma = \sigma(L)$, then we set $\mathcal{P}(L)(x) = x$. If x is a non-singular point of Σ, then we consider a sequence (L_i) of Legendrian links in $\mathcal{L}^\sigma_k$ and a sequence of points $x_i \in \sigma(L_i)$ such that $L_i \to L$ and $x_i \to x$. Let $\mathcal{P}(L)(x) = \lim_{i\to\infty} \mathcal{P}_0(L_i)(x_i)$. We claim that this definition is correct and the map $\mathcal{P}$ has the desired properties.

Let Γ be a set of non-singular points of $\Sigma = \sigma(L)$ that is the graph of a smooth function defined on a closed interval $I \subset M$. We denote by $W_\varepsilon(\Gamma)$ the ε-neighbourhood of Γ in $J^0(I)$. Let $\varepsilon > 0$ be small enough that the neighbourhood $W_\varepsilon(\Gamma)$ does not contain points of Σ outside Γ. A Legendrian link L' is said to be Γ-*convenient* if the intersection of its front with $W_\varepsilon(\Gamma)$ consists of non-singular points of the front and is the graph of a smooth function on I. Suppose that $L' \in \mathcal{L}^\sigma_k$ is a σ-generic Γ-convenient link and $\mathrm{P}' = \mathcal{P}_0(L')$ is a pseudo-involution of $\sigma(L')$. In this case the set $\mathrm{P}'(\sigma(L') \cap W_\varepsilon(\Gamma))$ is the graph of a continuous function on I. We denote this function by $f_{L'}$.

Lemma 5.2. *For each $\delta > 0$ there is a neighbourhood U of a Legendrian link L in $\mathcal{L}'_k$ consisting of Γ-convenient links and such that $\|f_{L_1} - f_{L_2}\|_{C^0} < \delta$ for all $L_1, L_2 \in \mathcal{L}^\sigma_k \cap U$.*

We first prove the following auxiliary assertion.

Lemma 5.3. *Let $L_0 \subset J^1(M)$ be a Legendrian link and let T^{L_0} be the set consisting of all points $q_0 \in M$ such that the line $\{q = q_0\}$ intersects the front $\sigma(L_0)$ only at non-singular points. Then T^{L_0} is open and dense in M.*

Proof. Obviously, T^{L_0} is open in M. By Sard's theorem, the set of critical values of the projection $L_0 \to M$ is closed and of measure zero. We claim that every closed interval $K \subset M$ consisting of non-singular values of this projection contains a point of the set T^{L_0}. Indeed, the set $L_0^K = L_0 \cap J^1(K)$ is a union of finitely many 1-graphs of smooth functions $f_1, \dots, f_m$ on K. Since the graphs of these functions cannot be tangent (because L_0 is embedded), it follows that any two graphs of this kind can meet at only finitely many points. Therefore, almost all points of K belong to T^{L_0}. Thus, T^{L_0} is dense in M. $\square$

Proof of Lemma 5.2. Every sufficiently small neighbourhood of L in $\mathcal{L}'_k$ consists of Γ-convenient links. We claim that for any $\delta_0 > 0$ and any point $s \in T^{L_0} \cap I$ there is a neighbourhood U_{s,δ_0} of L in $\mathcal{L}'_k$ such that $|f_{L_1}(s) - f_{L_2}(s)| < \delta_0$ for any $L_1, L_2 \in \mathcal{L}^\sigma_k \cap U_{s,\delta_0}$. Indeed, we choose first a connected neighbourhood U_{s,δ_0} such that s belongs to T^{L_0} for every L_0 in this neighbourhood. Suppose that $L_0 \in \mathcal{L}^\sigma_k \cap U_{s,\delta_0}$. We denote by $x_1^{L_0}, x_2^{L_0}, \dots, x_{2n}^{L_0}$ the points of the set $\sigma(L_0) \cap \{q = q_0\}$ enumerated in ascending order of u-coordinates. The pseudo-involution $\mathcal{P}_0(L_0)$ defines an involution θ of the set $\{1, \dots, 2m\}$ by the formula $\mathcal{P}_0(L_0)(x_j^{L_0}) = x_{\theta(j)}^{L_0}$. The involutions θ corresponding to different links $L_0 \in \mathcal{L}^\sigma_k \cap U_{s,\delta_0}$ coincide. This follows from Theorem 2.5 and the fact that any two links of this kind can always be connected by a σ-generic family of Legendrian links in U_{s,δ_0}. Since these involutions coincide, there is an index $j \in \{1, \dots, 2m\}$ such that $f_{L_0}(s)$ is equal to the u-coordinate of the point $x_j^{L_0}$ for each $L_0 \in \mathcal{L}^\sigma_k \cap U_{s,\delta_0}$. Thus, by reducing the neighbourhood U_{s,δ_0}, we can achieve the validity of the inequality $|f_{L_1}(s) - f_{L_2}(s)| < \delta_0$.

One can find a constant $C > 0$ and a neighbourhood U_C of the Legendrian link L in $\mathcal{L}'_k$ such that the absolute value of the p-coordinate of each element $L' \in U_C$ does not exceed C. In this case, $|f_{L_0}(s_1) - f_{L_0}(s_2)| \leqslant C|s_1 - s_2|$ for any $L_0 \in \mathcal{L}^\sigma_k \cap U_C$ and $s_1, s_2 \in I$. There are positive numbers ε_0 and δ_0 such that $\delta_0 + 2C\varepsilon_0 < \delta$. For any point $s \in T^{L_0} \cap I$ the inequality $|f_{L_1}(s_1) - f_{L_2}(s_1)| < \delta$ holds for any s_1 in the ε_0-neighbourhood of s in I and for any Legendrian links $L_1, L_2 \in \mathcal{L}^\sigma_k \cap U_{s,\delta_0}$. By Lemma 5.3, the set $T^{L_0} \cap I$ is dense in I, and hence it contains points $s_1, \dots, s_l$ whose ε_0-neighbourhoods cover the closed interval I. The desired neighbourhood U is the intersection of the neighbourhoods U_{s_i,δ_0}.

Let us return to the proof of Theorem 5.1. We claim that the above definition of the map $\mathcal{P}(L)\colon G_{\sigma(L)} \cup C_{\sigma(L)} \to \sigma(L)$ is correct. Suppose that x is a non-singular point of the front $\sigma(L)$, (L_i) is a sequence of Legendrian links in $\mathcal{L}^\sigma_k$, $x_i \in \sigma(L_i)$, $L_i \to L$, and $x_i \to x$. Consider a set Γ of non-singular points of $\sigma(L)$ that contains the point x and is the graph of a smooth function defined on some closed interval $I \subset M$. By Lemma 5.2, the sequence of functions $f_{L_i}\colon I \to \mathbb{R}$ converges in the space C^0 to some function f. Thus, the sequence $\mathcal{P}_0(L_i)(x_i)$ converges to $(\pi(x), f(\pi(x)))$ (where π stands for the projection $J^0(M) \to M$). Therefore, the map $\mathcal{P}$ is well defined.

The map $\mathcal{P}(L)$ satisfies the conditions imposed on a pseudo-involution. This follows by continuity considerations, because the maps $\mathcal{P}_0(L')$ are pseudo-involutions for any σ-generic Legendrian links L' close to L. We claim that the map $\Phi_{\mathcal{P}}$ is continuous at the point (x, L), where $x \in G_{\sigma(L)}$. By Lemma 5.2, it suffices to show that the point $\mathcal{P}(L')(x')$ is sufficiently close to the point $\mathcal{P}(L)(x)$ if the link $L' \in \mathcal{L}^\sigma_k$ is sufficiently close to L and x' is a non-singular point of $\sigma(L')$ sufficiently close to x. In turn, these conditions follow from Lemma 5.2. Finally, the continuity of the map $\Phi_{\mathcal{P}}$ at the point (x, L), where x is a cusp, follows from the fact that there is a small neighbourhood V of x in $J^0(M)$ such that the intersection of the front of L' with V can be obtained from $V \cap \sigma(L)$ by a C^∞-small diffeomorphism for any Legendrian link L' sufficiently close to L. The proof of Theorem 5.1 is complete. $\square$

§ 6. Combinatorics of decompositions of fronts

6.1. Resolution. Starting from a decomposition $D = \{\gamma_1, \dots, \gamma_N\}$ of a σ-generic front $\Sigma = \sigma(L) \subset J^0(M)$ (where $M = S^1$, $M = \mathbb{R}$, or $M = I$), we shall construct a topological manifold $R(D)$ (with boundary if $M = I$), the so-called *resolution* (the *D-resolution*) of Σ. The resolution $R(D)$ is obtained from $\bigcup_{i=1}^N \Lambda_i$ (where $\gamma_i\colon \Lambda_i \to \Sigma$ and the closed intervals and circles Λ_i are pairwise disjoint) by gluing together two ends $y \in \Lambda_j$ and $y' \in \Lambda_{j'}$ such that $\gamma_j(y) = \gamma_{j'}(y') = c$ for each cusp c of Σ. The maps γ_i define a natural continuous projection $\psi_D\colon R(D) \to \Sigma$ under which each crossing point of Σ has two pre-images and each non-crossing point has one pre-image.

6.2. Tree-like decompositions. Let $L \subset J^1(S^1)$ be a σ-generic Legendrian link and let D be a decomposition of its front $\Sigma \subset J^0(S^1)$. We denote by $\mathrm{Sw}_L(D) \subset L$ the pre-image of the set $\mathrm{Sw}(D)$ under the action of $\sigma|_L$.

To a decomposition D we assign a graph K_D as follows. The vertices of K_D correspond to the connected components of the space $R(D)$. For each point $x \in \mathrm{Sw}(D)$ we construct an edge of K_D joining the vertices (possibly coinciding) that correspond to the components containing the pre-images of x under the map ψ_D. A decomposition D is said to be *tree-like* if K_D is a tree. We say that a connected component S of the space $R(D)$ is a *k-component* if $\psi_D(S)$ contains exactly k cusps. A decomposition D is said to be *tame* if L is connected, $\chi(D) = 0$ (for the definition of the Euler characteristic $\chi(D)$, see 2.7), each connected component of $R(D)$ is either a 2-component or a 0-component, and there is at least one 0-component.

Lemma 6.1. *If D is a tame decomposition, then D is tree-like and the resolution space $R(D)$ has exactly one 0-component.*

Proof. To each oriented interval $J \subset L$ whose ends lie outside the set $\mathrm{Sw}_L(D)$ one can assign a path on the graph K_D, that is, a finite sequence of edges such that the beginning of each edge coincides with the end of the previous edge. This path is constructed as follows. The points of the set $J \cap \mathrm{Sw}_L(D)$ are ordered by using the orientation on J. Passing to the images of these points in Σ, we obtain a sequence of points in $\mathrm{Sw}(D)$. Corresponding to this sequence is a sequence of edges in K_D. Using this construction, one can readily see that the connectedness of the knot L implies the connectedness of the graph K_D. The condition $\chi(D) = 0$ means that the number of edges of K_D is equal to the number of 2-components of the space $R(D)$. Thus, the Euler characteristic $\chi(K_D)$ is equal to the number of 0-components, and therefore $\chi(K_D) \geq 1$. Since K_D is connected, it follows that $\chi(K_D) \leq 1$. Therefore, K_D is a tree and the resolution $R(D)$ has exactly one 0-component. $\square$

Suppose that a decomposition D is tree-like. If $\mathrm{Sw}(D)$ is non-empty, then the knot L is divided by the points of $\mathrm{Sw}_L(D)$ into pieces homeomorphic to a closed interval. Let S be a component of the space $R(D)$. We denote by $S_L \subset L$ the union of the pieces that are projected on $\psi_D(S) \subset \Sigma$ (if $\mathrm{Sw}(D)$ is empty and $S = R(D)$, then we set $S_L = L$). We say that a component S is *cyclic* if for each pair of points $y_1, y_2 \in \partial S_L$ with $\sigma(y_1) = \sigma(y_2)$ there is a closed interval $J \subset L$ with ends y_1 and y_2 such that $J \cap S_L = \{y_1, y_2\}$ (or, informally speaking, L is transformed into S by contracting each of these intervals to a point).

Lemma 6.2. *Suppose that $L \subset J^1(S^1)$ is a σ-generic Legendrian link and D is a tree-like decomposition of its front $\Sigma \subset J^0(S^1)$. Then L is connected and every component of the space $R(D)$ is cyclic. Moreover, for each switching point $x \in \mathrm{Sw}(D)$ and each component $S^* \subset R(D)$ the set S^*_L is entirely contained in one of the two closed intervals into which L is divided by the pre-images of x.*

Proof. We proceed by induction on the number $k = \#(\mathrm{Sw}(D))$. For $k = 0$ the assertion is obvious. Suppose that the assertion is proved for all tree-like decompositions D_0 such that $\#(\mathrm{Sw}(D_0)) = k - 1$. Let D be a tree-like decomposition with $\#(\mathrm{Sw}(D)) = k > 0$. Then the tree K_D has at least two univalent vertices. Therefore, there is a component $S \subset R(D)$ such that the set $\psi_D(S)$ contains exactly one switching crossing of the decomposition D. We denote this point by x_S and the pre-images of x_S in L by z_1 and z_2.

Let us consider a decomposition D_0 of Σ such that $\mathrm{Sw}(D_0) = \mathrm{Sw}(D) \setminus \{x_S\}$. Since K_D is obtained from K_{D_0} by contracting one of the edges into a point, D_0 is tree-like. The assertion of the lemma for the decomposition D follows from the assertion for D_0, and the assertion for D_0 holds by the induction assumption. Indeed, this results from the following description of the sets S_L, where S ranges over the components of $R(D)$: one of these sets is an interval $J_S \subset L$ with ends at the points z_1 and z_2 that contains no other points of the set $\mathrm{Sw}_L(D)$, and another is $S'_L \setminus J_S$, where S' is the component of $R(D_0)$ containing the pre-images of the point x_S; the remaining sets are of the form S''_L, where S'' is a connected component of the space $R(D_0)$. $\square$

6.3. Tame decompositions and intervals in a Legendrian link. Let $\gamma\colon \Lambda \to \Sigma$ be a section of the front Σ and let y be an interior point of the closed interval Λ. By the *half-sections* (with respect to the point y) we mean the restrictions of γ to the closed intervals into which the point y divides the interval Λ. By the *right half-section* (*left half-section*) we mean the map whose graph ends at the right (left) cusp of the front Σ.

