UNIRATIONALITY AND EXISTENCE OF INFINITELY TRANSITIVE MODELS

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ABSTRACT. We study unirational algebraic varieties and the fields of rational functions on them. We show that after adding a finite number of variables some of these fields admit an *infinitely transitive model*. The latter is an algebraic variety with the given field of rational functions and an infinitely transitive regular action of a group of algebraic automorphisms generated by unipotent algebraic subgroups. We expect that this property holds for all unirational varieties and in fact is a peculiar one for this class of algebraic varieties among those varieties which are rationally connected.

1. INTRODUCTION

This article aims to relate unirationality of a given algebraic variety with the property of being a homogeneous space with respect to unipotent algebraic group action. More precisely, let X be an algebraic variety defined over a field $\mathbf{k}$, and $\mathrm{Aut}(X)$ be the group of regular automorphisms of X. Let also $\mathrm{SAut}(X) \subseteq \mathrm{Aut}(X)$ be the subgroup generated by algebraic groups isomorphic to the additive group $\mathbb{G}_a$.

Definition 1.1 (cf. [1]). *We call variety X infinitely transitive if for any $k \in \mathbb{N}$ and any two collections of points $\{P_1, \ldots, P_k\}$ and $\{Q_1, \ldots, Q_k\}$ on X there exists an element $g \in \mathrm{SAut}(X)$ such that $g(P_i) = Q_i$ for all i. Similarly, we call X stably infinitely transitive if $X \times \mathbf{k}^m$ is infinitely transitive for some m.*

Recall that in Birational Geometry adding a number m of algebraically independent variables to the function field $\mathbf{k}(X)$ is referred to as *stabilization*. Geometrically this precisely corresponds to taking the product $X \times \mathbf{k}^m$ with the affine space. Note also that if X is infinitely transitive, then it is unirational, i.e., $\mathbf{k}(X) \subseteq \mathbf{k}(y_1, \ldots y_m)$ for some $\mathbf{k}$-transcendental elements y_i (see [1, Proposition 5.1]). This suggests to regard (stable) infinite transitivity as a birational property of X (in particular, we will usually assume the test variety X to be smooth and projective):

Definition 1.2. *We call variety X stably b-infinitely transitive if the field $\mathbf{k}(X)(y_1, \ldots y_m)$ admits an infinitely transitive model (not necessarily smooth or projective) for some m and $\mathbf{k}(X)$-transcendental elements y_i. If $m = 0$, we call X b-infinitely transitive.*

Example 1.3. The affine space $X := \mathbf{k}^{\dim X}$ is stably infinitely transitive (and infinitely transitive when $\dim X \geq 2$), see [9]. More generally, any rational variety is stably b-infinitely transitive, and it is b-infinitely transitive if the dimension ≥ 2.

Example 1.3 suggests that being stably b-infinitely transitive does not give anything interesting for rational varieties. In the present article, we put forward the following:

Conjecture 1.4. *Any unirational variety X is stably b-infinitely transitive.*

Thus, Conjecture 1.4 together with the above mentioned result from [1, Proposition 5.1] provides a (potential) characterization of unirational varieties among all those which are rationally connected. Note also that the class of rationally connected varieties contains all stably b-infinitely transitive varieties. We think that not every rationally connected variety is stably birationally infinitely transitive. In particular we expect that generic Fano hypersurfaces from the family considered by Kollar in [11] are not stably birationally infinitely transitive. These are generic smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$, $d > \dfrac{2}{3}(n+3)$. Our expectations are based on the Kollar's fundamental observation (see [11, Thm. (4.3)]) which yields strong restrictions on any surjective map of a uniruled variety of the same dimension on such a hypersurface.

Remark 1.5. Originally, the study of infinitely transitive varieties was initiated in the paper [9]. We also remark one application of these varieties to the Lüroth problem in [1], where a non-rational infinitely transitive variety was constructed. See [4] for the properties of locally nilpotent derivations (LNDs for short), [16] for the Makar-Limanov invariant, and [2], [5], [6], [10], [13], and [15] for other results, properties and applications of infinitely transitive (and related) varieties.

We are going to verify Conjecture 1.4 for some particular cases of X (see Theorems 2.1, 2.2 and Propositions 3.4, 3.5 and 3.6 below). At this stage, one should note that it is not possible to lose the stabilization assumption in Conjecture 1.4:

Example 1.6. Any three-dimensional algebraic variety X with an infinitely transitive model is rational. Indeed, let us take a one-dimensional algebraic subgroup $G \subset \mathrm{SAut}(X)$ acting on X with a free orbit. Then X is birationally isomorphic to $G \times Y$ (see Remark 2.16 below), where Y is a rational surface (since X is unirational). On the other hand, if $X := X_3 \subset \mathbb{P}^4$ is a smooth cubic hypersurface, then it is unirational but not rational (see [3]). However, Conjecture 1.4 is true as stated for X_3, because X_3 is stably b-infinitely transitive (see Proposition 3.4 below). In this context, it would be also interesting to settle down the case of the quartic hypersurface X_4 in $\mathbb{P}^4$ (or, more generally, in $\mathbb{P}^n$ for arbitrary n), which relates our subject to the old problem of (non-)unirationality of (generic) X_4 (cf. Remark 3.7 below).

Notations 1.7. Throughout the paper we keep up with the following:
- $\mathbf{k}$ is an algebraically closed field of characteristic zero and $\mathbf{k}^{\times}$ is the multiplicative group of $\mathbf{k}$;
- $X_1 \approx X_2$ denotes birational equivalence between two algebraic varieties X_1 and X_2;
- we abbreviate infinite transitivity (transitive, transitively, etc.) to inf. trans.

