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THE COLOMBEAU GENERALIZED NONLINEAR ANALYSIS AND THE SCHWARTZ LINEAR DISTRIBUTION THEORY

V. V. Chistyakov

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Introduction

The theory of generalized functions has a rather long history. The first generalized functions were introduced by Dirac in his classical papers [67, 68], where he employed heuristically his famous delta-function δ for the needs of quantum field theory. However, objects of the form δ^2, $\delta \cdot \delta', \ldots$, which were exploited in this theory had no rigorous mathematical meaning. Thus, a very nontrivial problem arose: how one can multiply generalized functions.

An interpretation of generalized functions as distributions, that is, as linear continuous functionals on a suitable space of test functions, was laid by Sobolev [184] (see also [185, 186]) in connection with problems of linear partial differential equations. The theory of distributions took its final form after Schwartz's monograph [179], published in 1950–1951 (here we cite a new 1973 edition), where various spaces of distributions were introduced and thoroughly studied. The theory of Sobolev spaces and that of Schwartz distributions became the main tools in the study of linear and nonlinear problems of mathematical physics; see Adams [2], Maz'ya [137], Ziemer [208], Bogolyubov and Parasiuk [22], Gel'fand and Shilov [80], Hörmander [88, 89], Antosik, Mikusiński, and Sikorski [5], Vladimirov [201], Bogolyubov, Logunov, Oksak, and Todorov [23].

Although the distribution theory give a rigorous sense to the Dirac δ function and to many other objects, the problem of multiplication of such objects remained to be unsolved. The lack of multiplication and other deficiencies of the distribution theory were discovered soon after its creation. In 1954, Schwartz [178] proved the "impossibility result," which affirmed the attitude to the problem of multiplication of distributions as unsolvable (for more details, see Sec. 7.2). The linear distribution theory was also found to be not sufficient for solution of rather simple linear partial differential equations with smooth coefficients: an example of such an equation without solutions was constructed by Lewy [115] (in linear extensions of the space of distributions the situation is quite similar: in the space of Sato's hyperfunctions, an example of an equation without solutions was found by Schapira [177]).

The situation changed radically in the late 70s and early 80s, when nonlinear theories of generalized functions containing distributions were created. Let us mention here the impressive works of Rosinger [167–169, 171], where a general theory of algebras of generalized functions was developed and, on its basis, nonlinear partial differential equations of a very general form were studied. In 1982, Colombeau [34–36, 41, 42] introduced a differential algebra of new generalized functions $\mathcal{G}(\Omega)$ having the following optimal properties: $\mathcal{G}(\Omega)$ contains the space of distributions $\mathcal{D}'(\Omega)$ as a linear subspace, partial derivatives in $\mathcal{G}(\Omega)$ extend the corresponding usual derivatives in $\mathcal{D}'(\Omega)$, the space $C^\infty(\Omega)$ of all infinitely differentiable functions is a subalgebra in $\mathcal{G}(\Omega)$, and the algebra $\mathcal{G}(\Omega)$ is invariant under smooth nonlinear operations of polynomial growth at infinity. This is the best situation for a differential algebra of generalized functions containing distributions since, by the Schwartz impossibility result mentioned above, no algebra of finite-times continuously differentiable functions can be a subalgebra in such an algebra. On the other hand, Colombeau's theory of generalized functions opens broad possibilities for finding solutions of various classes of linear and nonlinear differential equations.

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The purpose of our paper is to present the basic facts from Colombeau's nonlinear theory of generalized functions, which have achieved enormous success in the last decade. For its understanding it suffices to know only the classical differential and integral calculus and rudiments from the classical algebra. No knowledge in the distribution theory is assumed, since the main facts from this theory are presented in the framework of Colombeau's theory. This approach to the distribution theory was initiated by Colombeau [39] (see also Aragona and Biagioni [8]), and it is interesting to note that the distribution theory looks not so habitual in this approach as it does in classical courses of the distribution theory.

We adopt the elementary definition of the Colombeau algebra which was developed by Aragona and Colombeau [9] and Colombeau [39]. Unlike these works, we define an imbedding of the space $C(\Omega)$ of continuous functions, and then also the space of distributions $\mathcal{D}'(\Omega)$, into the algebra $\mathcal{G}(\Omega)$ somewhat differently: our definition is close to that used by Oberguggenberger [156] and is based on the elementary concept of convolution of a continuous function and a smooth function with compact support. We should note that Colombeau's theory of generalized functions is already available in monograph form — Biagioni [15], Colombeau [37, 39, 48], Oberguggenberger [156], and Rosinger [169] — and undoubtedly these works have had an influence on our presentation, too.

The present paper is divided into nine sections. The material of Sec. 1 is classical: it is shown that the space of smooth functions with compact supports (i.e., test functions) is large enough, and some properties of the convolution of functions are recalled. In Sec. 2, we define the Colombeau algebra of generalized functions on an open set in $\mathbb{R}^n$ and establish its main properties. In Sec. 3, we introduce an algebra of generalized numbers so that Colombeau's generalized functions assume values at individual points and can be integrated over compact sets. Also, in this section, we study solutions of algebraic equations within the framework of Colombeau's theory. In Sec. 4, we define nonlinear operations of polynomial growth over generalized functions, composition of generalized functions, and restriction of generalized functions to linear subspaces. In Sec. 5, we present distributions which are defined as those generalized functions in the sense of Colombeau which locally (on every relatively compact open subset) can be represented as partial derivatives of continuous functions. Using the integration theory for generalized functions developed in Sec. 3, we obtain the classical formulation of distributions in a way that is accepted in Schwartz's distribution theory. Then, in Sec. 6, we establish some of the classical properties of distributions which, in particular, allow us to display in Sec. 6.9 the natural character of the construction of the Colombeau algebra. The difficulties related to the problem of multiplication of distributions are described in Sec. 7, where, in particular, the Schwartz impossibility result is treated in more detail. As we have already mentioned, many classical operations (multiplication, composition, restriction, etc.) are necessarily changed in the algebra $\mathcal{G}(\Omega)$, so, in Sec. 8, all these operations are recovered by means of the following two specific concepts: the equality in the sense of generalized distributions and the equality in the sense of the association. Finally, in Sec. 9, we present some further properties of the association: multiplication by the Dirac δ function, characterization of the product of distributions in the Colombeau algebra (due to Jelínek [95]); also, the Heaviside generalized functions and the Dirac generalized functions are defined, and examples of discontinuous solutions to a first-order system in the conservative form are considered. In concluding our paper, we enumerate some recent papers (as known by the author) not mentioned in the body of this paper, which contribute to Colombeau's theory and related theories of generalized functions.

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Notation

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote, respectively, the sets of positive integers, nonnegative integers, integers, rational numbers, real numbers, and complex numbers. By $\mathbb{K}$ we denote either the field $\mathbb{R}$ or $\mathbb{C}$. If $n \in \mathbb{N}$, a point $x \in \mathbb{K}^n$ is usually written as $x = (x_1, \dots, x_n)$ with $x_j \in \mathbb{K}$, $j = 1, \dots, n$; the norm of x is $|x| = (\sum_{j=1}^{n} |x_j|^2)^{1/2}$, and the inner product of x and y in $\mathbb{K}^n$ is $x \cdot y = \sum_{j=1}^{n} x_j y_j$.

The sum of two nonempty sets X, $Y \subset \mathbb{R}^n$ is the set $X + Y = \{x + y \in \mathbb{R}^n \mid x \in X \text{ and } y \in Y\}$; if $X = \{x\}$, we write $X + Y = x + Y$. Analogously, $cX = \{cx \in \mathbb{R}^n \mid x \in X\}$ if $c \in \mathbb{R}$. A closed (open) ball in $\mathbb{R}^n$ of radius $r > 0$ centered at a point $x \in \mathbb{R}^n$ is denoted by $B_r(x) = \{y \in \mathbb{R}^n \mid |y - x| \leq r\}$ (resp. $B_r^\circ(x) = \{y \in \mathbb{R}^n \mid |y - x| < r\}$); we set $B_r = B_r(0)$ and $B_r^\circ = B_r^\circ(0)$, so that $B_r(x) = x + B_r$ and $B_r^\circ(x) = x + B_r^\circ$.

Given a set $X \subset \mathbb{R}^n$, we denote by $X^\circ = \text{int}X = \{x \in X \mid \exists r > 0 : B_r^\circ(x) \subset X\}$ the interior of X, by $X^c = \mathbb{R}^n \setminus X$ the complement of X in $\mathbb{R}^n$, by $\overline{X} = ((X^c)^\circ)^c$ the closure of X in $\mathbb{R}^n$, and by $\partial X = \overline{X} \setminus X^\circ$ the boundary of X. The distance from a point $x \in \mathbb{R}^n$ to a set $Y \subset \mathbb{R}^n$ is the number $\text{dist}(x, Y) = \inf_{y \in Y} |x - y|$, and analogously, $\text{dist}(X, Y) = \inf_{x \in X, y \in Y} |x - y| = \inf_{x \in X} \text{dist}(x, Y)$ is the distance between two sets X and Y.

The symbol Ω denotes usually a nonempty open subset of $\mathbb{R}^n$; $K \subset\subset \Omega$ means that K is a compact (i. e. closed and bounded) set contained in Ω, and $S \Subset \Omega$ means that S is a relatively compact open set such that its closure is contained in Ω.

A multi-index α is an element of $\mathbb{N}_0^n$ of the form $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$, $j = 1, \dots, n$. We set $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, and $0! = 1$. For $\alpha, \beta \in \mathbb{N}_0^n$ we write $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for all $j = 1, \dots, n$. If $x = (x_1, \dots, x_n)$, we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and if $\partial_j = \partial_{x_j} = \partial / \partial x_j$ is the operator of partial differentiation with respect to the variable x_j, $j = 1, \dots, n$, and $\partial = (\partial_1, \dots, \partial_n) = \nabla$ is the vector of the gradient, we also set $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} = \partial_x^\alpha$ with $\partial^0 = \text{id}$, the identity operator.

The set of all continuous functions $f : \Omega \longrightarrow \mathbb{K}$ is denoted by $C(\Omega; \mathbb{K}) = C^0(\Omega; \mathbb{K})$, which will simply be written as $C(\Omega)$ if it does not matter which of the sets $\mathbb{R}$ or $\mathbb{C}$ is meant by $\mathbb{K}$. Analogously, the set of all k-times ($k \in \mathbb{N}$) continuously differentiable functions on Ω is denoted by $C^k(\Omega)$. We also set $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$. Given the functions $f, g : \Omega \longrightarrow \mathbb{K}$, their sum $f + g$, the multiplication cf by a number $c \in \mathbb{K}$, and the product fg are defined pointwise, i.e., $(f + g)(x) = f(x) + g(x)$, $(cf)(x) = cf(x)$, and $(fg)(x) = f(x)g(x)$ for $x \in \Omega$. All the above function spaces are associative and commutative linear algebras over the field $\mathbb{K}$. The support, $\text{supp}\, f$, of a function $f : \Omega \longrightarrow \mathbb{K}$ is the smallest closed subset in Ω outside which f vanishes, so that the support of f is the closure of the set $\{x \in \Omega \mid f(x) \neq 0\}$ in Ω. The set of all continuous functions on Ω with compact supports is denoted by $C_c(\Omega)$.

The Banach space of all measurable functions on Ω for which the pth power ($1 \leq p < \infty$) of their absolute value is Lebesgue integrable is denoted by $L^p(\Omega)$, and the usual norm in this space is denoted by $\| \cdot \|_{L^p(\Omega)}$ or $\| \cdot \|_{p,\Omega}$; $L^\infty(\Omega)$ is the Banach space of all Lebesgue-measurable essentially bounded functions on Ω with usual norm denoted by $\| \cdot \|_{L^\infty(\Omega)}$ or $\| \cdot \|_{\infty,\Omega}$. We denote by $L^1_{\text{loc}}(\Omega)$ the linear space of all measurable locally Lebesgue integrable functions f on Ω such that the restriction $f|_K$ is in $L^1(K)$ for all $K \subset\subset \Omega$.

The Lebesgue integral $\int_\Omega f(x)\, dx$ over the support, $\text{supp}\, f$, of a function $f : \Omega \longrightarrow \mathbb{K}$ is often denoted by $\int f(x)\, dx$ or even by $\int f$. The last brief notation will also be used when the domain of integration is clear from the context.

Finally, to show that an expression A is defined by means of an expression B, we will write $A := B$ or $B =: A$.

Below, for the reader's convenience, we recall some classical facts from the smooth analysis, namely, Leibnitz's rule, Taylor's formula, the theorem on C^∞ partitions of the unity, and the Paley–Wiener theorem.

Leibnitz's rule. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $k \in \mathbb{N}$, and let $f,\ g \in C^k(\Omega)$. Then $fg \in C^k(\Omega)$, and for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le k$, the following Leibnitz's rule holds:*

$$\partial^\alpha(fg) = \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)(\partial^\beta g), \quad \textit{where} \quad \binom{\alpha}{\beta} := \frac{\alpha!}{(\alpha-\beta)!\,\beta!}. \quad \square$$

Taylor's formula. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $k \in \mathbb{N}$, $f \in C^k(\Omega)$, and let the closed line segment $[x, x+h] := \{\, x + th \in \mathbb{R}^n \mid t \in [0,1]\,\}$ be contained in Ω. Then the following Taylor's formula with the integral remainder term holds:*

$$f(x+h) - f(x) = \sum_{j=1}^{n} \int_0^1 (\partial_j f)(x+th)\,dt \cdot h_j \quad \text{(Hadamard's formula)}, \quad \text{if } k = 1, \text{ then,}$$

$$f(x+h) - f(x) = \sum_{|\alpha|=1}^{k-1} \frac{1}{\alpha!} (\partial^\alpha f)(x) \cdot h^\alpha + R_k(x,h) \quad \textit{if } k \ge 2,$$

where

$$R_k(x,h) = k \sum_{|\alpha|=k} \frac{1}{\alpha!} \int_0^1 (1-t)^{k-1} (\partial^\alpha f)(x+th)\,dt \cdot h^\alpha,$$

and for the remainder $R_k(x,h)$, we have the estimate

$$|R_k(x,h)| \le \sum_{|\alpha|=k} \frac{1}{\alpha!} \sup_{[x,x+h]} |\partial^\alpha f| \cdot |h^\alpha| \le \Big(\sum_{|\alpha|=k} \sup_{[x,x+h]} |\partial^\alpha f| \Big) \cdot |h|^k. \quad \square$$

Theorem on C^∞ partitions of the unity. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\{\Omega_i\}_{i\in I}$ be a family of open sets in $\mathbb{R}^n$ covering Ω, i.e., $\bigcup_{i\in I} \Omega_i = \Omega$. Then there exists a family of functions $\phi_i \in C^\infty(\Omega; \mathbb{R})$, $i \in I$, such that*

(a) $0 \le \phi_i \le 1$ *in* Ω, *and* $\operatorname{supp} \phi_i \subset \Omega_i$ *for all* $i \in I$;

(b) *the family* $\{\operatorname{supp} \phi_i\}_{i\in I}$ *is* locally finite, *i.e., for every $x \in \Omega$, there is a neighborhood $\mathcal{O}(x)$ of x in which only a finite number of functions ϕ_i are not identically equal to zero*;

(c) $\sum_{i\in I} \phi_i(x) = 1$ *for all* $x \in \Omega$.

The family $\{\phi_i\}_{i\in I}$ with properties (a), (b), (c) is said to be a *partition of the unity subordinated to the covering $\{\Omega_i\}_{i\in I}$ of Ω.* $\square$

The Paley–Wiener theorem. *Let $\varphi \in C^\infty(\mathbb{R}^n)$ have support in a ball B_R, $R > 0$. Then its Fourier-Laplace transform*

$$f(z) = \int_{\mathbb{R}^n} e^{-ix\cdot z} \varphi(x)\,dx, \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \qquad (*)$$

is an entire analytic function of n complex variables $z_j = x_j + iy_j$, $j = 1, \ldots, n$, satisfying the condition

$$\begin{gathered} \forall N \in \mathbb{N},\ \exists C_N > 0 \ \textit{such that} \ \forall z \in \mathbb{C}^n, \\ |f(z)| \le C_N (1+|z|)^{-N} e^{R|\operatorname{Im} z|}. \end{gathered} \qquad (**)$$

Conversely, if an entire analytic function f on $\mathbb{C}^n$ satisfies condition $(**)$, then there is a function $\varphi \in C^\infty(\mathbb{R}^n)$ supported in the ball B_R such that the representation $(*)$ holds. $\square$

1. Spaces of Smooth Functions

Denote by $\mathcal{D}(\mathbb{R}^n)$ the algebra (over the field $\mathbb{K}$) of all infinitely differentiable functions from $\mathbb{R}^n$ into $\mathbb{K}$ having compact supports,

$$\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n) := \{\, \varphi \in C^\infty(\mathbb{R}^n) \mid \operatorname{supp}\varphi \subset\subset \mathbb{R}^n \,\},$$

and for an arbitrary nonempty set $X \subset \mathbb{R}^n$, we set

$$\mathcal{D}(X) = C_c^\infty(X) := \{\, \varphi \in \mathcal{D}(\mathbb{R}^n) \mid \operatorname{supp}\varphi \subset X \,\}.$$

Functions from $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}(X)$ are usually called *test functions.* In this section, we show that the space of test functions is sufficiently large and recall some properties of the convolution of functions.

First, let $n = 1$. The function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $\phi(x) = 0$ if $x \le 0$ and $\phi(x) = \exp(-1/x)$ if $x > 0$, has the following properties: $0 \not\equiv \phi \in C^\infty(\mathbb{R})$, $0 \le \phi < 1$, ϕ increases on $(0,\infty)$, and $\operatorname{supp}\phi = [0,\infty)$. If, for $-\infty < a < b < \infty$, we set

$$\zeta_{a,b}(x) = \frac{\phi(b-x)}{\phi(x-a) + \phi(b-x)}, \quad x \in \mathbb{R},$$

then $\zeta_{a,b} \in C^\infty(\mathbb{R})$, $0 \le \zeta_{a,b} \le 1$, $\zeta_{a,b} = 1$ on $(-\infty, a]$, $\operatorname{supp}\zeta_{a,b} = (-\infty, b]$, and $\zeta_{a,b}$ decreases on $\mathbb{R}$. Now it is easy to construct nontrivial functions from the space $\mathcal{D}(\mathbb{R})$: for $-\infty < a < b \le c < d < \infty$; let

$$\zeta(x) = \zeta_{a,b,c,d}(x) = (1 - \zeta_{a,b}(x))\zeta_{c,d}(x), \quad x \in \mathbb{R};$$

then

$$\zeta \in \mathcal{D}(\mathbb{R}), \quad 0 \le \zeta \le 1, \quad \zeta = 1 \text{ on } [b,c] \quad \text{and} \quad \operatorname{supp}\zeta = [a,d].$$

If $n \in \mathbb{N}$ is arbitrary and $\psi(\lambda) = \zeta_{r,R}(|\lambda - x|)$, where λ, $x \in \mathbb{R}^n$ and $0 < r < R < \infty$, then $\psi \in \mathcal{D}(\mathbb{R}^n)$, $0 \le \psi \le 1$, $\psi = 1$ on $B_r(x)$, and $\operatorname{supp}\psi = B_R(x)$. Instead of $\zeta_{r,R}$, one could use the function ζ. A more general way to construct smooth functions with compact support on $\mathbb{R}^n$ is to consider the tensor product $\psi = \bigotimes_{j=1}^{n} \psi_j$ of functions $\{\psi_j\}_{j=1}^n \subset \mathcal{D}(\mathbb{R})$ which is defined by $\psi(x) = \prod_{j=1}^{n} \psi_j(x_j)$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Since the support of ψ is $\operatorname{supp}\psi = (\operatorname{supp}\psi_1) \times \cdots \times (\operatorname{supp}\psi_n) \subset\subset \mathbb{R}^n$, we have $\psi \in \mathcal{D}(\mathbb{R}^n)$.