Lemma 6.3. *Let D be a tame decomposition of a σ-generic Legendrian knot $L \subset J^1(S^1)$ and let S^0 be a 0-component of the resolution $R(D)$. Then each point $x \in \mathrm{Sw}(D)$ uniquely determines the following objects.*

(1) *A closed interval $J^*_x(D) \subset L$ with ends at the pre-images of x and containing no interior points of S^0_L.*

(2) *A section $\gamma_x\colon \Lambda \to \sigma(L)$ of D, where Λ is a closed interval, such that the image of γ_x contains x and is contained in the set $\sigma(J^*_x(D))$ (denote by γ^r_x (γ^l_x) the right (left) half-section of γ_x with respect to the pre-image of x in Λ).*

(3) *Disjoint closed intervals $J^r_x(D), J^l_x(D) \subset J^*_x(D)$ whose ends are taken by σ to the ends of the half-sections γ^r_x and γ^l_x, respectively.*

Let $J^*_x(D)$ and $J^*_{x'}(D)$ be any two distinct closed intervals of the above form. Either these intervals are disjoint or one of them is a subset of the interior of the other. The intervals $J^*_x(D)$ corresponding to distinct switching crossings x contained in the section $\psi_D(S^0)$ are disjoint, and L is the union of all these intervals and the set S^0_L. The sections γ_x corresponding to different switching crossings are different, and every cusp is an end of exactly one of these sections. The points of the intervals $J^r_x(D)$ and $J^l_x(D)$ that are sufficiently close to the ends of these intervals are taken by σ to points of the half-sections γ^r_x and γ^l_x, respectively.

Proof. The decomposition D is tree-like by Lemma 6.1. By Lemma 6.2, the set S^0_L is contained in one of the two closed intervals into which L is divided by the pre-images of x. We denote this interval by $I^*_x(D)$, and the other by $J^*_x(D)$. The interval $J^*_x(D) \subset L$ contains no interior points of S^0_L by definition. The ends of the interval $J^*_{x'}(D) \subset L$, where $x' \neq x$, belong to the set S'_L, where S' is some component of the space $R(D)$. It follows from Lemma 6.2 that these ends both belong to one of the intervals $I^*_x(D)$ and $J^*_x(D)$. Thus, we consider the following possibilities: the closed intervals $J^*_x(D)$ and $J^*_{x'}(D)$ are disjoint, or one of them is a subset of the interior of the other, or their union is L. However, the last case is impossible, since the interior points of the non-empty set S^0_L belong to neither of these intervals.

Let x be a point of the set $\mathrm{Sw}_0(D)$ of switching points belonging to the section $\psi_D(S^0)$. Then the interval $J_x^*(D) \subset L$ is the closure of a connected component of the complement to S_L^0 in L (because the ends of the interval belong to S_L^0). Thus, the intervals $J_x^*(D) \subset L$ corresponding to different points $x \in \mathrm{Sw}_0(D)$ are disjoint, and L is the union of all such intervals and the set S_L^0.

Since D is tree-like, each crossing point $x \in \mathrm{Sw}(D)$ belongs to the images of two different sections; denote these sections by γ_1 and γ_2. Let S^1 and S^2 be components of $R(D)$ such that the image of γ_i is contained in $\psi_D(S^i)$. By Lemma 6.2, each of the sets S_L^i is contained in one of the intervals $I_x^*(D)$ and $J_x^*(D)$. Each pre-image of x in L has one small half-neighbourhood contained in S_L^1 and the other in S_L^2. Thus, there is exactly one index $i \in \{1,2\}$ such that the closed interval $J_x^*(D)$ contains S_L^i. We write $\gamma_x = \gamma_i$ and $S^x = S^i$. Then the image of γ_x is a subset of $\sigma(J_x^*(D))$. It follows from the relation $S_L^i \subset J_x^*(D)$ that $S^i \neq S^0$. Therefore, S^i is a 2-component, and thus the domain of γ_x is a closed interval.

We claim that the sections γ_x and $\gamma_{x'}$ corresponding to different crossings are different. Suppose the contrary. Then $S^x = S^{x'}$, and the intersection of the intervals $J_x^*(D)$ and $J_{x'}^*(D)$ contains S_L^x, hence is non-empty. Thus, one of these intervals is a subset of the interior of the other. However, this contradicts the fact that the ends of both the intervals belong to S_L^x. Therefore, $\gamma_x \neq \gamma_{x'}$ and $S^x \neq S^{x'}$.

Since the decomposition D is tame, the number of switching crossings of D is equal to the number of 2-components of $R(D)$. Hence, the map $x \mapsto S^x$ defines a one-to-one correspondence between the switching crossings and the 2-components. Thus, for each cusp of the front $\sigma(L)$ there is exactly one switching crossing x such that the pre-image in $R(D)$ of the cusp belongs to the component S^x, and so x is the only switching crossing for which the cusp is an end of the section γ_x.

We denote by Z_L^r (by Z_L^l) the closure of the subset of L consisting of all points z such that $\sigma(z)$ is a non-singular point of $\sigma(L)$ belonging to the image of the half-section γ_x^r (γ_x^l, respectively). Suppose that a crossing point $x' \in \mathrm{Sw}(D)$ belongs to the image of the half-section γ_x^r and denote the set of all such crossings by Sw_x^r. Let $J_x^r(D)$ be the union of the set Z_L^r and all intervals $J_{x'}^*$ with $x' \in \mathrm{Sw}_x^r$. The ends of the interval $J_{x'}^*$ belong to the set Z_L^r, and its interior points do not belong to S_L^x by Lemma 6.2. The boundary points of Z_L^r are exactly the ends of the intervals $J_{x'}^*$ with $x' \in \mathrm{Sw}_x^r$ and the points mapped by the projection σ to the ends of the half-section γ_x^r. Hence, the set $J_x^r(D)$ is a closed interval, and the assertions of the lemma about its ends and their neighbourhoods are valid. The interval $J_x^r(D)$ is a subset of $J_x^*(D)$, because $Z_L^r \subset J_x^*(D)$ and each of the intervals $J_{x'}^*$ with $x' \in \mathrm{Sw}_x^r$ is a subset of $J_x^*(D)$ by Lemma 6.2 (since the ends of $J_{x'}^*$ are interior points of $J_x^*(D)$). One can similarly define the closed interval $J_x^l(D)$ and verify its properties. The intervals $J_x^r(D)$ and $J_x^l(D)$ are disjoint, because otherwise the cusp at which the right half-section ends would also belong to the left half-section, which is impossible. $\square$

§ 7. Hurwitz theorems for fronts

7.1. Generalized curvature map. Let $\lambda \in \mathbb{R}$. We consider the 1-forms $\omega = dq$ and $\beta_\lambda = dp + \lambda u\, dq$ on $J^1(S^1)$. For a given Legendrian link $L \subset J^1(S^1)$ we define the map $\mathrm{Curv}_{L,\lambda}\colon L \to \mathbb{R}\mathrm{P}^1 = \mathbb{R} \cup \{\infty\}$ by setting $\mathrm{Curv}_{L,\lambda}(z) = [\omega|_L(z) : \beta_\lambda|_L(z)]$

(that is, $\operatorname{Curv}_{L,\lambda}(z) = \infty$ for $\beta_\lambda|_L(z) = 0$ and $\omega|_L(z) = \operatorname{Curv}_{L,\lambda}(z)\beta_\lambda|_L(z)$ for $\beta_\lambda|_L(z) \neq 0$). This definition is correct (that is, the 1-forms $\beta_\lambda|_L(z), \omega|_L(z) \in T^*_z L$ do not vanish simultaneously), because the 1-forms $\alpha = du - p\,dq$, β_λ, and ω are linearly independent at each point. For $\lambda = 1$ this definition coincides with the definition of the map Curv_L given in 0.2. We define a map $F_{L,\lambda}\colon L \to \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ as follows. Let $F_{L,\lambda}(y) = \infty$ for $y \notin G_L$. In a neighbourhood of the point $y \in G_L$ the link L coincides with the 1-graph $j^1 f$ of a function $f\colon U \to \mathbb{R}$, where $U \subset S^1$. We then set $F_{L,\lambda}(y) = f''(q_y) + \lambda f(q_y)$, where q_y stands for the q-coordinate of y. Arguing as in the proof of Lemma 1.1, one can show that $F_{L,\lambda} = 1/\operatorname{Curv}_{L,\lambda}$.

7.2. Sturm points and Arnol'd points. A point $z \in L$ is called a *Sturm λ-point* if $F_{L,\lambda}(z) = 0$. We call critical points of the map $F_{L,\lambda}\colon L \to \mathbb{RP}^1$ *Arnol'd λ-points.* For each $\lambda \in \mathbb{R}$ the Sturm λ-points belong to G_L, because G_L consists of all points $z \in L$ such that $\omega|_L(z) \neq 0$. A point $z = (p_0, q_0, u_0) \in L$ is called a *critical point* of L if $p_0 = 0$.

7.3. Continuous sections of fronts. Let $\mathcal{L}_1$ be the connected component of the space of Legendrian links in $J^1(S^1)$ that contains the zero section of the vector bundle $J^1(S^1) \to S^1$. Let us consider the space $\mathcal{L}_1^+$ of Legendrian links that are obtained from elements $L \in \mathcal{L}_1$ by adding a component $V_c = \{p = 0, u = c\}$ for which the value of u on L is less than c.

The canonical projection $\mathcal{L}_1^+ \to \mathcal{L}_1$ is a fibration with contractible fibre. The component $\mathcal{L}_1^+$ contains the link $V_0 \cup V_1$, whose front admits exactly one pseudo-involution P_0, which is positive. According to Proposition 2.6, the front of a σ-generic link in $\mathcal{L}_1^+$ admits exactly one positive pseudo-involution, and the Euler characteristic of this pseudo-involution vanishes. Consider the map $L^+ \mapsto \mathcal{P}(L^+)$ in Theorem 5.1 that takes a link L^+ to a pseudo-involution of the front $\sigma(L^+)$. We define the map $\mathcal{H}\colon \mathcal{L}_1 \to C^0(S^1)$ as follows. If $L \in \mathcal{L}_1$, then we take an $L^+ = L \cup V_c \subset \mathcal{L}_1^+$ and define $\mathcal{H}(L)$ as the function whose graph is $\mathcal{P}(L^+)(V_c)$. One can readily see that $\mathcal{H}$ is well defined (that is, it does not depend on c). The following assertion results from Theorem 5.1.

Proposition 7.1. *The map $\mathcal{H}\colon \mathcal{L}_1 \to C^0(S^1)$ continuously assigns to any Legendrian link L a function $\mathcal{H}(L)$ whose graph $\Gamma_{\mathcal{H}(L)}$ is a subset of the front $\sigma(L)$.*

7.4. Canonical Maslov potential. Let us consider the cover $\mathcal{M}_1$ over $\mathcal{L}_1$ whose fibre over $L \in \mathcal{L}_1$ consists of integer-valued Maslov potentials on L.

Lemma 7.2. *The cover $\mathcal{M}_1$ admits a unique continuous section $L \mapsto \mu_L$ such that $\mu_{V_0} = 0$. For any generic knot $L \in \mathcal{L}_1$ the Maslov potential μ_L vanishes on the pre-images of non-singular points of $\sigma(L)$ belonging to $\Gamma_{\mathcal{H}(L)}$. For each pre-image $L^+ \in \mathcal{L}_1^+$ of L the pseudo-involution $\mathcal{P}(L^+)$ is a Maslov pseudo-involution with respect to the potential which coincides with μ_L on L and is equal to* 1 *on the complementary component.*

Proof. Let μ be a Maslov potential on $V_0 \cup V_1$ such that $\mu(V_0) = 0$ and $\mu(V_1) = 1$. To prove the first assertion of the lemma, it suffices to show that each Maslov potential μ' obtained from μ by a continuous extension along a loop in $\mathcal{L}_1^+$ coincides with μ.

But this follows from Theorem 2.5 and the relation $\mu'(V_1)=1$. The other assertions follow immediately from Theorem 2.5. □

7.5. Three generalizations of the Hurwitz theorem. Let $L\in\mathcal{L}_1$. We denote by $N_\lambda^{\mathrm{S}}(L)$ the set of Sturm λ-points on L at which the Maslov potential μ_L vanishes. We denote by $N_\lambda^{\mathrm{A}}(L)$ the set of Arnol'd λ-points on L.

The inner product of functions on S^1 is defined by the L^2-pairing, $\langle f,g\rangle=\int_{S^1}f(q)g(q)\,dq$. The following three theorems generalize the Hurwitz theorem and reduce to it in the case when L is the 1-graph of a smooth function on S^1.

Theorem 7.3. *Let $L\in\mathcal{L}_1$ be a Legendrian knot for which the function $\mathcal{H}(L)$ is orthogonal to each of the $2k+1$ functions $1,\cos q,\sin q,\dots,\cos kq,\sin kq$. Then $\#\bigl(N_\lambda^{\mathrm{S}}(L)\bigr)\geqslant 2k+2$ for any $\lambda\in\mathbb{R}$ and $\#\bigl(N_\lambda^{\mathrm{S}}(L)\bigr)\geqslant 2k+4$ for $\lambda=(k+1)^2$.*

Theorem 7.4. *Let $L\in\mathcal{L}_1$ be a Legendrian knot for which the function $\mathcal{H}(L)$ is orthogonal to each of the $2k$ functions $\cos q,\sin q,\dots,\cos kq,\sin kq$. Then the inequality $\#\bigl(N_\lambda^{\mathrm{A}}(L)\bigr)\geqslant 2k+2$ holds for any $\lambda\in\mathbb{R}$, and $\#\bigl(N_\lambda^{\mathrm{A}}(L)\bigr)\geqslant 2k+4$ for $\lambda=(k+1)^2$.*

Theorem 7.5. *Let $L\in\mathcal{L}_1$ be a Legendrian knot for which the function $\mathcal{H}(L)$ is orthogonal to each of the $2k$ functions $\cos q,\sin q,\dots,\cos kq,\sin kq$. Then L has at least $2k+2$ critical points.*

7.6. Remark on the zeros of higher derivatives. For any smooth function on the circle its nth derivative has at least two zeros, for any positive integer n. One can ask whether analogues of this assertion hold for Legendrian knots in the component $\mathcal{L}_1$.

Let $L\in J^1(S^1)$ be a Legendrian link. We define a function $h_{n,L}\colon G_L\to\mathbb{R}$. Suppose that $y\in G_L$ and f is a smooth function whose 1-graph coincides with L in a neighbourhood of y. We set $h_{n,L}(y)=f^{(n)}(q_y)$, where q_y is the q-coordinate of y, and we denote by $Q_n(L)$ the set of zeros of $h_{n,L}$. Let $\overline{Q}_n(L)$ be the set of points $y\in L$ such that there is a Legendrian link L' arbitrarily C^∞-close to L which contains a point $y'\in Q_n(L')$ close to the point y. Then $Q_n(L)\subset\overline{Q}_n(L)$ for each n, the set $\overline{Q}_1(L)$ is formed by the critical points of L, $Q_2(L)=\overline{Q}_2(L)$ is the set of Sturm 0-points of L, and $\overline{Q}_3(L)$ is the set of Arnol'd 0-points of L.

