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On Expansion of Zeta(3) in Continued Fraction[1]

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Abstract

We found a series of continued fractions for zeta(3) parametrized by some family of pairs of sequences F, G. Two members of this series are present here; they are different from Apéry-Nesterenko continued fraction.

Introduction

Let is given a difference equation

$$x_{\nu+1} - b_{\nu+1}x_\nu - a_{\nu+1}x_{\nu-1} = 0, \tag{1}$$

with $\nu \in \mathbb{N}_0$. We denote by

$$\{P_\nu(b_0,\, a_1,\, b_1,\, ...,\, a_\nu,\, b_\nu)\}_{\nu=-1}^{+\infty}$$

and

$$\{Q_\nu(b_0,\, a_1,\, b_1,\, ...,\, a_\nu,\, b_\nu)\}_{\nu=-1}^{+\infty}$$

the solutions of this equation with initial values

$$P_{-1} = 1,\, Q_{-1} = 0,\, P_0(b_0) = b_0,\, Q_0(b_0) = 1. \tag{2}$$

Then

$$\left\{ \frac{P_\nu(b_0,\, a_1,\, b_1,\, ...,\, a_\nu,\, b_\nu)}{Q_\nu(b_0,\, a_1,\, b_1,\, ...,\, a_\nu,\, b_\nu)} \right\}_{\nu=0}^{+\infty}$$

[1] Short version

is sequence of convergents of continued fraction

$$b_0 + \frac{a_1|}{|b_1} + ... + \frac{a_\nu|}{|b_\nu} + ... \, .$$

According to the famous result of R. Apéry [1],

$$\zeta(3) = \lim_{\nu\to\infty} \frac{v_\nu}{u_\nu}, \tag{3}$$

where $\{u_\nu\}_{\nu=1}^{+\infty}$ and $\{v_\nu\}_{\nu=1}^{+\infty}$ are solutions of difference equation

$$(\nu+1)^3 x_{\nu+1} - (34\nu^3 + 51\nu^2 + 27\nu + 5)x_\nu + \nu^3 x_{\nu-1} = 0, \tag{4}$$

with initial values $u_0 = 1$, $u_1 = 5$, $v_1 = 0$, $v_1 = 6$. The equality (4) is equivalent to the equality

$$\zeta(3) = b_0^\vee + \frac{a_1^\vee|}{|b_1^\vee} + \frac{a_2^\vee|}{|b_2^\vee} + ... + \frac{a_\nu^\vee|}{|b_\nu^\vee} + ... \tag{5}$$

with

$$b_0^\vee = 0,\ b_1^\vee = 5,\ a_1^\vee = 6,\ b_{\nu+1}^\vee = 34\nu^3 + 51\nu^2 + 27\nu + 5,\ a_{\nu+1}^\vee = -\nu^6,$$

where $\nu \in \mathbb{N}$. Yu.V. Nesterenko in [3] has offered the following expansion the number $2\zeta(3)$ in continuous fraction:

$$2\zeta(3) = 2 + \frac{1|}{|2} + \frac{2|}{|4} + \frac{1|}{|3} + \frac{4|}{|2}..., \tag{6}$$

with

$$b_0 = b_1 = a_2 = 2,\ a_1 = 1,\ b_2 = 4,$$

$$b_{4k+1} = 2k+2,\ a_{4k+1} = k(k+1),\ b_{4k+2} = 2k+4,\ a_{4k+2} = (k+1)(k+2)$$

for $k \in \mathbb{N}$,

$$b_{4k+3} = 2k+3,\ a_{4k+3} = (k+1)^2,\ b_{4k+4} = 2k+2,\ a_{4k+4} = (k+2)^2$$

for $k \in \mathbb{N}_0$. The halved of $4n-2$-th convergent of continued fraction (6) is equal to n-th convergent of continuous fraction (5). Elementary proof of Yu.V. Nesterenko expansion can be found in [9]. Making use of the method developed in our papers [7] – [8], we have found the followng expansions of the Number $\zeta(3)$ in contiuous fractions :