For a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varepsilon > 0$, and for λ, $x \in \mathbb{R}^n$, we set

$$\varphi_\varepsilon(\lambda) = \frac{1}{\varepsilon^n}\varphi\Big(\frac{\lambda}{\varepsilon}\Big), \qquad \check{\varphi}(\lambda) = \varphi(-\lambda), \qquad (\tau_x\varphi)(\lambda) = \varphi(\lambda - x),$$

$$\rho(\varphi) = \sup\{\, |\lambda| \mid \lambda \in \operatorname{supp}\varphi \,\};$$

here τ_x is the translation operator and $\rho(\varphi)$ is the radius of the smallest closed ball centered at the origin containing the support of φ, so that $\rho(\varphi) > 0$ if $\varphi \not\equiv 0$. The following relations for the functions defined above are easy consequences of their definitions:

Proposition 1.1. *For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varepsilon > 0$, λ, $x \in \mathbb{R}^n$, and for $\alpha \in \mathbb{N}_0^n$ we have*

(a) $(\tau_x\varphi_\varepsilon)(\lambda) = \varphi_\varepsilon(\lambda - x) = \varepsilon^{-n}\varphi((\lambda - x)/\varepsilon), \quad (\check{\varphi})^\vee = \varphi;$

(b) $\operatorname{supp}\tau_x\varphi_\varepsilon = x + \varepsilon\operatorname{supp}\varphi, \quad \operatorname{supp}\check{\varphi} = -\operatorname{supp}\varphi;$

(c) $\pm\operatorname{supp}\varphi \subset B_{\rho(\varphi)}, \quad \rho(\varphi_\varepsilon) = \varepsilon\rho(\varphi), \quad \rho(\check{\varphi}) = \rho(\varphi), \quad \rho(\partial^\alpha\varphi) \le \rho(\varphi);$

(d) $\partial^\alpha\varphi_\varepsilon = \varepsilon^{-|\alpha|}(\partial^\alpha\varphi)_\varepsilon, \quad \partial^\alpha\check{\varphi} = (-1)^{|\alpha|}(\partial^\alpha\varphi)^\vee;$

(e) $\displaystyle\int_{\mathbb{R}^n}(\tau_x\varphi_\varepsilon)(\lambda)\,d\lambda = \int\varphi, \quad \int_{\mathbb{R}^n}\check{\varphi}(\lambda)\,d\lambda = \int\varphi.$ □

We will need the following variant of the theorem on differentiation with respect to a parameter under the sign of the integral:

Proposition 1.2. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two nonempty open sets, $f \in C(Y)$ or $L^1_{\mathrm{loc}}(Y)$, and let $\Phi = \Phi(x,y) \in C^\infty(X \times Y)$ be such that*

$$\exists K \subset\subset Y \quad \forall x \in X \quad : \quad \operatorname{supp}\Phi(x,\cdot) \subset K. \tag{1.1}$$

If $F(x) = \int\limits_Y f(y)\Phi(x,y)\,dy$ for $x \in X$, then $F : X \longrightarrow \mathbb{K}$ satisfies the following properties:

$$F \in C^\infty(X) \quad \text{and} \quad \partial_x^\alpha F(x) = \int\limits_Y f(y)\partial_x^\alpha \Phi(x,y)\,dy \;\; \forall x \in X \;\; \forall \alpha \in \mathbb{N}_0^n.$$

If, moreover, $\Phi \in \mathcal{D}(X \times Y)$, or $f \in C_c(Y)$ and $\Phi(\cdot,y) \in \mathcal{D}(L)$ for some $L \subset\subset X$ and for all $y \in \operatorname{supp} f$, then $F \in \mathcal{D}(X)$.

Proof. Due to (1.1), the function $\Phi(x,\cdot)$ is in $\mathcal{D}(K)$; therefore, the (Lebesgue) integral $F(x)$ is well defined for all $x \in X$. Let $x \in X$, and let a number $r > 0$ be such that $B_r(x) \subset X$. Applying Taylor's formula to the function $\Phi(x,y)$ in the variable(s) x, multiplying by $f(y)$, and integrating with respect to $y \in Y$, for $h \in B_r^\circ$ we have

$$F(x+h) - F(x) = \sum_{j=1}^n \Big(\int\limits_Y f(y)\partial_{x_j}\Phi(x,y)\,dy\Big)h_j +$$
$$+ \int\limits_K f(y)\Big(2\sum_{|\alpha|=2}\frac{1}{\alpha!}\int\limits_0^1 (1-t)(\partial_x^\alpha\Phi)(x+th,y)\,dt \cdot h^\alpha\Big)\,dy. \tag{1.2}$$

The last integral is estimated by

$$\Big(\int\limits_K |f|\Big)\cdot\Big(\sum_{|\alpha|=2}\sup_{(\lambda,y)\in B_r(x)\times K}|(\partial_\lambda^\alpha\Phi)(\lambda,y)|\Big)\cdot|h|^2 = \mathrm{const}(x)\cdot|h|^2,$$

i.e., it is $o(|h|)$ as $h \longrightarrow 0$ in Landau's notation. Equality (1.2), together with the last remark, shows that the function F is differentiable at the point $x \in X$ and

$$\partial_{x_j}F(x) = \int\limits_Y f(y)\partial_{x_j}\Phi(x,y)\,dy, \quad j = 1,\ldots,n.$$

The repeated application of this result proves the first part of the proposition.

Now, if $\Phi \in \mathcal{D}(X \times Y)$, we set $S = \operatorname{supp}\Phi \subset\subset X \times Y$. Denote by S_X and S_Y the projections of the compact set S onto the sets X and Y, respectively:

$$S_X = \{\, x \in X \mid \exists y \in Y : (x,y) \in S \,\}, \quad S_Y = \{\, y \in Y \mid \exists x \in X : (x,y) \in S \,\},$$

so that $S_X \subset\subset X$, $S_Y \subset\subset Y$, and $S \subset S_X \times S_Y$. Since $\Phi(x,\cdot) \in \mathcal{D}(S_Y)$, condition (1.1) is fulfilled, and hence $F \in C^\infty(X)$. Let us show that $\operatorname{supp} F \subset S_X$. Indeed, if $x \notin S_X$, then for all $y \in Y$, the point (x,y) is not in S, whence $\Phi(x,y) = 0$, and therefore, $F(x) = 0$.

In the latter case, $F \in \mathcal{D}(L)$ since if $x \notin L$, then $\Phi(x,y) = 0$ for all $y \in \operatorname{supp} f$; thus $f(y)\Phi(x,y) = 0$ for all $y \in Y$, so that $F(x) = 0$. □

Before recalling the properties of the convolution of functions, let us introduce some notation. Let $\Omega \subset \mathbb{R}^n$ be an open set, $\varnothing \neq K \subset\subset \Omega$, and let $d := \operatorname{dist}(K,\partial\Omega)$. If $\Omega \neq \mathbb{R}^n$, then $\partial\Omega \neq \varnothing$ and $0 < d < \infty$; in the case $\Omega = \mathbb{R}^n$, we assume that $d = \infty$. For $\rho > 0$, define the compact and relatively compact ρ-neighborhoods of the compact set K by

$$K_\rho = \{\, x \in \mathbb{R}^n \mid \operatorname{dist}(x,K) \le \rho \,\} = K + B_\rho,$$

$$K_\rho^\circ = \operatorname{int} K_\rho = \{\, x \in \mathbb{R}^n \mid \operatorname{dist}(x, K) < \rho \,\} = K + B_\rho^\circ,$$

so that for $0 < \rho < d$, we have

$$K \subset\subset K_\rho^\circ \Subset K_\rho \subset\subset K_d^\circ \subset \Omega \quad \text{and} \quad \operatorname{dist}(K_\rho, \partial\Omega) = \operatorname{dist}(K_\rho^\circ, \partial\Omega) = d - \rho. \tag{1.3}$$

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, consider the set

$$\Omega(\varphi) = \{\, x \in \Omega \mid \tau_x\varphi \in \mathcal{D}(\Omega) \,\} = \{\, x \in \Omega \mid x + \operatorname{supp}\varphi \subset \Omega \,\},$$

and note that $\Omega(\varphi) = \mathbb{R}^n$ if $\Omega = \mathbb{R}^n$, and $\Omega(\varphi) := \Omega$ if $\varphi \equiv 0$. The set $\Omega(\varphi)$ is *open* in $\mathbb{R}^n$ (possibly empty) since if $\Omega \neq \mathbb{R}^n$ and $x \in \Omega(\varphi)$, then setting $d = \operatorname{dist}(K, \partial\Omega)$, where $K := \operatorname{supp}\tau_x\varphi \subset\subset \Omega$, we have $B_d^\circ(x) + \operatorname{supp}\varphi = \operatorname{supp}\tau_x\varphi + B_d^\circ = K_d^\circ \subset \Omega$, whence $B_d^\circ(x) \subset \Omega(\varphi)$. Moreover, the set $\Omega(\varphi)$ is, certainly, *nonempty* for small values of $\rho(\varphi)$:

$$\text{if } K \subset\subset \Omega \text{ and } \rho(\varphi) < d = \operatorname{dist}(K, \partial\Omega), \text{ then } K \subset \Omega(\varphi) \tag{1.4}$$

since $K + \operatorname{supp}\varphi \subset K_\rho$ for $\rho(\varphi) \le \rho < d$; this follows from

$$\operatorname{supp}\tau_x\varphi = x + \operatorname{supp}\varphi \subset B_{\rho(\varphi)}(x) \subset K_\rho \subset\subset \Omega \quad \forall x \in K. \tag{1.5}$$

In particular, it follows that $K \subset \Omega(\varphi_\varepsilon)$ for $0 < \varepsilon < \eta = d/\rho(\varphi)$ and for every $\varphi \not\equiv 0$ in $\mathcal{D}(\mathbb{R}^n)$, and $\bigcup_{\varepsilon>0} \Omega(\varphi_\varepsilon) = \Omega$. Also, we note that if Ω_1, $\Omega_2 \subset \mathbb{R}^n$ are open sets and $\Omega_1 \subset \Omega_2$, then $\Omega_1(\varphi) \subset \Omega_2(\varphi)$.

The *convolution* of functions $f \in C(\Omega)$ (or $f \in L^1_{\text{loc}}(\Omega)$) and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is defined on the open set $\Omega(\check{\varphi}) \neq \varnothing$ by the formula

$$(f * \varphi)(x) = \int_\Omega f(y)\varphi(x - y)\, dy = \int_{x - \operatorname{supp}\varphi} f(y)(\tau_x \check{\varphi})(y)\, dy, \quad x \in \Omega(\check{\varphi}). \tag{1.6}$$

Because of property (1.5), if $K \subset\subset \Omega$ and $\rho(\varphi) \le \rho < \operatorname{dist}(K, \partial\Omega)$, then $K \subset \Omega(\check{\varphi})$ and

$$(f * \varphi)(x) = \int_{B_\rho(x)} f(y)(\tau_x \check{\varphi})(y)\, dy = \int_{B_\rho} f(x - \mu)\varphi(\mu)\, d\mu, \quad x \in K. \tag{1.7}$$

If $\Omega = \mathbb{R}^n$, equality (1.7) holds for all $x \in \mathbb{R}^n$.

The main properties of the convolution are enumerated in the following proposition:

Proposition 1.3. *Let $f \in C(\Omega)$ or $f \in L^1_{\text{loc}}(\Omega)$, and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then*

$$f * \varphi \in C^\infty(\Omega(\check{\varphi})) \quad \text{and} \quad \partial^\alpha(f * \varphi) = f * (\partial^\alpha\varphi) \text{ on } \Omega(\check{\varphi}) \ \forall \alpha \in \mathbb{N}_0^n.$$

In addition, we have

(a) *If $f \in C(\Omega)$ and $K \subset\subset \Omega$, then $(f * \varphi_\varepsilon)(x) \longrightarrow \left(\int \varphi\right) f(x)$ as $\varepsilon \longrightarrow +0$ uniformly in $x \in K$, and, in particular, $\lim_{\varepsilon\to+0} \int (\tau_x \check{\varphi}_\varepsilon)(\lambda) f(\lambda)\, d\lambda = \left(\int \varphi\right) f(x)$ $\forall x \in \Omega$.*

(b) *If $f \in C_c(\Omega)$, then $f * \varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp} f * \varphi \subset \operatorname{supp} f + \operatorname{supp}\varphi =: L$; also, if $\rho(\varphi) < \operatorname{dist}(\operatorname{supp} f, \partial\Omega)$, then $L \subset\subset \Omega$, so that $f * \varphi \in \mathcal{D}(\Omega)$; the convergence in* (a) *is uniform on $\mathbb{R}^n$.*

(c) *If $f \in C^k(\Omega)$ for some $k \in \mathbb{N}$, then also $\partial^\alpha(f * \varphi) = (\partial^\alpha f) * \varphi$ on $\Omega(\check{\varphi})$, $|\alpha| \le k$.*

(d) *If $f \in L^p_{\text{loc}}(\Omega)$, where $1 \le p < \infty$, then $f * \varphi_\varepsilon \longrightarrow \left(\int \varphi\right) f$ in $L^p(K)$ as $\varepsilon \longrightarrow +0$ for every compact $K \subset\subset \Omega$.*

(e) *If* $f \in L^p_{\rm loc}(\Omega)$ *with* $1 \le p \le \infty$, $K \subset\subset \Omega$ *and* $\rho(\varphi) \le \rho < \operatorname{dist}(K, \partial\Omega)$, *then*

$$\sup_{x\in K} |(f * \varphi)(x)| \le \| f \|_{L^p(K_\rho)} \| \varphi \|_{L^{p'}(B_{\rho(\varphi)})},$$

where $p' = p/(p-1)$ *if* $1 < p < \infty$, $p' = \infty$ *if* $p = 1$, *and* $p' = 1$ *if* $p = \infty$.

(f) *If* $\Omega = \mathbb{R}^n$ *then* $f * \varphi = \varphi * f$ *and* $\tau_x(f * \varphi) = (\tau_x f) * \varphi = f * (\tau_x \varphi)$ $\forall x \in \mathbb{R}^n$.

Proof. Consider an exhaustion of the set $\Omega(\check{\varphi})$ by compact subsets $\{E_m\}_{m=1}^{\infty}$ such that $E_m \subset\subset \Omega(\check{\varphi})$, $E_m \subset E^{\circ}_{m+1}$, and $\bigcup\limits_{m=1}^{\infty} E^{\circ}_m = \Omega(\check{\varphi})$. As E_m one can take, for example, any set of the form

$$E_m = \{\, x \in \Omega(\check{\varphi}) \mid \operatorname{dist}(x, \partial(\Omega(\check{\varphi}))) \ge 1/m \quad \text{and} \quad |x| \le m \,\},$$

and if $\Omega = \mathbb{R}^n$, one can also take $E_m = B_m$. In Proposition 1.2, we set $X = E^{\circ}_m$, $Y = \Omega$, $\Phi(x,y) = \varphi(x-y) \in C^{\infty}(\mathbb{R}^{2n})$, and $K = E_m + \operatorname{supp} \check{\varphi}$; note that (1.1) is satisfied:

$$\operatorname{supp} \Phi(x, \cdot\,) = \operatorname{supp} \tau_x \check{\varphi} = x + \operatorname{supp} \check{\varphi} \subset K \subset\subset \Omega, \quad x \in E^{\circ}_m,$$

where the inclusion $K \subset \Omega$ is a consequence of the inclusion $E_m \subset \Omega(\check{\varphi})$ and the definition of the set $\Omega(\check{\varphi})$. Hence the restriction $(f * \varphi)|_{E^{\circ}_m}$ is in $C^{\infty}(E^{\circ}_m)$ for every $m \in \mathbb{N}$, whence $f * \varphi \in C^{\infty}(\Omega(\check{\varphi}))$. The formula for the differentiation of the convolution is obvious if we take into account that $\Omega((\partial^{\alpha}\varphi)^{\vee}) \supset \Omega(\check{\varphi})$.

Let us prove (a) and (d). To this end, fix $K \subset\subset \Omega$ and set $d = \operatorname{dist}(K, \partial\Omega)$, $\eta = \rho/\rho(\varphi)$ for some $0 < \rho < d$. Then (see also (1.4) and (1.5)) we have

$$K \subset \Omega(\check{\varphi}_{\varepsilon}) \quad \text{and} \quad B_{\varepsilon\rho(\varphi)}(x) \subset K_\rho \subset\subset \Omega \quad \text{for } \varepsilon \in (0, \eta) \text{ and } x \in K.$$

(a) If $f \in C(\Omega)$, then changing the variable $\mu = (x-y)/\varepsilon$ in the integral defining the convolution, for a fixed $x \in K$, we obtain

$$\Big|(f * \varphi_\varepsilon)(x) - \Big(\int \varphi\Big) f(x)\Big| = \Big|\int f(y) \frac{1}{\varepsilon^n} \varphi\Big(\frac{x-y}{\varepsilon}\Big)\, dy - f(x) \int \varphi\Big| =$$

$$= \Big|\int (f(x - \varepsilon\mu) - f(x))\varphi(\mu)\, d\mu\Big| \le \Big(\int\limits_{B_{\rho(\varphi)}} |\varphi|\Big) \cdot \sup_{\lambda \in B_{\varepsilon\rho(\varphi)}} |f(x+\lambda) - f(x)|\,.$$

The latter expression tends to zero as $\varepsilon \longrightarrow +0$ uniformly in $x \in K$ because of the uniform continuity of the function f on compact subsets of Ω (in particular, on K_ρ).

(d) If $f \in L^p_{\rm loc}(\Omega)$, then setting $r = \varepsilon\rho(\varphi)$ for brevity, for $x \in K$ and $\varepsilon \in (0, \eta)$, we have

$$\Big|(f * \varphi_\varepsilon)(x) - \Big(\int \varphi\Big) f(x)\Big| = \Big|\int\limits_{B_r(x)} (\tau_x \check{\varphi}_\varepsilon)(y)(f(y) - f(x))\, dy\Big| \le$$

$$\le \frac{1}{\varepsilon^n}\Big(\sup_{B_{\rho(\varphi)}} |\varphi|\Big) \int\limits_{B_r(x)} |f(y) - f(x)|\, dy = \frac{C(\varphi)}{\varepsilon^n} \int\limits_{B_r} |f(x-\mu) - f(x)|\, d\mu, \tag{1.8}$$

where $C(\varphi) = \sup_{B_{\rho(\varphi)}} |\varphi|$. If $p > 1$ and $p' := p/(p-1)$, Hölder's inequality implies

$$\Big|(f * \varphi_\varepsilon)(x) - \Big(\int \varphi\Big) f(x)\Big| \le \frac{C(\varphi)}{\varepsilon^n} \Big(\int\limits_{B_r} 1\, d\mu\Big)^{1/p'} \Big(\int\limits_{B_r} |f(x-\mu) - f(x)|^p d\mu\Big)^{1/p}$$

Raising to the pth power, integrating in $x \in K$, and applying Fubini's theorem, we find that

$$\int\limits_K \Big|(f * \varphi_\varepsilon)(x) - \Big(\int \varphi\Big) f(x)\Big|^p dx \le \frac{C(\varphi)^p}{\varepsilon^{np}} \Big(\int\limits_{B_r} 1\, d\mu\Big)^{p/p'} \int\limits_{B_r} d\mu \int\limits_K |f(x-\mu) - f(x)|^p dx \le$$

$$\leq \frac{C(\varphi)^p}{\varepsilon^{np}}\Big(\int_{B_r} 1\,d\mu\Big)^p \operatorname*{ess\,sup}_{\mu\in B_r}\int_K |f(x-\mu)-f(x)|^p d\mu.$$

If $p=1$, the latter inequality follows from (1.8) as above, without applying Hölder's inequality. Thus,

$$\| f*\varphi_\varepsilon - \Big(\int \varphi\Big) f \|_{L^p(K)} \leq \frac{C(\varphi)}{\varepsilon^n}\Big(\int_{B_r} 1\,d\mu\Big)\cdot \operatorname*{ess\,sup}_{\mu\in B_r} \| \tau_\mu f - f\|_{L^p(K)}.$$

It remains to note that $\int_{B_r} 1\,d\mu = \pi^{n/2}r^n/\Gamma(1+n/2)$ is the n-dimensional Lebesgue measure of the ball B_r with $r=\varepsilon\rho(\varphi)$, where $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$ $(x>0)$ is Euler's gamma function, and that $\operatorname{ess\,sup}_{\mu\in B_r}\|\tau_\mu f - f\|_{L^p(K)} \longrightarrow 0$ as $\varepsilon \longrightarrow +0$ due to the *continuity in the large* of the function $f\in L^p_{\rm loc}(\Omega)$ on compact subsets of Ω (in particular, on K_ρ).

(b) If $f\in C_c(\Omega)$, then noting that for all $y\in \operatorname{supp} f$,

$$\operatorname{supp}\Phi(\,\cdot\,,y) = \operatorname{supp}\tau_y\varphi = y+\operatorname{supp}\varphi \subset \operatorname{supp} f + \operatorname{supp}\varphi = L,$$

we have $\operatorname{supp} f*\varphi \subset L$ due to the second part of Proposition 1.2. If, in addition, $\rho(\varphi)$ is less than the value in the statement, then $L\subset\subset\Omega$ due to (1.3), so that $f*\varphi\in\mathcal{D}(\Omega)$. The uniform convergence of $f*\varphi_\varepsilon$ on $\mathbb{R}^n$ is proved as in (a) if we take into account that f is uniformly continuous on $\mathbb{R}^n$.

(c) is the formula for integration by parts

$$\partial^\alpha(f*\varphi)(x) = \int f(y)(\partial_x^\alpha\varphi)(x-y)\,dy = (-1)^{|\alpha|}\int f(y)\partial_y^\alpha\varphi(x-y)\,dy =$$
$$= \int(\partial_y^\alpha f)(y)\varphi(x-y)\,dy = ((\partial^\alpha f)*\varphi)(x).$$

If $1<p<\infty$, the inequality in (e) follows from Hölder's inequality and from (1.7); in all other cases, it is obvious. The first property in (f) follows from the formula of change of variables in the integral, and the second property is obvious if it is written in the integral form. □

From properties (a) and (d) of Proposition 1.3 it is seen that $f*\varphi_\varepsilon$ approximates the function f as $\varepsilon\longrightarrow +0$ if φ belongs to the set

$$\mathcal{A}_0(\mathbb{R}^n) = \Big\{\varphi\in\mathcal{D}(\mathbb{R}^n)\ \Big|\ \int_{\mathbb{R}^n}\varphi(x)\,dx = 1\Big\}. \tag{1.9}$$

This set plays the most important role in Colombeau's theory of generalized functions. We note that it is *not empty* since the set $\mathcal{A}_0(\mathbb{R})$ contains the function $\varphi(x) = c\zeta(x)$, $x\in\mathbb{R}$, where ζ was constructed above and $c=\big(\int\zeta\big)^{-1}$; if $n\in\mathbb{N}$, the tensor product of functions $\psi_1,\dots,\psi_n\in\mathcal{A}_0(\mathbb{R})$ belongs to $\mathcal{A}_0(\mathbb{R}^n)$. Furthermore, the set $\mathcal{A}_0(\mathbb{R}^n)$ contains functions with arbitrary small supports; in fact, if $r>0$ and $\varphi\in\mathcal{A}_0(\mathbb{R}^n)$, then by setting $\psi=\varphi_\varepsilon$, where $0<\varepsilon<r/\rho(\varphi)$, we have $\psi\in\mathcal{A}_0(\mathbb{R}^n)$ and $\rho(\psi)<r$. Note that the set $\mathcal{A}_0(\mathbb{R}^n)$ is neither a linear space nor an algebra, and that it is still nonempty if we replace the unit in its definition by any number from $\mathbb{K}$.