2. Varieties with many cancellations

2.1. **The set-up.** The goal of the present section is to prove the following:

Theorem 2.1. *Let $K := \mathbf{k}(X)$ for some (smooth projective) algebraic variety X of dimension n over $\mathbf{k}$. We assume there are n presentations (we call them cancellations (of K or X)) $K = K'(x_i)$ for some K'-transcendental elements x_i, algebraically independent over $\mathbf{k}$. Then there exists an inf. trans. model of $K(y_1, \ldots, y_n)$ for some K-transcendental elements y_i.*

Let us put Theorem 2.1 into a geometric perspective. Namely, the presentation $K = K'(x_i)$ reads as there exists a model of K, say X_i^n, with a surjective regular map $\pi_i : X_i^n \to Y_i^{n-1}$ and general fiber $\simeq \mathbb{P}^1$ such that π_i admits a section over an open subset in Y_i^{n-1}. Moreover, by resolving indeterminacies, we may assume $X_i^n := X$ fixed for all i. Then, since K admits n cancellations, n vectors, each tangent to a fiber of some π_i, span the tangent space to X at the general point. Indeed, we have a map to $\mathbb{P}^n$

$$X \dashrightarrow \mathbb{P}^n, \quad x \mapsto (1 : x_1(x) : \ldots : x_n(x)).$$

It is dominant since elements $x_1, \ldots, x_n$ are algebraically independent over $\mathbf{k}$, and the tangent map is surjective at the general point. So we obtain the geometric counterpart of Theorem 2.1:

Theorem 2.2. *Let X be a smooth projective variety of dimension n. Assume that there exist n morphisms $\pi_i : X \to Y_i$ satisfying the following:*

    (1) *Y_i is a (normal) projective variety such that π_i admits a section over an open subset in Y_i;*

    (2) *for the general point $\zeta \in X$ and the fiber $F_i = \mathbb{P}_i^1 := \pi_i^{-1}(\pi_i(\zeta)) \simeq \mathbb{P}^1$, vector fields $T_{F_1, \zeta}, \ldots, T_{F_n, \zeta}$ span the tangent space $T_{X, \zeta}$.*

Then X is stably b-inf. trans.

Note that existence of a section over an open subset on Y_i means (almost by definition) birational triviality of the fibration π_i.

In Sections 2.2 and 2.3 we illustrate our arguments by considering the cases when $\dim X = 1$ and 2, respectively. In higher dimensions we additionally need the following:

    (3) *for some ample line bundles H_i on Y_i and their pullbacks $\pi_i^* H_i$ to X, the $n \times n$-matrix (2.5) $(\pi_i^* H_i \cdot \mathbb{P}_i^1)$ is of maximal rank (in particular, the classes of $\pi_1^* H_1, \ldots, \pi_n^* H_n$ in $\mathrm{Pic}(X)$ are linearly independent).*

In particular, this means that the fibers $\mathbb{P}_1^1, \mathbb{P}_2^1, \ldots, \mathbb{P}_n^1$ are linearly independent in $H_2(X)$. In Sections 2.4, 2.10 and 2.11 we prove Theorem 2.2, assuming that the condition **(3)** is satisfied. Furthermore, adding new variables (i.e., forming the product of X and an affine space) and passing to a (good) birational model, we may assume that **(3)** holds, see Sections 2.12 and 2.13.

2.2. **One-dimensional case.** Variety $\mathcal{O}(m)_{\mathbb{P}^1}^{\times}$ (or, equivalently, $\mathcal{O}(-m)_{\mathbb{P}^1}^{\times}$) is just an affine cone minus the origin over a rational normal curve of degree m. Thus $\mathcal{O}(m)_{\mathbb{P}^1}^{\times}$ is a quasiaffine toric variety, so it is infinitely transitive by [2, Theorem 0.2(3)]. Indeed, we can use only those automorphisms which preserve the origin, i.e., for m-transitivity on $\mathbf{k}^2 \setminus \{0\}$ we use $(m + 1)$-transitivity on $\mathbf{k}^2$.

2.3. **Two-dimensional case.** Let us study now the next simplest case when $X = \mathbb{P}^1 \times \mathbb{P}^1$. Choose $H_2 := \mathcal{O}(1)$ on the first factor $\mathbb{P}^1$ and, similarly, $H_1 := \mathcal{O}(1)$ on the second factor $\mathbb{P}^1$. Now take the pullbacks $\pi_1^* H_1$ and $\pi_2^* H_2$ to X and throw away their zero sections. We obtain a toric bundle over X isomorphic to $(\mathbf{k}^2 \setminus \{0\}) \times (\mathbf{k}^2 \setminus \{0\})$. The latter is inf. trans. since $\mathbf{k}^2 \setminus \{0\}$ is (cf. the one-dimensional case above).

More generally, if one starts with $H_2 = \mathcal{O}(m_2)$ and $H_1 = \mathcal{O}(m_1)$ for some $m_i \geqslant 1$, then the resulting variety will be $((\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_1\mathbb{Z})) \times ((\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_2\mathbb{Z}))$. It is again inf-transitive being the product of two inf-transitive varieties. Indeed, for $m_i > 1$ the corresponding variety is just the smooth locus on the corresponding toric variety, and its inf-transitivity is shown in [2, Theorem 0.2(3)].