It follows from the theorems stated above that each of the sets $\overline{Q}_1(L)$, $\overline{Q}_2(L)$, and $\overline{Q}_3(L)$ contains at least two points for any $L\in\mathcal{L}_1$. One can readily see that the set $\overline{Q}_n(L)$ also contains at least two points for each odd $n\geq 3$ and each $L\in\mathcal{L}_1$. (Let us sketch the proof for a σ-generic L which is not a 1-graph. It follows from the relation $m(L)=0$ that there are at least two closed intervals $J\subset L$ such that their ends (but not their interior points) are projected on cusps of $\sigma(L)$ and the curve $\sigma(J)$ either enters both cusps from above or enters both from below. In this case the function $h_{n,L}$ tends to $+\infty$ when approaching one end of J and to $-\infty$ when approaching the other end. Thus, J contains a point of $Q_n(L)$.) However, there is an example of a σ-generic Legendrian knot $L\in\mathcal{L}_1$ (whose front has two cusps and one crossing point) such that the set $\overline{Q}_4(L)$ is empty. Similar examples can probably be constructed for all even $n>4$.

§ 8. Proof of generalizations of the Hurwitz theorem

8.1. A Legendrian link $L \subset J^1(S^1)$ is said to be λ-*generic* if it is σ-generic, the images of Sturm λ-points are non-singular points of $\sigma(L)$, and the differential of the map $F_{L,\lambda}$ is non-zero at each Sturm λ-point. We first prove the assertion of Theorem 7.3 for a λ-generic knot L.

We take a Legendrian link $L^+ = L \cup V_c \in \mathcal{L}_1^+$ mapped to the knot $L \in \mathcal{L}_1$ under the projection $\mathcal{L}_1^+ \to \mathcal{L}_1$. Let $\mathrm{P} = \mathcal{P}(L^+)$ be a positive pseudo-involution of $\sigma(L^+)$. The graph Γ_{H_L} of the function $H_L = \mathcal{H}(L)$ coincides with the set $\mathrm{P}(V_c) \subset \Sigma = \sigma(L)$. We denote by Y the projection of the set $\mathrm{Sw}(\mathrm{P}) \cap \Gamma_{H_L}$ on S^1. The piecewise smooth function H_L is smooth outside Y, and its left and right derivatives are different at each point of Y. Let $\mathcal{G}_Y$ be the set of distributions (elements of the space dual to $C^\infty(S^1)$) of the form $\varphi = \sum_{s\in Y} b_\varphi^s \delta_s + g_\varphi$, where δ_s is the Dirac delta function supported at s, b_φ^s is a non-zero number, and g_φ is a smooth function on $S^1 \setminus Y$ with regular zeros that has non-zero left and right limits at each point $s \in Y$. (More precisely, the result of applying the distribution φ to $f \in C^\infty(S^1)$ is equal to $\sum_{s\in Y} b_\varphi^s f(s) + \int_{S^1\setminus Y} f(q)g(q)\,dq$.)

8.2. Changes of sign. Let us define the sign of a distribution $\varphi \in \mathcal{G}_Y$ at a point $q \in S^1$ as the sign of the number b_φ^q for $q \in Y$ and as the sign of $g_\varphi(q)$ for $q \notin Y$. We say that a function $\varphi \in \mathcal{G}_Y$ *changes sign from the right* (*from the left*) at $s \in Y$ if the sign of b_φ^s differs from the sign of the right (left) limit of g_φ at the point s. Thus, φ can change its sign at a point $s \in Y$ twice. We say that φ *changes sign at* $q \in S^1 \setminus Y$ if g_φ changes sign at q.

If a knot L is λ-generic, then applying the differential operator $D_\lambda = \frac{d^2}{dq^2} + \lambda$ to the function H_L gives a distribution in $\mathcal{G}_Y$, and we have $g_{D_\lambda H_L}(q) = {H_L}''(q) + \lambda H_L(q)$ for $q \in S^1 \setminus Y$. Arguing as in the proof of Lemma 1.1, one can readily show that if $D_\lambda H_L(q_0) = 0$ for $q_0 \notin Y$, then $F_{L,\lambda}(z_0) = 0$, where $z_0 \in L$ is such that $\sigma(z_0) = (q_0, H_L(q_0))$. It follows from Lemma 7.2 that $z_0 \in N_\lambda^{\mathrm{S}}(L)$.

8.3. Hurwitz theorem for distributions.

Lemma 8.1. *If a distribution* $\varphi \in \mathcal{G}_Y$ *vanishes at the functions* $1, \cos q, \sin q, \dots,$ $\cos lq$, $\sin lq$, *then it changes sign at least* $2l+2$ *times.*

Proof. Let Y_i, $i \in \{1,2\}$, be the set of points at which $D_\lambda H_L$ changes its sign exactly i times. Then $Y_2 \subset Y$. Suppose that

$$2l' = \#(Y_1) + 2\#(Y_2) < 2l + 2.$$

By multiplying appropriate trigonometric polynomials of degree 1, we can construct a trigonometric polynomial Q of degree l' which vanishes exactly on the set $Y_1 \cup Y_2$, changes sign at the points of Y_1, and does not change sign at the points of Y_2. Reversing the sign of Q if necessary, we can assume that $D_\lambda H_L$ and Q have the same sign at each point of the set $S^1 \setminus (Y_1 \cup Y_2)$. We claim that

$$\langle D_\lambda H_L, Q\rangle = \sum_{s\in Y} b^s_{D_\lambda H_L} Q(s) + \int_{S^1\setminus Y} Q(q) g_{D_\lambda H_L}(q)\,dq > 0.$$

Indeed, it follows from the construction of Q that the second summand on the right-hand side is positive and the first summand is equal to

$$\sum_{s\in Y\setminus(Y_1\cup Y_2)} b^s_{D_\lambda H_L} Q(s) \geqslant 0.$$

Hence, $D_\lambda H_L$ does not annihilate Q, a contradiction. $\square$

8.4. Application of Lemma 8.1. The space T_k of trigonometric polynomials of degree at most k is invariant under the action of the differential operator D_λ. Since D_λ is selfadjoint and H_L is orthogonal to T_k, we conclude that $D_\lambda H_L$ vanishes on functions in T_k. If $\lambda = (k+1)^2$, then the functions $\sin(k+1)q$ and $\cos(k+1)q$ belong to the kernel of D_λ, and hence $D_\lambda H_L$ vanishes on these functions. Therefore, under the assumptions of Theorem 7.3, the distribution $D_\lambda H_L$ (for a λ-generic L) always has at least $2k+2$ changes of sign, and at least $2k+4$ changes of sign if $\lambda = (k+1)^2$.

For each change of sign of the distribution $D_\lambda H_L$ occurring at some point of Y we shall find a point of the set $N^{\mathrm{S}}_\lambda(L)$. The Sturm λ-points found in this way will be pairwise different and will differ from the points corresponding to the zeros of $g_{D_\lambda H_L}$. This will complete the proof of Theorem 7.3 in the case of a λ-generic knot.

8.5. Pseudo-involutions and tame decompositions. Let $L^+ = L \cup V_c$ be a pre-image of L in $\mathcal{L}_1^+$ under the projection $\mathcal{L}_1^+ \to \mathcal{L}_1$. The pseudo-involution $\mathrm{P}^+ = \mathcal{P}(L^+)$ defines a decomposition D_{P^+} of the front $\sigma(L) \cup \sigma(V_c)$. Let D_L denote the decomposition of the front $\Sigma = \sigma(L)$ obtained from D_{P^+} by removing the section $\sigma(V_c)$. The decomposition D_L does not depend on the choice of L^+. The definition of a tame decomposition was given in 6.2.

Lemma 8.2. *The decomposition D_L is tame.*

Proof. If P is a pseudo-involution of a σ-generic front $\Sigma' \subset J^0(S^1)$, then each component of the resolution space $R(D_{\mathrm{P}})$ is either a 2-component or a 0-component. This holds because if two sections of D_{P} have the same right ends, then their left ends also coincide. Therefore, each component of $R(D_L)$ is either a 2-component or a 0-component. The space $R(D_L)$ has a 0-component which projects to Γ_{H_L}. It follows from Theorem 2.5 that $\chi(D_L) = \chi(\mathrm{P}^+) = 0$. $\square$

8.6. Search for zeros of $F_{L,\lambda}$ corresponding to generalized changes of sign. The graph Γ_{H_L} is the image of a unique (by Lemma 6.1) 0-component S^0 of the space $R(D_L)$ under the action of ψ_{D_L}. For any point $s \in Y$ we denote by $\hat{s} = (s, H_L(s)) \in \mathrm{Sw}_0(D_L) = \Gamma_{H_L} \cap \mathrm{Sw}(D_L)$ the corresponding break point of the graph. Since D_L is tame, Lemma 6.3 assigns closed intervals $J^r_{\hat{s}}(D_L), J^l_{\hat{s}}(D_L) \subset L$ to each point $\hat{s}$.

Lemma 8.3. *Let $s \in Y$. If $D_\lambda H_L$ changes sign at s from the left, then the interval $J^r_{\hat{s}}(D_L)$ contains a point of the set $N^{\mathrm{S}}_\lambda(L)$. If $D_\lambda H_L$ changes sign at s from the right, then the interval $J^l_{\hat{s}}(D_L)$ contains a point of $N^{\mathrm{S}}_\lambda(L)$.*

By Lemma 8.3, corresponding to each change of sign for the distribution $D_\lambda H_L$ at a point $s \in Y$ is a point of $N^{\mathrm{S}}_\lambda(L)$. According to Lemma 6.3, the points of

$N^{\mathrm{S}}_{\lambda}(L)$ corresponding to different changes of sign are different and do not belong to Γ_{H_L}. Hence, the number of points in the set $N^{\mathrm{S}}_{\lambda}(L)$ is not less than the number of changes of sign for the distribution $D_\lambda H_L$, and the assertion of Theorem 7.3 holds for a λ-generic knot. Before proving Lemma 8.3, we formulate and prove Lemma 8.4, which is also used below in the proof of Lemma 8.12.

8.7. Maslov index and the curvature. Let $e\colon \mathbb{R}\to\mathbb{R}\mathrm{P}^1$ be the universal covering. We fix some function $\mathrm{i}_e\colon \mathbb{R}\setminus e^{-1}(\{\infty\})\to\mathbb{Z}$ which is constant on each open interval making up $\mathbb{R}\setminus e^{-1}(\{\infty\})$ and such that $\mathrm{i}_e(x_2)=\mathrm{i}_e(x_1)+1$ for each point $y\in e^{-1}(\{\infty\})$ and each pair of points $x_1,x_2\in\mathbb{R}\setminus e^{-1}(\{\infty\})$ close to y for which $e(x_1)<0$ and $e(x_2)>0$ ($e(x_1),e(x_2)\in\mathbb{R}\mathrm{P}^1\setminus\{\infty\}$). Suppose that $l\colon [0,T]\to L$ is a path (a continuous map) such that $l(0),l(T)\in G_L$. Let us consider a lifting $\widetilde{l}_\lambda$ of the map $F_{L,\lambda}\circ l\colon [0,T]\to\mathbb{R}\mathrm{P}^1$, that is, a continuous map $\widetilde{l}_\lambda\colon [0,T]\to\mathbb{R}$ such that $e\circ\widetilde{l}_\lambda=F_{L,\lambda}\circ l$. We write $\mathrm{i}_\lambda(l)=\mathrm{i}_e(\widetilde{l}_\lambda(T))-\mathrm{i}_e(\widetilde{l}_\lambda(0))$. One can readily see that the number $\mathrm{i}_\lambda(l)$ does not depend on the choice of the lifting. For the definition of the Maslov index $m(l)$ of a path l, see 2.5.

Lemma 8.4. $\mathrm{i}_\lambda(l)=m(l)$.

It also follows from Lemma 8.4 that the Maslov number of a Legendrian knot L is equal to the degree (winding number) of the map $F_{L,\lambda}$ under a certain choice of orientations.

Lemma 8.5. *Let x be a cusp of the front $\Sigma=\sigma(L)$. Then for each point $y\in G_\Sigma$ sufficiently close to x the value of $F_{L,\lambda}$ on the pre-image of y in L is positive if y belongs to the upper branch of Σ entering the cusp and negative if y belongs to the lower branch.*

Proof. The assertion follows from the definition of $F_{L,\lambda}$ and from the following fact: when the cusp is approached, the second derivative of the function whose graph is some branch entering the cusp tends to $+\infty$ for the upper branch and to $-\infty$ for the lower branch. $\square$

Proof of Lemma 8.4. It suffices to prove the assertion for a generic smooth path $l\colon [0,T]\to L$. Let l_t be the path $l_t=l|_{[0,t]}\colon [0,t]\to L$. By Lemma 8.5, the numbers $\mathrm{i}_\lambda(l_t)$ and $m(l_t)$ change by the same amount as t passes through any value t_0 such that $l(t_0)\in L\setminus G_L$. This completes the proof. $\square$

Lemma 8.6. *Suppose that $L\subset J^1(S^1)$ is a Legendrian link and $l\colon [0,T]\to L$ is a path such that $l(0),l(T)\in G_L$. If the numbers $F_{L,\lambda}(l(0))$ and $F_{L,\lambda}(l(T))$ have different signs and $m(l)=0$, then there is an $t_1\in[0,T]$ such that $F_{L,\lambda}(l(t_1))=0$ and $m(l_1)=0$, where l_1 stands for the restriction of l to $[0,t_1]$.*

Proof. Let us consider the closed interval $Z\subset\mathbb{R}\subset\mathbb{R}\mathrm{P}^1$ with ends at the points $F_{L,\lambda}(l(0))$ and $F_{L,\lambda}(l(T))$. It follows from the condition on the signs that $0\in Z$. Consider a lifting $\widetilde{l}_\lambda\colon [0,T]\to\mathbb{R}$ of the map $F_{L,\lambda}\circ l\colon [0,T]\to\mathbb{R}\mathrm{P}^1$. It follows from the condition $m(l)=0$ and Lemma 8.4 that the points $\widetilde{l}_\lambda(0)$ and $\widetilde{l}_\lambda(T)$ belong to the same connected component of $\mathbb{R}\setminus e^{-1}(\{\infty\})$. Thus, the closed interval Z' with ends at the points $\widetilde{l}_\lambda(0)$ and $\widetilde{l}_\lambda(T)$ is taken to Z by the projection e. There is a point $r\in Z'$ such that $e(r)=0\in\mathbb{R}\mathrm{P}^1$. Since r lies between the ends of the interval Z', there is a $t_1 \in [0,T]$ such that $\widetilde{l}_\lambda(t_1) = r$. Then $F_{L,\lambda}(l(t_1)) = 0$. Moreover, $\mathrm{i}_e(\widetilde{l}_\lambda(0)) = \mathrm{i}_e(\widetilde{l}_\lambda(t_1))$, and hence $m(l_1) = 0$ by Lemma 8.4. $\square$

8.8. Proof of Lemma 8.3. We prove the assertion for a change of sign from the left (the proof for a change of sign from the right is similar). Let $J^r \subset L$ be a closed interval obtained from $J^r_{\hat{s}}(D_L)$ by removing small neighbourhoods of the ends. Let a^r be the end of J^r close to the end of $J^r_{\hat{s}}(D_L)$ that is projected on a cusp and let x^r be the end of J^r close to z^r.