The following construction in Proposition 1.4 gives a large number of examples of test functions in the space $\mathcal{D}(\Omega)$, where $\Omega\subset\mathbb{R}^n$ is an open set.

Proposition 1.4. *Let $K\subset\subset\Omega$, $d:=\operatorname{dist}(K,\partial\Omega)$, and let $\varphi\in\mathcal{A}_0(\mathbb{R}^n)$ be such that $\rho(\varphi)<d/3$. If $\chi_{2\rho}$ is the characteristic function of the compact set $K_{2\rho}$ (that is, $\chi_{2\rho}(x)=1$ if $x\in K_{2\rho}$ and $\chi_{2\rho}(x)=0$ if $x\notin K_{2\rho}$), where $\rho(\varphi)\leq\rho<d/3$, then the function $\psi=\chi_{2\rho}*\varphi$ satisfies the following properties:*

$$\psi\in\mathcal{D}(K_{3\rho})\subset\mathcal{D}(\Omega) \quad and \quad \psi=1 \quad on \quad K_\rho\supset K;$$

if, in addition, $\varphi\geq 0$, then $0\leq\psi\leq 1$ on $\mathbb{R}^n$.

Proof. By Proposition 1.3, the function ψ is in $C^\infty(\mathbb{R}^n)$, and also

$$\operatorname{supp}\psi \subset \operatorname{supp}\chi_{2\rho} + \operatorname{supp}\varphi \subset K_{2\rho} + B_{\rho(\varphi)} \subset K_{3\rho} \subset K_d^\circ \subset \Omega,$$

so that $\psi \in \mathcal{D}(K_{3\rho})$. Further, for $x \in K_\rho$, we have $B_{\rho(\varphi)}(x) \subset K_\rho + B_{\rho(\varphi)} \subset K_{2\rho}$; therefore, $\chi_{2\rho}(y) = 1$ for all $y \in B_{\rho(\varphi)}(x)$ and $x \in K_\rho$. Consequently,

$$\psi(x) = \int\limits_{B_{\rho(\varphi)}(x)} \chi_{2\rho}(y)(\tau_x \check{\varphi})(y)\,dy = \int (\tau_x \check{\varphi})(y)\,dy = \int \varphi = 1, \quad x \in K_\rho.$$

If, furthermore, $\varphi \geq 0$, then, in addition, we have

$$0 \leq \psi(x) = \int \chi_{2\rho}(y)\varphi(x-y)\,dy \leq \int \varphi(x-y)\,dy = 1, \quad x \in \mathbb{R}^n. \qquad \square$$

2. The Colombeau Algebra on an Open Set

2.1. Index sets. For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, we define the *αth moment of the function* φ by

$$M^\alpha(\varphi) = \int\limits_{\mathbb{R}^n} \lambda^\alpha \varphi(\lambda)\,d\lambda, \tag{2.1}$$

so that (by the change of variables formula for integration)

$$M^\alpha(\varphi_\varepsilon) = \varepsilon^{|\alpha|} M^\alpha(\varphi)\ \forall\, \varepsilon > 0 \quad \text{and} \quad M^\alpha(\check{\varphi}) = (-1)^{|\alpha|} M^\alpha(\varphi). \tag{2.2}$$

In the set $\mathcal{A}_0(\mathbb{R}^n) = \{\varphi \in \mathcal{D}(\mathbb{R}^n) \mid M^0(\varphi) = 1\}$, which was already defined earlier in (1.9), we choose the subsets

$$\mathcal{A}_q(\mathbb{R}^n) = \{\varphi \in \mathcal{A}_0(\mathbb{R}^n) \mid M^\alpha(\varphi) = 0 \quad \forall\, \alpha \in \mathbb{N}_0^n,\ 1 \leq |\alpha| \leq q\}, \quad q \in \mathbb{N},$$

which will be called the *index sets.* These sets play the crucial role in Colombeau's generalized analysis, so they will be studied first of all. The following lemma determines some of the frequently used properties of sets $\mathcal{A}_q(\mathbb{R}^n)$ and gives a constructive way of forming their elements. The usefulness of the special structure of the sets $\mathcal{A}_q(\mathbb{R}^n)$ will be clarified somewhat later.

Lemma 2.1. *The sets $\mathcal{A}_q(\mathbb{R}^n)$ are nonempty, nondecreasing, have empty intersection, and if $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$, then φ_ε, $\check{\varphi} \in \mathcal{A}_q(\mathbb{R}^n)$.*

Proof. 1. First, we consider the case $n = 1$. Let $\varphi_0 \in \mathcal{D}(\mathbb{R})$ be such that $\int \varphi_0(\lambda)\,d\lambda \neq 0$. For $q \in \mathbb{N}$, set $\varphi = \sum\limits_{k=0}^{q} a_k \varphi_0^{(k)}$, and let us find conditions on the numbers $a_0, \ldots, a_q \in \mathbb{K}$ under which $\varphi \in \mathcal{A}_q(\mathbb{R})$. Note that if i, $k \in \mathbb{N}_0$, integrating by parts, we have

$$\int \lambda^i \varphi_0^{(k)}(\lambda)\,d\lambda = \begin{cases} (-1)^k \dfrac{i\,!}{(i-k)\,!} \displaystyle\int \lambda^{i-k} \varphi_0(\lambda)\,d\lambda =: C_k(i,\varphi_0) & \text{if } k \leq i, \\ 0 & \text{if } k > i, \end{cases}$$

and

$$C_i(i,\varphi_0) = (-1)^i\, i\,! \int \varphi_0(\lambda)\,d\lambda \neq 0. \tag{2.3}$$

For $i \in \mathbb{N}_0$, $i \leq q$, it follows that

$$M^i(\varphi) := \int \lambda^i \varphi(\lambda)\,d\lambda = \sum_{k=0}^{q} a_k \int \lambda^i \varphi_0^{(k)}(\lambda)\,d\lambda = \sum_{k=0}^{i} a_k C_k(i,\varphi_0). \tag{2.4}$$

Fix $\varphi_0 \in \mathcal{A}_0(\mathbb{R})$, and let $a_0 = 1$. Due to (2.3), we choose recursively the numbers $a_1, \dots, a_q$ as follows: first choose a_1 such that $M^1(\varphi) = 0$, then choose a_2 such that $M^2(\varphi) = 0, \dots$, and finally choose a_q such that $M^q(\varphi) = 0$. Then $\varphi \in \mathcal{A}_q(\mathbb{R})$.

2. For an arbitrary $n \in \mathbb{N}$, consider the function $\psi = \bigotimes_{j=1}^{n} \psi_j$ which is the tensor product of functions $\psi_1, \dots, \psi_n \in \mathcal{A}_q(\mathbb{R})$. By Fubini's theorem, we find that $M^\alpha(\psi) = \prod_{j=1}^{n} M^{\alpha_j}(\psi_j)$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$; this immediately yields $\psi \in \mathcal{A}_q(\mathbb{R}^n)$.

3. The inclusions $\mathcal{A}_q(\mathbb{R}^n) \supset \mathcal{A}_{q+1}(\mathbb{R}^n)$ for every $q \in \mathbb{N}$ are immediate from the definition of the index sets. The fact that the functions φ_ε and $\check{\varphi}$ belong to the set $\mathcal{A}_q(\mathbb{R}^n)$ ensures that $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ is a consequence of (2.2).

4. Let us show that $\bigcap_{q \in \mathbb{N}} \mathcal{A}_q(\mathbb{R}^n) = \varnothing$. Assume the contrary, i.e., that a function φ is in $\mathcal{A}_q(\mathbb{R}^n)$ for all $q \in \mathbb{N}$, so that $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $M^0(\varphi) = 1$, and $M^\alpha(\varphi) = 0$ for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \geq 1$. Consider the Fourier transform $\hat{\varphi}$ of φ given by

$$\hat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

where $x \cdot \xi = \sum_{j=1}^{n} x_j\xi_j$ and $i = \sqrt{-1}$ is the imaginary unity. By the Paley–Wiener theorem, $\hat{\varphi}(\xi)$ is an entire analytic function in the variable ξ, so, in particular, it has the Taylor expansion at $\xi = 0$ of the form

$$\hat{\varphi}(\xi) = \hat{\varphi}(0) + \sum_{|\alpha|=1}^{\infty} \frac{1}{\alpha!}(\partial^\alpha\hat{\varphi})(0) \cdot \xi^\alpha, \quad \xi \in \mathbb{R}^n,$$

and, moreover,

$$\hat{\varphi}(\xi) \longrightarrow 0 \quad \text{as} \quad |\xi| \longrightarrow \infty. \tag{2.5}$$

Noting that $\hat{\varphi}(0) = M^0(\varphi) = 1$ and $(\partial^\alpha\hat{\varphi})(0) = (-i)^{|\alpha|}M^\alpha(\varphi) = 0$, $|\alpha| \geq 1$, we come to the equality $\hat{\varphi}(\xi) = \hat{\varphi}(0) = 1$ for all $\xi \in \mathbb{R}^n$; this contradicts (2.5). □

Remark 2.2. From the proof of Lemma 2.1, one sees that the sets $\mathcal{A}_q(\mathbb{R}^n)$ contain (a) real-valued functions (take real-valued φ_0), (b) functions φ with arbitrarily small support (since $\operatorname{supp}\varphi \subset \operatorname{supp}\varphi_0$), (c) even functions (choose a function $\varphi_0 \in \mathcal{A}_0(\mathbb{R})$ to be even and set $a_k = 0$ for odd k; then the function φ from Lemma 2.1 will be even, too, and $\varphi \in \mathcal{A}_q(\mathbb{R})$ provided a_k are suitably chosen for even k), and (d) functions φ satisfying the condition $\varphi(0) = 1$ (choose $\varphi_0 \in \mathcal{A}_0(\mathbb{R})$ such that $\varphi_0 = 1$ in a neighborhood of zero).

Note, in addition, that if $q \geq 2$, then real-valued elements of $\mathcal{A}_q(\mathbb{R}^n)$ cannot assume only nonnegative values. □

In the sequel, by a *differential algebra* we shall mean any algebra A (over the field $\mathbb{K}$) with product denoted by $\cdot$, say, for which there exists at least one linear mapping $D : A \longrightarrow A$ satisfying *Leibnitz's rule* for the differentiation of a product: $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$ $\forall a, b \in A$. Such a mapping D is called a *differential operator* (or a *derivation*) in A. If $D^0 := \mathrm{id}_A$ is the identity mapping of A and $D^k := D(D^{k-1})$ for $k \in \mathbb{N}$, then the following general Leibnitz's rule holds in the differential algebra A:

$$D^k(a \cdot b) = \sum_{i=0}^{k} \binom{k}{i} D^{k-i}(a) \cdot D^i(b), \qquad \binom{k}{i} := \frac{k!}{i!(k-i)!}.$$

2.2. The definition of the Colombeau algebra. Now we turn to the definition of the differential algebra $\mathcal{G}(\Omega)$ of Colombeau's generalized functions on an open set $\Omega \subset \mathbb{R}^n$. As a starting point, consider the infinite product

$$\mathcal{E}[\Omega] = \left(C^\infty(\Omega)\right)^{\mathcal{A}_0(\mathbb{R}^n)},$$

consisting of all mappings $u : \mathcal{A}_0(\mathbb{R}^n) \longrightarrow C^\infty(\Omega)$. The value $u(\varphi) \in C^\infty(\Omega)$ of an element $u \in \mathcal{E}[\Omega]$ on a function $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ calculated at a point $x \in \Omega$ will be written as $u(\varphi, x) := u(\varphi)(x)$. In this way, the set $\mathcal{E}[\Omega]$ can be considered as the set of mappings $u : \mathcal{A}_0(\mathbb{R}^n) \times \Omega \longrightarrow \mathbb{K}$ such that $u(\varphi, \cdot) \in C^\infty(\Omega)$ for all $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. The set $\mathcal{E}[\Omega]$ is a differential algebra under the componentwise operations of addition, multiplication by a number, product, and partial differentiation:

$$\begin{aligned}(c_1 u + c_2 v)(\varphi, x) &= c_1 u(\varphi, x) + c_2 v(\varphi, x), \qquad c_1,\, c_2 \in \mathbb{K},\\ (u \cdot v)(\varphi, x) &= u(\varphi, x) v(\varphi, x),\\ (\partial^\alpha u)(\varphi, x) &= \partial_x^\alpha(u(\varphi, x)), \qquad \alpha \in \mathbb{N}_0^n,\end{aligned}$$

for $u,\, v \in \mathcal{E}[\Omega]$, $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, and $x \in \Omega$. In particular, the following Leibnitz's rule holds in $\mathcal{E}[\Omega]$:

$$\partial^\alpha(u \cdot v) = \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} u) \cdot (\partial^\beta v), \quad \binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\,\beta!}, \quad \alpha \in \mathbb{N}_0^n.$$

The algebra $C^\infty(\Omega)$ is contained in $\mathcal{E}[\Omega]$ as the subset of those elements of $\mathcal{E}[\Omega]$ which do not depend on the first variable $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. In other words, the map $\sigma : C^\infty(\Omega) \longrightarrow \mathcal{E}[\Omega]$, defined by $\sigma(f)(\varphi, x) = f(x)$ for $f \in C^\infty(\Omega)$, $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ and $x \in \Omega$, is a homomorphic imbedding of $C^\infty(\Omega)$ into the algebra $\mathcal{E}[\Omega]$ preserving partial derivatives (commuting with partial derivatives):

$$\partial^\alpha \sigma(f) = \sigma(\partial^\alpha f) \quad \text{in} \quad \mathcal{E}[\Omega], \quad \alpha \in \mathbb{N}_0^n.$$

In the sequel, for the sake of brevity, we set $f(\varphi, x) := \sigma(f)(\varphi, x)$ if $f \in C^\infty(\Omega)$.

Summing up what we have said above, we conclude that

> $\mathcal{E}[\Omega]$ is an associative and commutative differential algebra (with the unit element $1 := \sigma(1) \in \mathcal{E}[\Omega]$ with respect to the multiplication) containing the algebra $C^\infty(\Omega)$ as a differential subalgebra.

Although the algebra $\mathcal{E}[\Omega]$ is almost the desired object for a good nonlinear theory of generalized functions, it is "too wide" and its elements can grow too fast (in a sense to be made precise in the sequel). So, we define a subalgebra $\mathcal{E}_M[\Omega]$ in $\mathcal{E}[\Omega]$ (M is not to be dissociated in $\mathcal{E}_M[\Omega]$) of *moderate* (polynomially growing in $1/\varepsilon$) elements as follows:

$$\begin{aligned}\mathcal{E}_M[\Omega] = \{\, u \in \mathcal{E}[\Omega] \,|\, &\forall K \subset\subset \Omega \ \forall \alpha \in \mathbb{N}_0^n \ \exists N \in \mathbb{N} :\\ &\forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) \ \exists c > 0, \eta > 0 :\\ &\forall \varepsilon \in (0, \eta) : \sup_{x \in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \le c\varepsilon^{-N} \,\}.\end{aligned} \tag{2.6}$$

In the definition of $\mathcal{E}_M[\Omega]$, we use the convention that a letter following the quantifier $\exists$ can, in general, depend on all the letters encountered in the preceding quantifiers. Equivalently we can write $\forall K \subset\subset \Omega$ $\forall \alpha \in \mathbb{N}_0^n \, \exists N_1,\, N_2 \in \mathbb{N}$ such that $\forall \varphi \in \mathcal{A}_{N_1}(\mathbb{R}^n) \, \exists c > 0,\, \eta \in (0,1)$ such that $\forall x \in K \ \forall \varepsilon \in (0, \eta)$ we have the inequality $|\partial^\alpha u(\varphi_\varepsilon, x)| \le c\varepsilon^{-N_2}$; then we obtain N as in (2.6) if we set $N = \max\{N_1, N_2\}$. Using Leibnitz's rule in $\mathcal{E}[\Omega]$, it is easy to verify that the set $\mathcal{E}_M[\Omega]$ is, indeed, a differential algebra with respect to the operations as in $\mathcal{E}[\Omega]$, and, moreover, the map σ is, in fact, a homomorphic imbedding of the algebra $C^\infty(\Omega)$ into $\mathcal{E}_M[\Omega]$. In addition, the algebra $\mathcal{E}_M[\Omega]$ is invariant with respect to partial differentiation operators:

$$\partial^\alpha \mathcal{E}_M[\Omega] \subset \mathcal{E}_M[\Omega], \quad \alpha \in \mathbb{N}_0^n.$$

The next step is to define an ideal $\mathcal{N}[\Omega]$ in the algebra $\mathcal{E}_M[\Omega]$. Denote by Γ the set of all increasing sequences $\gamma : \mathbb{N} \longrightarrow (0, \infty)$ such that $\gamma(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. The set Γ has the following property: if

$\gamma_1, \dots, \gamma_m \in \Gamma$, then $\min\{\gamma_1, \dots, \gamma_m\} \in \Gamma$ as well. We define the set $\mathcal{N}[\Omega]$ of *null* elements in $\mathcal{E}[\Omega]$ as follows:

$$\begin{aligned} \mathcal{N}[\Omega] = \{ u \in \mathcal{E}[\Omega] \,|\, \forall K \subset\subset \Omega \ \forall \alpha \in \mathbb{N}_0^n \ \ \exists N \in \mathbb{N}, \gamma \in \Gamma : \\ \forall q \in \mathbb{N}, q \geq N, \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n) \ \exists c > 0, \eta > 0 : \\ \forall \varepsilon \in (0, \eta) : \sup_{x \in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \leq c\varepsilon^{\gamma(q) - N} \}. \end{aligned} \tag{2.7}$$

Equivalently, one can write a bound in (2.7) of the form $c\varepsilon^{\gamma(q)}$ (by properly changing N and γ). From this definition it is seen that $\mathcal{N}[\Omega] \subset \mathcal{E}_M[\Omega]$, and, moreover, by using Leibnitz's rule in $\mathcal{E}[\Omega]$ and the property of Γ, one can check that the algebra $\mathcal{N}[\Omega]$ is, in fact, an *ideal* in $\mathcal{E}_M[\Omega]$, i.e.,

$$\mathcal{N}[\Omega] \cdot \mathcal{E}_M[\Omega] \quad \text{and} \quad \mathcal{E}_M[\Omega] \cdot \mathcal{N}[\Omega] \subset \mathcal{N}[\Omega].$$

Furthermore, $\mathcal{N}[\Omega]$ is invariant under partial derivative operators:

$$\partial^\alpha \mathcal{N}[\Omega] \subset \mathcal{N}[\Omega], \qquad \alpha \in \mathbb{N}_0^n.$$

Let us mention a special property of the ideal $\mathcal{N}[\Omega]$ consisting in the fact that the "convergence to zero" being constructed in its definition is such that all its elements and their derivatives tend to zero faster than any power of ε provided that $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$ with N sufficiently large:

$$\begin{aligned} &\forall u \in \mathcal{N}[\Omega] \ \forall k \in \mathbb{N} \ \ \forall K \subset\subset \Omega \ \forall \alpha \in \mathbb{N}_0^n \ \ \exists N \in \mathbb{N} \text{ such that} \\ &\forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) : \sup_{x \in K} |\partial^\alpha u(\varphi_\varepsilon, x)| = o(\varepsilon^k) \quad \text{as} \quad \varepsilon \longrightarrow +0. \end{aligned} \tag{2.8}$$

The Colombeau algebra of new generalized functions on an open set $\Omega \subset \mathbb{R}^n$ is defined as the quotient (= factor) algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_M[\Omega]/\mathcal{N}[\Omega].$$

In other words, on $\mathcal{E}_M[\Omega]$ the equivalence relation $\sim$ is introduced as follows: $u \sim v$ if and only if $u - v \in \mathcal{N}[\Omega]$, so that a generalized function U in the sense of Colombeau is the equivalence class

$$U = [u] := \{ v \in \mathcal{E}_M[\Omega] \mid v \sim u \} = u + \mathcal{N}[\Omega]$$

of some element $u \in \mathcal{E}_M[\Omega]$, which is called a *representative* of the generalized function U. In what follows, we shall use the *convention* that elements of $\mathcal{G}(\Omega)$ will be denoted (as a rule) by capital letters, their representatives (members of $\mathcal{E}_M[\Omega]$) by the corresponding small letters. Algebraic operations and partial differentiation in $\mathcal{G}(\Omega)$ are defined for generalized functions $U = [u]$, $V = [v] \in \mathcal{G}(\Omega)$ by means of their representatives in the standard way:

$$\begin{aligned} c_1 U + c_2 V &= [c_1 u + c_2 v], \qquad c_1, c_2 \in \mathbb{K}, \\ U \cdot V &= [u \cdot v], \\ \partial^\alpha U &= [\partial^\alpha u], \qquad \alpha \in \mathbb{N}_0^n. \end{aligned}$$

These operations are well defined (in the sense that their result does not depend on representatives of equivalence classes) since, as we have mentioned above, $\mathcal{N}[\Omega]$ is a two-sided ideal in the algebra $\mathcal{E}_M[\Omega]$, and both these sets are invariant under partial derivatives. Taking into account the linearity of operators ∂^α and Leibnitz's rule in $\mathcal{E}_M[\Omega]$, we are sure that $\mathcal{G}(\Omega)$ is a *differential algebra* (over the field $\mathbb{K}$).