Theorem A. *The following equalities hold*

$$2\zeta(3) = b_0^{(*1)} + \frac{a_1^{(*1)}|}{|b_1^{(*1)}} + ... + \frac{a_\nu^{(*1)}|}{|b_\nu^{(*1)}} + ..., \tag{7}$$

$$2\zeta(3) = b_0^{(*0)} + \frac{a_1^{(*0)}|}{|b_2^{(*0)}} + \ldots + \frac{a_\nu^{(*0)}|}{|b_\nu^{(*0)}} + \ldots, \tag{8}$$

with $b_0^{(*1)} = 3$, $a_1^{(*1)} = -81$, $a_\nu^{(*1)} = -(\nu-1)^3\nu^3(4\nu^2-4\nu-3)^3$ *for* $\nu \geq 2$,

$$b_\nu^{(*1)} = 4(68\nu^6 - 45\nu^4 + 12\nu^2 - 1) \text{ for } \nu \geq 1,$$

$a_\nu^{(*0)} = -(\nu^2-\nu)^3(4\nu^2-4\nu-3)(8\nu^4+16\nu^3-8\nu-3)(8\nu^4-48\nu^3+96\nu^2-72\nu+13)/81$ *for* $\nu \geq 2$, $b_0^{(*0)} = 7/3$, $a_1^{(*0)} = -13/3$, $b_\nu^{(*0)} = 4(136\nu^8 - 504\nu^6 + 305\nu^4 - 84\nu^2 + 9)/9$ *for* $\nu \geq 1$.

I give here a sketch of proof of Theorem A. My research based on the results about difference systems connected with Mejer's functions; I have talk about these results on conference in memory of professor N.M.Korobov (see [8]).

Sketch of proof of Theorem A

Step 1. Auxiliary functions.

Let z satisfies to the following conditions,

$$|z| > 1, -3\pi/2 < \arg(z) \leq \pi/2, \log(z) = \ln(|z|) + i\arg(z), \tag{9}$$

and δ is the following differentiation $z\frac{\partial}{\partial z}$. Let α is nonnegative integer. My first auxiliary function is a finite sum

$$f_{\alpha,1}^{*\vee}(z,\nu) := f_{\alpha,1}^{*}(z,\nu) := \sum_{k=0}^{\nu+\alpha}(z)^k \binom{\nu+\alpha}{k}^2 \binom{\nu+k}{\nu}^2. \tag{10}$$

Let us consider the rational function given by the equality

$$R(\alpha,t,\nu) = \left(\prod_{j=1}^{\nu}(t-j)\right) \Bigg/ \left(\prod_{j=0}^{\nu+\alpha}(t+j)\right). \tag{11}$$

My second and fourth auxiliary function are sums of the following series

$$f_{\alpha,2}^{*}(z,\nu) = \sum_{t=1}^{+\infty} z^{-t}((\nu+\alpha)!/\nu!)^2(R(\alpha,t,\nu))^2, \tag{12}$$

$$f_{\alpha,4}^{*}(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t}\frac{(\nu+\alpha)!^2}{\nu!^2}\left(\frac{\partial}{\partial t}(R^2)\right)(\alpha,t,\nu). \tag{13}$$

Finally my third auxiliary function is defined as follows:

$$f^*_{\alpha,3}(z,\nu)=(\log(z))f^*_{\alpha,2}(z,\nu)+f^*_{\alpha,0,4}(z,\nu). \tag{14}$$

We consider also the functions $f_{\alpha,k}(z,\nu)$, $k=1,\,2,\,3,\,4$ connected with previous functions by means of the equalities

$$f_{\alpha,k}(z,\nu)=(\nu!/(\nu+\alpha)!)^2 f^*_{\alpha,k}(z,\nu) \tag{15}$$

where $k=1,\,2,\,3,\,4$, $\nu\in\mathbb{N}_0$. After expanding of the following rational function $((\nu+\alpha)!/\nu!)^2(-t)^r(R(\alpha,t,\nu))^2$ into partial fractions relatively t, and some transformations we come to the equality

$$\delta^r f^*_{\alpha,2+j}(z,\nu)-j(\log(z))\delta^r f^*_{\alpha,2}(z,\nu)= \tag{16}$$
$$\left(\sum_{i=1}^{2}(1-j+ij)\beta^{*(r)}_{\alpha,i}(z;\nu)L_{i+j}(1/z)\right)-\beta^{(r)}_{\alpha,3+j}(z;\nu),$$