Remark 2.3. The product of two (quasiaffine or affine) inf-transitive varieties is inf-transitive. Indeed, we call variety X *flexible* if the tangent space at every smooth point on X is generated by the tangent vectors to the orbits of one-parameter unipotent subgroups in $\mathrm{Aut}(X)$. It was shown in [1] that for affine X being flexible is equivalent to inf-transitivity. But clearly the product of two flexible varieties is again flexible.

2.4. **Construction of an inf. transitive model in the simplest case.** Recall the setting. In the notation of Theorem 2.2, we choose very ample line bundles H_i on each Y_i, $i = 1, \ldots, n$, take their pullbacks $\pi_i^* H_i$ to X, put $m_{ij} := (\pi_i^* H_i)|_{\mathbb{P}_j^1}$, and form the *intersection matrix*

$$(2.5) \qquad M_n = M_n(X) = (m_{ij})_{1 \leqslant i,j \leqslant n}, \quad m_{ij} = (\pi_i^* H_i)|_{\mathbb{P}_j^1}.$$

Clearly, for all i we have $(\pi_i^* H_i)|_{\mathbb{P}_i^1} = 0$; however, for $i \neq j$, $(\pi_i^* H_i)|_{\mathbb{P}_j^1} > 0$, being equal the restriction of H_i to an image of a generic $\mathbb{P}^1$ via p_i (i.e. the restriction of a bundle on the variety Y_i). The matrix M_n defines a linear map from a subgroup of the Pickard group $\mathrm{Pic}\, X$ to $\mathbb{Z}^n$. In this section we suppose that the classes of $\mathbb{P}_1^1, \ldots, \mathbb{P}_n^1$ in $H_2(X)$ are linearly independent, and also that $\det M_n \neq 0$. Our goal is to construct a quasiaffine variety $\mathfrak{T}_X$, $\mathfrak{T}_X \approx X \times k^N$ for some N, equipped with a collection of projections to quasiaffine varieties $\bar{Y}_i$ with generic fibers being equal to $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m\mathbb{Z})$, and such that an open subset of $\mathfrak{T}_X$ is inf-transitive, cf. Section 2.3. The existence of a good open subset will be shown in Section 2.10.

To start with, let us set

$$\bar{Y}_i := \text{the affine cone } \mathcal{O}_{Y_i}(H_i)^\times \text{ minus the origin}$$

over Y_i embedded via H_i, $1 \leqslant i \leqslant n$. It is a quasiaffine variety.

2.5.1. *Technical step – adding one more coordinate.* We already embedded Y_i into affine varieties, now we also need to embed X. For this purpose, we take a very ample line bundle H_0 on X, replace X with $X' = X \times \mathbb{P}^1$, and Y_i with $Y_i' = Y_i \times \mathbb{P}^1$. Let also $Y_0 = X$, clearly we have $X' \to Y_0 = X$, which makes the situation absolutely symmetric with respect to indices $0, 1, \ldots, n$. We modify the set of H_is in the following way: for every $i > 0$, we construct H_i' on Y_i' being the sum of the trivial lift of H_i from Y_i and $\mathcal{O}(1)$ on the new $\mathbb{P}^1$ (in fact here we can take any $\mathcal{O}(n_i)$). Now the intersection matrix $M_{n+1}(X')$ takes the form

$$M_{n+1}(X') = \begin{pmatrix} 0 & k_1 & \ldots & k_n \\ 1 & & & \\ \vdots & & M_n & \\ 1 & & & \end{pmatrix}.$$

Here $k_i := H_0 \cdot \mathbb{P}_i^1$. We further denote X' just by X and $n + 1 = \dim X'$ just by n, keeping in mind that one of our projections is just a trivial projection. We also assume that one column of our matrix contains only 1s (and one 0 on the diagonal).

2.5.2. *The construction of $\mathfrak{T}_X$.* We construct a vector bundle

$$H_1 \times_X \pi_2^* H_2 \times_X \pi_3^* H_3 \times_X \ldots \times_X \pi_n^* H_n$$

and furthermore a toric bundle

$$(2.6) \qquad \mathfrak{T}_X = (H_1)^\times \times_X (\pi_2^* H_2)^\times \times_X \ldots \times_X (\pi_n^* H_n)^\times.$$

We denote by δ the canonical projection $\mathfrak{T}_X \to X$. Line operations in the intersection matrix (2.5) correspond to base changes in this toric bundle (Neron-Severi torus). For our convenience, we fix below the following set of line bundles $L_1, \ldots, L_n \in \langle H_1, \pi_2^* H_2, \ldots, \pi_n^* H_n \rangle$:

(i) each of them should be primitive in the lattice $\mathbb{Z}(H_1, \pi_2^* H_2, \ldots, \pi_n^* H_n)$,

(ii) all in total, they should be linearly independent in the lattice

$$\mathbb{Z}(H_1, \pi_2^* H_2, \ldots, \pi_n^* H_n).$$

They can be chosen in the following way. There is a map

$$\mathbb{Z}(H_1, \ldots, H_n) \xrightarrow{M_n} \mathbb{Z}^n \xrightarrow{i\text{th coordinate}} \mathbb{Z}.$$

Its kernel has dimension $n - 1$, and H_i itself belongs to the kernel. So in fact there is a map $\mathbb{Z}(H_1, \ldots, \check{H}_i, \ldots, H_n) \to \mathbb{Z}$, and L_i is any covector defining this map. All in total, they can be chosen linearly independent.

2.6.3. *Construction of local 2-dimensional coordinates.* Recall that we denote by $\bar{Y}_i$ the total space of $H_i^\times \to Y_i$.