The *homomorphic imbedding* of the algebra $C^\infty(\Omega)$ into the algebra $\mathcal{G}(\Omega)$ is effected via the mapping

$$\imath : C^\infty(\Omega) \longrightarrow \mathcal{G}(\Omega), \quad \imath(f) = [f] = f(\cdot\,,\cdot) + \mathcal{N}[\Omega] \quad \text{for} \quad f \in C^\infty(\Omega), \tag{2.9}$$

where $f(\varphi, x) = f(x)$ for $(\varphi, x) \in \mathcal{A}_0(\mathbb{R}^n) \times \Omega$. In particular, the injectivity of this mapping means that the equality relation $=$ in $\mathcal{G}(\Omega)$ generalizes the usual pointwise equality of functions from $C^\infty(\Omega)$, the equality $[f] \cdot [g] = [fg]$ means that the pointwise product of functions f and g in $C^\infty(\Omega)$ is preserved in $\mathcal{G}(\Omega)$, and the equality $\partial^\alpha[f] = [\partial^\alpha f]$ shows that the operators ∂^α in $\mathcal{G}(\Omega)$ restricted to $C^\infty(\Omega)$ coincide with the usual partial derivatives in $C^\infty(\Omega)$. Summing up, we have

$\mathcal{G}(\Omega)$ is an associative and commutative differential algebra in which the constant function $1 \in C^\infty(\Omega)$ is the unit element, $\imath(1) = 1 + \mathcal{N}[\Omega] \in \mathcal{G}(\Omega)$, which is not equal to zero, $\imath(0) = \mathcal{N}[\Omega] \in \mathcal{G}(\Omega)$, and the algebra $C^\infty(\Omega)$ is a differential subalgebra in $\mathcal{G}(\Omega)$ (via (2.9)).

We show by examples of elements of $\mathcal{G}(\Omega)$ that the algebra $\mathcal{E}_M[\Omega]$ and its ideal $\mathcal{N}[\Omega]$ are nontrivial:

Proposition 2.3. *$\mathcal{E}_M[\Omega] \subsetneqq \mathcal{E}[\Omega]$, and $\mathcal{N}[\Omega]$ is not an ideal in $\mathcal{E}[\Omega]$.*

Proof. 1. Let $\Omega = \mathbb{R}^n$ for simplicity. First we consider an interesting object $\delta \in \mathcal{G}(\mathbb{R}^n)$ which, as we will see later, represents the Dirac δ function. Define its representative by

$$u_\delta(\varphi, x) = \varphi(-x) = (\tau_x\varphi)(0), \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n), \quad x \in \mathbb{R}^n.$$

Clearly, $u_\delta \in \mathcal{E}[\mathbb{R}^n]$. If $K \subset\subset \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$, then $(\partial^\alpha u_\delta)(\varphi_\varepsilon, x) = (-1)^{|\alpha|}\varepsilon^{-n-|\alpha|}(\partial^\alpha\varphi)(-x/\varepsilon)$ and setting $N = n + |\alpha|$, for $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$, we find that

$$|(\partial^\alpha u_\delta)(\varphi_\varepsilon, x)| \le (\sup_{B_{\rho(\varphi)}} |\partial^\alpha\varphi|)\varepsilon^{-N} \equiv c\,\varepsilon^{-N}$$

for all $x \in K$ and $\varepsilon \in (0,1)$. Hence $u_\delta \in \mathcal{E}_M[\mathbb{R}^n]$.

2. Now if $v(\varphi, x) = e^{\varphi(-x)}$, then $v \notin \mathcal{E}_M[\mathbb{R}^n]$ since $\forall q \in \mathbb{N}$, $\exists \varphi \in \mathcal{A}_q(\mathbb{R}^n)$ such that $\varphi(0) = 1$ (see Remark 2.2(d)), and hence $v(\varphi_\varepsilon, 0) = e^{1/\varepsilon^n}$. (In other words, v is, so to say, a representative of e^δ; however, this composition does not make sense in $\mathcal{G}(\mathbb{R}^n)$. Nevertheless, we will see below that $e^{i\delta} \in \mathcal{G}(\mathbb{R}^n)$, where $i = \sqrt{-1}$.)

3. Consider an element $U \in \mathcal{G}(\mathbb{R}^n)$ with representative $u(\varphi) = e^{-1/\rho(\varphi)}$ which does not depend on x; $\rho(\varphi)$ was introduced in Sec. 1. Since by Proposition 1.1(c) $\rho(\varphi_\varepsilon) = \varepsilon\rho(\varphi)$, $u(\varphi_\varepsilon)$ decreases as $\varepsilon \longrightarrow +0$ faster than any power ε^q, $u \in \mathcal{N}[\mathbb{R}^n]$ (or $U = 0$). On the other hand, if $u^{-1}(\varphi) = 1/u(\varphi)$, then u^{-1} is in $\mathcal{E}[\mathbb{R}^n]$ but not in $\mathcal{E}_M[\mathbb{R}^n]$ since $u^{-1}(\varphi_\varepsilon)$ grows faster than any power $(1/\varepsilon)^N$ as $\varepsilon \longrightarrow +0$. Since $u \cdot u^{-1} = 1 \notin \mathcal{N}[\mathbb{R}^n]$, it follows that $\mathcal{N}[\mathbb{R}^n]$ is not an ideal in $\mathcal{E}[\mathbb{R}^n]$. (Note that the algebra $\mathcal{E}_M[\mathbb{R}^n]$ of moderate elements, which is completely similar to $\mathcal{E}[\mathbb{R}^n]$, was introduced by Colombeau in order that the set $\mathcal{N}[\mathbb{R}^n]$ be an ideal in $\mathcal{E}_M[\mathbb{R}^n]$.) □

The definitions of the algebra $\mathcal{E}_M[\Omega]$ and the ideal $\mathcal{N}[\Omega]$ therein are rather complicated, and they were given without prior considerations. Their motivation and the natural character of the algebra $\mathcal{G}(\Omega)$ become clear when we try to imbed the space of continuous functions $C(\Omega)$ into $\mathcal{G}(\Omega)$. To emphasize the main ideas we start with the most simple case where $\Omega = \mathbb{R}^n$.

We assign the mapping $u_f : \mathcal{A}_0(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ defined by

$$u_f(\varphi) = f * \check{\varphi}, \qquad \varphi \in \mathcal{A}_0(\mathbb{R}^n) \tag{2.10}$$

to every continuous function $f \in C(\mathbb{R}^n)$ or locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. From Proposition 1.3 it follows that $u_f \in \mathcal{E}[\mathbb{R}^n]$. If $K \subset\subset \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$, then setting $N = n + |\alpha|$ and taking into account Propositions 1.1(d) and 1.3(e), for $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$, we have

$$\begin{aligned} |(\partial^\alpha u_f)(\varphi_\varepsilon, x)| &= |(-\varepsilon)^{-|\alpha|}(f * (\partial^\alpha\varphi)_\varepsilon^\vee)(x)| \le \\ &\le \varepsilon^{-n-|\alpha|}\| f \|_{L^1(K_\rho)}\| \partial^\alpha\varphi \|_{L^\infty(B_{\rho(\varphi)})} \equiv c\,\varepsilon^{-N}, \quad x \in K, \end{aligned} \tag{2.11}$$

where $0 < \varepsilon < \eta = \rho/\rho(\varphi)$, and $\rho \in (0, \infty)$ is fixed. Thus, $u_f \in \mathcal{E}_M[\mathbb{R}^n]$.

It follows that the mapping $\jmath : C(\mathbb{R}^n) \longrightarrow \mathcal{G}(\mathbb{R}^n)$ given by

$$\jmath(f) \equiv U_f := [u_f] = u_f + \mathcal{N}[\mathbb{R}^n], \qquad f \in C(\mathbb{R}^n), \tag{2.12}$$

is well defined (and analogously for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$). The mapping $\jmath$ is *linear*; this follows from the linearity of the mapping $f \mapsto u_f$ and the definition of operations in $\mathcal{G}(\mathbb{R}^n)$, and it is *injective* due to the implication $u_f \in \mathcal{N}[\mathbb{R}^n] \Longrightarrow f = 0$, which, in turn, follows from property (2.8) of $\mathcal{N}[\mathbb{R}^n]$ and Propositions 1.3(a), (d), and 1.1(e). Furthermore, $\jmath$ commutes with partial derivation operators ∂^α on the space $C^k(\mathbb{R}^n)$, where $k \in \mathbb{N}$:

$$\partial^\alpha \jmath(f) = \jmath(\partial^\alpha f) \quad \text{in} \quad \mathcal{G}(\mathbb{R}^n) \quad \text{for} \quad f \in C^k(\mathbb{R}^n) \quad \text{and} \quad |\alpha| \le k;$$

this is a consequence of Proposition 1.3(c).

One of the main properties of the ideal $\mathcal{N}[\mathbb{R}^n]$ (and the reason why it has been chosen in $\mathcal{E}_M[\mathbb{R}^n]$) is that both imbeddings (2.9) and (2.12) of $C^\infty(\mathbb{R}^n)$ into $\mathcal{G}(\mathbb{R}^n)$ coincide:

Proposition 2.4. *If $f \in C^\infty(\mathbb{R}^n)$, then $u_f - f \in \mathcal{N}[\mathbb{R}^n]$, so that $\jmath|_{C^\infty(\mathbb{R}^n)} = \imath$.*

Proof. Let $K \subset\subset \mathbb{R}^n$ and $\alpha = 0$. Applying Taylor's formula to the function f up to order $q \in \mathbb{N}$, for $x \in K$, $\varepsilon > 0$, and $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, one has

$$(f * \check{\varphi}_\varepsilon)(x) - f(x) = \int (f(x+\varepsilon\mu) - f(x))\varphi(\mu)\, d\mu =$$

$$= \sum_{|\beta|=1}^{q} \frac{\varepsilon^{|\beta|}}{\beta!}(\partial^\beta f)(x) \int \mu^\beta \varphi(\mu)\, d\mu +$$

$$+\varepsilon^{q+1} \cdot \sum_{|\beta|=q+1} \frac{q+1}{\beta!} \int\limits_{B_{\rho(\varphi)}} \int\limits_0^1 (1-t)^q (\partial^\beta f)(x + t\varepsilon\mu)\, dt \cdot \mu^\beta \varphi(\mu)\, d\mu.$$

If $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$, the first sum vanishes, and if $0 < \varepsilon < \eta = \rho/\rho(\varphi)$ with $\rho \in (0,\infty)$ fixed, then we have the following estimate for the second sum:

$$\varepsilon^{q+1} \cdot \sum_{|\beta|=q+1} \Big(\sup_{K_\rho} |\partial^\beta f|\Big) \int\limits_{B_{\rho(\varphi)}} |\mu^\beta \varphi(\mu)|\, d\mu \equiv c\,\varepsilon^{q+1}.$$

Analogous arguments can be applied to any partial derivative of the form $\partial^\alpha(u_f - f)$ if we take into account that $\partial^\alpha u_f = u_{\partial^\alpha f}$. □

In (2.9), we have seen that $C^\infty(\mathbb{R}^n)$ is a subalgebra in $\mathcal{G}(\mathbb{R}^n)$, so that, in particular, $\imath(fg) = \imath(f) \cdot \imath(g)$ in $\mathcal{G}(\mathbb{R}^n)$ if f and g are in $C^\infty(\mathbb{R}^n)$. By Proposition 2.4, the mapping $\jmath$ is a homomorphism of algebras $C^\infty(\mathbb{R}^n)$ and $\mathcal{G}(\mathbb{R}^n)$ as well, and if f and g are infinitely differentiable functions, then

$$\jmath(fg) = \imath(fg) = \imath(f) \cdot \imath(g) = \jmath(f) \cdot \jmath(g) \quad \text{in} \quad \mathcal{G}(\mathbb{R}^n).$$

This implies the less obvious inclusion

$$u_{fg} - u_f \cdot u_g \in \mathcal{N}[\mathbb{R}^n], \quad f, g \in C^\infty(\mathbb{R}^n). \tag{2.13}$$

However, the inclusion (2.13) does not take place in general if the functions f and g are only in $C^k(\mathbb{R}^n)$ with $k < \infty$. This is shown below by examples.

Examples 2.5. (1) For any (finite) $k \in \mathbb{N}_0$, the algebra $C^k(\mathbb{R}^n)$ is not a subalgebra in $\mathcal{G}(\mathbb{R}^n)$ (relative to the inclusion (2.12)), so that the product $\mathcal{G}(\mathbb{R}^n) \cdot \mathcal{G}(\mathbb{R}^n)$ does not generalize the product $C^k(\mathbb{R}^n) \cdot C^k(\mathbb{R}^n)$.

Proof. Consider the following two functions of one real variable:

$$f(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x^{k+1} & \text{if } x \ge 0, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x^{k+1} & \text{if } x \le 0, \\ 0 & \text{if } x \ge 0. \end{cases}$$

Then $fg = 0$ in $C^k(\mathbb{R})$, so that $\jmath(fg) = 0$ in $\mathcal{G}(\mathbb{R})$. On the other hand,

$$(u_f \cdot u_g)(\varphi_\varepsilon, x) = \int f(x+\varepsilon\mu)\varphi(\mu)\, d\mu \cdot \int g(x+\varepsilon\mu)\varphi(\mu)\, d\mu; \tag{2.14}$$

this implies

$$(u_f \cdot u_g)(\varphi_\varepsilon, 0) = \varepsilon^{2k+2} \int_0^\infty \mu^{k+1}\varphi(\mu)\,d\mu \cdot \int_{-\infty}^0 \mu^{k+1}\varphi(\mu)\,d\mu.$$

To prove that $u_f \cdot u_g \notin \mathcal{N}[\mathbb{R}]$, by property (2.8), it suffices to show that

$$\begin{gathered}\forall q \in \mathbb{N},\ q \geq k+1,\ \exists \varphi \in \mathcal{A}_q(\mathbb{R}) \text{ such that} \\ -\int_{-\infty}^0 \mu^{k+1}\varphi(\mu)\,d\mu = \int_0^\infty \mu^{k+1}\varphi(\mu)\,d\mu = \frac{1}{2}.\end{gathered} \tag{2.15}$$

To this end, we fix two functions φ_0, $\psi_0 \in \mathcal{A}_0(\mathbb{R})$ such that $\operatorname{supp}\varphi_0 \subset (0,\infty)$, $\operatorname{supp}\psi_0 \subset (-\infty,0)$, and for $q \geq k+1$ we set $\varphi_1 = \varphi_0 + \sum_{j=1}^q a_j\varphi_0^{(j)}$, $\psi_1 = \psi_0 + \sum_{j=1}^q b_j\psi_0^{(j)}$. As in Lemma 2.1 (using the same notation), due to (2.4) and (2.3), we choose successively numbers a_j and b_j for $j = 1,\ldots,q$ in such a way that $M^i(\varphi_1) = M^i(\psi_1) = 0$ if $1 \leq i \leq q$, $i \neq k+1$, and $M^{k+1}(\varphi_1) = -M^{k+1}(\psi_1) = 1$. Then $\varphi = (\varphi_1+\psi_1)/2 \in \mathcal{A}_q(\mathbb{R})$ for $q \geq k+1$, and φ satisfies the equalities in (2.15).

The case of functions in $C^k(\mathbb{R}^n)$ of n real variables reduces to the one above if we set $\tilde{f}(x_1,\ldots,x_n) = f(x_1)$, $\tilde{g}(x_1,\ldots,x_n) = g(x_1)$, and consider functions $\tilde{\varphi} \in \mathcal{A}_q(\mathbb{R}^n)$ in the tensor product form $\tilde{\varphi}(x_1,\ldots,x_n) = \varphi(x_1)\cdots\varphi(x_n)$, where $\varphi \in \mathcal{A}_q(\mathbb{R})$.

(2) Here we show that the product in the algebra $\mathcal{G}(\mathbb{R})$ does not generalize the product of the type $C^\infty(\mathbb{R}) \cdot C^k(\mathbb{R})$. Consider the functions $f(x) = x^k$ and $g(x) = x^k|x|$, $x \in \mathbb{R}$. Taking (2.14) into account, we have

$$(u_{fg} - u_f \cdot u_g)(\varphi_\varepsilon, 0) = \varepsilon^{2k+1}\Big(\int \mu^{2k}|\mu|\varphi(\mu)\,d\mu - \int \mu^k\varphi(\mu)\,d\mu \cdot \int \mu^k|\mu|\varphi(\mu)\,d\mu\Big).$$

If $\varphi \in \mathcal{A}_k(\mathbb{R})$, then $\int \mu^k\varphi(\mu)\,d\mu = 0$, and it remains to note that due to (2.15), for $q \geq 2k+1$, there exists a function $\varphi \in \mathcal{A}_q(\mathbb{R})$ such that $\int \mu^{2k}|\mu|\varphi(\mu)\,d\mu = 1$. Hence $u_{fg} - u_f \cdot u_g \notin \mathcal{N}[\mathbb{R}]$, and $\jmath(fg) \neq \jmath(f)\cdot\jmath(g)$ in $\mathcal{G}(\mathbb{R})$. □

The discrepancy between the classical pointwise product of continuous functions and their product in the algebra $\mathcal{G}(\mathbb{R}^n)$ might seem, at first sight, as a deficiency of the product in $\mathcal{G}(\mathbb{R}^n)$. However, due to the Schwartz impossibility result (Sec. 7.2) a certain incoherence between the two products is unavoidable. On the other hand, in the Colombeau theory, this incoherence will be removed (Theorem 8.10) by means of a weaker kind of equality, called the association, between the elements of $\mathcal{G}(\mathbb{R}^n)$, which is not so strict as the algebraic equality in $\mathcal{G}(\mathbb{R}^n)$. The last property is the characteristic feature of the Colombeau theory, which gives this theory an unusual power. The splitting of equality into several types (mainly three) is an inevitable cost for the desire to recover the classical products.

The space $C(\mathbb{R}^n)$ is imbedded into the algebra $\mathcal{G}(\mathbb{R}^n)$ via the mappings in (2.12) and (2.10). Alternatively, one might define another imbedding $\check{\jmath}\colon C(\mathbb{R}^n) \longrightarrow \mathcal{G}(\mathbb{R}^n)$ as follows: if $f \in C(\mathbb{R}^n)$, let $\check{\jmath}(f)$ be the equivalence class of the mapping $\{f * \varphi\}_{\varphi\in\mathcal{A}_0(\mathbb{R}^n)}$. Clearly, the mapping $\check{\jmath}$ possesses all the properties satisfied by the mapping $\jmath$, and in particular, $\check{\jmath}\,|_{C^\infty(\mathbb{R}^n)} = \imath$. Therefore $\check{\jmath}$ might also be taken as a canonical imbedding of $C(\mathbb{R}^n)$ into $\mathcal{G}(\mathbb{R}^n)$. However, one has to keep in mind that the mappings $\jmath$ and $\check{\jmath}$ are not to be used simultaneously, since $\check{\jmath} \neq \jmath$. To see this, let $f(x) = x^{2k-1}|x|$, $x \in \mathbb{R}$, and $k \in \mathbb{N}$ be fixed. Since

$$(f * \check{\varphi}_\varepsilon)(x) - (f * \varphi_\varepsilon)(x) = \int f(x+\varepsilon\mu)(\varphi(\mu) - \check{\varphi}(\mu))\,d\mu,$$

we have

$$(f * \check{\varphi}_\varepsilon - f * \varphi_\varepsilon)(0) = 2\varepsilon^{2k}\int \mu^{2k-1}|\mu|\varphi(\mu)\,d\mu.$$

By (2.15), for $q \geq 2k$ one can find $\varphi \in \mathcal{A}_q(\mathbb{R})$ such that $\int \mu^{2k-1}|\mu|\varphi(\mu)\,d\mu = 1$. Therefore, $\{f * \check{\varphi} - f * \varphi\}_{\varphi\in\mathcal{A}_0(\mathbb{R})} \notin \mathcal{N}[\mathbb{R}^n]$, and $\jmath(f) \neq \check{\jmath}(f)$ in $\mathcal{G}(\mathbb{R})$. Nevertheless, functions $\check{\jmath}(f)$ and $\jmath(f)$ are equal in $\mathcal{G}(\mathbb{R})$ in a weaker sense of the association (Definition 8.6).