where δ is operator $z\frac{\partial}{\partial z}$, $j=0,\,1$, $r=0,\,1,\,2,\,3$, $|z|>1$, $\alpha\in\mathbb{N}$, $s\in\mathbb{Z}$,

$$L_s(1/z)=\sum_{n=1}^{\infty}1/(z^n n^s) \tag{17}$$

are polylogarithms and $\beta^{*(r)}_{\alpha,0,i}(z;\nu)$, $\beta^{*(r)}_{\alpha,0,3+j}(z;\nu)$ are polynomials of z with rational coefficients.

Step 2. Pass to the difference system

The considered auxiliary functions $f^{\vee}_{\alpha,k}(z,\nu)$ are generalized hypergeometric functions known as Mejer's functions. They satisfy the following differential equation

$$D_\alpha(z,\nu,\delta)f^{\vee}_{\alpha,k}(z,\nu)=0, \tag{18}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$, $k\in\mathfrak{K}_0=\{1,\,2,\,3\}$,

$$D_\alpha(z,\nu,\delta)=z(\delta-\nu-\alpha)^2(\delta+\nu+1)^2-\delta^4. \tag{19}$$

is differential operator with differentiation $\delta:=z\frac{\partial}{\partial z}$. It follows from the general properties of the Mejer functions that

$$(\delta+\nu+1)^2 f_{\alpha,k}(z,\nu)=(\delta-\nu-1-\alpha)^2 f_{\alpha,k}(z,\nu+1), \tag{20}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$, $k\in\mathfrak{K}_0$. Since,

$$(1-1/z)^{-1}D_\alpha(z,\nu,\delta)=\delta^4-\sum_{k=1}^{4}b_{\alpha,k}\delta^{k-1},$$

we can in standard way come to the differential system

$$\delta X_{\alpha,k}(z;\nu) = B_\alpha(z;\nu)X_{\alpha,k}(z;\nu), \tag{21}$$

where $k = 1, 2, 3, |z| > 1, \nu \in \mathbb{N}_0$,

$$B_\alpha(z;\nu) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \\ b_{\alpha,1}(z;\nu) & b_{\alpha,2}(z;\nu) & b_{\alpha,3}(z;\nu) & b_{\alpha,4}(z;\nu) \end{pmatrix},$$

$$X_{\alpha,k}(z;\nu) = \begin{pmatrix} f^*_{\alpha,k}(z,\nu) \\ \delta f^*_{\alpha,k}(z,\nu) \\ \delta^2 f^*_{\alpha,k}(z,\nu) \\ \delta^3 f^*_{\alpha,k}(z,\nu) \end{pmatrix}$$

where $k = 1, 2, 3, |z| > 1$. In view of (19),

$$D_\alpha(z, -\nu - \alpha - 1, \delta) = D_\alpha(z, \nu, \delta). \tag{22}$$

Therefore we can put

$$X_{\alpha,k}(z; -\nu - 1 - \alpha) = X_{\alpha,k}(z;\nu), \tag{23}$$

where $\nu \in \mathbb{N}_0$ and consider $X_{\alpha,k}(z;\nu)$ on

$$\nu \in M^{***}_\alpha = ((-\infty, -1 - \alpha] \cup [0, +\infty) \cap \mathbb{Z},$$

Finally, we use the equations (18), (20) and (21) to obtain the following difference system.

Theorem 1. *The column $X_{\alpha,k}(z;\nu)$ satisfies to the equation*

$$\nu^5 X_{\alpha,k}(z;\nu - 1) = A^*_\alpha(z;\nu)X_{\alpha,k}(z;\nu), \tag{24}$$

where $\nu \in M^*_\alpha = (-\infty, -1 - \alpha] \cup [1, +\infty)) \cap \mathbb{Z}$, $k = 1, 2, 3$, $|z| > 1$; $A^*_\alpha(z;\nu)$ is 4×4-matrix, all elements of which are polynomials in $\mathbb{Q}[z, \nu, \alpha]$. Moreover, all these polynomials have degree 1 relatively z, and the matrix $A^*_\alpha(z;\nu)$ can be represented in the form

$$A^*_\alpha(z;\nu) = A^*_\alpha(1;\nu) + (z - 1)V^*_\alpha(\nu),$$

where the matrix $V^*_\alpha(\nu)$ does not depend from z.