Lemma 2.4. *For each i, there is a fibration $\varphi_i : \mathfrak{T}_X \to \bar{Y}_i$ such that its general fiber equals $((\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_i\mathbb{Z})) \times T_i^{n-2}$, where $T_i^{n-2} \simeq (\mathbf{k}^\times)^{n-2}$.*

Proof. Choose a basis $\langle H_i, L_i, H_1', \ldots, H_{n-2}' \rangle$ and a linear map $\mathbb{Z}^n \to \mathbb{Z}^2$ which is just taking the two first coordinates in the new basis. Its kernel will correspond precisely to a $(n-2)$-dimensional torus, the bundle H_i will provide us with the affine cone $\bar{Y}_i$ over Y_i, and the bundle L_i restricted to $\mathbb{P}_i^1$ will form a quasiaffine fiber of form $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m\mathbb{Z})$ over a general point of $\bar{Y}_i$. $\qquad \square$

We have a commutative diagram.

$$(2.7) \qquad \begin{array}{ccc} \mathfrak{T}_X & \xrightarrow{T_i^{n-2}} & L_i^\times \times H_i^\times \\ \downarrow & & \downarrow {\scriptstyle (\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/n_i\mathbb{Z}) \times \ldots} \\ X & \longrightarrow & Y_i \end{array}$$

$$(2.8) \qquad \text{where} \quad \mathfrak{T}_X \xrightarrow{T_i^{n-2} \times L_1^\times \times \mathbb{P}^1} \bar{Y}_i \longrightarrow Y_i$$

This realization will be intensively used below. Note that the fibration is trivial over any open subset U in Y_i such that all the fibers of π_i are $\mathbb{P}^1$s over U and the restriction of all Hs are generic on these fibers, and respectively over $\bar{Y}_i$. So if one fixes a finite number of points $P_1, \ldots, P_s$ in $\bar{Y}_i$, we can choose an open subset U' in $\bar{Y}_i$ containing the fibers passing through all these points (since it is quasiaffine).

Lemma 2.5. *At the general point x on $\mathfrak{T}_X$, local coordinates on $((\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_i\mathbb{Z}))$-fibers from Lemma 2.4, $i = 1, \ldots, n$, form a system of local coordinates on $\mathfrak{T}_X$ at x.*

Proof. In the notation of Lemma 2.4, tangent space to each fiber of φ_i is spanned by a pair of the tangent vectors to $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_i\mathbb{Z})$ and by tangent vectors to T_i^{n-2}. By the condition **(2)** of Theorem 2.2 and by non-degeneracy of matrix M_n, the tangent vectors to $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_i\mathbb{Z})$ are linearly independent, which proves the assertion. $\quad \square$

2.8.4. *Quasiaffineness of $\mathfrak{T}_X$.* Here we exploit the projections (2.8) and the technique from the proof of Lemma 2.4.

Lemma 2.6. *The variety $\mathfrak{T}_X$ is quasiaffine.*

Proof. The bundle H_1 gives an embedding of X to a projective space $\mathbb{P}^{N_1}$, and every H_i, $i = 2, \ldots, n$, embeds Y_i to a $\mathbb{P}^{N_i}$. The variety $\mathfrak{T}_X$ is now $\{(x, l_2, \ldots, l_n)\}$ such that $x \in X$, $l_i \in \operatorname{cone}(\pi_i(x))$ in $\mathbb{A}^{N_1 + N_2 + \ldots + N_n}$. $\qquad\square$

Note that $\mathfrak{T}_X \to X$ is a principal toric bundle which has a section (the diagonal), and all the fibers are isomorphic to $(\mathbf{k}^\times)^n$ (see formula (2.6)). In particular, we have $\mathfrak{T}_X \approx X \times \mathbf{k}^n$.

2.8.5. *Idea of further proof.*

Proposition 2.7. *The variety $\mathfrak{T}_X$ is stably b-inf. trans.*

Its proof will be given in Section 2.11. We use the ideas from [9], [2] and [1] to move an m-tuple of general (in the sence of Section 2.10) points to another such m-tuple.

2.9. **Stratification on X.** Let $q \in X$ be an arbitrary point. We denote by $X(q)$ the locus of all points on X connected to q by a sequence of smooth fibers $\mathbb{P}^1_i$ of the projections π_i, $1 \leqslant i \leqslant n$.

Lemma 2.8. *Let Z be an irreducible subvariety of X. Consider all smooth fibers $\mathbb{P}^1_i$ passing through the points of Z and the union Z' of all such fibers. Then either $\dim Z' > \dim Z$ or all smooth fibers $\mathbb{P}^1_i$ which contain points in Z are actually contained in the closure $\bar{Z}$.*

Proof. If the curve $\mathbb{P}^1_i$ intersects Z but is not contained in Z then the curves in the same family intersect an open subvariety in Z since the subvariety $\tilde{X}_i$ consisting of curves $\mathbb{P}^1_i$ is an open subvariety of X. Hence in the latter case $\dim Z' > Z$. Otherwise all the smooth fibers $\mathbb{P}^1_i$ which contain points in Z are actually contained in the closure of Z. Note that the same holds even if a line $\mathbb{P}^1_i$ intersects the closure $\bar{Z}$ but is not contained in $\bar{Z}$. $\qquad\square$

Corollary 2.9. *Every point in $X(q)$ is connected to q by a chain of $\mathbb{P}^1_i$ of length at most n^2.*

Proof. Indeed, let $X_p(q)$ be a subvariety obtained after adding the points connected by the chains of curves of length at most p. It is a union of algebraic subvarieties of X of dimension $\leqslant p$.