2.3. Imbedding of $C(\Omega)$ into $\mathcal{G}(\Omega)$. We now consider an imbedding of the space $C(\Omega)$ into the algebra $\mathcal{G}(\Omega)$ in the case of an arbitrary open set $\Omega \subset \mathbb{R}^n$. For functions $f \in C_c(\Omega)$ with compact support, such an imbedding is given via the mapping $\jmath_c$ from $C_c(\Omega)$ into $\mathcal{G}(\Omega)$ defined by the formula $\jmath_c(f) = u_f + \mathcal{N}[\Omega]$, where $u_f(\varphi) = (f * \check{\varphi})|_\Omega$, $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. To extend this imbedding onto the space $C(\Omega)$, we proceed as follows. Let $\{\Omega_\rho\}$ be an exhaustion of the set Ω by compact subsets Ω_ρ of the form

$$\Omega_\rho = \{\, x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \geq \rho \quad\text{and}\quad |x| \leq 1/\rho \,\}, \qquad \rho > 0.$$

(If Ω is bounded, then the condition $|x| \leq 1/\rho$ is redundant, and if $\Omega = \mathbb{R}^n$, the first condition in the definition of Ω_ρ is optional.) Let χ_ρ be the characteristic function of the set Ω_ρ ($\chi_\rho \equiv 0$ if $\Omega_\rho = \varnothing$). Given $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, we set (cf. also Proposition 1.4)

$$\ell(\varphi) = \chi_{2\rho(\varphi)} * \varphi \in C^\infty(\mathbb{R}^n). \tag{2.16}$$

Since $\rho(\varphi) < 2\rho(\varphi) \leq \operatorname{dist}(\Omega_{2\rho(\varphi)}, \partial\Omega)$, we have $\ell(\varphi) \in \mathcal{D}(\Omega)$ (possibly, $\ell(\varphi) \equiv 0$). Moreover, if $K \subset\subset \Omega$ and $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ are such that $\rho(\varphi) \leq \tilde{\rho}_K$ with

$$\tilde{\rho}_K = \frac{1}{4}\min\{\, \operatorname{dist}(K, \partial\Omega),\ (1 + \sup_{x\in K}|x|)^{-1} \,\},$$

then $\operatorname{supp}\ell(\varphi) \subset \Omega_{\rho(\varphi)}$, $K_{\rho(\varphi)} \equiv K + B_{\rho(\varphi)} \subset \Omega_{3\rho(\varphi)}$, and $\ell(\varphi) \equiv 1$ on $\Omega_{3\rho(\varphi)}$. We define the imbedding $\jmath : C(\Omega) \longrightarrow \mathcal{G}(\Omega)$ in the same way as in (2.12):

$$\jmath(f) = u_f + \mathcal{N}[\Omega], \quad \text{where} \quad u_f(\varphi) = (\ell(\varphi)f) * \check{\varphi}, \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n). \tag{2.17}$$

The main observation here is that if $K \subset\subset \Omega$ and $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, then there exists $\eta = \eta(K, \varphi) > 0$ (for example, $\eta = \tilde{\rho}_K/\rho(\varphi)$) such that $K \subset\subset \Omega(\check{\varphi}_\varepsilon)$ and

$$u_f(\varphi_\varepsilon, x) = (f * \check{\varphi}_\varepsilon)(x), \quad x \in K, \quad \varepsilon \in (0, \eta). \tag{2.18}$$

It is easily seen that the mapping $\jmath$ shares the same properties as (2.12), namely, $\jmath$ is a linear imbedding commuting with partial derivatives on the space $C^k(\Omega)$ and coinciding with the mapping $\imath$ from (2.9) on the algebra $C^\infty(\Omega)$.

An imbedding of the space $L^1_{\mathrm{loc}}(\Omega)$ is defined analogously.

Note that in (2.17) there is an arbitrariness in the definition of u_f: instead of the mapping $\rho(\varphi)$ allowing us to extract ε from φ_ε, one could use some other mapping, for example, the diameter of the support of φ, $d(\varphi) = \sup\{\, |x - y| \mid x, y \in \operatorname{supp}\varphi \,\}$, or the integral of the form $I(\varphi) = \int |\varphi|^2$, with $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, so that $d(\varphi_\varepsilon) = \varepsilon d(\varphi)$ and $I(\varphi_\varepsilon) = \varepsilon^{-n} I(\varphi)$, $\varepsilon > 0$; on the other hand, one could use another exhaustion of the set Ω. However, this arbitrariness disappears in the factor $\mathcal{G}(\Omega)$ and the imbedding $\jmath$, so that this imbedding, in general, is not different from the simpler one in the case $\Omega = \mathbb{R}^n$. Let us also mention that objects in $\mathcal{G}(\Omega)$ are completely determined by values of their representatives on elements of the form (φ_ε, x) for small ε and x in compact sets. We will see below in this section that $\mathcal{G}$ is a sheaf of differential algebras and that there exists a unique extension $\jmath$ of the mapping $\jmath_c$ which is a sheaf morphism $C \longrightarrow \mathcal{G}$.

2.4. Sheaf of algebras of generalized functions $\mathcal{G}$. We turn now to the study of *local properties* of Colombeau's generalized functions from $\mathcal{G}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^n$. Let $U = u + \mathcal{N}[\Omega] \in \mathcal{G}(\Omega)$ be a generalized function with a representative $u \in \mathcal{E}_M[\mathbb{R}^n]$. We define the *restriction* of U to an open subset $G \subset \Omega$ (through a representative of U) as follows: $U|_G := u|_G + \mathcal{N}[G] \in \mathcal{G}(G)$, where $(u|_G)(\varphi) := u(\varphi)|_G$ for

$\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. Clearly, any restriction mapping is a homomorphism of differential algebras. Furthermore, the new restriction mapping generalizes the usual classical restriction mapping of continuous functions: $\jmath(f)|_G = \jmath(f|_G)$ if $f \in C(\Omega)$ since $(u_f)|_G - u_{(f|_G)} \in \mathcal{N}[G]$ by property (2.18) (on compact subsets of G this difference vanishes for small enough $\varepsilon > 0$).

The following theorem expresses the fact that $\mathcal{G}$ is a *sheaf* (of sections over $\mathbb{R}^n$) of differential algebras.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set. We have the following properties:*

(a) *if $U \in \mathcal{G}(\Omega)$, then $U|_\Omega = U$ in $\mathcal{G}(\Omega)$;*

(b) *if $U \in \mathcal{G}(\Omega)$ and $E \subset G \subset \Omega$ are open sets, then $(U|_G)|_E = U|_E$ in $\mathcal{G}(E)$;*

(c) *if $\{\Omega_i\}_{i\in I}$ is a family of open subsets of Ω and U, $V \in \mathcal{G}(\Omega)$ are such that $U|_{\Omega_i} = V|_{\Omega_i}$ in $\mathcal{G}(\Omega_i)$ for all $i \in I$, then $U|_{\tilde\Omega} = V|_{\tilde\Omega}$ in $\mathcal{G}(\tilde\Omega)$, where $\tilde\Omega = \bigcup_{i\in I} \Omega_i$;*

(d) *if $\{\Omega_i\}_{i\in I}$ is a family of open subsets of Ω and $U_i \in \mathcal{G}(\Omega_i)$, $i \in I$, is a* compatible *family of generalized functions (in the sense that $U_i|_{\Omega_i\cap\Omega_j} = U_j|_{\Omega_i\cap\Omega_j}$ in $\mathcal{G}(\Omega_i \cap \Omega_j)$ for all i, $j \in I$ such that $\Omega_i \cap \Omega_j \neq \varnothing$), then there is a unique generalized function $U \in \mathcal{G}(\bigcup_{i\in I} \Omega_i)$ such that $U|_{\Omega_i} = U_i$ in $\mathcal{G}(\Omega_i)$ for all $i \in I$.*

Proof. The first two properties, (a) and (b), easily follow from the definition of the restriction. It suffices to prove (c) for $V = 0$. Given a compact set $K \subset\subset \bigcup_{i\in I} \Omega_i$, there are a finite subset $J \subset I$ and compact sets $K_j \subset \Omega_j$ for $j \in J$ such that $K = \bigcup_{j\in J} K_j$. It follows that if $u \in \mathcal{E}_M[\Omega]$ is a representative of U and $u|_{\Omega_i} \in \mathcal{N}[\Omega_i]$ $(i \in I)$, then, using the definition of ideal (2.7) and the property of the set Γ, we find that $u \in \mathcal{N}[\bigcup_{i\in I} \Omega_i]$. Thus, $U = 0$ in $\mathcal{G}(\bigcup_{i\in I} \Omega_i)$.

(d) The uniqueness of the generalized function U follows from (c). We now prove its existence. Let $\tilde\Omega = \bigcup_{i\in I} \Omega_i$, and let $\{\phi_i\}_{i\in I} \subset C^\infty(\tilde\Omega; \mathbb{R})$ be a partition of the unity subordinated to the covering $\{\Omega_i\}_{i\in I}$ of $\tilde\Omega$, so that $\operatorname{supp}\phi_i \subset \Omega_i$ $\forall i \in I$, the family $\{\operatorname{supp}\phi_i\}_{i\in I}$ is locally finite on $\tilde\Omega$, and $\sum_{i\in I} \phi_i(x) = 1$ $\forall x \in \tilde\Omega$. Define a representative u of the desired generalized function U by means of representatives u_i of functions U_i $(i \in I)$ as follows: $u(\varphi) = \sum_{i\in I} \phi_i u_i(\varphi)$ for $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. Each product $\phi_i u_i(\varphi) =: f_i$ is defined on the whole $\tilde\Omega$ ($f_i(x) = \phi_i(x)u_i(\varphi, x)$ if $x \in \Omega_i$, and $f_i(x) = 0$ if $x \in \tilde\Omega \setminus \Omega_i$), and the sum $u(\varphi)(x)$ is finite for x in compact subsets of $\tilde\Omega$ due to the local finiteness of the family of supports of ϕ_i. Since $u_i \in \mathcal{E}_M[\Omega_i]$, it follows that $u \in \mathcal{E}_M[\Omega]$. Set $U = u + \mathcal{N}[\Omega] \in \mathcal{G}(\Omega)$. It remains to show that $U|_{\Omega_i} = U_i$ or, what is the same, $u|_{\Omega_i} - u_i \in \mathcal{N}[\Omega_i]$ for all $i \in I$. Let $K \subset\subset S \Subset \Omega_i$. By the local finiteness of the family of supports of ϕ_i, there is a finite subset $J \subset I$ such that $\sum_{j\in J} \phi_j = 1$ on S, and

$$u(\varphi) = \phi_i u_i(\varphi) + \sum_{j\in J\setminus\{i\}} \phi_j u_j(\varphi) \quad \text{on} \quad S \quad \forall \varphi \in \mathcal{A}_0(\mathbb{R}^n);$$

this yields

$$u(\varphi) - u_i(\varphi) = \sum_{j\in J\setminus\{i\}} \phi_j(u_j(\varphi) - u_i(\varphi)) \quad \text{on} \quad S.$$

On the compact set K, only members with indices j for which $K \cap \operatorname{supp}\phi_j \neq \varnothing$ give a contribution to the sum. But $K \cap \operatorname{supp}\phi_j \subset\subset \Omega_i \cap \Omega_j$, and it remains to take into account the compatibility assumption $(u_i - u_j)|_{\Omega_i\cap\Omega_j} \in \mathcal{N}[\Omega_i \cap \Omega_j]$ and the definition of the ideal (2.7). □

Let us consider the imbedding of the space of continuous functions into the algebra of generalized functions in more detail. To this end we will use a more precise notation showing explicitly the dependence of imbeddings on Ω: the canonical imbedding of $C_c(\Omega)$ into $\mathcal{G}(\Omega)$ will be denoted by $\jmath_{c,\Omega}$, and by $\jmath_\Omega$ the imbedding of $C(\Omega)$ into $\mathcal{G}(\Omega)$; note that the mapping ℓ defined above depends on Ω as well. The coherence of restrictions in $\mathcal{G}(\Omega)$ and in $C(\Omega)$ means that imbeddings and restrictions commute; this is written precisely as follows:

$$\jmath_\Omega(f)|_G = \jmath_G(f|_G), \quad f \in C(\Omega), \qquad G \subset \Omega, \ G \text{ is open.} \tag{2.19}$$

In other words, the commutativity of imbeddings and restrictions expresses the fact that the imbeddings $\jmath_\Omega$ define a sheaf morphism $\jmath : C \longrightarrow \mathcal{G}$ (in the category of linear spaces over $\mathbb{K}$). Following Oberguggenberger [156, § 9], we will show that such a sheaf morphism is unique: if $\tilde{\jmath} : C \longrightarrow \mathcal{G}$ is another sheaf morphism such that $\tilde{\jmath}_\Omega|_{C_c(\Omega)} = \jmath_{c,\Omega}$ for all open sets $\Omega \subset \mathbb{R}^n$, then $\tilde{\jmath} = \jmath$. Let $G \Subset \Omega$ be arbitrary, and $\psi \in \mathcal{D}(\Omega)$ be such that $\psi = 1$ on G (Proposition 1.4). By the sheaf morphism property (2.19), we have the equality $\jmath_\Omega(f)|_G = \jmath_\Omega(\psi f)|_G$ for all $f \in C(\Omega)$, and an analogous equality holds for $\tilde{\jmath}$. Since $\jmath_\Omega(\psi f) = \jmath_{c,\Omega}(\psi f) = \tilde{\jmath}(\psi f)$, where the second equality is satisfied by assumption, we find that $\tilde{\jmath}_\Omega(f)|_G = \jmath_\Omega(f)|_G$ for all relatively compact open subsets $G \subset \Omega$. Thus, $\tilde{\jmath}_\Omega = \jmath_\Omega$ for all open sets $\Omega \subset \mathbb{R}^n$; this yields $\tilde{\jmath} = \jmath$.

> So, $\mathcal{G}$ is a sheaf (of sections over $\mathbb{R}^n$) of differential algebras, and there is only one sheaf morphism (of linear spaces over $\mathbb{K}$) $\jmath : C \longrightarrow \mathcal{G}$ which extends the canonical imbedding $\jmath_{c,\Omega} : C_c(\Omega) \longrightarrow \mathcal{G}(\Omega)$ on every space of sections; $\jmath$ commutes with derivatives, and the restriction of $\jmath$ to C^∞ (which is $\imath$) is a morphism of differential algebras.

We say that a generalized function $U \in \mathcal{G}(\Omega)$ is *null on an open subset* $G \subset \Omega$ if its restriction $U|_G = 0$ in $\mathcal{G}(G)$, and we say that two generalized functions $U, V \in \mathcal{G}(\Omega)$ are *equal on* G if their difference $U - V$ is null on G.

The *support* of a generalized function $U \in \mathcal{G}(\Omega)$ is defined as the complement in Ω to the largest open subset $\Omega_0(U)$ of Ω on which U is null:

$$\operatorname{supp} U = \Omega \setminus \Omega_0(U), \tag{2.20}$$

where $\Omega_0(U)$ is the union of all open subsets $G \subset \Omega$ such that $U = 0$ on G. By property (c) of Theorem 2.6, we have: $U = 0$ on $\Omega_0(U) = \Omega \setminus \operatorname{supp} U$. This new concept of the support generalizes the corresponding concept for continuous functions:

$$\operatorname{supp} \jmath(f) = \operatorname{supp} f, \quad f \in C(\Omega);$$

this is an immediate consequence of (2.19) (for brevity, we again use the notation in (2.17) instead of $\jmath_\Omega$). The set of all the Colombeau generalized functions on Ω with compact supports will be denoted by $\mathcal{G}_c(\Omega)$:

$$\mathcal{G}_c(\Omega) = \{\, U \in \mathcal{G}(\Omega) \mid \operatorname{supp} U \subset\subset \Omega \,\}.$$

Proposition 2.7. (a) *$\mathcal{G}_c(\Omega)$ is a subalgebra in $\mathcal{G}(\Omega)$;*

(b) *if $U = u + \mathcal{N}[\Omega] \in \mathcal{G}_c(\Omega)$, and a function $\zeta \in \mathcal{D}(\Omega)$ is such that $\zeta = 1$ in a neighborhood of* $\operatorname{supp} U$, *then $\zeta \cdot u \in \mathcal{E}_M[\Omega]$ is also a representative of U, i.e., we have the equality $U = \zeta \cdot U$ in $\mathcal{G}(\Omega)$.*

Proof. (a) This part of the proposition follows from the inclusions

$$\operatorname{supp}(U + V) \subset (\operatorname{supp} U) \cup (\operatorname{supp} V), \quad \operatorname{supp}(U \cdot V) \subset (\operatorname{supp} U) \cap (\operatorname{supp} V),$$

which are valid for $U, V \in \mathcal{G}(\Omega)$.

(b) Let $G \subset \Omega$ be an open neighborhood of the compact set $S = \operatorname{supp} U$ on which $\zeta = 1$. Then $1 - \zeta = 0$ on G, and $U = 0$ on S^c, so that $(1 - \zeta) \cdot U = 0$ on G and on S^c, whence $(1 - \zeta) \cdot U = 0$ on $G \cup S^c = \Omega$ by Theorem 2.6(c). □

Examples 2.8. (1) Consider the element $\delta^m \in \mathcal{G}(\mathbb{R}^n)$, where $m \in \mathbb{N}$ is fixed, with representative given by $u(\varphi,x) = (\varphi(-x))^m$, $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ (see also the proof of Proposition 2.3). Let us show that $\operatorname{supp} \delta^m = \{0\}$. In fact, if $K \subset\subset \mathbb{R}^n \setminus \{0\}$ and $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, then $u(\varphi_\varepsilon, x) = 0$ for $0 < \varepsilon < \operatorname{dist}(K,0)/\rho(\varphi)$ and $x \in K$ since $\operatorname{supp} \varphi \subset \mathbb{R}^n \setminus \{-x/\varepsilon\}$ (or $\operatorname{supp} \tau_x\varphi_\varepsilon \subset \mathbb{R}^n \setminus \{0\}$) for such ε and x. Thus, $\delta^m = 0$ on $\mathbb{R}^n \setminus \{0\}$. On the other hand, $\delta^m \neq 0$ in $\mathcal{G}(\mathbb{R}^n)$; this follows from $u(\varphi_\varepsilon, 0) = (\varepsilon^{-n}\varphi(0))^m$ and Remark 2.2(d). Hence, $\delta^m \in \mathcal{G}_c(\mathbb{R}^n)$.

(2) Consider a generalized function $U \in \mathcal{G}(\mathbb{R}^n)$ with the representative $u(\varphi,x) = x^m\varphi(-x)$, $\varphi \in \mathcal{A}_0(\mathbb{R})$, $x \in \mathbb{R}$, where $m \in \mathbb{N}$ (this function is the product of the classical function x^m and the Dirac delta function $\delta(x)$ in the algebra $\mathcal{G}(\mathbb{R})$). As in Example (1), $U = 0$ on $\mathbb{R} \setminus \{0\}$; $U \neq 0$ on $\mathbb{R}$ since otherwise the mth derivative of its representative $u^{(m)}$ would be in $\mathcal{N}[\mathbb{R}]$, but from Leibnitz's rule

$$u^{(m)}(\varphi_\varepsilon, x) = \sum_{i=0}^{m} \frac{(m!)^2}{(i!)^2(m-i)!} \frac{(-1)^i}{\varepsilon^{i+1}} x^i \varphi^{(i)}\Big(-\frac{x}{\varepsilon}\Big);$$

this implies $u^{(m)}(\varphi_\varepsilon, 0) = m!\varphi(0)/\varepsilon$, and, as above, it remains to take into account Remark 2.2(d). So, $U \in \mathcal{G}_c(\mathbb{R})$ and $\operatorname{supp} U = \{0\}$. □

2.5. The translation operator in $\mathcal{G}(\mathbb{R}^n)$. We conclude this section by a few remarks on the *translation operator* in $\mathcal{G}(\mathbb{R}^n)$. Given a generalized function $U \in \mathcal{G}(\mathbb{R}^n)$ with a representative $u \in \mathcal{E}_M[\mathbb{R}^n]$ and a point $a \in \mathbb{R}^n$, we set $(\tau_a u)(\varphi, x) := u(\varphi, x-a)$ for $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and note that $\tau_a u \in \mathcal{E}[\mathbb{R}^n]$; furthermore, $\tau_a u \in \mathcal{E}_M[\mathbb{R}^n]$ (resp. $\tau_a u \in \mathcal{N}[\mathbb{R}^n]$) if $u \in \mathcal{E}_M[\mathbb{R}^n]$ (resp. $u \in \mathcal{N}[\mathbb{R}^n]$). The *translation* $\tau_a U$ is defined by $\tau_a U = \tau_a u + \mathcal{N}[\mathbb{R}^n]$. The new concept of translation $\tau_a : \mathcal{G}(\mathbb{R}^n) \longrightarrow \mathcal{G}(\mathbb{R}^n)$ generalizes exactly the translation operator for continuous functions: if $f \in C(\mathbb{R}^n)$, then $\tau_a \jmath(f) = \jmath(\tau_a f)$ in $\mathcal{G}(\mathbb{R}^n)$; this follows from the equality $\tau_a(u_f) = u_{\tau_a f}$ for the representative $u_f \in \jmath(f)$. As in the classical case, one has

$$\operatorname{supp} \tau_a U = a + \operatorname{supp} U, \quad U \in \mathcal{G}(\mathbb{R}^n), \quad a \in \mathbb{R}^n.$$

3. The Algebra of Generalized Numbers

In this section, we define an algebra of generalized numbers $\overline{\mathbb{K}}$, so that Colombeau's generalized functions from $\mathcal{G}(\Omega)$ have pointwise values (from $\overline{\mathbb{K}}$), and one can integrate these generalized functions over compact subsets of Ω.