The switching crossing $\hat{s}$ belongs to the images of two sections in D_L. The image of one of these sections is Γ_{H_L}, and the other coincides with the section $\gamma_{\hat{s}}\colon \Lambda \to \Sigma$ defined in Lemma 6.3. Let us consider non-singular points of Σ close to $\hat{s}$. By definition, the Maslov potential μ_L vanishes on the pre-images in L of non-singular points of Σ belonging to Γ_{H_L}. Hence, μ_L vanishes on the pre-image in L of a small neighbourhood of $\hat{s}$. Since D_L is a Maslov decomposition, the potential μ_L vanishes on the pre-images in L of non-singular points of Σ belonging to $\gamma_{\hat{s}}(\Lambda)$. Since $\sigma(a^r)$, $\sigma(x^r) \in \gamma_{\hat{s}}(\Lambda)$, μ_L vanishes at the ends of J^r. We show that the numbers $F_{L,\lambda}(x^r)$ and $F_{L,\lambda}(a^r)$ have different signs. After this, the proof of Lemma 8.3 will be completed by applying Lemma 8.6 to a path parameterizing the interval J^r.

Suppose that the distribution $D_\lambda H_L$ is negative at the point s, that is, the graph Γ_{H_L} has an 'upward' break at $\hat{s}$ (the proof is similar if the sign is positive). Since $D_\lambda H_L$ changes sign from the left at the point s, the function $F_{L,\lambda}$ is positive on a neighbourhood of the point z^r. In particular, $F_{L,\lambda}(x^r) > 0$. Consider the point $d \in \Gamma_{H_L}$ whose q-coordinate coincides with that of $\sigma(x^r)$. The point $\sigma(x^r)$ is above the point d (that is, it has a greater u-coordinate), because the graph Γ_{H_L} forms an 'upward' break at $\hat{s}$. Since P^+ is positive, the point $\mathrm{P}^+(\sigma(x^r))$ is above the point $\sigma(x^r)$. Therefore, for each generic $y \in \gamma_{\hat{s}}(\Lambda)$ the point $\mathrm{P}^+(y)$ is above the point y. Hence, the section $\gamma_{\hat{s}}$ enters the right cusp $\sigma(a^r)$ of the front Σ from below. Thus, $F_{L,\lambda}(a^r) < 0$ by Lemma 8.5. This completes the proof of Lemma 8.3. $\square$

8.9. Reduction to the case of a λ-generic knot. Let us prove the assertion of Theorem 7.3 for arbitrary (non-generic) Legendrian knots $L \in \mathcal{L}_1$. By a small perturbation we can transform L into a λ-generic knot without increasing the number of points in the set $N^{\mathrm{S}}_\lambda(L)$ (we assume that this set is finite). For a smooth function $\eta \in C^\infty(S^1)$ we denote by Ψ_η the contactomorphism

$$(p,q,u) \mapsto (p + \eta'(q), q, u + \eta(q)).$$

The assertion of the theorem is reduced to the case of a λ-generic knot by using the following two lemmas.

Lemma 8.7. *There is a λ-generic Legendrian knot $L_0 \in \mathcal{L}_1$ arbitrarily C^∞-close to L and such that $\#N^{\mathrm{S}}_\lambda(L_0) \leqslant \#N^{\mathrm{S}}_\lambda(L)$.*

Lemma 8.8. *If $L_0 \in \mathcal{L}_1$ is a λ-generic Legendrian knot sufficiently C^∞-close to L, then there is a function $\eta \in C^\infty(S^1)$ such that the Legendrian link $L_1 = \Psi_\eta(L_0)$ satisfies the following conditions*: (1) $\#N^{\mathrm{S}}_\lambda(L_1) = \#N^{\mathrm{S}}_\lambda(L_0)$; (2) H_{L_1} *is orthogonal to the functions* $1, \sin q, \cos q, \dots, \sin kq, \cos kq$.

Proof of Lemma 8.7. We first prove the following assertion.

Lemma 8.9. *Let $F_{L,\lambda}(x) = 0$ and let U be a neighbourhood of the point x in $J^1(S^1)$ such that x is the only zero of $F_{L,\lambda}$ on the set $L \cap U$. There is a Legendrian link $L_U \in \mathcal{L}_1$ arbitrarily C^∞-close to L, coinciding with L outside U, and such that the following conditions hold*: (1) *if $F_{L,\lambda}$ does not change sign at x, then $F_{L_U,\lambda}$ has no zeros on $L_U \cap U$*; (2) *if $F_{L,\lambda}$ changes sign at x, then there is exactly one point $x' \in L_U \cap U$ such that $F_{L_U,\lambda}(x') = 0$; moreover, $dF_{L_U,\lambda}(x') \neq 0$ and $\mu_{L_U}(x') = \mu_L(x)$.*

Proof. In a small neighbourhood of the point $x = (p_0, q_0, u_0)$ the knot L coincides with the 1-graph of a function f defined on an interval $W \subset S^1$. We choose a compactly supported function $g\colon W \to \mathbb{R}$ as follows: if $f'' + \lambda f$ does not change sign at q_0, then g is positive and constant on a neighbourhood of q_0, and if $f'' + \lambda f$ changes sign at q_0, then $g(q)$ is equal to $q - q_0$ on a neighbourhood of q_0. Let L_t be the Legendrian manifold obtained from L by replacing the 1-graph of f by the 1-graph of $f + tg$. For a sufficiently small positive number ε the knots belonging to one of the families $\{L_\varepsilon\}$ and $\{L_{-\varepsilon}\}$ satisfy the assertion of the lemma (the values of the Maslov potentials coincide because the knots are C^∞-close). □

Let us choose disjoint neighbourhoods U of all the points of the set $N^S_\lambda(L)$ and apply Lemma 8.9. All zeros of the map $F_{L',\lambda}$ for the resulting knot L' are regular. Therefore, by using an arbitrarily C^∞-small perturbation we can make L' λ-generic without changing the number of points in the set $N^{\mathrm{S}}_\lambda(L')$. The proof of Lemma 8.7 is complete. □

Proof of Lemma 8.8. Using Lemma 5.3, we choose a closed interval $I \subset S^1$ such that the set $\sigma(L_0) \cap J^0(I)$ consists of non-singular points of the front $\sigma(L_0)$, and its pre-image in L_0 contains no points of the set $N^{\mathrm{S}}_\lambda(L')$.

Each continuous function η on S^1 determines a linear function η^* on the space T_k of trigonometric polynomials of degree at most k by using the L^2-pairing, namely, $\eta^*(f) = \int_{S^1} \eta(q) f(q)\, dq$. The following simple assertion is well known.

Lemma 8.10. *There are functions $\eta_1, \dots, \eta_{2k+1} \in C^\infty(S^1)$ supported in I such that the functionals $\eta_1^*, \dots, \eta_{2k+1}^*$ are linearly independent.*

Consider the linear functional $H^*_{L_0}$ on the space T_k. According to Lemma 8.10, there is exactly one linear combination η of the functions $\eta_1^*, \dots, \eta_{2k+1}^*$ such that $\eta^* = -H^*_{L_0}$. We set $L_1 = \Psi_\eta(L_0)$. Since L_0 and L are C^0-close, the functions H_{L_0} and H_L are C^0-close (by Proposition 7.1), and hence η is C^∞-small. Therefore, the knots L_1 and L are C^∞-close if L_0 and L are sufficiently C^∞-close. Hence, $N^{\mathrm{S}}_\lambda(L_1) = N^{\mathrm{S}}_\lambda(L_0)$, because L_1 and L_0 are C^∞-close to L and can differ only on the set $\{(p, q, u) \mid q \in I\}$.

To complete the proof of Lemma 8.8, it remains to show that $H^*_{L_1} = 0$. This is a corollary to the following assertion.

Lemma 8.11. *For each $L' \in \mathcal{L}_1$ and each $\eta \in C^\infty(S^1)$*

$$H_{\Psi_\eta(L')} = H_{L'} + \eta.$$

Proof. Let U be a dense subset of S^1 formed by regular values of the projection $L' \to S^1$. Consider the family of knots $L'_t = \Psi_{t\eta}(L')$, $t \in [0, 1]$. The function

$H_{L'_t} - t\eta$ depends continuously on t but does not really depend on t, because it can take only finitely many values at each point of U. Thus, $H_{\Psi_\eta(L')} - \eta = H_{L'}$. $\square$

This completes the proof of Theorem 7.3.

8.10. Proof of Theorem 7.4. Let us consider the Legendrian knot $L_c = \Psi_c(L) \in \mathcal{L}_1$ obtained from L by shifting by c along the u-coordinate. If $c = -\int_{S^1} H_L(q)\,dq$, then H_{L_c} is orthogonal to the functions 1, $\cos q$, $\dots$, $\sin kq$ by Lemma 8.11, and hence the knot L_c satisfies the conditions of Theorem 7.3. This shift takes the Arnol’d λ-points of L to Arnol’d λ-points of L_c. Therefore, the assertion of the theorem is a corollary to Theorem 7.3 and the following lemma, which is a variant of Rolle’s theorem.

Lemma 8.12. *Each closed interval $J \subset L$ whose ends belong to $N^{\mathrm{S}}_\lambda(L)$ has at least one interior point belonging to $N^{\mathrm{A}}_\lambda(L)$.*

Proof. Consider a smooth path $l\colon [0,1] \to L$ diffeomorphically parametrizing the interval J. Since the ends of J belong to $N^{\mathrm{S}}_\lambda(L)$, it follows that $m(l) = 0$. By Lemma 8.4, it follows that $\mathrm{i}_\lambda(l) = 0$. Therefore, every lifting $\widetilde{l}_\lambda\colon [0,1] \to \mathbb{R}$ of the map $F_{L,\lambda} \circ l\colon [0,1] \to \mathbb{R}\mathrm{P}^1$ satisfies the condition $\widetilde{l}_\lambda(0) = \widetilde{l}_\lambda(1)$. By Rolle’s theorem, there is a $t \in \,]0,1[$ such that $d\widetilde{l}_\lambda(t) = 0$, and hence $dF_{L,\lambda}(l(t)) = 0$. Thus, $l(t)$ is an Arnol’d λ-point. $\square$

8.11. Proof of Theorem 7.5. We assume that the knot $L \subset \mathcal{L}_1$ has finitely many critical points. Arguing as in the proof of Theorem 7.3, we can reduce Theorem 7.5 to the case in which L is σ-generic and the images of critical points of L under the action of σ are non-singular points of $\Sigma = \sigma(L)$.

Let a function $H_L\colon S^1 \to \mathbb{R}$ and a set $Y = Y(L) \subset S^1$ be defined as in the proof of Theorem 7.3. The derivative H'_L of H_L is a smooth function defined on $S^1 \setminus Y$ and having non-zero one-sided limits at the points of Y. The function H'_L is L^2-orthogonal to the trigonometric polynomials of degree $\leqslant k$. Then H'_L changes sign at least $2k$ times (when we also take into account the changes of sign at the points of Y). This follows from the Hurwitz theorem (the proof of Lemma 8.1 is valid).

If H'_L changes sign at a point $s_0 \in S^1 \setminus Y$, then the point of L projected onto the point $(s_0, H_L(s_0))$ is critical. According to Lemma 8.2 and Lemma 6.3, each point $s \in Y$ determines a closed interval $J^*_{\hat{s}}(D_L) \subset L$, where $\hat{s} = (s, H_L(s)) \in \Gamma_{H_L}$. If H'_L changes sign at $s \in Y$, then the values of the p-coordinate at the ends of $J^*_{\hat{s}}(D_L)$ have different signs. Hence, the interval $J^*_{\hat{s}}(D_L)$ contains at least one critical point of L. It follows from Lemma 6.3 that all the critical points thus constructed are distinct. $\square$

§ 9. The Arnol’d conjectures

9.1. Proof of Theorem 0.1. According to Lemma 1.1, the points at which the Legendrian link $L \in ST^*\mathbb{R}^2$ is tangent to the fibres of the projection ρ are exactly the zeros of F_L. Since the function F_L coincides with the function $F_{L,1}$ defined in 7.1, the points at which L is tangent to the fibres of ρ are the Sturm 1-points of L (the zeros of the map $F_{L,1}$).

We can assume that the origin of the plane is inside the fronts $\rho(L_0)$ and $\rho(L_1)$. Using the identification between $ST^*\mathbb{R}^2$ and $J^1(S^1)$ (see Fig. 3), we see that the front $\sigma(L_0)$ is the graph of a negative function $f_0 \in C^\infty(S^1)$ and the front $\sigma(L_1)$ is the graph of a positive function $f_1 \in C^\infty(S^1)$. According to Proposition 7.1, the family $\{L_t\}$ of Legendrian links determines a family $\{H_t = \mathcal{H}(L_t)\}$ of continuous functions on S^1. Clearly, $f_0 = H_0$ and $f_1 = H_1$. Therefore, there is a $t_0 \in [0,1]$ such that $\int_{S^1} H_{t_0}(q)\,dq = 0$. Applying Theorem 7.3 to L_{t_0} for $\lambda = 1$ and $k = 0$, we see that the knot L_{t_0} is tangent to the fibres of the projection ρ at least at four points. The main assertion of Theorem 0.1 is thus proved.

Suppose now that the family $\{L_t\}$ is generic in the following sense: the only bifurcations occurring in the family $\{\rho(L_t)\}$ of fronts are those shown in Fig. 1 a–e. We claim that among the fronts $\{\rho(L_t)\}$ there is a front having at least four non-degenerate cusps. Indeed, it was already shown that there is a knot L_{t_0} such that the projection $\rho|_{L_{t_0}}$ has at least four critical points. If t_0 is not a point of bifurcation, then the assertion is proved. If t_0 is a point of bifurcation, then the fronts $\{\rho(L_t)\}$ have at least four non-degenerate cusps if t belongs to at least one of the (two) sufficiently small half-neighbourhoods of t_0. $\square$

9.2. Proof of Theorem 0.3. The vertices of the Legendrian link L are exactly the Arnol'd 1-points defined in 7.2. Applying Theorem 7.4 (for $\lambda = 1$ and $k = 0$), we prove Theorem 0.3. $\square$

9.3. Tame knots. Analogues of Theorem 0.1 and Theorem 0.3 hold for some other components of the space of Legendrian knots in $ST^*\mathbb{R}^2 = J^1(S^1)$.

For any connected component $\mathcal{L}_0$ of the space of Legendrian links in $J^1(S^1)$ one can define a space $\mathcal{L}_0^+$ of Legendrian links in the same way that the space $\mathcal{L}_1^+$ was constructed from $\mathcal{L}_1$ in §7, namely, a link $L^+ \in \mathcal{L}_0^+$ is obtained from a link $L \in \mathcal{L}_0$ by adding a component $V_c = \{u = c, p = 0\}$ whose front lies above the front $\sigma(L)$. There is a natural projection $\mathcal{L}_0^+ \to \mathcal{L}_0$.

Let $L^+ \in \mathcal{L}_0^+$ be a σ-generic link. A pseudo-involution P^+ of $\sigma(L^+)$ is said to be *tame* if it is positive and $\chi(\mathrm{P}^+) = 0$. A component $\mathcal{L}_0$ of the space of Legendrian knots is said to be *tame* if the front of some (and hence every (by Theorem 2.5)) σ-generic link in $\mathcal{L}_0^+$ admits a tame pseudo-involution. A Legendrian knot is said to be *tame* if it belongs to a tame component.