Then by adding the curves from all n families of $\mathbb{P}^1_i$ we either increase the dimension of every component of maximal dimension, or one of them $X_p^0(q)$ is invariant, i.e. all smooth fibers $\mathbb{P}^1_i$ which contain points in $X_p^0(q)$ are actually contained in the closure of $X_p^0(q)$. Note that in the latter case since $q \in X_p^0(q)$, all other components are contained in $X_p^0(q)$, and hence $X_p^0(q) = X(q)$. Thus after adding lines from different families we obtain either $X(q)$ or a variety $X_{p+n}(q)$ with maximal component of greater dimension. Thus we will need at most n^2 lines to get $X(q)$. $\qquad\square$

Remark 2.10. If we started with a generic point $q \in X$, then it follows from Lemma 2.8 that $\dim X(q) = n$. Indeed, the condition **(2)** of Theorem 2.2 implies that the tangent vectors to the smooth $\mathbb{P}^1_i$-fibers in q generate the full tangent space in q, and if $X(q)$ was of lower dimension then the tangent space would also be of lower dimension.

Remark 2.11. The bound in Corollary 2.9 is not effective. By a more thorough examination one can show that the sequence

$$X_0(q) \subseteq X_1(q) \subseteq \ldots$$

stabilizes earlier than at the n^2th step.

Corollary 2.12. *We can apply the same in reverse. Consider $x \in X(q)$. Then all points in X which are connected to $x \in X(q)$ can be connected by a chain of length at most $n^2 + n$.*

Proof. Indeed, for any such point x' we have $X_{n^2}(x')$ of dimension n and hence contains an open subvariety in X. It may take at most n $\mathbb{P}^1_i$s to connect to x. $\qquad\square$

Thus the variety obtained from general points in X in n^2 steps coincides with the subvariety of all points in X connected to general point by a chain of smooth lines.

2.10. **Construction of a big open subset in $\mathfrak{T}_X$.** Now we pass from stratification on X to stratification on $\mathfrak{T}_X$. We stratify $\mathfrak{T}_X$ in the following way: we take toric preimages for every strata in X. For our needs we take the toric preimage of $X(q)$ for a general point $q \in X$. Note that for every fiber $\mathbb{P}^1_i$ of π_i its preimage is $(((\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m\mathbb{Z})) \times T^{n-2}$, and for every chain $P_1 - P_2 - \ldots - P_k$ connecting two points in X there is a chain $\bar{P}_1 - \bar{P}_2 - \ldots - \bar{P}_k$ in $\mathfrak{T}_X$ such that every two adjacent points belong to the same $((\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m\mathbb{Z}))$ for one of the projections,

2.11. **Proof of Proposition 2.7.** Let a variety $\mathfrak{T}_X$ be as in (2.6). For each i we have a fibration with the quasiaffine base and fiber being $((\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_i\mathbb{Z}) \times T^{n-2}$, see (2.8).

Definition 2.13. *For the points $C_1, \ldots, C_r$ in the base of projection (2.8), let $\mathrm{Stab}_{C_1,\ldots,C_r}$ be the subgroup in $\mathrm{SAut}(\mathfrak{T}_X)$ preserving all the fibers of the projection and fixing pointwise the fibers above $C_1, \ldots, C_r$.*

Now we need some technique concerning locally nilpotent derivations (LNDs). To lift automorphisms, we need to extend an LND on $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m\mathbb{Z})$ to a LND on $\mathfrak{T}_X$ (i.e. to a locally nilpotent derivation of the algebra $\mathbf{k}[\mathfrak{T}_X]$). More precisely, suppose that we chose a fiber of form $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m\mathbb{Z})$ of the projection (2.8) and some other fibers that we want to fix. We can project all these subvarieties to $\bar{Y}_i$ and then take a regular function on $\bar{Y}_i$ which equals 1 at the projection of the first fiber and 0 in the projections of other fibers. If we multiply the LND by this function (obviously belonging to the kernel of the derivation) and trivially extend it to the toric factor, we will obtain a rational derivation well-defined on an open subset U of $\mathfrak{T}_X$ corresponding to the smooth locus of the corresponding π_i. Now we can take a regular function on $\mathfrak{T}_X$ (lifted from a regular function on $\bar{Y}_i$) such that its zero locus contains the singular locus of the projection, multiply the derivation by some power of this function and obtain a regular LND on $\mathfrak{T}_X$.

For a given m-tuple of points $P_1, P_2, \ldots, P_m$, we need the following lemma:

Lemma 2.14. *In the notation as above (2.8), let C_0 be a point on a base such that the fiber over this point is general, and $P_1, \ldots, P_s$ be some points from this fiber with different projections to $\bar{Y}_i$. Let also $C_1, \ldots, C_r$ be some other points of the base. Then the subgroup $\mathrm{Stab}_{C_1,\ldots,C_r}$ acts infinitely transitively on the fiber over C_0, i.e. can map $P_1, \ldots, P_s$ in any other subset in the same fiber.*

Proof. By [2, Theorem 0.2(3)], the fiber is infinitely transitive. For every one-parameter unipotent subgroup of automorphisms on this fiber, we can lift it to $\mathfrak{T}_X$, fixing pointwise a given finite collection of fibers, see above. $\qquad\square$

Now it remains to prove infinite transitivity for $\mathfrak{T}_X$. There are two ways to show it.