3.1. The definition of the algebra $\overline{\mathbb{K}}$. In the infinite product $\mathcal{E}_0 = \mathbb{K}^{\mathcal{A}_0(\mathbb{R}^n)}$ consisting of all mappings $u : \mathcal{A}_0(\mathbb{R}^n) \longrightarrow \mathbb{K}$, which is an associative and commutative algebra over the field $\mathbb{K}$ relative to componentwise operations, consider a *subalgebra $\mathcal{E}_{0,M}$ of moderate* elements

$$\begin{aligned}\mathcal{E}_{0,M} = \{\, u \in \mathcal{E}_0 \,|\, \exists N \in \mathbb{N} :\\ \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)\ \exists c > 0,\ \eta > 0 :\\ \forall \varepsilon \in (0,\eta) : |u(\varphi_\varepsilon)| \le c\varepsilon^{-N} \,\}.\end{aligned} \tag{3.1}$$

Define an *ideal* $\mathcal{N}_0$ in $\mathcal{E}_{0,M}$ of *null* elements as follows:

$$\begin{aligned}\mathcal{N}_0 = \{\, u \in \mathcal{E}_0 \,|\, \exists N \in \mathbb{N},\ \gamma \in \Gamma :\\ \forall q \in \mathbb{N},\ q \ge N,\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n)\ \exists c > 0,\ \eta > 0 :\\ \forall \varepsilon \in (0,\eta) : |u(\varphi_\varepsilon)| \le c\varepsilon^{\gamma(q)-N} \,\}.\end{aligned} \tag{3.2}$$

As we have seen in Proposition 2.3 (steps 2 and 3 of the proof), the algebra $\mathcal{E}_{0,M}$ is a *proper* subset of $\mathcal{E}_0$, and the set $\mathcal{N}_0$ *is not an ideal* in $\mathcal{E}_0$. Analogously to the property (2.8) of the ideal $\mathcal{N}[\Omega]$, the set $\mathcal{N}_0$ has the property

$$\begin{aligned}&\forall u \in \mathcal{N}_0\ \forall k \in \mathbb{N}\ \ \exists N \in \mathbb{N} \text{ such that}\\ &\forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) : |u(\varphi_\varepsilon)| = o(\varepsilon^k) \quad \text{as} \quad \varepsilon \to +0.\end{aligned} \tag{3.3}$$

The *algebra of generalized numbers* $\overline{\mathbb{K}}$ is defined as an associative and commutative quotient algebra (over the field $\mathbb{K}$):

$$\overline{\mathbb{K}} = \mathcal{E}_{0,M}/\mathcal{N}_0. \tag{3.4}$$

The generalized number $Z \in \overline{\mathbb{K}}$, which is the equivalence class of an element $u \in \mathcal{E}_{0,M}$, usually will be denoted by $Z = \overline{u} := u + \mathcal{N}_0$. Note that, in general, the algebra $\overline{\mathbb{K}}$ depends on the dimension $n = \dim \mathbb{R}^n$, so that there are different algebras of generalized numbers (this, however, is of no significance for what follows). Let us take a closer look at the connection between the algebras $\overline{\mathbb{C}}$ and $\overline{\mathbb{R}}$; for this we will indicate the dependence of $\mathcal{E}_0$, (3.1), and (3.2) on $\mathbb{C}$ and $\mathbb{R}$. Both these algebras are defined in (3.4) starting respectively with the sets $\mathcal{E}_0$ of the form $\mathcal{E}_0(\mathbb{C}) = \mathbb{C}^{\mathcal{A}_0(\mathbb{R}^n)}$ and $\mathcal{E}_0(\mathbb{R}) = \mathbb{R}^{\mathcal{A}_0(\mathbb{R}^n)}$, so that $\mathcal{E}_0(\mathbb{C}) = \mathcal{E}_0(\mathbb{R}) + i\mathcal{E}_0(\mathbb{R})$ with $i = \sqrt{-1}$; similar equalities take place for algebras of moderate elements and ideals of null elements. If $Z = u + \mathcal{N}_0(\mathbb{C}) \in \overline{\mathbb{C}}$ is a generalized complex number with a representative $u = u_1 + iu_2 \in \mathcal{E}_{0,M}(\mathbb{C})$, where $u_1 = \operatorname{Re} u$, $u_2 = \operatorname{Im} u \in \mathcal{E}_{0,M}(\mathbb{R})$, then the *real* and *imaginary* parts of Z are defined as generalized real numbers by

$$\operatorname{Re} Z := \operatorname{Re} u + \mathcal{N}_0(\mathbb{R}) \in \overline{\mathbb{R}} \quad \text{and} \quad \operatorname{Im} Z := \operatorname{Im} u + \mathcal{N}_0(\mathbb{R}) \in \overline{\mathbb{R}},$$

so that $Z = \operatorname{Re} Z + i \operatorname{Im} Z$ for $Z \in \overline{\mathbb{C}}$, or $\overline{\mathbb{C}} = \overline{\mathbb{R}} + i\,\overline{\mathbb{R}}$. The *conjugate* generalized number is denoted by $Z^* = \operatorname{Re} Z - i \operatorname{Im} Z$, and, as usual, the inclusion $Z \in \overline{\mathbb{R}}$ is equivalent to $Z^* = Z$ or $Z = \operatorname{Re} Z$ in $\overline{\mathbb{C}}$. Note also that the usage of sets $\mathcal{A}_0(\mathbb{R}^n; \mathbb{C})$ or $\mathcal{A}_0(\mathbb{R}^n; \mathbb{R})$ instead of $\mathcal{A}_0(\mathbb{R}^n)$ leads to different algebras of generalized numbers $\overline{\mathbb{C}}$ and $\overline{\mathbb{R}}$ (if no confusion arises, we will as usual write $\mathcal{A}_0(\mathbb{R}^n)$ instead of the two sets mentioned above).

The mapping defined by

$$\imath_0 : \mathbb{K} \longrightarrow \overline{\mathbb{K}}, \qquad \imath_0(z) = \overline{z} = z + \mathcal{N}_0 \quad \text{for} \quad z \in \mathbb{K}, \tag{3.5}$$

is a *homomorphic imbedding* of the algebra $\mathbb{K}$ into the algebra $\overline{\mathbb{K}}$.

Convention. Elements Z of the image set $\imath_0(\mathbb{K})$ will be called *ordinary numbers from* $\mathbb{K}$, and if $Z = \imath_0(z)$ for some $z \in \mathbb{K}$, then z is uniquely determined (due to (3.3)), and we write $Z = z$ in $\overline{\mathbb{K}}$. □

Property (3.3) of the ideal $\mathcal{N}_0$ suggests an idea of defining the following (very important for the whole subsequent theory) equivalence relation on $\overline{\mathbb{K}}$, called the *association*, which is weaker than the equality in $\overline{\mathbb{K}}$:

Definition 3.1. We say that a generalized number $Z \in \overline{\mathbb{K}}$ *is associated to zero*, and we write $Z \approx 0$, if for some (and, hence, for any) representative $u \in Z$ the following holds:

$$\exists\, N \in \mathbb{N}: \quad \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n): \quad u(\varphi_\varepsilon) \longrightarrow 0 \text{ in } \mathbb{K} \text{ as } \varepsilon \longrightarrow +0.$$

Two generalized numbers Z_1, $Z_2 \in \overline{\mathbb{K}}$ are said to be *associated* (to each other), denoted by $Z_1 \approx Z_2$, if $Z_1 - Z_2 \approx 0$. The equivalence relation $\approx$ on $\overline{\mathbb{K}}$ is called the *association* on $\overline{\mathbb{K}}$. We say that a generalized number $Z \in \overline{\mathbb{K}}$ *has an associated* (ordinary) *number* $z \in \mathbb{K}$ if $Z \approx \imath_0(z)$; this will be written briefly as $Z \approx z$ in $\overline{\mathbb{K}}$. Clearly, an associated number is uniquely determined (if it exists). The set of all generalized numbers having associated ordinary numbers is denoted by

$$\overline{\mathbb{K}}_{\mathrm{A}} = \{\, Z \in \overline{\mathbb{K}} \mid \exists\, z \in \mathbb{K} : Z \approx z \,\}. \quad \square$$

Examples 3.2. (1) The generalized number $0 \in \overline{\mathbb{K}}$ has $0 \in \mathbb{K}$ as an associated number; this follows from (3.3).

(2) Any ordinary number from $\mathbb{K}$ is associated to itself (since z is a representative of $\imath_0(z)$ for any $z \in \mathbb{K}$).

(3) Equal generalized numbers are associated but not vice versa: if P is a generalized number with the representative $\{\rho(\varphi)\}_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)}$, then $\rho(\varphi_\varepsilon) = \varepsilon\rho(\varphi)$ for $\varepsilon > 0$; hence $\mathrm{P} \approx 0$ and $\mathrm{P} \neq 0$ in $\overline{\mathbb{K}}$. Thus the generalized number P is not an ordinary number (one can view it as an "infinitely small" positive number).

(4) Not every generalized number has an associated number: indeed, if Z has the representative $u(\varphi) = \varphi(0)$ for $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ (Z is the value of the Dirac function $\delta \in \mathcal{G}(\mathbb{R}^n)$ at the point $0 \in \mathbb{R}^n$, hence is denoted by

$Z = \delta(0)$), then $u(\varphi_\varepsilon) = \varphi(0)/\varepsilon^n$, $\varepsilon > 0$, so that $Z = \delta(0) \notin \overline{\mathbb{K}}_A$ in view of Remark 2.2(d). The generalized number $\delta(0)$ is not an ordinary number (it might be considered as an "infinitely large" positive number). □

The association relation $\approx$ on $\overline{\mathbb{K}}$ generalizes the usual equality in $\mathbb{K}$: if $z_1, z_2 \in \mathbb{K}$, then $z_1 \approx z_2 \iff z_1 = z_2$. The association is compatible with (preserved by) the linear operations, but not with multiplication in $\overline{\mathbb{K}}$ (as opposed to the algebraic equality $=$ in $\overline{\mathbb{K}}$): in fact, $\mathrm{P} \approx 0$, but $\mathrm{P} \cdot \delta(0) \not\approx 0 = 0 \cdot \delta(0)$ (and even $\mathrm{P} \cdot \delta(0) \notin \overline{\mathbb{K}}_A$) since a representative of the number $\mathrm{P} \cdot \delta(0)$ is $(\rho \cdot u)(\varphi_\varepsilon) = \rho(\varphi)\varphi(0)/\varepsilon^{n-1}$, and it is left to note that given $c \in \mathbb{K}$ and $q \in \mathbb{N}$, there is a $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ such that $\varphi(0) = c$ and $\rho(\varphi) = 1$.

The set $\overline{\mathbb{K}}_A$ is a subalgebra in $\overline{\mathbb{K}}$ (if $Z_1, Z_2 \in \overline{\mathbb{K}}_A$ and $Z_1 \approx z_1 \in \mathbb{K}$, $Z_2 \approx z_2 \in \mathbb{K}$, then $Z_1 + Z_2 \approx z_1 + z_2$ and $Z_1 \cdot Z_2 \approx z_1 z_2$), and the mapping $\imath_0$ from (3.5) is, in fact, a homomorphic imbedding of the algebra $\mathbb{K}$ into the algebra $\overline{\mathbb{K}}_A$. On the other hand, a mapping which assigns the associated ordinary number $z \in \mathbb{K}$ to a generalized number $Z \in \overline{\mathbb{K}}_A$ is a homomorphism of algebras $\overline{\mathbb{K}}_A$ and $\mathbb{K}$ (this mapping is not injective).

The algebra of generalized numbers $\overline{\mathbb{K}}$ is *not a field*. To prove this, consider an element Z with the representative $u(\varphi) = \int_{\mathbb{R}} |\mu|\varphi(\mu)\, d\mu$, $\varphi \in \mathcal{A}_0(\mathbb{R})$, and show that it is not invertible in $\overline{\mathbb{K}}$. Since $u(\varphi_\varepsilon) = \varepsilon u(\varphi)$, then $Z \approx 0$; but $Z \neq 0$ because, in view of (2.15), for a given $q \in \mathbb{N}$, there is a $\varphi \in \mathcal{A}_q(\mathbb{R})$ such that $u(\varphi) = 1$. Let $v \in \mathcal{E}_{0,M}$ be a representative of a generalized number $V \in \overline{\mathbb{K}}$ such that $V \cdot Z = 1$ in $\overline{\mathbb{K}}$. Then there exists $N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_N(\mathbb{R})$ we have $v(\varphi_\varepsilon)u(\varphi_\varepsilon) \longrightarrow 1$ as $\varepsilon \longrightarrow +0$. The latter, however, is impossible, for the following implication: if $\varphi \in \mathcal{A}_N(\mathbb{R})$ and $\operatorname{supp}\varphi \subset (0,\infty)$, then $u(\varphi) = 0$. Another example of a noninvertible generalized number is the number with representative equal to 1 if $\operatorname{supp}\varphi \subset (0,\infty)$, and equal to 0 otherwise ($\varphi \in \mathcal{A}_0(\mathbb{R})$).

3.2. Point values of generalized functions. Now we define the concept of a "point value" of a generalized function $U \in \mathcal{G}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set. Let $u \in \mathcal{E}_M[\Omega]$ be a representative of U, and let $x \in \Omega$ be a given point. For $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, set $u_x(\varphi) := u(\varphi, x)$, and note that $u_x \in \mathcal{E}_0$; furthermore, $u_x \in \mathcal{E}_{0,M}$ (resp. $\mathcal{N}_0$) if $u \in \mathcal{E}_M[\Omega]$ (resp. $\mathcal{N}[\Omega]$). It follows that the following definition is correct: the generalized number $U(x) := \overline{u_x} = u_x + \mathcal{N}_0 \in \overline{\mathbb{K}}$ is called the *value of the generalized function $U \in \mathcal{G}(\Omega)$ at the point $x \in \Omega$*. We note that the concept introduced is *local* in the sense that if $O_x \subset \Omega$ is an open neighborhood of x, then $U(x) = (U|_{O_x})(x)$ in $\overline{\mathbb{K}}$. This new concept generalizes exactly the concept of pointwise values for C^∞ functions: if $f \in C^\infty(\Omega)$, then $\imath(f)(x) = f(x)$ in $\overline{\mathbb{K}}$ (or more explicitly $\imath(f)(x) = \imath_0(f(x))$), see the convention in Sec. 3.1). For continuous functions (more precisely, for functions from $C(\Omega) \setminus C^\infty(\Omega)$), the classical point values are recovered only by means of the association:

Proposition 3.3. *If $f \in C(\Omega)$, then $\jmath(f)(x) \approx f(x)$ in $\overline{\mathbb{K}}$ for all $x \in \Omega$.*

Proof. Let $x \in \Omega$. In view of (2.18), for $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, there is an $\eta > 0$ such that for the representative u_f of the generalized function $\jmath(f) \in \mathcal{G}(\Omega)$ and the representative $u_{f,x}$ of the generalized number $\jmath(f)(x) \in \overline{\mathbb{K}}$, one has $u_{f,x}(\varphi_\varepsilon) = u_f(\varphi_\varepsilon, x) = f * \check{\varphi}_\varepsilon\,(x)$ for all $\varepsilon \in (0,\eta)$. By Proposition 1.3(a), it follows that $u_{f,x}(\varphi_\varepsilon) \longrightarrow f(x)$ as $\varepsilon \longrightarrow +0$, and the latter means that the generalized number $\jmath(f)(x)$ has $f(x) \in \mathbb{K}$ as an associated ordinary number. □

Examples 3.4. (1) Consider the element $\delta^m \in \mathcal{G}(\mathbb{R}^n)$, $m \in \mathbb{N}$, from Example 2.8(1). If $x \in \mathbb{R}^n \setminus \{0\}$, then $\delta^m(x) = (\delta^m|_{\mathbb{R}^n\setminus\{0\}})(x) = 0$ in $\overline{\mathbb{K}}$, whereas $\delta^m(0) \in \overline{\mathbb{K}} \setminus \overline{\mathbb{K}}_A$ (as in Example 3.2(4)), so that $\delta^m(0)$ is not an ordinary number. The fact that $\delta^m(0) \neq 0$ is "intuitively" consistent with $\operatorname{supp}\delta^m = \{0\}$. However, the following example shows that this is not always true.

(2) Generalized functions from $\mathcal{G}(\mathbb{R}^n)$ *are not determined* by their values at points $x \in \mathbb{R}^n$. The function $U \in \mathcal{G}(\mathbb{R})$ with the representative $u(\varphi, x) = x^m\varphi(-x)$ from Example 2.8(2) satisfies unusual properties: $U \neq 0$ in $\mathcal{G}(\mathbb{R})$, $U|_{\mathbb{R}\setminus\{0\}} = 0$ in $\mathcal{G}(\mathbb{R}\setminus\{0\})$, $U(x) = 0$ in $\overline{\mathbb{K}}$ for *all* $x \in \mathbb{R}$, and $\operatorname{supp} U = \{0\}$.

(3) For less smooth functions, compared to C^∞, generalized values at individual points cannot coincide any more with their respective classical values in the strong algebraic sense (only in the sense of association,

cf. Proposition 3.3). Consider a function $f \in C^k(\mathbb{R})$ such that $f(x) = 0$ if $x \leq 0$, and $f(x) = x^{k+1}$ if $x \geq 0$, where $k \in \mathbb{N}_0$ (see Example 2.5(1)). Set $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. Since $f|_{\mathbb{R}_0} \in C^\infty(\mathbb{R}_0)$, using (2.19), we have

$$\jmath(f)(x) = (\jmath(f)|_{\mathbb{R}_0})(x) = \jmath(f|_{\mathbb{R}_0})(x) = f(x) \quad \text{in} \quad \overline{\mathbb{K}} \quad \forall x \in \mathbb{R}_0.$$

On the other hand, for the representative u_f of $\jmath(f)$ at the point $x = 0$, we have

$$u_{f,0}(\varphi_\varepsilon) = u_f(\varphi_\varepsilon, 0) = \varepsilon^{k+1} \int_0^\infty \mu^{k+1} \varphi(\mu)\, d\mu, \quad \varphi \in \mathcal{A}_0(\mathbb{R}), \quad \varepsilon > 0.$$

From (2.15) it follows that $u_{f,0} \notin \mathcal{N}_0$, that is, the value $\jmath(f)(0) \neq 0$ does not coincide with the classical value $f(0) = 0$. Nevertheless, $\jmath(f)(0) \approx 0 = f(0)$. From this example and Example 2.5(1), we see that the "phenomena of noncoincidence" with classical values in both cases are very similar. The concept of association was introduced by Colombeau in order to *recover* classical values from the corresponding generalized values.

(4) Let $f : (a,b) \longrightarrow \mathbb{K}$ be a function, continuous on the open interval (a,b) outside a point $x \in (a,b)$ at which it has distinct finite one-sided limits from the left $f(x-0)$ and from the right $f(x+0)$. The representative $u_{f,x}$ of its generalized value $\jmath(f)(x)$, in view of (2.18), has the following property: for $\varphi \in \mathcal{A}_0(\mathbb{R})$, there is $\eta > 0$ such that

$$\begin{gathered} u_{f,x}(\varphi_\varepsilon) = (f * \check{\varphi}_\varepsilon)(x) = \Big(\int_{-\infty}^0 + \int_0^\infty \Big) f(x + \varepsilon\mu) \varphi(\mu)\, d\mu \longrightarrow \\ \longrightarrow f(x-0) \int_{-\infty}^0 \varphi(\mu)\, d\mu + f(x+0) \int_0^\infty \varphi(\mu)\, d\mu \quad \text{as} \quad (0,\eta) \ni \varepsilon \longrightarrow +0. \end{gathered} \tag{3.6}$$

If we show that

$$\forall c \in \mathbb{K} \quad \forall q \in \mathbb{N} \quad \exists \varphi \in \mathcal{A}_q(\mathbb{R}) \quad : \quad \int_0^\infty \varphi(\mu)\, d\mu = c, \tag{3.7}$$

then (3.6) implies $\jmath(f)(x) \in \overline{\mathbb{K}} \setminus \overline{\mathbb{K}}_{\mathrm{A}}$, so that $\jmath(f)(x)$ is not an ordinary number from $\mathbb{K}$. From the point of view of the classical analysis, the value $\jmath(f)(x)$ can be considered as *undetermined* (as long as it can be arbitrary), whereas in the algebra $\overline{\mathbb{K}}$ this value is quite well defined but does not have an associated ordinary number.

To prove (3.7), choose functions ψ_1, $\psi_2 \in \mathcal{A}_0(\mathbb{R})$ such that $\operatorname{supp}\psi_1 \subset (0,\infty)$ and $\operatorname{supp}\psi_2 \subset (-\infty, 0)$, and in Lemma 2.1, put $\varphi_0 = c\psi_1 + (1-c)\psi_2$. The function $\varphi_0 \in \mathcal{A}_0(\mathbb{R})$, in addition, satisfies the conditions $\operatorname{supp}\varphi_0 \subset\subset \mathbb{R} \setminus \{0\}$ and $\int_0^\infty \varphi_0(\lambda)\, d\lambda = c$. Then the function φ from the proof of Lemma 2.1 satisfies (3.7). □

For every open set $\Omega \subset \mathbb{R}^n$ there are natural inclusions $\mathcal{E}_0 \subset \mathcal{E}[\Omega]$, $\mathcal{E}_{0,M} \subset \mathcal{E}_M[\Omega]$, and $\mathcal{N}_0 \subset \mathcal{N}[\Omega]$. We have an imbedding of $\overline{\mathbb{K}}$ into $\mathcal{G}(\Omega)$ defined as follows: to a generalized number $Z = \{u(\varphi)\}_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)} + \mathcal{N}_0 \in \overline{\mathbb{K}}$, we associate the generalized function $U = \{u(\varphi)\}_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)} + \mathcal{N}[\Omega] \in \mathcal{G}(\Omega)$.

Using the latter imbedding, we say that $U \in \mathcal{G}(\Omega)$ is a *constant generalized function* if $U = Z$ in $\mathcal{G}(\Omega)$ for some $Z \in \overline{\mathbb{K}}$, or equivalently, if there is a representative of U which does not depend on $x \in \Omega$. For such a generalized function U, we have $U(x) = Z$ for all $x \in \Omega$. However, as we have seen in Example 3.4(2), a generalized function which has the same value at all points can *not be* a constant generalized function. As in the classical analysis we have the following characterization of constant generalized functions:

Proposition 3.5. *If $U \in \mathcal{G}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a connected open set, then U is a constant generalized function on Ω iff $\partial_j U = 0$ in $\mathcal{G}(\Omega)$ for all $j = 1, \ldots, n$.*

Proof. Fixing $x_0 \in \Omega$, let us show that $U = U(x_0)$ in $\mathcal{G}(\Omega)$. Let $u \in \mathcal{E}_M[\Omega]$ be a representative of U. 1. If $r > 0$ is such that the open ball $B_r^\circ(x_0)$ is contained in Ω, by Hadamard's formula, for all $x \in B_r^\circ(x_0)$ and $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, we have

$$u(\varphi, x) - u(\varphi, x_0) = \sum_{j=1}^{n} \int_0^1 (\partial_j u)(\varphi, (1-t)x_0 + tx)\, dt \cdot (x_j - x_{0j}).$$

Since $\partial_j U = 0$ on $B_r^\circ(x_0)$, the right-hand side of the above equality belongs to $\mathcal{N}[B_r^\circ(x_0)]$, whence $U|_{B_r^\circ(x_0)} = U(x_0)$ in $\mathcal{G}(B_r^\circ(x_0))$, where $U(x_0)$ is defined by its representative $\{u(\varphi, x_0)\}_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)}$.