Exact expressions of elements of the matrix $A^*_\alpha(z;\nu)$ can be found in [10]. Here we consider the case $\alpha = 1$. In this case elements of the matrix $A^*_1(z;\nu)$ are polynomials in $\mathbb{Q}[z, \nu]$. Exact form of the matrix $A^*_1(1;\nu)$ we specify below (see (32)). The matrix $A^*_\alpha(z;\nu)$ has the following property:

$$-\nu^5(\nu + \alpha)^5 E_4 = A^*_\alpha(z; -\nu - \alpha)A_\alpha,^*(z;\nu), \tag{25}$$

where E_4 is the 4×4 unit matrix, $z \in \mathbb{C}$, $\nu \in \mathbb{C}$. The equality (25) was very helpful for us, when we check our calculations.

Step 3. Reducing the obtained system to the difference system of second order in he case $\alpha = 1$.

This is key moment in our research, it leads to our results. In the case $\alpha = 1$, situation simplifies. The above system in this case is reducible and our task can be reduced to the consideration of system of second order. Let

$$\tau = \tau_1(\nu) = \nu + 1,\ \mu = \mu_1(\nu) = (\nu+1)^2. \tag{26}$$

then

$$\frac{1}{z}D_\alpha(z,\nu,\delta) = (1-1/z)\delta^4 + \sum_{k=0}^{3} r_{\alpha,k+1}(\nu)\delta^k, \tag{27}$$

where

$$r_1(\nu) = \mu_1(\nu)^2 = (\nu+1)^4 = \tau^4,\ r_2(\nu) = 0,$$
$$r_3(\nu) = -2\mu_1(\nu) = -2(\nu+1)^2,\ r_4(\nu) = 0,$$

and let us consider the row

$$R(\nu) = (r_1(\nu),\ r_2(\nu),\ r_(\nu),\ r_4(\nu)). \tag{28}$$

Let E_4 denotes 4×4-unit matrix, and let $C(\nu)$ is result of replacement of 1-th row of the matrix E_4 by the row in (28). Let further $D(\nu)$ denotes the adjoint matrix to the matrix $C(\nu)$. Then

$$C(\nu)D(\nu) = \mu^2 E_4,\ C(-\nu-2) = C(\nu),\ D(-\nu-2) = D(\nu). \tag{29}$$

Let me to introduce the matrix $A_1^{**}(z,\nu)$, which is connected with above matrix $A_1^{*}(z,\nu)$. All elements this matrix $A_1^{**}(z,\nu)$ are polynomials in $\mathbb{Q}[z,\nu]$ and they have degree 1 relatively z also. So, this matrix $A_1^{**}(z,\nu)$ can be represented also in the form

$$A_1^{**}(z;\nu) = A_1^{**}(1;\nu) + (z-1)V_1^{**}(\nu), \tag{30}$$

where the matrix $V_1^{**}(\nu)$ does not depend from z, and

$$A_1^{**}(1;\nu) = \begin{pmatrix} (\nu+1)^4\nu^5 & 0 & 0 & 0 \\ a^{**}_{1,2,1}(1;\nu) & a^{**}_{1,2,2}(1;\nu) & a^{**}_{1,2,3}(1;\nu) & 0 \\ a^{**}_{1,3,1}(1;\nu) & a^{**}_{1,3,2}(1;\nu) & a^{**}_{1,3,3}(1;\nu) & 0 \\ a^{**}_{1,4,1}(1;\nu) & a^{**}_{1,4,2}(1;\nu) & a^{**}_{1,4,3}(1;\nu) & (\nu+1)^4\nu^5 \end{pmatrix} \tag{31}$$

with

$$a^{**}_{1,2,1}(1;\nu) = -\tau^2(\tau-1)(2\tau-1)(6\tau^2-4\tau+1),$$
$$a^{**}_{1,2,2}(1;\nu) = \tau^5(\tau-1)(\tau^3+2(2\tau-1)^3),$$