2.11.1. *Way 1.*

Lemma 2.15. *For $m + 1$ points $P_1, P_1', P_2, P_3, \ldots, P_m$ projecting to the chosen above open subset in X, there exists an automorphism mapping P_1 to P_1' and preserving all the other points.*

Proof. There always exists a small automorphism which moves the initial set to a set where for all i all the $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_i\mathbb{Z})$-coordinates of the given points are different. Let us connect the projections of P_1 and P_1' by a chain of smooth $\mathbb{P}_i^1$-curves in X. We denote by $Q_1, \ldots, Q_s$ the intersection points of these curves, $Q_i \in X$, $Q_1 = \delta(P_1)$, $Q_s = \delta(P_1')$. For $i = 2, \ldots, (s-1)$ we take some lifts $R_i \in \mathfrak{T}_X$ of these points in such a way that all their $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m_i\mathbb{Z})$-coordinates do not coincide with the corresponding coordinates of the previous points. Let $R_1 := P_1$ and $R_s = P_1'$. For every i, $1 \leqslant i \leqslant (s-1)$, we want to map R_i to R_{i+1} by an automorphism of $\mathfrak{T}_X$ preserving all the other points in the given set. We may assume that R_i and R_{i+1} belong to one two-dimensional fiber of form $(\mathbf{k}^2 \setminus \{0\})/(\mathbb{Z}/m\mathbb{Z})$ of one of the projections to $\bar{Y}_j \times T_j^{n-2}$ (the toric fibration is generated by L_is, and we can densify the sequence of R_is if needed to change only one L_i-direction at every step to fulfill this condition). Every such two-dimensional fiber is inf-transitive. Now we need to lift the corresponding automorphism to $\mathfrak{T}_X$. We need two following observations. First, if we are lifting a curve with respect to the projection π_i, then the resulting automorphism is well defined over the singular fibers and is trivial there. Second, all the two-dimensional fibers belonging to the same fiber of φ_i move together, and if several P_j belong to the same fiber as the R_i which we are moving, then we use that their projections to the two-dimensional fiber are different and also different from the projection of R_{i+1}, and we use inf-transitivity (not only 1-transitivity) of the corresponding fiber. Now for every i we lift the corresponding automorphism of the 2-dimensional fiber to an automorphism of $\mathfrak{T}_X$ from the corresponding subgroup Stab fixing the points from the other fibers of δ, and multiply all these automorphisms. It does not change $P_2, \ldots, P_m$ and maps P_1 to P_1'. This ends the proof. □

Now infinite transitivity follows easily: to map $P_1, P_2, \ldots, P_m$ to $Q_1, Q_2, \ldots, Q_m$, we map P_1 to Q_1 fixing $P_2, \ldots, P_m, Q_2, \ldots, Q_m$, etc. .

2.11.2. *Way 2.* The other way to finish the proof is as follows. It is enough to show 1-transitivity while fixing some other points of a given finite set. Let us consider automorphisms of bounded degree fixing $P_2, \ldots, P_m$ and the orbits of P_1 and P_1' under this group. Clearly, by flexibility every orbit is an open subset in $\mathfrak{T}_X$, and every two dominant subsets should have a nonempty intersection. So there is a common point, which means that P_1 can be mapped to P_1' by a subgroup in $\mathrm{SAut}(\mathfrak{T}_X)$ fixing $P_2, \ldots, P_m$.

Remark 2.16. Conversely, in view of Theorem 2.2, given a b-inf. trans. variety X there exist $\dim X$ cancellations of X. Indeed, for general point $\zeta \in X$ we can find $\dim X$ tangent vectors spanning $T_{X,\zeta}$, such that each vector generates a copy of $\mathbb{G}_a =: G_i \subseteq \mathrm{SAut}(X)$, $1 \leq i \leq n$. Let $\mathfrak{G} \subseteq \mathrm{SAut}(X)$ be the subgroup generated by the groups $G_2, \ldots, G_n$. Then we have $X \approx G_1 \times \mathfrak{G} \cdot \zeta$.

2.12. **Increasing the rank of the corresponding subgroup in H_2.** We want to treat here the case when $\mathrm{rk}\langle \mathbb{P}_1^1, \ldots, \mathbb{P}_n^1 \rangle$ is t, $t < n$, as of a subgroup in $H_2(X)$.

Remark 2.17. Here we precise the ancient construction of $\mathfrak{T}_X$. Indeed, if the rank is not maximal, then the toric bundle contains a trivial part, and we need to get rid of it. One way is to change it with the trivial vector bundle part. However here we give another construction which uses stabilization.

Lemma 2.18. *There is a stabilization X' of X such that $\dim X' - t(X') < (n - t)$.*

Proof. We assume that the cycles $\mathbb{P}^1_i$ are dependent and in particular that an integer multiple of $\mathbb{P}^1_n$ is contained in the envelope of $\mathbb{P}^1_i$, $i < n$, on X. There is a natural projection $p_{n,n+1} : X \times \mathbb{P}^1 \to Y_n$ with a generic fiber $\mathbb{P}^1 \times \mathbb{P}^1$. Let us take $\mathbb{P}^1 \times \mathbb{P}^1$ and blow it up at 3 points. Thus we will have $\mathbb{P}^2$ with 5 blown up points. For every 4 points there is a pencil of conics passing through four points. Indeed, if we fix two smooth conics, there is a pencil of conics passing through the intersection. So on $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at three points we can choose two different 4-tuples of points on $\mathbb{P}^2$ and define two projections $\bar{\pi}_i : Bl_{Q_1,Q_2,Q_3}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1$. Now we can extend them to $\mathbb{P}^1 \times \mathbb{P}^1 \times B$ by blowing up three constant sections and similarly extend projections. The projections $\bar{\pi}_1$, $\bar{\pi}_2$ provide cancellations with new $\bar{\mathbb{P}}^1_i$, $i = 1, 2$, independent with generic $\mathbb{P}^1_j$, $j \neq 1, 2$ on the blown up X. We denote the resulting variety by $\tilde{X}$, it is a smooth model of $X \times \mathbb{P}^1$. Here the rank $t^1 = t + 2$. $\qquad\square$

2.13. **Increasing the rank of the matrix M.** For a variety X with a given set of cancellations and corresponding bundles, we constructed (2.5) a matrix M_n of restrictions. To prove birational stable infinite transitivity, we need the rank of this matrix to be full. The aim of this section is to prove the following lemma.