2. Since Ω is connected, for every $x \in \Omega$, there is a continuous function $h_x : [0,1] \longrightarrow \Omega$ such that $h_x(0) = x_0$ and $h_x(1) = x$. Since the image $K_x = h_x([0,1])$ is compact in Ω, its covering by open balls of radius $d_x = \operatorname{dist}(K_x, \partial\Omega) > 0$ centered at points of K_x admits a finite subcovering. Applying the argument of step 1 to the balls from this finite subcovering and noting that on the intersection of two of these balls the function U is the same, for every $x \in \Omega$, we shall find an $r_x > 0$ such that $U = U(x_0)$ on $B_{r_x}^\circ(x)$. Since the family $\{B_{r_x}^\circ(x)\}_{x \in \Omega}$ covers Ω, in view of Theorem 2.6(c), we conclude that $U = U(x_0)$ on Ω.

Remark. If $U(x_0) \in \mathbb{K}$ for some $x_0 \in \Omega$, then U is a constant generalized function equal to the classical value $U(x_0)$. □

3.3. The integral of a generalized function over a compact set. Now we define an integral of a generalized function $U \in \mathcal{G}(\Omega)$ over a compact set $K \subset \Omega$. If $u \in \mathcal{E}_M[\Omega]$ is a representative of U, set $I_K(\varphi) = \int_K u(\varphi, x)\, dx$ for $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, and note that $I_K \in \mathcal{E}_0$; furthermore, $I_K \in \mathcal{E}_{0,M}$ (resp. $\mathcal{N}_0$) if $u \in \mathcal{E}_M[\Omega]$ (resp. $\mathcal{N}[\Omega]$). Thus, the following definition is correct: the generalized number $\int_K U := \overline{I_K} = I_K + \mathcal{N}_0 \in \overline{\mathbb{K}}$ is called an *integral of* $U \in \mathcal{G}(\Omega)$ *over* K. Sometimes it is convenient to show explicitly the integration variable; in this case we write $\int_K U(x)\, dx = \int_K U$. Note that the introduced concept of an integral is "local" in the sense that if O_K is an open neighborhood of K, then $\int_K U = \int_K (U|_{O_K})$ in $\overline{\mathbb{K}}$. For C^∞ functions, the new concept exactly generalizes the classical one: if $f \in C^\infty(\Omega)$, then $\int_K \imath(f) = \int_K f$ in $\overline{\mathbb{K}}$. For continuous functions (and locally integrable as well), the classical integral over a compact set is recovered by means of the association:

Proposition 3.6. *If $f \in C(\Omega)$ or $f \in L^1_{\mathrm{loc}}(\Omega)$, then $\int_K \jmath(f) \approx \int_K f$ in $\overline{\mathbb{K}}$ for all $K \subset\subset \Omega$.*

Proof. In view of property (2.18), given a compact set K and a function $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, there is $\eta > 0$ such that for the representative u_f of the generalized function $\jmath(f)$, we have $u_f(\varphi_\varepsilon, x) = (f * \check{\varphi}_\varepsilon)(x)$ for $x \in K$ and $\varepsilon \in (0, \eta)$. Then Proposition 1.3(a, d) yields

$$\int_K u_f(\varphi_\varepsilon, x)\, dx = \int_K (f * \check{\varphi}_\varepsilon)(x)\, dx \longrightarrow \int_K f(x)\, dx \quad \text{as} \quad \varepsilon \longrightarrow +0,$$

Q.E.D. □

If U is a generalized function with compact support, i.e., $U \in \mathcal{G}_c(\Omega)$, we can define the integral of U over the whole set Ω

$$\int_\Omega U := \int_K U, \quad \text{where} \quad K \subset\subset \Omega \quad \text{and} \quad K^\circ = \operatorname{int} K \supset \operatorname{supp} U. \tag{3.8}$$

The left-hand side in this definition does not depend on K with the mentioned properties: in fact, if compact sets K_1 and K_2 are contained in Ω and are such that $K_1^\circ \supset \operatorname{supp} U$ and $K_2^\circ \supset \operatorname{supp} U$, then choosing $\zeta \in \mathcal{D}((K_1 \cap K_2)^\circ)$ such that $\zeta = 1$ in a neighborhood of $\operatorname{supp} U$ and using Proposition 2.7(b), we find that

$$\int_{K_1} U = \int_{K_1} \zeta \cdot U = \int_{K_1 \cap K_2} \zeta \cdot U = \int_{K_2} \zeta \cdot U = \int_{K_2} U.$$

Examples 3.7. Consider the element $\delta \in \mathcal{G}(\mathbb{R}^n)$ having the representative $u_\delta(\varphi, x) = \varphi(-x)$ with $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

(1) In one dimension ($n = 1$), let us compute the value of the integral $\int_a^b \delta(x)\,dx \in \overline{\mathbb{K}}$ for $a, b \in \mathbb{R}$, $a < b$. A representative of this generalized number is of the form

$$I(\varphi_\varepsilon) = \int_a^b u_\delta(\varphi_\varepsilon, x)\,dx = \int_a^b \frac{1}{\varepsilon}\varphi\Big(-\frac{x}{\varepsilon}\Big)\,dx = \int_{-b/\varepsilon}^{-a/\varepsilon} \varphi(\mu)\,d\mu, \quad \varepsilon > 0.$$

If $a < b < 0$ or $0 < a < b$, then $\operatorname{supp}\varphi \subset \mathbb{R} \setminus [-b/\varepsilon, -a/\varepsilon]$ for small $\varepsilon > 0$, whence $I(\varphi_\varepsilon) = 0$, so that $\int_a^b \delta(x)\,dx = 0$ is an ordinary number. If $a < 0 < b$, then $\operatorname{supp}\varphi \subset (-b/\varepsilon, -a/\varepsilon)$ for small $\varepsilon > 0$, whence $I(\varphi_\varepsilon) = 1$, so that $\int_a^b \delta(x)\,dx = 1$ is also an ordinary number. In particular, it follows that $\int_{\mathbb{R}} \delta(x)\,dx = 1$. If $a < 0 = b$ (or $a = 0 < b$), then $I(\varphi_\varepsilon) = \int_0^\infty \varphi(\mu)\,d\mu$ (resp. $I(\varphi_\varepsilon) = \int_{-\infty}^0 \varphi(\mu)\,d\mu$) for small $\varepsilon > 0$, hence $\int_a^0 \delta(x)\,dx$ (resp. $\int_0^b \delta(x)\,dx$) lies in $\overline{\mathbb{K}} \setminus \overline{\mathbb{K}}_{\mathrm{A}}$, so that these generalized numbers are not ordinary numbers (as in Example 3.4(4) they can be viewed as undetermined). Note that the defining conditions in (3.8) are *important* in view of

$$1 = \int_{\mathbb{R}} \delta(x)\,dx \neq \int_{\operatorname{supp}\delta} \delta(x)\,dx = \int_{\{0\}} \delta(x)\,dx = 0.$$

(2) In dimension $n = 1$, we have $\int_{\mathbb{R}} x^m \delta(x)\,dx = 0$ in $\overline{\mathbb{K}}$, $m \in \mathbb{N}$; this follows from the definition of sets $\mathcal{A}_q(\mathbb{R})$.

(3) Analogously to Example (1), $\int_{\mathbb{R}^n} \delta(x)\,dx = 1$; however, $\int_{\mathbb{R}^n} \delta^2(x)\,dx \in \overline{\mathbb{K}} \setminus \overline{\mathbb{K}}_{\mathrm{A}}$ since a representative of the latter number is $I(\varphi_\varepsilon) = \dfrac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \varphi^2(\mu)\,d\mu$. □

From the definition of the integral of a generalized function, one sees that the classical formulas of *integration by parts, change of variables in the integral, change of order of integration*, etc. hold for integration of generalized functions since these formulas hold for representatives of generalized functions. For example, if $U \in \mathcal{G}_c(\Omega)$ and $V \in \mathcal{G}(\Omega)$, then for all $\alpha \in \mathbb{N}_0^n$ we have the integration by parts formula:

$$\int_\Omega (\partial^\alpha U) \cdot V = (-1)^{|\alpha|} \int_\Omega U \cdot (\partial^\alpha V) \quad \text{in} \quad \overline{\mathbb{K}}. \tag{3.9}$$

If $U \in \mathcal{G}(\Omega)$, $\psi \in \mathcal{D}(\Omega)$, or $U \in \mathcal{G}_c(\Omega)$, $\psi \in C^\infty(\Omega)$, then $U \cdot \psi \in \mathcal{G}_c(\Omega)$, and in view of (3.8), the following integral is well defined:

$$\int_\Omega U \cdot \psi = \int_{\operatorname{supp}\psi} U \cdot \psi \in \overline{\mathbb{K}}, \tag{3.10}$$

where the dot $\cdot$ denotes the product in $\mathcal{G}(\Omega)$. For what follows, it is interesting to note that if U is a continuous function, then the generalized number (3.10) is, in fact, a classical number; this is shown in the following proposition, which refines Proposition 3.6:

Proposition 3.8. (a) *If $f \in C(\Omega)$ or $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$, then*

$$\int_\Omega \jmath(f) \cdot \psi = \int_\Omega f\psi \quad in \quad \overline{\mathbb{K}},$$

where the equality is understood in the sense of the convention in Sec. 3.1. An analogous assertion holds for $f \in C_c(\Omega)$ and $\psi \in C^\infty(\Omega)$.

(b) *If $f \in C_c(\Omega)$, then $\int_\Omega \jmath(f) = \int_\Omega f$ in $\overline{\mathbb{K}}$. The same equality holds for functions $f \in L^1(\Omega)$ with compact supports.*

Proof. (a) Let $K \subset\subset \Omega$ be such that $\operatorname{supp}\psi \subset K^\circ = \operatorname{int}K$. Given $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, there is an $\eta > 0$ such that $\operatorname{supp}\psi + \varepsilon B_{\rho(\varphi)} \subset\subset K^\circ$, $K \subset \Omega(\check{\varphi}_\varepsilon)$, and $u_f(\varphi_\varepsilon, x) = (f * \check{\varphi}_\varepsilon)(x)$ for all $x \in K$ and $\varepsilon \in (0, \eta)$, where u_f is the usual representative of $\jmath(f)$. For the representative $I(\varphi_\varepsilon)$ of the generalized number $\int_\Omega \jmath(f) \cdot \psi$, we have

$$I(\varphi_\varepsilon) = \int_\Omega u_f(\varphi_\varepsilon, x)\psi(x)\,dx = \int_{\operatorname{supp}\psi} \Big(\int_{B_{\rho(\varphi)}} f(x + \varepsilon\mu)\varphi(\mu)\,d\mu \Big) \psi(x)\,dx =$$
$$= \int_{B_{\rho(\varphi)}} \varphi(\mu)\,d\mu \int_{\operatorname{supp}\psi} f(x + \varepsilon\mu)\psi(x)\,dx, \quad \varepsilon \in (0, \eta),$$

where, in the latter equality, we have used the Fubini theorem. Changing variables in the second integral $y = x + \varepsilon\mu$ (with $\mu \in \operatorname{supp}\varphi$ fixed) and using again the Fubini theorem, we find that

$$I(\varphi_\varepsilon) = \int_K \int_{B_{\rho(\varphi)}} f(y)\varphi(\mu)\psi(y - \varepsilon\mu)\,d\mu\,dy.$$

Expanding the function $y \longmapsto \psi(y - \varepsilon\mu)$ according to the Taylor formula up to order $q \in \mathbb{N}$, we obtain

$$I(\varphi_\varepsilon) - \int_\Omega f(y)\psi(y)\,dy = \sum_{|\alpha|=1}^{q} \frac{(-\varepsilon)^{|\alpha|}}{\alpha!} \Big(\int_K f(y)(\partial^\alpha\psi)(y)\,dy \Big) \Big(\int \mu^\alpha \varphi(\mu)\,d\mu \Big) +$$
$$+ (-\varepsilon)^{q+1} \sum_{|\alpha|=q+1} \frac{q+1}{\alpha!} \int_K \int_{B_{\rho(\varphi)}} f(y) \int_0^1 (1-t)^q (\partial^\alpha\psi)(y - t\varepsilon\mu)\,dt \cdot \mu^\alpha\varphi(\mu)\,d\mu\,dy.$$

It follows (as in the proof of Proposition 2.4) that $I - \int_\Omega f\psi \in \mathcal{N}_0$; this is what was required. The other possibility for functions f and ψ from (a) is considered similarly.

(b) It suffices to consider $K \subset\subset \Omega$ such that $\operatorname{int}K \supset \operatorname{supp} f$, and $\psi \in \mathcal{D}(\Omega)$ with $\psi = 1$ on K, and apply (a). □

3.4. Algebraic equations. Elements of $\overline{\mathbb{K}}$ which are not ordinary numbers are sometimes called "wild constants." They were introduced in order that the usual operations on classical functions could be carried over to generalized functions. However, the presence of wild constants can bring to existence new nonclassical solutions to classical equations. In this connection, we will consider solutions in $\overline{\mathbb{K}}$ of algebraic equations.

Consider, for example, the equation $P(x) = (x-a)(x-b) = 0$, $x \in \mathbb{K}$, where a and $b \in \mathbb{K}$ are given numbers such that $a \neq b$. A generalized number $Z \in \overline{\mathbb{K}}$ with representative $u(\varphi) = a$ if $(2n+1)^{-1} < \rho(\varphi) \leq (2n)^{-1}$, $u(\varphi) = b$ if $(2n)^{-1} < \rho(\varphi) \leq (2n-1)^{-1}$, $n \in \mathbb{N}$, and $u(\varphi) =$ (arbitrary number) if $\rho(\varphi) > 1$, where $\varphi \in \mathcal{A}_0(\mathbb{R})$, is a solution to $P(Z) = 0$ in $\overline{\mathbb{K}}$ which does not equal to any of the classical solutions $x = a$ and $x = b$ of this equation. Sometimes the presence of these new solutions is undesirable; below we will consider a variant $\overline{\mathbb{K}}^1$ of the algebra $\overline{\mathbb{K}}$ in which polynomial equations have only classical solutions.

Proposition 3.9. *If $Z \in \overline{\mathbb{K}}$, $m \in \mathbb{N}$, and $Z^m = 0$ in $\overline{\mathbb{K}}$, then $Z = 0$ in $\overline{\mathbb{K}}$.*

Proof. By assumption, for a representative $u \in \mathcal{E}_{0,M}$ of Z, we have that $\exists\, N \in \mathbb{N}$, $\gamma \in \Gamma$ such that $\forall q \geq N$, $\forall \varphi \in \mathcal{A}_q(\mathbb{R})$ $\exists c > 0$, $\eta > 0$ such that for all $\varepsilon \in (0,\eta)$, we have a bound $|u(\varphi_\varepsilon)|^m \leq c\varepsilon^{\gamma(q)}$ (see the text after (2.7)). Then $|u(\varphi_\varepsilon)| \leq c^{1/m}\varepsilon^{\gamma(q)/m}$, so that $u \in \mathcal{N}_0$. □

Now we consider an algebra of generalized functions $\mathcal{G}^1(\mathbb{R})$ and the corresponding algebra of generalized number $\overline{\mathbb{K}}^1$. Denote by $\mathcal{E}^1[\mathbb{R}]$ the algebra of mappings $u \in \mathcal{E}[\mathbb{R}]$ with the following property: for all functions φ^1, $\varphi^2 \in \mathcal{A}_0(\mathbb{R})$ the mapping

$$\mathbb{R} \times (0,\infty) \times \mathbb{R} \ni (\tau, \varepsilon, x) \longmapsto u((1-\tau)\varphi^1_\varepsilon + \tau\varphi^2_\varepsilon, x) \in \mathbb{K}$$

is infinitely differentiable in variables (τ,ε,x). Define a subalgebra $\mathcal{E}^1_M[\mathbb{R}]$ in $\mathcal{E}^1[\mathbb{R}]$ and an ideal $\mathcal{N}^1[\mathbb{R}]$ in $\mathcal{E}^1_M[\mathbb{R}]$ in just the same way as we have defined $\mathcal{E}_M[\mathbb{R}]$ and $\mathcal{N}[\mathbb{R}]$ above, but with the additional requirement that numbers c and η in these definitions can be chosen independently of functions φ from the closed interval $[\varphi^1,\varphi^2]$ in the set $\mathcal{A}_N(\mathbb{R})$, where $[\varphi^1,\varphi^2] = \{\,(1-t)\varphi^1 + t\varphi^2 \mid t \in [0,1]\,\}$ for φ^1, $\varphi^2 \in \mathcal{A}_N(\mathbb{R})$. Now we set $\mathcal{G}^1(\mathbb{R}) = \mathcal{E}^1_M[\mathbb{R}]/\mathcal{N}^1[\mathbb{R}]$. Excluding the dependence on $x \in \mathbb{R}$ in these definitions and denoting the corresponding sets in the Colombeau construction by $\mathcal{E}^1_0$, $\mathcal{E}^1_{0,M}$, and $\mathcal{N}^1_0$, we obtain the algebra of generalized numbers $\overline{\mathbb{K}}^1 := \mathcal{E}^1_{0,M}/\mathcal{N}^1_0$. The specific character of algebras $\overline{\mathbb{K}}^1$ and $\mathcal{G}^1(\mathbb{R})$ is that in their definition, the continuous dependence of representatives of elements of these algebras is built in not only on x, but also on the parameters τ and ε.

Proposition 3.10. *Let P be a nonzero polynomial in one variable with coefficients in $\mathbb{C}$. A generalized number $Z \in \overline{\mathbb{C}}^1$ is a solution to the equation $P(Z) = 0$ in $\overline{\mathbb{C}}^1$ if and only if Z is a classical root of the polynomial P.*

Proof. We prove the necessity. Let $P(x) = \prod_{j=1}^{m}(x_j - a_j)$, where $a_j \in \mathbb{C}$ $(j = 1, \ldots, m)$, and let $u \in \mathcal{E}^1_{0,M}$ be a representative of Z. From the equality $P(Z) = 0$ we find that $\exists\, N \in \mathbb{N}$, $\gamma \in \Gamma$ such that $\forall \varphi \in \mathcal{A}_q(\mathbb{R})$ with $q \geq N$, there exist numbers $c > 0$, $\eta > 0$ such that

$$\left|\prod_{j=1}^{n}(u(\varphi_\varepsilon) - a_j)\right| \leq c\varepsilon^{\gamma(q)} \qquad \forall \varepsilon \in (0,\eta). \tag{3.11}$$

It follows that for every $\varepsilon \in (0,\eta)$, there is an index $j(\varepsilon) \in \{1,\ldots,m\}$ for which $|u(\varphi_\varepsilon) - a_{j(\varepsilon)}| \leq c^{1/m}\varepsilon^{\gamma(q)/m}$. But $u(\varphi_\varepsilon)$ depends continuously on ε; hence $j(\varepsilon)$ does not depend on ε; this means that $j(\varepsilon)$ coincides with some j_0 for all (small) ε. Let us show that j_0 does not depend on φ as well. Let $\varphi = (1-\tau)\varphi^1 + \tau\varphi^2$ for some φ^1, $\varphi^2 \in \mathcal{A}_q(\mathbb{R})$. Then inequality (3.11) holds with c and η independent of τ; it follows that for $\tau \in [0,1]$ and $\varepsilon \in (0,\eta)$, there exists an index $j(\tau,\varepsilon) \in \{1,\ldots,m\}$ such that

$$|u((1-\tau)\varphi^1_\varepsilon + \tau\varphi^2_\varepsilon) - a_{j(\tau,\varepsilon)}| \leq c^{1/m}\varepsilon^{\gamma(q)/m}.$$

Taking into account the continuity in (τ, ε) of the expression $u(\dots)$, we conclude that the index $j(\tau, \varepsilon)$ does not depend on (τ, ε), and in particular, this index is the same for all $\varphi^1, \varphi^2 \in \mathcal{A}_q(\mathbb{R})$. Therefore, $u - a_{j_0} \in \mathcal{N}_0^1$; this means that Z is a classical root a_{j_0} of the polynomial P. □

For generalized functions $U \in \mathcal{G}^1(\mathbb{R})$, we have the following analog of Proposition 3.10:

Proposition 3.11. *Let P be a nonzero polynomial in one variable with coefficients in $\mathbb{C}$. A generalized function $U \in \mathcal{G}^1(\mathbb{R})$ is a solution to the equation $P(U) = 0$ in $\mathcal{G}^1(\mathbb{R})$ if and only if U is a constant generalized function which is equal to a classical root of the polynomial P.*

Proof. Let $\{a_j\}_{j=1}^m \subset \mathbb{C}$ be the classical roots of the polynomial P, $a_j \neq a_k$ if $j \neq k$, and let $u \in \mathcal{E}_M^1[\mathbb{R}]$ be a representative of U. From the equality $P(U) = 0$ we find that for a compact set $K \subset \mathbb{R}$, there are $N \in \mathbb{N}$ and $\gamma \in \Gamma$ such that $\forall \varphi \in \mathcal{A}_q(\mathbb{R})$ with $q \geq N$, there are $c > 0$ and $\eta > 0$ such that $|P(u(\varphi_\varepsilon, x))| \leq c\varepsilon^{\gamma(q)}$ $\forall x \in K$, $\forall \varepsilon \in (0, \eta)$. Taking into account the multiplicity of roots of P, we obtain

$$|u(\varphi_\varepsilon, x) - a_{j_0}| \leq c_1 \varepsilon^{\gamma_1(q)}, \quad x \in K, \quad \varepsilon \in (0, \eta_1), \tag{3.12}$$

for some (other) $c_1 > 0$, $\eta_1 > 0$, and $\gamma_1 \in \Gamma$, where a_{j_0} can depend on ε, φ, and x. But in view of the continuity of $u(\varphi_\varepsilon, x)$ in ε, φ, and x, we conclude that a_{j_0} does not depend on ε, φ, and x. Now we have to obtain the bounds for the representative $u(\varphi_\varepsilon, x)$ and its derivatives in x (denoted below by primes). For the first derivative, we have

$$\partial_x P(u(\varphi_\varepsilon, x)) = P'(u(\varphi_\varepsilon, x)) \cdot u'(\varphi_\varepsilon, x);$$

this implies the necessary bound for $|u'(\varphi_\varepsilon, x)|$ when $P'(a_{j_0}) \neq 0$. If $P'(a_{j_0}) = 0$ and $P''(a_{j_0}) \neq 0$, then from the equality

$$\partial_x^2 P(u(\varphi_\varepsilon, x)) = P''(u(\varphi_\varepsilon, x)) \cdot (u'(\varphi_\varepsilon, x))^2 + P'(u(\varphi_\varepsilon, x)) \cdot u''(\varphi_\varepsilon, x)$$

we obtain again the necessary bound for $|u'(\varphi_\varepsilon, x)|$ (here we use (3.12) in order to get rid of the last term). Applying induction on the multiplicity of the root a_{j_0}, finally, we obtain $|u'(\varphi_\varepsilon, x)| \leq c_2 \varepsilon^{\gamma_2(q)}$, $x \in K$, $\varepsilon \in (0, \eta_2)$, for some $c_2 > 0$, $\eta_2 > 0$, and $\gamma_2 \in \Gamma$. Arguing analogously in case of derivatives in x of higher order, we obtain bounds of the same kind, so that $u - a_{j_0} \in \mathcal{N}^1[\mathbb{R}]$. □

In each of the algebras $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}^1(\mathbb{R})$, the following proposition holds.