$$a^{**}_{1,2,3}(1;\nu) = -3\tau^4(\tau-1)(2\tau-1)^3,$$
$$a^{**}_{1,3,1}(1;\nu) = \tau^2(\tau-1)^2(2\tau-1)(4\tau^2-3\tau+1),$$
$$a^{**}_{1,3,2}(1;\nu) = -2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3),$$
$$a^{**}_{1,3,3}(1;\nu) = \tau^4(\tau-1)^2((\tau-1)^3+2(2\tau-1)^3),$$
$$a^{**}_{1,4,1}(1;\nu) = -\tau^2(\tau-1)^3(2\tau-1)(2\tau^2-2\tau+1),$$
$$a^{**}_{1,4,2}(1;\nu) = \tau^5(\tau-1)^3(2\tau-1)(4\tau^2-5\tau+3),$$
$$a^{**}_{1,4,3}(1;\nu) = -\tau^4(\tau-1)^3(2\tau-1)(6\tau^2-8\tau+3).$$

I describe now the connection between matrices $A^{**}_1(z;\nu)$ and $A^*_1(z;\nu)$. We have

$$(\nu(\nu+1))^4 A^*_1(z,\nu) = D(\nu-1)A^{**}_1(z,\nu)C(\nu), \tag{32}$$

$$A^{**}_1(z,\nu) = C(\nu-1)A^*_1(z,\nu)D(\nu). \tag{33}$$

Let

$$Y_{1,k}(z;\nu) = \begin{pmatrix} y_{1,1,k}(z;\nu) \\ y_{1,2,k}(z;\nu) \\ y_{1,3,k}(z;\nu) \\ y_{1,4,k}(z;\nu) \end{pmatrix} = C(\nu)X_{1,k}(z;\nu), \tag{34}$$

where $k = 1, 2, 3$, $|z| > 1$, $\nu \in M^{***}_1 = ((-\infty,-2]\cup[0,+\infty))\cap\mathbb{Z}$. Then

$$Y_{1,k}(z;-\nu-2) = Y_{1,k}(z;\nu), \tag{35}$$

$$\mu_1(\nu)^2\nu^5 Y_{1,k}(z;\nu-1) = A^{**}_1(z,\nu)Y_{1,k}(z;\nu), \text{ where} \tag{36}$$

$k = 1, 2, 3$, $|z| > 1$, $\nu \in M^*_1 = ((-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z}$. Replacing in (36)

$$\nu \in M^*_1 \text{ by } \nu := -\nu-2 \in M^{**}_1 = ((-\infty,-3]\cup[0,+\infty))\cap\mathbb{Z},$$

and taking in account (35) we obtain the equality

$$\mu_1(\nu)^2(\nu+2)^5 Y_{1,k}(z;\nu+1) = -A^{**}_1(z,-\nu-2)Y_{1,k}(z;\nu), \tag{37}$$

where $k = 1, 2, 3$, $|z| > 1$, $\nu \in M^{**}_1 = ((-\infty,-3]\cup[0,+\infty))\cap\mathbb{Z}$.

We will tend $z \in (1,+\infty)$ to 1. Therefore we must study the behavior of our auxiliary functions, when we tend $z \in (1,+\infty)$ to 1. Then

$$t^r R(1,t,\nu)^2 = \left(\prod_{j=1}^{\nu}(t-j)^2\right) \Big/ \left(\prod_{j=0}^{\nu+1}(t+j)^2\right) = t^{r-4} + t^{r-5}O(1) \ (t\to\infty)$$

$$t^r\left(\frac{\partial}{\partial t}(R^2)\right)(1,t,\nu)=t^{r-5}O(1)\ (t\to\infty)$$

for $r=0,\ 1,\ 2,\ 3,\ 4$. Therefore

$$(z-1)\delta^r f_{1,2}(z,\nu)=$$

$$\sum_{t=1}^{+\infty} z^{-t}(-t)^r(R(\alpha,t,\nu))^2=(z-1)O(1)\ln(1-1/z)\to 0\ (z\to 1+0)$$