Lemma 2.19. *Let the rank of the subgroup generated by $\mathbb{P}^1_i$ in $H_2(X, Z)$ be n, and let matrix $M_n = M(X) = (m_{i,j})$ be as in (2.5) and its rank be $s < n = \dim X$. Then there exists a birational model $\tilde{X}$ for $X \times \mathbb{P}^1$ with $n + 1$ projections corresponding to cancellations and a family H^1_i, $i = 1, 2, \ldots, (n + 1)$, such that $s_1 \geq n + 2$ for the new matrix $M(\tilde{X})$.*

Proof. By Lemma 2.18, we may assume that all the classes $[\mathbb{P}^1_1], [\mathbb{P}^1_2], \ldots, [\mathbb{P}^1_n]$ are independent in $H_2(X)$. If $s < n = \dim X$, then due to Hodge duality there is a divisor with nonzero positive pairings with all the fibers $\mathbb{P}^1_i$, i.e. there is an ample divisor H_{n+1} on X such that it defines an element in the lattice $\mathbb{Z}^n$ which is not contained in $M(\mathbb{Z}(H_1, H_2, \ldots, H_n))$ (here we identify H_i with the elements of the standard basis in $\mathbb{Z}^n$). Let us define in this case $\pi^1_i : X \times \mathbb{P}^1 \to (Y_i \times \mathbb{P}^1)$; take $H^1_i = H_i + \mathcal{O}_{\mathbb{P}^1}(n_i)$ for some positive numbers n_i; $\pi_{n+1} : X \times \mathbb{P}^1 \to X$ the trivial projection; and H_{n+1} chosen above. Then if the restriction of H_{n+1} on $\mathbb{P}^1_i$ is $\mathcal{O}(t_i)$, the new matrix $M(\tilde{X})$ is as follows:

$$M(\tilde{X}) = \begin{pmatrix} & & & n_1 \\ & M(X) & & \vdots \\ & & & n_n \\ t_1 & \ldots & t_n & 0 \end{pmatrix}.$$

Note that all the diagonal elements $m_{i,i} = 0$. The matrix $M(\tilde{X})$ in this case has rank $s + 2$ for some choice of n_i. Indeed, the last row of $M(\tilde{X})$ is independent with other rows by the assumption on H_{n+1}. Now we can add n_i in such a way that the rank of $M(\tilde{X})$ will be $(\text{rk } M + 1) + 1$ (if $\text{rk } M < n$). Hence $\text{rk } M_1 = s + 2$ in this case. $\qquad\square$

Corollary 2.20. *In finite number of steps (not more than $2n$), using Lemmas 2.18 and 2.19, we obtain a model $\tilde{X}$ of $X \times \mathbb{P}^r$ with $\text{rk } M(\tilde{X}) = \dim(X \times \mathbb{P}^r) = \dim(\tilde{X})$.*

3. EXAMPLES

Here we collect several examples and properties of (stably) b-inf. trans. varieties.

3.1. **Quotients.** Let us start with the projective space $\mathbb{P}^n$, $n \geq 2$, and a finite group $G \subset PGL_{n+1}(\mathbf{k})$. Notice that the quotient $\mathbb{P}^n/G$ is stably b-inf. trans. Indeed, let us replace G by its finite central extension $\tilde{G}$ acting linearly on $V := \mathbf{k}^{n+1}$, so that $V/\tilde{G} \approx \mathbb{P}^n/G \times \mathbb{P}^1$. Further, form the product $V \times V$ with the diagonal $\tilde{G}$-action, and take the quotient $V' := (V \times V)/\tilde{G}$. Then, projecting on the first factor we get $V' \approx V \times V/\tilde{G}$, and similarly for the second factor. This implies that V' admits $2n + 2$ cancellations (cf. Theorem 2.1). Hence V' is stably b-inf. trans. by Theorem 2.2. The argument just used can be summarized as follows:

Lemma 3.1. *Let $X \to S$ be a $\mathbb{P}^m$-fibration for some $m \in \mathbb{N}$. Then the product $X \times_S X \approx X \times \mathbf{k}^m$ admits $2m$ algebraically independent cancellations over S.*

Proof. Note that $X \times_S X$ has two projections (left and right) onto X, both having a section (the diagonal $\Delta_X \subset X \times_S X$), hence the corresponding $\mathbb{P}^m$-fibrations are birational (over S) to $X \times \mathbf{k}^m$. This gives $2m$ algebraically independent cancellations over S. $\square$

Corollary 3.2. *Assume that X carries a collection of distinct birational structures of $\mathbb{P}^{m_i}$-bundles, $\pi_i : X \to S_i$, with the condition that the tangent spaces of generic fibers of π_i span the tangent space of X at the generic point. Then X is stably b-inf. trans.*

Proof. Indeed, after multiplying by the maximum of m_i we may assume that all $\mathbb{P}^{m_i}$-bundles provide with at least $2m_i$ different cancellations (see Lemma 3.1). We can now apply Theorem 2.2. $\square$