Lemma 9.1. *Let $L \subset J^1(S^1)$ be a tame σ-generic knot. Then the Maslov number $m(L)$ of L vanishes. Suppose that P^+ is a tame pseudo-involution of $\sigma(L^+)$, where $L^+ = L \cup V_c$ is a lifting of L to $\mathcal{L}_0^+$. Then there is a unique integer-valued Maslov potential μ_{P^+} on L^+ taking the value 1 on V_c and such that P^+ is a Maslov pseudo-involution with respect to μ_{P^+}.*

Proof. Let us consider the decomposition D_{P^+} of $\sigma(L^+)$ associated with P^+. After removing the section $\sigma(V_c)$, we obtain a decomposition D_- of the front $\sigma(L)$. Arguing as in the proof of Lemma 8.2, we show that D_- is tame in the sense of the definition given in 6.2. We claim that the Maslov index of the path l_x diffeomorphically parameterizing the interval $J_x^*(D_-)$ defined in Lemma 6.3 vanishes for each $x \in \mathrm{Sw}(D_-)$. Indeed, by Lemma 6.3, each cusp of $\sigma(L)$ is an end of exactly one section of the form $\gamma_{x'}$ with $x' \in \mathrm{Sw}(D_-)$. All pre-images in $R(D)$ of the cusps that are ends of $\gamma_{x'}$ belong to the same component of the resolution $R(D)$. It follows from Lemma 6.2 that the two pre-images of these cusps in L either both belong or both do not belong to $J_x^*(D_-)$. Since the decomposition D_- was obtained from a pseudo-involution, the section $\gamma_{x'}$ either enters both cusps from above or enters both cusps from below. Hence, when the Maslov index of the path l_x is computed, these cusps are taken into account with opposite signs. Therefore, $m(l_x) = 0$.

We write $\Gamma = \mathrm{P}^+(\sigma(V_c))$. It follows from Lemma 6.3 that the Maslov number of L is the sum over all $x \in \mathrm{Sw}(D_-) \cap \Gamma$ of the Maslov indices of the paths l_x, computed with certain signs. Thus, $m(L) = 0$.

Suppose that $y_0 \in \Gamma$ is a non-singular point of the front $\sigma(L)$, z_0 is the pre-image of y_0 in L, and $z_1 \in V_c$ is a point such that $\mathrm{P}^+(y_0) = \sigma(z_1)$. The Maslov potential μ_{P^+} is uniquely determined by the condition that it takes the value 1 on V_c and the value 0 at z_0. For each switching crossing x we have $m(l_x) = 0$, and hence the values of μ_{P^+} on the pre-images of x in L coincide. Since $\mu_{\mathrm{P}^+}(z_1) = \mu_{\mathrm{P}^+}(z_0) + 1$, P^+ is a Maslov pseudo-involution with respect to μ_{P^+} by Proposition 2.4. In turn, this implies that μ_{P^+} vanishes on the pre-images in L of the non-singular points of $\sigma(L)$ belonging to Γ. Hence, the definition of μ_{P^+} does not depend on the choice of y_0. □

9.4. The Arnol'd conjectures for tame knots.

Theorem 9.2. *Let $\{L_{t\in[0,1]}\}$ be a smooth path in a tame component of the space of Legendrian knots such that the restriction of u to L_0 is negative and its restriction to L_1 is positive. Then there is a point $t_0 \in [0,1]$ such that the Legendrian knot L_{t_0} is tangent to the fibres of the projection ρ at least at four points.*

Theorem 9.3. *Every tame Legendrian knot has at least four vertices.*

9.5. Example. We present an example of a tame Legendrian knot with a smooth ρ-front but not in the component $\mathfrak{L}_1$ studied above.

We first describe a convenient geometric way to construct from a front $\sigma(L) \subset J^0(S^1)$ a front $\rho(L_1) \subset \mathbb{R}^2$ of some Legendrian knot L_1 belonging to the same connected component of $\mathfrak{L}$ to which the knot L belongs. The front $\rho(L_1)$ is the image of $\sigma(L)$ under the embedding of the cylinder $J^0(S^1)$ in $\mathbb{R}^2$, that is, $(q,u) \mapsto (e^u \cos q, e^u \sin q)$, and $\rho(L_1)$ is co-oriented from the origin.

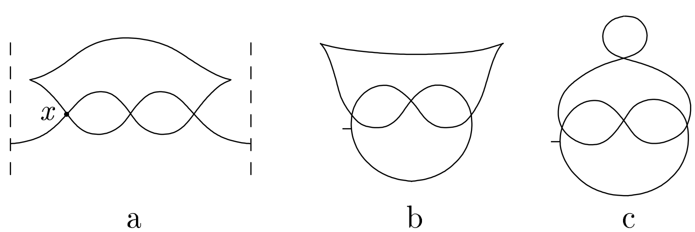


Figure 13

This embedding takes the σ-front of L' shown in Fig. 13 a to a front diffeomorphic to that shown in Fig. 13 b. Then it is easy to see (by constructing an appropriate sequence of bifurcations of fronts on the plane) that the knot $L \subset ST^*\mathbb{R}^2$ with front $\rho(L)$ shown in Fig. 13 c belongs to the connected component $\mathfrak{L}'$ of $\mathfrak{L}$ containing the knot L'. This component is tame since $\sigma(L')$ admits a tame pseudo-involution P determined by the condition $\mathrm{Sw}(\mathrm{P}) = \{x\}$. The components $\mathcal{L}'$ and $\mathcal{L}_1$ differ, because the knot L' is not isotopic in the class of smooth embeddings to a section of $J^1(S^1) \to S^1$. By Theorem 9.3, every Legendrian knot in $\mathcal{L}'$ has at least four vertices.

9.6. Hurwitz theorems for tame knots. The key role in the proof of Theorems 9.2 and 9.3 is played by Theorems 9.4 and 9.5, which are analogues of Theorems 7.3 and 7.4 for tame components. Let L be a σ-generic Legendrian knot in a tame component $\mathcal{L}_0$. We consider a lifting $L^+ = L \cup V_c$ of L to $\mathcal{L}_0^+$. By Theorem 2.5, the front $\sigma(L^+)$ admits at least one tame pseudo-involution, say P. We denote by $H_{L,\mathrm{P}} \colon S^1 \to \mathbb{R}$ the continuous function whose graph coincides with $\mathrm{P}(\sigma(V_c)) \subset \sigma(L)$ and by $\mathcal{R}(L)$ the set of functions $H\colon S^1 \to \mathbb{R}$ such that $H = H_{L,\mathrm{P}}$ for some tame pseudo-involution P. The set $\mathcal{R}(L)$ is always finite. In general, the number of elements in $\mathcal{R}(L)$ can differ for knots L belonging to different components of $\mathcal{L}_0 \setminus \mathcal{D}$.

For any Legendrian link $L \in \mathcal{L}_0 \cap \mathcal{D}$ we define a set $\mathcal{R}(L)$ consisting of the continuous functions on S^1 whose graphs are continuous sections of $\sigma(L)$, namely, a function H belongs to $\mathcal{R}(L)$ if and only if there exist a sequence (L_i) of knots in $\mathcal{L}_0 \setminus \mathcal{D}$ which C^∞-converges to L and a sequence of functions $H_i \in \mathcal{R}(L_i)$ which C^0-converges to H. Arguing as in the proof of Theorem 5.1, one can show that $\mathcal{R}(L)$ is non-empty and finite.

It follows from Lemma 7.2 that every tame Legendrian knot admits an integer-valued Maslov potential which is unique up to an additive constant. The following theorems generalize Theorems 7.3 and 7.4.

Theorem 9.4. *Let L be a tame knot such that there is a function $H \in \mathcal{R}(L)$ orthogonal to the $2k+1$ functions $1, \cos q, \sin q, \dots, \cos kq, \sin kq$. Then for each $\lambda \in \mathbb{R}$ there are at least $2k+2$ Sturm λ-points on L at which any integer-valued Maslov potential takes the same values. If $\lambda = (k+1)^2$, then the link L has at least $2k+4$ Sturm λ-points at which any integer-valued Maslov potential takes the same values.*

Theorem 9.5. *Let L be a tame knot such that there is a function $H \in \mathcal{R}(L)$ orthogonal to the $2k$ functions $\cos q, \sin q, \dots, \cos kq, \sin kq$. Then $\#\bigl(N^{\mathrm{A}}_\lambda(L)\bigr) \geqslant 2k+2$ and $\#\bigl(N^{\mathrm{A}}_{(k+1)^2}(L)\bigr) \geqslant 2k+4$ for each $\lambda \in \mathbb{R}$.*

The proofs of these theorems practically repeat the proofs of Theorems 7.3 and 7.4 literally. The only difference is that the proof of Theorem 9.4 for non-generic knots uses the following continuity property instead of Proposition 7.1: for any knot $L \in \mathcal{L}_0$ and any function $H \in \mathcal{R}(L)$, if L' is sufficiently C^∞-close to L, then $\mathcal{R}(L')$ contains a function that is sufficiently C^0-close to H.

9.7. Dual projective spaces and Legendrian projections. We pass to the consideration of some problems formulated by Arnol'd about fronts on the projective plane.

For any manifold N the contact manifold ST^*N is a natural two-sheeted covering over the contact manifold PT^*N of the non-co-oriented contact elements in N. The natural projection $\rho^\times\colon PT^*N \to N$ is Legendrian. Suppose that

$N = \mathbb{R}\mathrm{P}^n$, and consider the projective space $(\mathbb{R}\mathrm{P}^n)^\vee$ dual to $\mathbb{R}\mathrm{P}^n$, that is, the manifold of hyperplanes ($(n-1)$-dimensional projective subspaces) in the space $\mathbb{R}\mathrm{P}^n$. Along with the projection $\rho^\times\colon PT^*\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$, there is a natural projection $\rho^\vee\colon PT^*\mathbb{R}\mathrm{P}^n \to (\mathbb{R}\mathrm{P}^n)^\vee$. The projection $\rho^\vee$ takes a contact element to the hyperplane to which it is tangent.

9.8. Cusps and inflection points of fronts on the projective plane. We restrict ourselves to the case $n = 2$. One can readily see that if x is a point of a generic Legendrian link $L \subset PT^*\mathbb{R}\mathrm{P}^2$, then the front $\rho^\times(L)$ has a cusp at $\rho^\times(x)$ if and only if the front $\rho^\vee(L)$ has an inflection point at $\rho^\vee(x)$. Conversely, the front $\rho^\times(L)$ has an inflection point at $\rho^\times(x)$ if and only if the front $\rho^\vee(L)$ has a cusp at $\rho^\vee(x)$.

9.9. Non-co-oriented fronts on the plane. We state a conjecture related to the Arnol'd conjecture on three inflection points in [3]: if $\{L_t\}$ is a generic path in the space of Legendrian knots in $PT^*\mathbb{R}\mathrm{P}^2$ such that L_0 is a fibre of $\rho^\times$, then the front $\rho^\times(L_1)$ has at least three cusps (and hence the front $\rho^\vee(L_1)$ has at least three inflection points). It is unknown whether or not this conjecture is true. We can only prove a weaker assertion about fronts of Legendrian submanifolds of $PT^*\mathbb{R}^2 \subset PT^*\mathbb{R}\mathrm{P}^2$.

Let us consider a connected component $\mathcal{L}_1^\times$ of the space of Legendrian knots in $PT^*\mathbb{R}^2$ such that $\mathcal{L}_1^\times$ contains a fibre of the natural projection $\rho^\times\colon PT^*\mathbb{R}^2 \to \mathbb{R}^2$.

Theorem 9.6. *Every Legendrian knot $L \in \mathcal{L}_1^\times$ is tangent to the fibres of the projection $\rho^\times$ at least at three points. For any generic fibre $L \in \mathcal{L}_1^\times$ the front $\rho^\times(L)$ has at least three cusps.*

Proof. We denote by $\overline{\mathcal{L}}$ the space of Legendrian links in $J^1(S^1)$ invariant under the action of the contactomorphism $\Delta\colon J^1(S^1) \to J^1(S^1)$ given by the rule $(p, q, u) \mapsto (-p, q + \pi, -u)$. The pre-images in $ST^*\mathbb{R}^2$ of the Legendrian links in $PT^*\mathbb{R}^2$ are exactly the Legendrian links in $\overline{\mathcal{L}}$ (we use the identification between $ST^*\mathbb{R}^2$ and $J^1(S^1)$). Let $\overline{\mathcal{L}}_1$ denote the set of Δ-invariant Legendrian knots in $\mathcal{L}_1$. Since the component $\mathcal{L}_1$ contains the Legendrian knot $V_0 = \{u = p = 0\}$ which is a lifting to $J^1(S^1)$ of the fibre of the projection $\rho^\times$ over the origin, Theorem 9.6 results from the following theorem.

Theorem 9.7. *Every Legendrian knot $\overline{\mathcal{L}}_1$ admits at least six Sturm* 1*-points.*

Proof. Let $L \in \overline{\mathcal{L}}_1$. We consider the function H_L defined in Proposition 7.1.

Lemma 9.8. *If $L \in \overline{\mathcal{L}}_1$, then the function H_L is odd, in the sense that $H_L(q+\pi) = -H_L(q)$ for any $q \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$.*

Theorem 9.7 follows from Lemma 9.8 and Theorem 7.3. Indeed, Lemma 9.8 implies that H_L is orthogonal to the functions $1, \cos 2q, \sin 2q$. We write $\eta(q) = a_1 \sin q + a_2 \cos q$. The Legendrian knot $L_\eta = \Psi_\eta(L)$ belongs to the space $\overline{\mathcal{L}}_1$. It follows from Lemma 8.11 that $H_{L_\eta} = H_L + \eta$, and we can choose the numbers a_1 and a_2 such that H_{L_η} is orthogonal to the functions $1, \cos q, \sin q, \cos 2q, \sin 2q$. By Theorem 7.3, the knot L_η has at least six Sturm 1-points. Since the contactomorphism Ψ_η defines a one-to-one map of the Sturm 1-points of L to the Sturm 1-points of L_η, the knot L also has at least six Sturm 1-points.