for $r=0,\ 1,\ 2,\ 3,$

$$(z-1)\delta^4 f_{1,2}(z,\nu)=$$

$$\sum_{t=1}^{+\infty} z^{-t}(-t)^4(R(\alpha,t,\nu))^2=1+(z-1)O(1)\ln(1-1/z)\to 1\ (z\to 1+0)$$

$$(z-1)\delta^r f_{1,4}(z,\nu)=$$

$$-\sum_{t=1}^{+\infty} z^{-t}(-t)^r\left(\frac{\partial}{\partial t}(R^2)\right)(1,t,\nu)=(z-1)O(1)\to 0\ (z\to 1+0)$$

for $r=0,\ 1,\ 2,\ 3,\ 4$ and

$$(z-1)\delta^r f_{1,3}(z,\nu)=(z-1)(\log(z))\delta^r f_{1,2}(z,\nu)+$$

$$(z-1)r\delta^{r-1}f_{1,2}(z,\nu)+(z-1)\delta^r f_{1,4}(z,\nu)\to 0\ (z\to 1+0)$$

for $r=0,\ 1,\ 2,\ 3,\ 4$. Clearly,

$$y_{1,j+1,k}(z,\nu)=\delta^j f_{1,k}(z,\nu)\text{ for }j=1,\ 2,\ 3\,k=1,\ 2,\ 3,\ |z|>1,\ \nu\in\mathbb{N}_0.$$

Further we have

$$y_{1,1,k}(1,\nu):=\lim_{z\to 1+0} y_{1,1,k}(z,\nu)=-\lim_{z\to 1+0}(1-1/z)\delta^4 f_{1,k}(z,\nu)= \tag{38}$$

$$(k-1)(k-3),\ \text{ where } k=1,\ 2,\ 3,\ ,\nu\in\mathbb{N}_0.$$

If we consider the second and third equations in (36) with $k=1,3$ and tend $z\in(1,+\infty)$ to 1, then, in view of (38) and (31), we obtain equations

$$\mu_1(\nu)^2\nu^5\delta^i f_{1,0,k}(1,\nu-1)=\sum_{j=1}^{2} a^{**}_{1,0,i+1,j+1}(1;\nu)(\delta^j f_{1,0,k})(1,\nu) \tag{39}$$

where $i=1,\ 2,\ k=1,\ 3,\ \nu\in M_1^*=((-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z}$. Let are given

$$F=\{F(\nu)\}_{\nu=-\infty}^{+\infty}\text{ and }G=\{G(\nu)\}_{\nu=-\infty}^{+\infty}\text{ such that}$$

$$F(-\nu-2)=F(\nu),\ G(-\nu-2)=G(\nu),\ F(\nu)\in\mathbb{Q},\ G(\nu)\in\mathbb{Q}$$

for $\nu \in \mathbb{Z}$. Then, in view of (35),

$$y^{**}_{F,G,k}(z,-\nu-2) = y^{**}_{F,G,k}(z,\nu) := F(\nu)\delta f_{1,0,k}(1,\nu) + G(\nu)\delta^2 f_{1,0,k}(z,\nu)$$

for $k = 1,3$, $\nu \in M_1^{***} = ((-\infty,-2] \cup [0,+\infty)) \cap \mathbb{Z}$. Let

$$a^{***}_{F,G,j}(z;\nu) = F(\nu-1)a^{**}_{1,0,2,j}(1,\nu) + G(\nu-1)a^{**}_{1,0,3,j}(1,\nu)$$

for $j = 1, 2, 3$ and $\nu \in M_1^{*} = ((-\infty,-2] \cup [1,+\infty)) \cap \mathbb{Z}$. In view of (39),

$$\sum_{j=1}^{2} a^{***}_{F,G,j+1}(1;\nu)(\delta^j f_{1,0,k})(1,\nu) = \mu_1(\nu)^2\nu^5 y^{***}_{F,G}(z,\nu-1), \tag{40}$$

for $k = 1, 3$, $\nu \in M_1^{*} = ((-\infty,-2] \cup [1,+\infty)) \cap \mathbb{Z}$. Replacing in (40)

$$\nu \in M_1^{*} \text{ by } \nu := -\nu-2 \in M_1^{**} = ((-\infty,-3] \cup [0,+\infty)) \cap \mathbb{Z},$$