Remark 3.3. It seems plausible that given an inf. trans. variety X and a finite group $G \subset \mathrm{Aut}(X)$, variety X/G is stably b-inf. trans. (though the proof of this fact requires a finer understanding of the group $\mathrm{SAut}(X)$). At this stage, note also that if G is cyclic, then there exists a G-fixed point on X. Indeed, since X is unirational (cf. Section 1), it has trivial algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$ (see [12]). Then, if the G-action is free on X, we get $G \subset \pi_1^{\mathrm{alg}}(X/G) = \{1\}$ for X/G smooth unirational, a contradiction. This fixed-point-non-freeness property of X relates X to homogeneous spaces, and it would be interesting to investigate whether this is indeed the fact, i.e., in particular, does X, after stabilization and passing to birational model, admit a uniformization which is a genuine (finite dimensional) algebraic group?[1]

3.2. **Cubic hypersurfaces.** Let $X_3 \subset \mathbb{P}^{n+1}$, $n \geq 2$, be a smooth cubic. Then

Proposition 3.4. *X_3 is stably b-inf. trans.*

Proof. Smooth cubic X_3 contains a two-dimensional family of lines which span $\mathbb{P}^4$. Let $L \subset X_3$ be a line and $\pi : X_3 \dashrightarrow \mathbb{P}^{n-1}$ the linear projection from L. Let us resolve the indeterminacies of π by blowing up X_3 at L. We arrive at a smooth variety X_L together with a morphism $\pi_L : X_L \to \mathbb{P}^{n-1}$ whose general fiber is $\mathbb{P}^1$ ($\simeq$ a conic in $\mathbb{P}^2$). Varying $L \subset X_3$, we then apply Lemma 3.1 and Corollary 3.2 to get that X_3 is stably b-inf. trans. $\square$

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[1] This question was suggested by J.-L. Colliot-Thélène in connection with Conjecture 1.4. However, there are reasons to doubt the positive answer, since, for example, it would imply that X is (stably) birationally isomorphic to G/H, where both G, H are (finite dimensional) reductive algebraic groups. Even more, up to stable birational equivalence we may assume that $X = G'/H'$, where H' is a finite group and G' is the product of a general linear group, Spin groups and exceptional Lie groups. The latter implies, among other things, that there are only countably many stable birational equivalence classes of unirational varieties, but we could not develop a rigorous argument to bring this to contradiction.

3.3. **Quartic hypersurfaces.** Let $X_4 \subset \mathbb{P}^n$, $n \geq 4$, be a quartic hypersurface with a line $L \subset X_4$ of double singularities. Then

Proposition 3.5. X_4 *is stably b-inf. trans.*

Proof. Consider the cone $\mathfrak{X}_4 \subset \mathbb{P}^{n+1}$ over X_4. Then $\mathfrak{X}_4$ contains a plane Π of double singularities. Pick up a (generic) line $L' \subset \Pi$ and consider the linear projection $\mathfrak{X}_4 \dashrightarrow \mathbb{P}^{n-1}$ from L'. This induces a conic bundle structure on $\mathfrak{X}_4$, similarly as in the proof of Proposition 3.4, and varying L' in Π as above we obtain that $\mathfrak{X}_4$ is stably b-inf. trans. Then, since $\mathfrak{X}_4 \approx X_4 \times \mathbf{k}$, Proposition 3.5 follows. $\qquad \square$

3.4. **Complete intersections.** Let $X_{2\cdot2\cdot2} \subset \mathbb{P}^6$ be the smooth complete intersection of three quadrics. Then

Proposition 3.6. $X_{2\cdot2\cdot2}$ *is stably b-inf. trans.*

Proof. The threefold $X_{2\cdot2\cdot2}$ contains at least a one-dimensional family of lines. Let $L \subset X_{2\cdot2\cdot2}$ be a line and $X_L \to X_{2\cdot2\cdot2}$ the blowup of L. Then the threefold X_L carries the structure of a conic bundle (see [7, Ch. 10, Example 10.1.2, (ii)]). Now, varying L and applying the same arguments as in the proof of Proposition 3.4, we obtain that $X_{2\cdot2\cdot2}$ is stably b-inf. trans. $\qquad \square$

Remark 3.7. Fix $n, r \in \mathbb{N}$, $n \gg r$, and a sequence of integers $0 < d_1 \leq \ldots \leq d_m$, $m \geq 2$. Let us assume that $(n - r)(r + 1) \geq \sum_{i=1}^{m} \binom{d_i + r}{r}$. Consider the complete intersection $X := H_1 \cap \ldots \cap H_m$ of hypersurfaces $H_i \subset \mathbb{P}^n$ of degree d_i. Then it follows from the arguments in [14] that X contains a positive dimensional family of linear subspaces $\simeq \mathbb{P}^r$. Moreover, X is unirational, provided X is generic. It would be interesting to adopt the arguments from the proofs of Propositions 3.4, 3.5 and 3.6 to this more general setting in order to prove that X is stably b-inf. trans.

Remark 3.8. Propositions 3.4, 3.5 and 3.6 (cf. Remark 3.7) provide an alternative method of proving unirationality of complete intersections (see [7, Ch. 10] for recollection of classical arguments). Note also that (generic) $X_{2\cdot2\cdot2}$ is non-rational (see for example [17]), and (non-)rationality of the most of other complete intersections considered above is not known. At the same time, verifying stable b-inf. trans. property of other (non-rational) Fano manifolds (cf. [7, Ch. 10, Examples 10.1.3, (ii), (iii), (iv)]) is out of reach for our techniques at the moment.

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