Proof of Lemma 9.8. It suffices to prove the assertion for generic knots; one can extend it to non-generic knots by continuity using Proposition 7.1. Let L be a generic Legendrian knot invariant with respect to the involution Δ. We choose a $c>0$ such that $\sigma(V_c)$ lies above $\sigma(L)$ and $\sigma(V_{-c})$ lies below $\sigma(L)$. Let $L_+ = L\cup V_c$ and $L_- = L\cup V_{-c}$, and let $B(L_\pm)$ be the set of positive pseudo-involutions of the front $\sigma(L_\pm)$. We define a one-to-one map $T\colon B(L_+)\to B(L_-)$ by the following condition: $T(\mathrm{P})(x) = \mathrm{P}(x)$ if $x\notin\sigma(V_c)$ and $\mathrm{P}(x)\notin\sigma(V_c)$. Let $\Delta'\colon J^0(S^1)\to J^0(S^1)$ be given by the rule $(q,u)\mapsto(q+\pi,-u)$. We define a one-to-one map $T_\Delta\colon B(L_+)\to B(L_-)$ by the formula $T_\Delta(\mathrm{P})(x) = \Delta'\mathrm{P}\Delta'(x)$. By Proposition 2.6, the set $B(L_+)$ contains exactly one pseudo-involution; denote it by P. The assertion of the lemma reduces to the invariance of the section $\mathrm{P}(\sigma(V_c))\subset\sigma(L)$ under the action of the involution Δ'. This invariance follows from the equality

$$\Delta'(\mathrm{P}(\sigma(V_c)) = (T^{-1}T_\Delta(\mathrm{P}))(\sigma(V_c)) = \mathrm{P}(\sigma(V_c)).$$

The proof of Lemma 9.8 and of Theorem 9.7 is thus complete. $\square$

9.10. Persistence of cusps. If L is a Legendrian knot in $ST^*\mathbb{R}^2$ and if $m(L)=0$, then this cusp can be deformed in the class of Legendrian immersions to a Legendrian knot whose ρ-front is a smooth immersed curve. This assertion is no longer valid if self-intersections are forbidden, namely, for any k there is a component $\mathcal{L}^k$ of the space of Legendrian knots such that the Maslov number of each Legendrian knot in $\mathcal{L}^k$ vanishes and the ρ-front of each generic knot in $\mathcal{L}^k$ has at least $2k$ cusps ([14], [15]). One can also readily prove this assertion by using the approach of the present paper.

§ 10. Critical points of Legendrian links

A point (p,q,u) of a Legendrian link $L\subset J^1(M)$ is said to be *critical* if $p=0$. This definition generalizes the definition of a critical point of a function on M. Let $M=S^n$. A point $z\in L$ is critical if and only if the vector $\rho(z)\in\mathbb{R}^{n+1}$ is orthogonal to the hyperplane tangent to $\rho(L)$ at the point $\rho(z)$ (with $J^1(S^n)$ and $ST^*\mathbb{R}^{n+1}$ identified). Therefore, the critical points of L correspond to the normals to the front $\rho(L)$ that pass through the origin.

10.1. Positive links and their critical points. Let $M=S^1$. A σ-generic Legendrian link $L\subset J^1(S^1)$ is said to be *positive* if the front of the link $L^+ = L\cup V_c$ (such that $\sigma(V_c)$ lies above $\sigma(L)$) admits a positive pseudo-involution. A component $\mathcal{L}_0$ of the space of Legendrian links is said to be *positive* it there is a positive σ-generic link in $\mathcal{L}_0^+$ (in which case, by Theorem 2.5, all σ-generic links in $\mathcal{L}_0^+$ are positive). A Legendrian link is said to be *positive* if it belongs to a positive component. For a positive Legendrian link L we introduce the set $\mathcal{R}'(L)$ of continuous functions on S^1 in the same way that the set $\mathcal{R}(L)$ was defined in 9.6 for a tree-like knot L. The only difference is that the definition now involves all positive pseudo-involutions (and not just tame pseudo-involutions) of the front $\sigma(L\cup V_c)$. The set $\mathcal{R}'(L)$ is non-empty and finite. The following theorem generalizes Theorem 7.5.

Theorem 10.1. *Let L be a positive Legendrian link such that there is a function $H \in \mathcal{R}'(L)$ orthogonal to the $2k$ functions $\cos q, \sin q, \ldots, \cos kq, \sin kq$. Then L has at least $2k+2$ critical points.*

In particular, every positive Legendrian link has at least two critical points. We note that there is an example of a positive link that is a disjoint union of two knots for each of which there is a Legendrian knot without critical points and in the same component (of the space of Legendrian knots) as the given knot.

A critical point z of a Legendrian link $L \subset J^1(S^1)$ is said to be *non-degenerate* if $\sigma(z)$ is a non-singular point of $\sigma(L)$ and the restriction of p to L has a non-degenerate zero at z. A Legendrian link $L \subset J^1(S^1)$ (and its front $\sigma(L)$) is said to be C-*generic* if L is σ-generic and all its critical points are non-degenerate. The following assertion plays the crucial role in the proof of Theorem 10.1.

Proposition 10.2. *Let L be a* C*-generic positive Legendrian link and let $H \in \mathcal{R}'(L)$. Then the number of critical points of L is not less than the number of local extrema of the function H.*

Before passing to the proofs of Proposition 10.2 and Theorem 10.1, we give necessary definitions and prove some auxiliary assertions.

10.2. Internal and external critical points. A critical point $z \in L$ is said to be a *maximum* (*minimum*) *point* if the restriction of the coordinate u to L has a local maximum (local minimum) at z. Let P be a pseudo-involution of a C-generic front $\sigma(L)$. A critical point z is said to be P-*external* if either z is a maximum point and the point $\mathrm{P}(\sigma(z))$ is below the point $\sigma(z)$ or z is a minimum point and the point $\mathrm{P}(\sigma(z))$ is above the point $\sigma(z)$. A critical point is said to be P-*internal* if it is not P-external. We denote by the symbol B_L the number of critical points of L and by $B^{\mathrm{E}}_{L,\mathrm{P}}$ ($B^{\mathrm{I}}_{L,\mathrm{P}}$) the number of P-external (P-internal, respectively) critical points of the link L.

10.3. Smoothing. Let X be a subset of the set of crossing points of a C-generic front $\sigma(L)$ of a Legendrian link L. By a *smoothing* of L with respect to X we mean a (non-unique) Legendrian link L' whose front is constructed as follows. For each point $x \in X$ we choose a small neigbourhood $U_x \subset J^0(S^1)$ of x. The front $\sigma(L')$ of a Legendrian link L' coincides with $\sigma(L)$ outside the union U of the neighbourhoods U_x. The intersection of $\sigma(L')$ with each of the neighbourhoods U_x consists of two smooth non-intersecting branches. We choose these branches in such a way that either the pre-image in L' of each of them contains a single critical point (if the p-coordinates of the pre-images of x in L have different signs) or these pre-images contain no critical points at all (if the p-coordinates of the pre-images of x in L have the same sign). If P is a positive pseudo-involution of $\sigma(L)$ and $X \subset \mathrm{Sw}(\mathrm{P})$, then there is exactly one positive pseudo-involution P' of $\sigma(L')$ such that $\mathrm{P}'(y) = \mathrm{P}(y)$ for each non-singular point y of $\sigma(L')$ satisfying the conditions $y \notin U$ and $\mathrm{P}(y) \notin U$. The pseudo-involution P' is said to be the *smoothing of the pseudo-involution* P with respect to the set X.

Lemma 10.3. *Suppose that* P *is a positive pseudo-involution of some* C*-generic front* $\sigma(L)$ *and let* $X \subset \mathrm{Sw}(\mathrm{P})$*. Then*

$$B^{\mathrm{E}}_{L',\mathrm{P}'} - B^{\mathrm{I}}_{L',\mathrm{P}'} \geqslant B^{\mathrm{E}}_{L,\mathrm{P}} - B^{\mathrm{I}}_{L,\mathrm{P}},$$

where L' *and* P' *are obtained from* L *and* P *by smoothing with respect to the set* X*.*

Proof. The set of critical points of the Legendrian link L' that belong to no pre-image of any neighbourhood $U_x \subset J^0(S^1)$ for any $x \in X$ coincides with the set of critical points of L. Moreover, the P-internal critical points are P'-internal and the P-external points are P'-external. The pre-image in L' of each neighbourhood U_x either contains none of the critical points of L' or contains exactly two critical points. In the latter case, since the pseudo-involution P is positive, either both the critical points are P'-external, or one of them is P'-external and the other is P'-internal. □

Lemma 10.4. *Suppose that* $L_0 \subset J^1(S^1)$ *is a* C*-generic link and* P_0 *is a positive pseudo-involution of* $\sigma(L_0)$*. In this case the number of* P_0*-internal critical points is not less than the number of* P_0*-external critical points.*

Proof. Suppose that the set $\mathrm{Sw}(\mathrm{P}_0)$ is empty. Then the number of P_0-internal critical points is equal to the number of P_0-external critical points. This holds because the P_0-internal and P_0-external critical points alternate on each connected component of the link L_0.

Suppose now that the set $\mathrm{Sw}(\mathrm{P}_0)$ is non-empty. Consider a smoothing L_0' of the Legendrian link L_0 with respect to the set $\mathrm{Sw}(\mathrm{P}_0)$. We denote by P_0' the positive pseudo-involution of the front $\sigma(L_0')$ obtained from P_0' by the smoothing. Then the set $\mathrm{Sw}(\mathrm{P}_0')$ is empty and $0 = B^{\mathrm{E}}_{L_0',\mathrm{P}_0'} - B^{\mathrm{I}}_{L_0',\mathrm{P}_0'} \geqslant B^{\mathrm{E}}_{L_0,\mathrm{P}_0} - B^{\mathrm{I}}_{L_0,\mathrm{P}_0}$ by Lemma 10.3. □

10.4. Proof of Proposition 10.2. We denote by X the subset of break points in the graph $\Gamma \subset J^0(S^1)$ of the function H. Consider the smoothing L' of L with respect to X. Let P be a positive pseudo-involution of the front $\sigma(L \cup V_c)$ (where $\sigma(V_c)$ is above $\sigma(L)$) such that $\Gamma = \sigma(V_c)$. Then $X \subset \mathrm{Sw}(P)$. We denote by P' the smoothing of the pseudo-involution P with respect to X. Consider the Legendrian link L'' whose front is $\sigma(L') \setminus \mathrm{P}'(V_c)$. By restricting P', we obtain a positive pseudo-involution P'' of $\sigma(L'')$.

We denote by W the set of critical points of L'' that are critical points of L, by B_H the number of critical points of L projecting to Γ, and by b the number of points at which the function H is non-smooth and has a local extremum. We must show that $B_L \geqslant B_H + b$. When we apply smoothing, we obtain a single critical point of L'' from every non-smooth local extremum of H. All critical points of L'' not belonging to W can be obtained in this way. Since P is positive, all critical points of L'' not belonging to W are P''-external. Therefore, $\#(W) \geqslant B^{\mathrm{I}}_{L'',\mathrm{P}''}$ and $B^{\mathrm{E}}_{L'',\mathrm{P}''} \geqslant b$. It follows from these inequalities and the inequality $B^{\mathrm{I}}_{L'',\mathrm{P}''} \geqslant B^{\mathrm{E}}_{L'',\mathrm{P}''}$ given by Lemma 10.4 that $\#(W) \geqslant b$. Since $B_L = B_H + \#(W)$, it follows that $B_L \geqslant B_H + b$. □

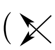
10.5. Remarks on the Maslov indices of critical points. Suppose that L is a C-generic tame Legendrian link, P is a tame pseudo-involution of $\sigma(L \cup V_c)$,

μ_{P} is the Maslov potential on L associated with P (see Lemma 9.1), and $H \in \mathcal{R}(L)$ is the continuous function on S^1 whose graph is the set $\mathrm{P}(\sigma(V_c))$. Let $\nu_i(L)$ be the number of critical points in L at which μ_{P} takes the value i. One can show that the following stronger version of Proposition 10.2 holds: if H has $2k$ local extrema, then $\nu_{-1}(L) + \nu_0(L) \geqslant k$, $\nu_1(L) + \nu_0(L) \geqslant k$, and $\nu_{-1}(L) + \nu_0(L) + \nu_1(L) \geqslant 2k$.

10.6. Proof of Theorem 10.1. Arguing as in the proof of Theorem 7.3, one can show (we omit the details) that it suffices to prove the assertion for a C-generic Legendrian link L. The function $\frac{d}{dq}H$ is smooth on $S^1 \setminus Y$, where $Y \subset S^1$ is a finite set. At each point of Y the derivative H' of H has non-zero left and right limits. The function H' is L^2-orthogonal to the trigonometric polynomials of degree at most k. In this case H' changes sign at least $2k+2$ times (counting also the sign changes at the points of Y). This follows from the Hurwitz theorem (the proof of Lemma 8.1 can be used). Therefore, H has at least $2k+2$ local extrema. The theorem now follows from Proposition 10.2. $\square$

§ 11. Invariants of Legendrian links

11.1. Classical invariants. The theory of pseudo-involutions enables one to define invariants of Legendrian links in $J^1(M)$. (The invariants are locally constant functions on the space $\mathcal{L}$ of Legendrian links.) To simplify the exposition, we restrict ourselves to the case of Legendrian knots in $J^1(\mathbb{R})$. Let us recall the definitions of the classical invariants of a (non-oriented) Legendrian knot. These invariants are, first, the smooth type of the knot, second, the Maslov number $m(L)$ defined in §2, and third, the Thurston–Bennequin number[2] $\beta(L)$, which is defined as follows. Let us choose an orientation on L (the result does not depend on the choice of it). The Thurston–Bennequin number $\beta(L)$ is the linking number between the knot L and a knot L' constructed by slightly shifting L along the u-coordinate. For a generic Legendrian knot the number $\beta(L)$ can be computed by counting the right cusps and the crossings of the front with appropriate signs:

$\beta(L) = \#($  $) + \#($  $) - \#($  $) - \#($  $) - \#($  $).$

The Maslov number and the Thurston–Bennequin number are the only finite-order invariants (in the sense of Vassiliev) which cannot be reduced to finite-order invariants of smooth knots [17].

11.2. Pseudo-involutions and invariants. We denote by μ_L a unique (up to a constant) $\mathbb{Z}/m(L)\mathbb{Z}$-valued Maslov potential on a Legendrian knot $L \subset J^1(\mathbb{R})$. For any σ-generic Legendrian knot L we denote by $\mathrm{PI}(L)$ the number of positive Maslov pseudo-involutions with respect to μ_L of the front $\sigma(L)$, and by $\mathrm{PI}'(L)$ the number of all positive pseudo-involutions of $\sigma(L)$. It follows from Proposition 2.6 that each of the functions PI and PI$'$ takes the same value on all σ-generic Legendrian knots belonging to a given connected component of the space of Legendrian knots. Therefore, when extending the functions PI and PI$'$ to non-generic knots, we obtain invariants of Legendrian knots.

[2] *Russian Editor's note*: This invariant is also called the *Bennequin–Tabachnikov number*; see, for instance, S. Chmutov, V. Goryunov, and H. Murakami, "Regular Legendrian knots and the HOMFLY polynomial of immersed plane curves," *Math. Ann.* **317** (2000), no. 3, 389–413.

11.3. Example. Let us consider Legendrian links L and L' whose fronts $\Sigma = \sigma(L)$ and $\Sigma' = \sigma(L')$ are shown in Fig. 14. Their classical invariants coincide, namely, $m(L) = m(L') = 0$ and $\beta(L) = \beta(L') = 1$, and both knots are of smooth type 5_2. We claim that they are distinguished by the invariant PI and hence belong to different components of the space $\mathfrak{L}$. The fact that these components are different was proved earlier by using another approach in [11] and independently by Eliashberg (unpublished).