and taking in account the equality (35) we obtain the equalities

$$\sum_{j=1}^{2} a^{***}_{F,G,j+1}(1;-\nu-2)(\delta^j f_{1,0,k})(1,\nu) = -\mu_1(\nu)^2(\nu+1)^5 y^{**}_{F,G}(z,\nu+1), \tag{41}$$

where $k = 1, 3$ and $\nu \in M_1^{**} = ((-\infty,-3] \cup [0,+\infty)) \cap \mathbb{Z}$. Let

$$\vec{w}_{F,G,j}(\nu) = \begin{pmatrix} a^{***}_{F,G,j+1}(1;-\nu-2) \\ F(\nu)(2-j) + G(\nu)(j-1) \\ a^{***}_{F,G,j+1}(1;\nu-1) \end{pmatrix},$$

where $j = 1,2$, $\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}$, $W_{F,G}(\nu) =$

$$\begin{pmatrix} \vec{w}_{F,G,1}(\nu) & \vec{w}_{F,G,2}(\nu) \end{pmatrix} = \begin{pmatrix} a^{***}_{F,G,2}(1;-\nu-2) & a^{***}_{F,G,3}(1;-\nu-2) \\ F(\nu) & G(\nu) \\ a^{***}_{F,G,2}(1;\nu) & a^{***}_{F,G,3}(1;\nu) \end{pmatrix},$$

$$Y_k^{***}(\nu) = \begin{pmatrix} \delta f_{1,0,k}(1,\nu) \\ \delta^2 f_{1,0,k}(1,\nu) \end{pmatrix}, Y^{****}_{F,G,k}(\nu) = \begin{pmatrix} -\mu_1(\nu)^2(\nu+2)^5 y^{**}_{F,G,k}(z,-\nu-3) \\ y^{**}_{F,G,k}(z,\nu) \\ \mu_1(\nu)^2\nu^5 y^{**}_{F,G}(z,\nu-1) \end{pmatrix}$$

for $k = 1, 3$, $\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}$. In view of (40) – (41),

$$Y^{****}_{F,G,k}(\nu) = W_{F,G}(\nu)Y_k^{***}(\nu).$$

Let further

$$\vec{w}_{F,G,3}(\nu) = \begin{pmatrix} w_{F,G,3,1}(\nu) \\ w_{F,G,3,2}(\nu) \\ w_{F,G,3,3}(\nu) \end{pmatrix} = [\vec{w}_{F,G,1}(\nu), \vec{w}_{F,G,2}(\nu)]$$

is vector product of $\vec{w}_{F,G,1}(\nu)$ and $\vec{w}_{F,G,2}(\nu)$, and let $\bar{w}_{F,G,3}(\nu) = (\vec{w}_{F,G,3}(\nu))^t$ is the row conjugate to the column $\vec{w}_{F,G,3}(\nu)$. Then we have the following equalities for the scalar products $(\vec{w}_{F,G,3}(\nu), \vec{w}_{F,G,j}(\nu))$:

$$\bar{w}_{F,G,3}(\nu)\vec{w}_{F,G,j}(\nu) = (\vec{w}_{F,G,3}(\nu), \vec{w}_{F,G,j}(\nu)) = 0,$$

where $j = 1, 2$ and $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$. Therefore

$$\bar{w}_{F,G,3}(\nu)W_{F,G}(\nu) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \tag{42}$$

where $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$. In view of (39) (41) and (42),

$$\bar{w}_{F,G,3}(\nu)Y^{*****}_{F,G,k}(\nu) = \bar{w}_{F,G,3}(\nu)W_{F,G}(\nu)Y^{***}_k(\nu) = 0,$$

where $k = 1, 3$ and $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$.

So, for given F and G we came to difference equation of second order, which leads to our results. First we take $F(\nu) = 1$ and $G(\nu) = 0$ for all $\nu \in \mathbb{Z}$. Then we obtain the first expansion specified in Theorem A. After that we take $F(\nu) = 1/3$ and $G(\nu) = 2/3$ for all $\nu \in \mathbb{Z}$ Then we obtain the second expansion specified in Theorem A.

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