The front Σ admits exactly two positive pseudo-involutions. These pseudo-involutions P_0 and P_1 are determined by the conditions $\mathrm{Sw}(\mathrm{P}_0) = \{x_1, x_2, x_5, x_6\}$ and $\mathrm{Sw}(\mathrm{P}_1) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. The front Σ' also admits exactly two positive pseudo-involutions. These pseudo-involutions P'_0 and P'_1 are determined by the conditions $\mathrm{Sw}(\mathrm{P}'_0) = \{y_1, y_4, y_5\}$ and $\mathrm{Sw}(\mathrm{P}'_1) = \{y_1, y_2, y_3, y_4, y_5\}$. Therefore, $\mathrm{PI}'(L) = \mathrm{PI}'(L') = 2$. All crossings of Σ are Maslov points, and y_1, y_4, y_5 are the only Maslov crossings of Σ' (with respect to an integer-valued Maslov potential). Hence, the pseudo-involutions P_0, P_1, P'_0 have the Maslov property and P'_1 does not. Thus, $\mathrm{PI}(L) = 2$ and $\mathrm{PI}(L') = 1$.

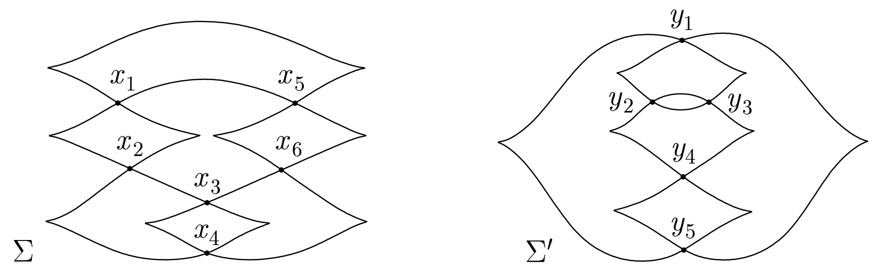


Figure 14

No examples are known in which the invariant PI' distinguishes Legendrian knots with coinciding classical invariants.

11.4. Refining the invariants. Actually, the theory of pseudo-involutions can provide more refined invariants related to the invariants PI and PI'. Let m and k be non-negative integers. For a generic Legendrian knot L we define the number $\mathrm{PI}_{m,k}(L)$ as follows. If m is a divisor of $m(L)$, then $\mathrm{PI}_{m,k}(L)$ is the number of positive pseudo-involutions P of the front $\sigma(L)$ that are Maslov pseudo-involutions with respect to the potential $\mu_L \otimes \mathbb{Z}/m\mathbb{Z}$ (taking values in $\mathbb{Z}/m\mathbb{Z}$) for which $\chi(\mathrm{P}) = 1-k$. In the particular case $m = 1$ this definition means that the value of $\mathrm{PI}_{1,k}(L)$ is the number of positive pseudo-involutions P of $\sigma(L)$ for which $\chi(\mathrm{P}) = 1 - k$. Finally, let $\mathrm{PI}_{m,k}(L) = 0$ if m is not a divisor of $m(L)$.

It follows from Proposition 2.6 that each of the functions $\mathrm{PI}_{m,k}(L)$ is an invariant for the Legendrian knots. We define the Poincaré polynomials PI^L_m as follows:

$$\mathrm{PI}^L_m(t) = \sum_k \mathrm{PI}_{m,k}(L)t^k.$$

We note that

$$\mathrm{PI}(L) = \mathrm{PI}^L_{m(L)}(1), \qquad \mathrm{PI}'(L) = \mathrm{PI}^L_1(1).$$

Let $L \subset J^1(\mathbb{R})$ be a σ-generic Legendrian knot. Then by Proposition 2.6, corresponding to every σ-generic loop in $\mathcal{L}$ beginning at the knot L is a one-to-one map of the set of positive pseudo-involutions of the front $\sigma(L)$ into itself. These maps form the monodromy group acting on the positive pseudo-involutions of $\sigma(L)$. As was shown in §4, this group can be non-trivial. Using the action of the monodromy group, one can define additional invariants of Legendrian knots (for example, by counting the orbits of the action). We note that the set of positive pseudo-involutions counted by the number $\mathrm{PI}_{m,k}(L)$ is invariant with respect to the action of the monodromy group.

11.5. Euler characteristic and the Thurston–Bennequin number.

Lemma 11.1. *Let* P *be a positive pseudo-involution of the front* $\sigma(L)$ *of a* σ*-generic Legendrian knot* $L \subset J^1(\mathbb{R})$*. Then* $\chi(\mathrm{P}) \equiv \beta(L) \pmod 2$.

Thus, $\mathrm{PI}_{m,k}(L) = 0$ if the number $k + \beta(L)$ is even.

Proof of Lemma 11.1. The Thurston–Bennequin number $\beta(L)$ is congruent modulo 2 to the sum of the number of right cusps and the number of crossing points of the front $\Sigma = \sigma(L)$. The Euler characteristic $\chi(\mathrm{P})$ is equal to the difference between the number of right cusps and the number of switching crossings of the pseudo-involution P. Hence, it suffices to show that the number of non-switching crossings of P is even. We denote by D the decomposition of Σ associated with the pseudo-involution P. Every component of the resolution space $R(D)$ (see 6.1) is homeomorphic to a circle. Let S and S' be two different components. Consider their images $\psi_D(S)$ and $\psi_D(S')$ under the projection $\psi_D \colon R(D) \to \Sigma$. We claim that the number of non-switching crossings of P at which $\psi_D(S)$ intersects $\psi_D(S')$ is even. This follows from the fact that the intersection index modulo 2 of the cycles $\psi_D(S)$ and $\psi_D(S')$ vanishes, the contribution of every non-switching crossing in $\psi_D(S) \cap \psi_D(S')$ to the intersection index is equal to 1, and the contribution of every switching crossing in $\psi_D(S) \cap \psi_D(S')$ to the intersection index is equal to 0. Since every non-switching crossing point of the front Σ belongs to the images of exactly two components of the space $R(D)$, it follows that the total number of non-switching points is even, which completes the proof of the lemma. □

11.6. Connected sum and the invariants $\mathbf{PI}_{m,k}$. Suppose that L_1 and L_2 are Legendrian knots in $J^1(\mathbb{R})$ whose fronts $\Sigma_1 = \sigma(L_1)$ and $\Sigma_2 = \sigma(L_2)$ are shown in Fig. 15. By the connected sum of L_1 and L_2 we mean the Legendrian knot L whose front is shown in Fig. 15 on the right. One can show that the operation of connected sum introduces a uniquely defined operation on the connected components in the space of oriented Legendrian knots, that is, in the corresponding components one can choose knots of desired shape, and the connected component containing the resulting knot does not depend on the choice of the knots.

Proposition 11.2. *Let a Legendrian knot L be the connected sum of Legendrian knots* $L_1, L_2 \subset J^1(\mathbb{R})$*. Then* $\mathrm{PI}_1^L(t) = \mathrm{PI}_1^{L_1}(t) \cdot \mathrm{PI}_1^{L_2}(t)$ *and* $\mathrm{PI}_2^L(t) = \mathrm{PI}_2^{L_1}(t) \cdot \mathrm{PI}_2^{L_2}(t)$*, and if* $m(L_1) = m(L_2) = 0$ *(and hence* $m(L) = 0$*), then* $\mathrm{PI}_0^L(t) = \mathrm{PI}_0^{L_1}(t) \cdot \mathrm{PI}_0^{L_2}(t)$.

Proof. Let P_1 (P_2) be a positive pseudo-involution of the front $\sigma(L_1)$ ($\sigma(L_2)$, respectively). Then there is a unique positive pseudo-involution P of $\sigma(L)$ such that

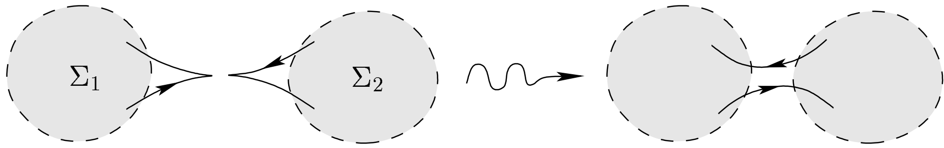


Figure 15

$\mathrm{Sw}(\mathrm{P}) = \mathrm{Sw}(\mathrm{P}_1) \cup \mathrm{Sw}(\mathrm{P}_2)$. Moreover, every positive pseudo-involution of $\sigma(L)$ can be obtained by using this construction. Hence, $\mathrm{PI}_1^L(t) = \mathrm{PI}_1^{L_1}(t) \cdot \mathrm{PI}_1^{L_2}(t)$. Let x be a crossing point of $\sigma(L)$. The pre-images of x in L can be joined in L by a path entirely contained either in L_1 or in L_2. Therefore, if $m(L_1) = m(L_2) = 0$, then x is a Maslov point for L if and only if x is a Maslov point for $L_1 \cup L_2$. The assertion about the Maslov property of x always holds for Maslov potentials taking values in $\mathbb{Z}/2\mathbb{Z}$, because the Maslov numbers of Legendrian knots are even. This completes the proof. $\square$

11.7. A refinement of the continuation theorem for links. In more complicated cases (in which the link consists of several components or the base M is a circle), the following construction of the quotient by a pseudo-involution can be used to construct additional invariants.

Let P be a pseudo-involution of a σ-generic front $\Sigma \subset J^0(M)$, where $M = I$, $M = \mathbb{R}$, or $M = S^1$. The resolution space $R(D_\mathrm{P})$ is equipped with a natural continuous involution i_P transposing the pre-images of the points x and $\mathrm{P}(x)$ for $\{x, \mathrm{P}(x)\} \subset \Sigma \setminus X_\Sigma$. Consider the quotient space $R(D_\mathrm{P})/i_\mathrm{P}$. For $y \in R(D_\mathrm{P})$ we denote by y/i_P the image of y in $R(D_\mathrm{P})/i_\mathrm{P}$. The 'quotient space' Σ/P is obtained from $R(\mathrm{P})/i_\mathrm{P}$ by gluing together two points y/i_P and y'/i_P such that $\psi_{D_\mathrm{P}}(y) = \psi_{D_\mathrm{P}}(y') = x$ for each switching crossing $x \in \mathrm{Sw}(\mathrm{P})$. The projection $\Sigma \to M$ defines a natural continuous map $\mathrm{H}_\mathrm{P} \colon \Sigma/\mathrm{P} \to M$. The space Σ/P is a one-dimensional cell complex with vertices of two kinds, namely, univalent vertices obtained from the cusps of Σ (and the boundary points if $M = I$) and four-valent vertices obtained from the switching crossings of P.

Let P_i be a pseudo-involution of a σ-generic front Σ_i for $i \in \{1, 2\}$. We say that the pseudo-involutions P_1 and P_2 are *homotopically similar* if there is a homotopy equivalence $h \colon \Sigma_1/\mathrm{P}_1 \to \Sigma_2/\mathrm{P}_2$ such that the maps $\mathrm{H}_{\mathrm{P}_2} \circ h$ and $\mathrm{H}_{\mathrm{P}_1}$ are homotopy equivalent. Using the explicit continuation of pseudo-involutions described in the proof of Theorem 2.5, one can establish the following assertion.

Proposition 11.3. *Under the assumptions of Theorem* 2.5, *the map* $\mathrm{P}_a \mapsto \mathrm{P}_b$ *takes any pseudo-involution to a homotopically similar pseudo-involution.*

For connected Legendrian links in $J^1(I)$ or $J^1(\mathbb{R})$ two pseudo-involutions are homotopically similar if and only if their Euler characteristics are equal. In this situation Proposition 11.3 gives no new information. In other cases one can readily construct additional invariants of Legendrian links by using Proposition 11.3.

§ 12. Generating families and pseudo-involutions

In this section we discuss the relationships between the theory of generating families for Legendrian submanifolds and the theory of pseudo-involutions.

12.1. Generating families. Let us briefly recall the construction of a generating family for a Legendrian manifold (for details, see [5]). Let M and W be smooth manifolds and let $F\colon M \times W \to \mathbb{R}$ be a smooth function. For any point $q \in M$ we consider the set $B_q \subset \{q\} \times W$ formed by the critical points of the restriction of F to $\{q\} \times W$. We write $B_F = \bigcup_{q\in M} B_q \subset M \times W$. Suppose that the rank of the matrix (F_{wq}, F_{ww}) of second derivatives is maximal (equal to the dimension of M) at each point of B_F (this condition holds for any generic function F). Then the set $B_F \subset M \times W$ is a smooth submanifold whose dimension is equal to that of M, and the restriction to B_F of the map $(q, w) \xmapsto{l_F} (q, d_M(F(q, w)), F(q, w))$ (where d_M stands for the first differential along M) defines a Legendrian immersion of B_F into $J^1(M)$. In this case F is called a *generating family* (or a *generating function*) of the (immersed) Legendrian submanifold $L_F = l_F(B_F)$.

We now describe a special class $\mathcal{G}$ of generating families and show that each generic function $F \in \mathcal{G}$ determines a combinatorial structure on the front $\sigma(L_F)$. In the case $\dim M = 1$ this structure is equivalent to introducing a positive pseudo-involution defined on the front.

We assume for simplicity that the manifold M is closed. Let W have the form $W = W_0 \times \mathbb{R}^N$, where W_0 is closed and $N \geq 1$. The class $\mathcal{G}$ consists of the generating families which can be represented as sums of a non-zero linear function on $\mathbb{R}^N$ and a compactly supported function on W. Families of this kind are said to be *linear at infinity.* The functions on $W = W_0 \times \mathbb{R}^N$ which can be represented as sums of a non-zero linear function on $\mathbb{R}^N$ and a compactly supported function are also said to be *linear at infinity.*

Suppose that $F \in \mathcal{G}$ and the point $q_0 \in M$ is such that $F(q_0, \cdot\,)\colon W \to \mathbb{R}$ is a strictly Morse function, that is, all its critical values are different. This condition holds for any generic point. Let us consider the front $\Sigma_F = \sigma(L_F)$ of the Legendrian manifold L_F, that is, the image of L_F under the projection $\sigma\colon J^1(M) \to J^0(M)$ given by the rule $(p, q, u) \mapsto (q, u)$. This front consists of the points (q_0, u_0) such that u_0 is a critical value of the function $F(q_0, \cdot\,)\colon W \to \mathbb{R}$.

12.2. Morse complexes. Each strictly Morse function f on $W = W_0 \times \mathbb{R}^N$ that is linear at infinity determines a partition of its set of critical values into pairs (the partition also depends on the choice of the coefficient field involved in the construction and playing the role of a parameter). This partition into pairs is constructed by using a Morse complex which we are going to describe. We fix a metric g_0 on $W = W_0 \times \mathbb{R}^N$ which is the product of the Euclidean metric on $\mathbb{R}^N$ and some metric on W_0. Let g be a metric on W representable in the form $g = g_0 + h$, where h is a compactly supported tensor 2-form on W. We denote by the symbol $\mathcal{M}_g(x', x'')$ the set of trajectories $\gamma\colon \mathbb{R} \to M$ of the anti-gradient vector field $-\nabla_g f$ that tend to x' as $t \to -\infty$ and to x'' as $t \to +\infty$. We write $\widehat{\mathcal{M}}_g(x', x'') = \mathcal{M}_g(x', x'')/\mathbb{R}$, where $\mathbb{R}$ acts by shifting the parameter.

We suppose that the metric g is such that the trajectories in the set $\widehat{\mathcal{M}}_g(x', x'')$ with $\mathrm{ind}(x') = \mathrm{ind}(x'')+1$ (the symbol 'ind' stands for the Morse index of a critical point) are stable under small perturbations of g. This condition holds for generic metrics. Let $x_1, \dots, x_K$ be the critical points of f indexed in ascending order of the critical values. We consider a free graded $\mathbb{Z}$-module $C_f = \mathbb{Z} \otimes \{x_1, \dots, x_K\}$ with generators x_i such that the degree of any generator is equal to its Morse index.

We introduce an operator $\partial_g\colon C_f \to C_f$ by defining its action on the generators by the formula

$$\partial_g(x_k) = \sum_{\operatorname{ind}(x_j)=\operatorname{ind}(x_k)-1} \#\bigl(\widehat{\mathcal{M}}_g(x_k,x_j)\bigr)x_j,$$

where the trajectories of the set $\widehat{\mathcal{M}}_g(x_k,x_j)$ are counted with properly defined signs. (Strictly speaking, in order to define these signs uniquely, one must fix some orientations; here the possible ambiguity leads to a transformation of the operators ∂_g by means of an automorphism of C_f that changes the signs of some generators.) We note that the element $\partial_g(x_j)$ is a linear combination of the generators $x_1,\dots,x_{j-1}$. As is well known, $\partial_g^2 = 0$ (see, for example, [22]). The homology $\ker\partial_g/\operatorname{im}\partial_g$ vanishes, since it coincides with the integer homology of the pair of topological spaces $\bigl(\{f \le c\},\{f \le -c\}\bigr)$, where c is a sufficiently large positive number.

Let us consider the group $\operatorname{Aut}_{\mathrm{T}}(C_f)$ of upper-triangular automorphisms of the $\mathbb{Z}$-module C_f. This group consists of the graded automorphisms preserving each submodule of the form $\mathbb{Z}\otimes\{x_1,\dots,x_j\}\subset C_f$, where $j\in\{1,\dots,K\}$. One can show that distinct differentials ∂_{g_1} and ∂_{g_2} corresponding to distinct admissible metrics g_1 and g_2 are related by the rule

$$\partial_{g_2} = A\partial_{g_1}A^{-1},$$

where A is an element of the group $\operatorname{Aut}_{\mathrm{T}}(C_f)$ (and A depends on g_1 and g_2).

12.3. *M*-differentials and the combinatorics of the generators of an *M*-complex. The following algebraic definition axiomatizes the properties of the complex (C_f,∂_g) (that is, differential graded module). Let $\mathbb{E}$ be a commutative ring and let C be a free graded $\mathbb{E}$-module with a distinguished system of generators $(x_1,\dots,x_K)$. We say that an $\mathbb{E}$-linear map $\partial\colon C\to C$ of degree -1 is an *M-differential* if $\partial^2=0$ and $\partial(x_j)\in\mathbb{E}\otimes\{x_1,\dots,x_{j-1}\}$ for each $j\in\{1,\dots,K\}$; in this case we call the pair (C,∂) an *M-complex*. We consider the group $\operatorname{Aut}_{\mathrm{T}}(C)$ of upper-triangular automorphisms of C that preserve each of the submodules $\mathbb{E}\otimes\{x_1,\dots,x_j\}$, $j\in\{1,\dots,K\}$. We say that two M-differentials $\partial_1,\partial_2\colon C\to C$ are *equivalent* if there is an $A\in\operatorname{Aut}_{\mathrm{T}}(C)$ such that $\partial_2 = A\partial_1A^{-1}$. An example of an M-complex is given by the pair $(C_f\otimes\mathbb{E},\partial_g\otimes\mathbb{E})$; the M-differentials $\partial_{g_1}\otimes\mathbb{E}$ and $\partial_{g_2}\otimes\mathbb{E}$ constructed from different metrics g_1 and g_2 are equivalent.

An M-differential ∂ is said to be *elementary* if it satisfies the following two conditions: (1) for each $j\in\{1,\dots,K\}$ either we have $\partial(x_j)=0$ or there is an index $m\in\{1,\dots,K\}$ such that $\partial(x_j)=x_m$; (2) if $\partial(x_j)=\partial(x_{j'})=x_m$, then $j=j'$.

Theorem 12.1 ([8], [6]). *Let (C,∂) be an M-complex over a field $\mathbb{E}$. Then the M-differential ∂ is equivalent to exactly one of the elementary M-differentials on C.*

Each elementary M-differential ∂ defines the following combinatorial structure on the set of generators: some of the generators represent a basis in the homology of the complex (C,∂) and the other generators are partitioned into pairs (x_i,x_j) such that $\partial(x_i)=x_j$. If $\mathbb{E}$ is a field, then, by Theorem 12.1, an arbitrary M-differential defines a structure of the same kind on the set of generators, and these structures coincide for equivalent M-differentials.

We note that the assertion of Theorem 12.1 fails for $\mathbb{E} = \mathbb{Z}$ (already in the existence part). By tensoring an M-complex with different fields $\mathbb{E}$, we obtain different combinatorial structures on the set of generators in general. One can show that the assertion of the theorem holds also for $\mathbb{E} = \mathbb{Z}$ if these structures coincide for all fields of the form $\mathbb{Z}/p\mathbb{Z}$.

Let f be a strictly Morse function that is linear at infinity. We fix the field $\mathbb{E}$ once and for all. By Theorem 12.1, the critical points (and hence the critical values) of the function f are partitioned into pairs in a canonical way (that is, independent of the choice of an admissible metric g), since the homology of the complex $(C_f \otimes \mathbb{E}, \partial_g \otimes \mathbb{E})$ vanishes.

One can show that the differential-algebra definition of the partition into pairs is equivalent to the following topological definition. Two critical values $a > b$ of the function f form a pair if and only if the following condition holds for each sufficiently small $\varepsilon > 0$:

$$\begin{aligned}\dim H_*(\{f \leqslant a+\varepsilon\}, \{f \leqslant b+\varepsilon\}) &= \dim H_*(\{f \leqslant a-\varepsilon\}, \{f \leqslant b-\varepsilon\}) \\ &= \dim H_*(\{f \leqslant a-\varepsilon\}, \{f \leqslant b+\varepsilon\}) + 1 \\ &= \dim H_*(\{f \leqslant a+\varepsilon\}, \{f \leqslant b-\varepsilon\}) + 1.\end{aligned}$$

12.4. From a generating family to a pseudo-involution. Let an embedded Legendrian submanifold L_F be defined by a generating family $F \in \mathcal{G}$. Suppose that a point $q_0 \in M$ is such that the line $\{q = q_0\} \subset J^0(M)$ intersects the front $\Sigma_F = \sigma(L_F)$ only at non-singular points of the front. Then $F(q_0, \cdot\,)$ is a strictly Morse function. Thus, the set $\Sigma^{q_0} = \pi^{-1}(q_0) \cap \Sigma_F$ is equipped with a free involution $i_F^{q_0}$. It follows from the topological definition of the partition into pairs that the map $i_F^{q_0}$ depends continuously on q_0, and moreover, the collection of maps $i_F^{q_0}$ can be extended to a continuous map $\mathrm{P}_F^G\colon G_{\Sigma_F} \to \Sigma_F$, where G_{Σ_F} stands for the set of non-singular points of Σ_F.

Suppose now that $M = S^1$. We claim that the continuous map P_F^G arising from the partition into pairs can be extended to a pseudo-involution P_F of the front Σ_F. Indeed, it suffices to show that the branches entering a cusp are transposed by the map P_F^G and the branches that intersect transversely cannot be transposed by P_F^G. Let us consider two non-singular points (q, u_1) and (q, u_2) of Σ_F that are close to a singular point $(q_0, u_0) \in \Sigma_F$. Suppose that $u_1 > u_2$. It follows from Morse theory that the number $\dim H_*\bigl(\{F(q_1, \cdot\,) \leqslant u_1 + \varepsilon\}, \{F(q_1, \cdot\,) \leqslant u_2 - \varepsilon\}\bigr)$ is 0 if (q_0, u_0) is a cusp and is 2 if (q_0, u_0) is a crossing. After these remarks the assertion follows readily from Morse theory and the above topological definition of a pair of critical values.

The link L_F is equipped with a natural Maslov potential $\mu_F\colon G_{L_F} \to \mathbb{Z}$ whose value at any point $x = (p_0, q_0, u_0) \in G_{L_F}$ is equal to the Morse index of the corresponding critical point of the function $F(q_0, \cdot\,)$ with critical value u_0. The pseudo-involution P_F is a Maslov pseudo-involution with respect to μ_F. This follows from the fact that if two critical values a and b, $a > b$, of the function $F(q_0, \cdot\,)$ form a pair, then the Morse index of the critical point with the critical value a is greater by 1 than the Morse index of the critical point with the critical value b. The pseudo-involution P_F is positive. This can be verified by studying the bifurcations of partitions of the critical values into pairs for generic one-parameter families of functions on W (see [6]). The very definition of positive pseudo-involution (and that of Maslov pseudo-involution) is obtained by axiomatizing combinatorial structures arising on the front of a Legendrian submanifold defined by a generating family. One can also show that for any σ-generic link L and any positive pseudo-involution P of $\sigma(L)$ which is a Maslov pseudo-involution with respect to an integer-valued Maslov potential μ there is a generating family $F \in \mathcal{G}$ such that $L = L_F$, $\mathrm{P} = \mathrm{P}_F$, and μ differs from μ_F by a constant (Pushkar', unpublished).

12.5. Comparing families of pseudo-involutions. Let $F\colon M \times W_0 \times \mathbb{R}^N \to \mathbb{R}$ be a generating family for a Legendrian submanifold of $J^1(M)$. We say that a generating family $\widetilde{F}$ is a *stabilization of* F if $\widetilde{F}$ is of the form

$$\widetilde{F}\colon M \times W_0 \times \mathbb{R}^N \times \mathbb{R}^k \to \mathbb{R}, \qquad \widetilde{F}(x,y) = F(x) + Q(y),$$

where $x \in M \times W_0 \times \mathbb{R}^N$, $y \in \mathbb{R}^k$, and Q is a non-degenerate quadratic form on $\mathbb{R}^k$. Two generating families in $\widetilde{G}$ are said to be *equivalent* if they have stabilizations defined on the same manifold $M \times W_0 \times \mathbb{R}^N$ and these stabilizations can be transformed one into another by a diffeomorphism of $M \times W_0 \times \mathbb{R}^N$ fibred over M. One can readily see that if $M = S^1$, then the pseudo-involutions arising from equivalent generating families coincide.

Suppose that $\{L_{t\in[0,1]}\}$ is a smooth family of Legendrian submanifolds of $J^1(M)$ and L_0 is determined by a generating family $F \in \mathcal{G}$. Then, arguing as in [10], [13], [21], and [9], one can prove that there is a family $\{F_{t\in[0,1]}\}$ of generating families in $\mathcal{G}$ such that $L_{F_t} = L_t$ and F_0 is equivalent to F. It should be noted that if $\dim M = 1$, then the existence proof for such a family was implicitly contained already in [18]. Arguing as in [25] and [24], one can show that if $\{\widetilde{F}_t\}$ is another family satisfying the same conditions as $\{F_t\}$, then the generating family $\widetilde{F}_t$ is equivalent to the generating family F_t for any $t \in [0,1]$.

Suppose that $M = S^1$ and the family L_t of Legendrian links is σ-generic. Then the family $\{F_t\}$ gives rise to a continuous family $\{\mathrm{P}_{F_t}\}$ of positive pseudo-involutions of the fronts $\sigma(L_t)$. If $\{\widetilde{F}_t\}$ is another family satisfying the same conditions as $\{F_t\}$, then the family of pseudo-involutions generated by $\{\widetilde{F}_t\}$ coincides with $\{\mathrm{P}_{F_t}\}$. Thus, the theory of generating families determines a canonical continuation of the pseudo-involution P_F. This observation was the starting point for the formulation of Theorem 2.5.

Theorem 2.5 defines a continuation of the pseudo-involution $\mathrm{P}_0 = \mathrm{P}_{F_0}$ to a family $\{\mathrm{P}_{t\in[0,1]}\}$ of pseudo-involutions of the fronts $\sigma(L_t)$. Let us compare the families of pseudo-involutions given by $\{\mathrm{P}_t\}$ and $\{\mathrm{P}_{F_t}\}$. If the front $\sigma(L_0)$ admits exactly one positive pseudo-involution, then the families $\{\mathrm{P}_t\}$ and $\{\mathrm{P}_{F_t}\}$ coincide by Theorem 2.5. In particular, this implies that the continuous section $\Gamma_{\mathcal{H}(L)}$ of the front $\sigma(L)$ coincides with the minimax section defined in [9] for any $L \in \mathcal{L}_1$. In general, the family $\{\mathrm{P}_{F_t}\}$ can be non-characteristic, in contrast to $\{\mathrm{P}_t\}$.

Restricting ourselves to the consideration of a family $\{F_t\}$ for which the path $\{L_{F_t}\}$ intersects the discriminant $\mathcal{D}$ exactly once, we note that, according to the results in §4, a continuous non-characteristic family of pseudo-involutions can exist only for bifurcations of types III and XX. Both for bifurcations of type III and for bifurcations of type XX there are examples of families $\{F_t\}$ such that the family $\{\mathrm{P}_{F_t}\}$ is non-characteristic. (Moreover, in the study of bifurcations of Morse complexes for two-parameter families of functions, one can show that for $\mathbb{E} = \mathbb{Z}/2\mathbb{Z}$ and for any bifurcation of type III_3 (see 3.13) the family $\{\mathrm{P}_{F_t}\}$ is always non-characteristic, and hence never coincides with the family $\{\mathrm{P}_t\}$.) Since the family $\{\mathrm{P}_{F_t}\}$ can be non-characteristic, without additional considerations one cannot replace Theorem 2.5 by results of the theory of generating families in the proof of the Arnol'd conjectures.

One can deduce from properties of generating families that $\mathrm{P}_{F_1} = \mathrm{P}_{F_0}$ for any family $\{F_t\}$ determining a loop $\{L_t\}$ that is contractible in $\mathcal{L}$. As shown in §4, the equality $\mathrm{P}_1 = \mathrm{P}_0$ does not always hold for the family $\{\mathrm{P}_t\}$.

We also note that two different families $\{F_t\}$ and $\{\widetilde{F}_t\}$ determining the same family $\{L_t\}$ of Legendrian links can determine different families $\{\mathrm{P}_{F_t}\}$ and $\{\mathrm{P}_{\widetilde{F}_t}\}$ of pseudo-involutions even if $\mathrm{P}_{F_0} = \mathrm{P}_{\widetilde{F}_0}$.

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