Proposition 3.12. *For $U \in \mathcal{G}(\mathbb{R})$, the equalities $x \cdot U = 0$ and $U = 0$ in the algebra $\mathcal{G}(\mathbb{R})$ are equivalent.*

Proof. We have $xy = 0 \Longrightarrow xy' + y = 0 \Longrightarrow x^2 y' = 0 \Longrightarrow x^2 (y')^3 = 0$. If $u \in \mathcal{E}_M[\mathbb{R}]$ is a representative of U, then, in the abbreviated notation, it follows that $|x^2 (u'(\varphi_\varepsilon, x))^3| \leq c\varepsilon^{\gamma(q)}$. In view of the integrability of the function $|x|^{-2/3}$ at the point $x = 0$, we find that $|u(\varphi_\varepsilon, x)| \leq c_1 \varepsilon^{\gamma_1(q)}$ (if $x \geq 0$ one should use the formula $u(\varphi_\varepsilon, x) = u(\varphi_\varepsilon, 1) + \int_1^x u'(\varphi_\varepsilon, \lambda)\, d\lambda$, and if $x \leq 0$ one should replace 1 by -1). In order to obtain the same bound for the derivative $u'(\varphi_\varepsilon, x)$, we start from the equality $x^2 y' = 0$; setting $z = y'$, as in the beginning of the proof, we have $x^3 (z')^4 = 0$. Using the integrability of the function $|x|^{-3/4}$ at $x = 0$ and arguing as above, we obtain the necessary bound for $u'(\varphi_\varepsilon, x)$. To estimate the second derivative $u''(\varphi_\varepsilon, x)$, we use the implications $x^2 y' = 0 \Longrightarrow 2xy' + x^2 y'' = 0 \Longrightarrow x^3 y'' = 0$, and argue starting from the equality $x^3 y'' = 0$ instead of $x^2 y' = 0$; setting $z = y''$, we have $x^3 z = 0 \Longrightarrow 3x^2 z + x^3 z' = 0 \Longrightarrow 3x^3 z + x^4 z' = 0 \Longrightarrow x^4 (z')^5 = 0$, and so on. □

Proposition 3.13. *The solutions U to the equation $U^2 = x^2$ in $\mathcal{G}^1(\mathbb{R})$ are only two C^∞ functions $+x$ or $-x$ (continuous functions $|x|$ and $-|x|$ are not the solutions to the equation in $\mathcal{G}^1(\mathbb{R})$). The equation $U^4 = x^2$ has no solutions U in $\mathcal{G}^1(\mathbb{R})$.*

Proof. From the equation $y^2 = x^2$ we have $yy' = x \Longrightarrow y^2y' = yx \Longrightarrow x^2y' = yx \Longrightarrow x(xy' - y) = 0$. By Proposition 3.12, it follows that $xy' = y$. The latter equality, together with $(y - x)(y + x) = 0$, gives us $x^2(y' - 1)(y' + 1) = 0$. Again, by Proposition 3.12, $(y' - 1)(y' + 1) = 0$. From Proposition 3.11 we find that $y' = 1$ or $y' = -1$ in $\mathcal{G}^1(\mathbb{R})$. By Proposition 3.5, we obtain $y = x + c_1$ or $y = -x + c_2$ for some generalized numbers c_1 and c_2. Finally, the equation $y^2 = x^2$ yields $c_1 = c_2 = 0$. □

A more general result on the regularity of solutions to algebraic equations which includes all the previous results is contained in the following assertion.

Theorem 3.14. *Let $P(x, y)$ be a nonzero polynomial in two variables x and y. If $I \subset \mathbb{R}$ is an open interval and $U \in \mathcal{G}^1(I)$, then $P(x, U) = 0$ in $\mathcal{G}^1(I)$ if and only if U is a classical C^∞ solution of this equation on I.* □

We do not prove this theorem here, and refer the reader to the papers from which most of the material of Sec. 3.4 was taken: Biagioni [15, 1.10.8], Colombeau [47], Marzouk [130], and Marzouk and Perrot [131].

4. Nonlinear Properties of Generalized Functions

The algebraic structure of $\mathcal{G}(\Omega)$ allows us to perform polynomial nonlinear operations in $\mathcal{G}(\Omega)$ (as usual $\Omega \subset \mathbb{R}^n$ is an open set). In this section, we are going to show that in $\mathcal{G}(\Omega)$, a large class of nonpolynomial nonlinear operations much more general than the multiplication have a sense. Such operations are useful in connection with the solution in $\mathcal{G}(\Omega)$ of nonlinear partial differential equations. Moreover, we define the concept of composition of generalized functions and restrictions of generalized functions to linear subspaces.

4.1. Nonlinear operations from $\mathcal{O}_M(\mathbb{K}^p)$. For $p \in \mathbb{N}$, denote by $\mathcal{O}_M(\mathbb{R}^p)$ the algebra of functions from $C^\infty(\mathbb{R}^p)$ *slowly increasing at infinity*,

$$\mathcal{O}_M(\mathbb{R}^p) = \{ F \in C^\infty(\mathbb{R}^p) \,|\, \forall \alpha \in \mathbb{N}_0^p \ \exists c > 0,\ m \in \mathbb{N} : \\ \forall x \in \mathbb{R}^p : |(\partial^\alpha F)(x)| \le c(1 + |x|)^m \}.$$

Clearly, $\mathcal{O}_M(\mathbb{R}^p)$ contains all polynomials and is a differential subalgebra in $C^\infty(\mathbb{R}^p)$ with respect to the pointwise operations which, moreover, is invariant relative to the partial differential operators:

$$\partial^\alpha \mathcal{O}_M(\mathbb{R}^p) \subset \mathcal{O}_M(\mathbb{R}^p), \quad \alpha \in \mathbb{N}_0^p.$$

For example, if $f(x) = \sin x$, $g(x) = \cos x$, $h(x) = e^{ix}$ $(i = \sqrt{-1})$, $x \in \mathbb{R}$, then we have $f, g \in \mathcal{O}_M(\mathbb{R}; \mathbb{R})$, and $h \in \mathcal{O}_M(\mathbb{R}; \mathbb{C})$. Another example is the "product in $\mathbb{C}$," which is an operation from $\mathcal{O}_M(\mathbb{R}^4; \mathbb{C})$. Having in mind the latter example, we identify $\mathbb{C}$ and $\mathbb{R}^2$ so that we have $\mathcal{O}_M(\mathbb{K}^p) = \mathcal{O}_M(\mathbb{R}^p)$ if $\mathbb{K} = \mathbb{R}$, and $\mathcal{O}_M(\mathbb{K}^p) = \mathcal{O}_M(\mathbb{R}^{2p})$ if $\mathbb{K} = \mathbb{C}$.

Let (as usual) $U_1, \dots, U_p \in \mathcal{G}(\Omega)$ be $\mathbb{K}$-valued generalized functions. For the corresponding representatives $u_1, \dots, u_p \in \mathcal{E}_M[\Omega]$ of these generalized functions and $F \in \mathcal{O}_M(\mathbb{K}^p)$, we set

$$(F(u_1, \dots, u_p))(\varphi, x) := F(u_1(\varphi, x), \dots, u_p(\varphi, x)), \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n), \quad x \in \Omega;$$

define the generalized function $F(U_1, \dots, U_p) \in \mathcal{G}(\Omega)$ by the equality

$$F(U_1, \dots, U_p) = F(u_1, \dots, u_p) + \mathcal{N}[\Omega].$$

The fact that this equality defines correctly the mapping

$$F : (\mathcal{G}(\Omega))^p = \underbrace{\mathcal{G}(\Omega) \times \dots \times \mathcal{G}(\Omega)}_{p \text{ times}} \longrightarrow \mathcal{G}(\Omega)$$

is verified in the following proposition.

Proposition 4.1. *Assume that* u_j, $\tilde{u}_j \in \mathcal{E}_M[\Omega]$, $j = 1, \dots, p$. *Then*

(a) $F(u_1, \dots, u_p) \in \mathcal{E}_M[\Omega]$;

(b) *if* $u_j - \tilde{u}_j \in \mathcal{N}[\Omega]$, $j = 1, \dots, p$, *then* $F(u_1, \dots, u_p) - F(\tilde{u}_1, \dots, \tilde{u}_p) \in \mathcal{N}[\Omega]$.

Proof. (a) Obviously, $F(u_1, \dots, u_p) \in \mathcal{E}[\Omega]$. Since $F \in \mathcal{O}_M(\mathbb{K}^p)$, there are $c_1 > 0$ and $m \in \mathbb{N}$ such that for all $x \in \mathbb{K}^p$ we have

$$|F(x)| \le c_1(1 + |x|_p)^m, \quad \text{where} \quad |x|_p = |x|_{\mathbb{K}^p}. \tag{4.1}$$

If $K \subset\subset \Omega$, then $\exists N \in \mathbb{N}$ such that $\forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)$ $\exists c > 0$, $\eta > 0$ such that

$$|u_j(\varphi_\varepsilon, x)| \le c\varepsilon^{-N}, \quad x \in K, \quad \varepsilon \in (0, \eta), \quad j = 1, \dots, p. \tag{4.2}$$

It follows that for all $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$, $x \in K$, and $\varepsilon \in (0, \eta)$, in view of (4.1) and (4.2), we have

$$|F(u_1(\varphi_\varepsilon, x), \dots, u_p(\varphi_\varepsilon, x))| \le c_2\varepsilon^{-Nm},$$

where the constant $c_2 > 0$ depends only on c, c_1, p, and m.

For the first-order derivatives of $F(u_1, \dots, u_p)$ (which corresponds to a multi-index of length $|\alpha| = 1$) we have by the chain rule

$$\partial_{x_k} F(u_1(\varphi, x), \dots, u_p(\varphi, x)) = \sum_{j=1}^{p} (\partial_j F)(u_1(\varphi, x), \dots, u_p(\varphi, x)) \cdot (\partial_{x_k} u_j)(\varphi, x),$$

where $(\partial_j F)(y_1, \dots, y_p)$ denotes the partial derivative of F with respect to the variable y_j. Using bounds for $\partial_j F$ from the definition of $\mathcal{O}_M(\mathbb{K}^p)$, and bounds for u_j and $\partial_{x_k} u_j$ from the definition of $\mathcal{E}_M[\Omega]$, for the derivative $\partial_{x_k} F(u_1, \dots, u_p)(\varphi_\varepsilon, x)$, we obtain a bound of the form $c\varepsilon^{-N}$ uniformly in $x \in K \subset\subset \Omega$.

The general case of arbitrary multi-indices $\alpha \in \mathbb{N}_0^n$ follows from *Faa di Bruno's formula*, which is read as follows.

Faa di Bruno's formula. *Let* $\Delta \subset \mathbb{R}^p$ *and* $\Omega \subset \mathbb{R}^n$ *be open sets,* $F \in C^\infty(\Delta; \mathbb{K})$, *and let* $f \in C^\infty(\Omega; \Delta)$, *so that* $f(x) = (f_1(x), \dots, f_p(x)) \in \Delta$ *for all* $x \in \Omega$. *Then for all* $\alpha \in \mathbb{N}_0^n$, $|\alpha| \ge 1$, *and* $x \in \Omega$, *Faa di Bruno's formula holds:*

$$\partial_x^\alpha [F(f(x))] = \partial_x^\alpha [F(f_1(x), \dots, f_p(x))] =$$

$$= \sum_{N=1}^{|\alpha|} \sum_{\alpha^1, \dots, \alpha^N} \frac{\alpha!}{\alpha^1! \cdots \alpha^N! N!} \sum_{i_1=1}^{p} \cdots \sum_{i_N=1}^{p} (\partial_{i_1} \cdots \partial_{i_N} F)(f(x)) \cdot (\partial^{\alpha^1} f_{i_1})(x) \cdots (\partial^{\alpha^N} f_{i_N})(x),$$

where the second sum is taken over all ordered collections of multi-indices $(\alpha^1, \dots, \alpha^N)$, $\alpha^i \in \mathbb{N}_0^n$, $|\alpha^i| \ge 1$, $i = 1, \dots, N$, *such that* $\alpha^1 + \dots + \alpha^N = \alpha$.

In particular, if $|\alpha| = 1$, *this formula is the usual formula for the differentiation of a composition function of several variables (the chain rule):*

$$\partial_{x_k}[F(f(x))] = \sum_{j=1}^{p} (\partial_j F)(f(x)) \cdot (\partial_{x_k} f_j)(x), \quad k = 1, \dots, n. \qquad \square$$

(b) As in (a), the proof is performed by induction on the order $|\alpha|$ of the multi-index $\alpha \in \mathbb{N}_0^n$. If $|\alpha| = 0$, by the mean-value theorem, we have

$$\begin{aligned} &|F(u_1(\varphi_\varepsilon, x), \dots, u_p(\varphi_\varepsilon, x)) - F(\tilde{u}_1(\varphi_\varepsilon, x), \dots, \tilde{u}_p(\varphi_\varepsilon, x))| \le \\ &\le \sum_{j=1}^{p} \sup_{0 \le \theta \le 1} |(\partial_j F)(\{u_k + \theta(\tilde{u}_k - u_k)\}_{k=1}^{p}(\varphi_\varepsilon, x))| \cdot |(u_j - \tilde{u}_j)(\varphi_\varepsilon, x)|. \end{aligned} \tag{4.3}$$

If $K \subset\subset \Omega$, there are $N \in \mathbb{N}$, $m \in \mathbb{N}$, and $\gamma \in \Gamma$ such that if $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ with $q \geq N$, then there are $c > 0$ and $\eta > 0$ such that

$$
\begin{aligned}
&|(\partial_j F)(\lambda)| \leq c(1 + |\lambda|_p)^m, \quad \lambda \in \mathbb{K}^p, \\
&|u_j(\varphi_\varepsilon, x)| \leq c\varepsilon^{-N}, \\
&|(u_j - \tilde{u}_j)(\varphi_\varepsilon, x)| \leq c\varepsilon^{\gamma(q)-N},
\end{aligned} \tag{4.4}
$$

for all $x \in K$, $\varepsilon \in (0, \eta)$, and $j = 1, \ldots, p$. From here and from (4.3), for all $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$, $x \in K$, and $\varepsilon \in (0, \eta)$, we find that

$$
|F(\{u_k(\varphi_\varepsilon, x)\}_{k=1}^p) - F(\{\tilde{u}_k(\varphi_\varepsilon, x)\}_{k=1}^p)| \leq c_1 \varepsilon^{\gamma(q)-(m+1)N},
$$

where the constant $c_1 > 0$ depends on c, p, and m. This is the necessary bound in the case $|\alpha| = 0$. If $|\alpha| = 1$, for $i = 1, \ldots, n$, we have

$$
\begin{aligned}
&\partial_{x_i}\Big(F(\{u_k(\varphi_\varepsilon, x)\}_{k=1}^p) - F(\{\tilde{u}_k(\varphi_\varepsilon, x)\}_{k=1}^p)\Big) = \\
&= \sum_{j=1}^p \Big((\partial_j F)(\{u_k(\varphi_\varepsilon, x)\}_{k=1}^p) - (\partial_j F)(\{\tilde{u}_k(\varphi_\varepsilon, x)\}_{k=1}^p)\Big) \cdot (\partial_{x_i} u_j)(\varphi_\varepsilon, x) + \\
&\quad + \sum_{j=1}^p (\partial_j F)(\{\tilde{u}_k(\varphi_\varepsilon, x)\}_{k=1}^p) \cdot (\partial_{x_i}(u_j - \tilde{u}_j)(\varphi_\varepsilon, x)).
\end{aligned} \tag{4.5}
$$

From here we obtain the necessary bound for the difference in (4.5). In the general case of $\alpha \in \mathbb{N}_0^n$ one should use Faa di Bruno's formula. □

The nonlinear operation $F : (\mathcal{G}(\Omega))^p \longrightarrow \mathcal{G}(\Omega)$ defined above exactly generalizes the usual nonlinear operation over C^∞ functions: if $F \in \mathcal{O}_M(\mathbb{K}^p)$ and $\{f_j\}_{j=1}^p \subset C^\infty(\Omega)$, then $F(\imath(f_1), \ldots, \imath(f_p)) = \imath(F(f_1, \ldots, f_p))$ in $\mathcal{G}(\Omega)$; here we have used the imbedding (2.9). In the case where functions f_j are only continuous, the classical function $F(f_1, \ldots, f_p)$ is recovered by means of the concept of the association, which is defined below for generalized functions from $\mathcal{G}(\Omega)$ (Theorem 8.12).

For generalized numbers from $\overline{\mathbb{K}}$ one can define nonlinear operations more general than the multiplication as well. The corresponding construction is, in this case, as for generalized functions if we exclude everywhere the dependence of mappings on $x \in \Omega$: if $F \in \mathcal{O}_M(\mathbb{K}^p)$, and generalized numbers $Z_1, \ldots, Z_p \in \overline{\mathbb{K}}$ have corresponding representatives $u_1, \ldots, u_p \in \mathcal{E}_{0,M}$, we set $(F(u_1, \ldots, u_p))(\varphi) = F(u_1(\varphi), \ldots, u_p(\varphi))$, $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, and the generalized number $F(Z_1, \ldots, Z_p) \in \overline{\mathbb{K}}$ is defined by

$$
F(Z_1, \ldots, Z_p) = F(u_1, \ldots, u_p) + \mathcal{N}_0.
$$

Thus, the nonlinear operation $F : (\overline{\mathbb{K}})^p \longrightarrow \overline{\mathbb{K}}$ is well defined and generalizes the nonlinear operation over ordinary numbers from $\mathbb{K}$: if $z_1, \ldots, z_p \in \mathbb{K}$, then $F(\imath_0(z_1), \ldots, \imath_0(z_p)) = \imath_0(F(z_1, \ldots, z_p))$ in $\overline{\mathbb{K}}$ (see (3.5)).

Calculations in $\overline{\mathbb{K}}$ are recovered via the association:

Proposition 4.2. *Let $F \in \mathcal{O}_M(\mathbb{K}^p)$, $\{Z_j\}_{j=1}^p \subset \overline{\mathbb{K}}$, $\{z_j\}_{j=1}^p \subset \mathbb{K}$, and let $Z_j \approx z_j$ in $\overline{\mathbb{K}}$ for $j = 1, \ldots, p$. Then $F(Z_1, \ldots, Z_p) \approx F(z_1, \ldots, z_p)$ in $\overline{\mathbb{K}}$.*

Proof. For $j = 1, \ldots, p$ and for a representative $u_j \in \mathcal{E}_{0,M}$ of the generalized number Z_j, from the association $Z_j \approx z_j$ we find $N \in \mathbb{N}$ independent of j such that

$$
u_j(\varphi_\varepsilon) \longrightarrow z_j \quad \text{as} \quad \varepsilon \longrightarrow +0, \quad \varphi \in \mathcal{A}_N(\mathbb{R}^n), \quad j = 1, \ldots, p.
$$

Hence $F(u_1(\varphi_\varepsilon), \ldots, u_p(\varphi_\varepsilon)) \longrightarrow F(z_1, \ldots, z_p)$ as $\varepsilon \longrightarrow +0$ for φ as above. □

Note that nonlinear operations in $\mathcal{G}(\Omega)$ and $\overline{\mathbb{K}}$ are coherent with pointvalues of generalized functions, which is explicitly given in the following assertion.

Proposition 4.3. *If $F \in \mathcal{O}_M(\mathbb{K}^p)$ and $U_1, \ldots, U_p \in \mathcal{G}(\Omega)$, then*

$$(F(U_1, \ldots, U_p))(x) = F(U_1(x), \ldots, U_p(x)) \quad \text{in} \quad \overline{\mathbb{K}} \quad \forall x \in \Omega.$$

This follows immediately from the corresponding definitions. □

Let us consider in more detail real-valued generalized functions. Let $\delta \in \mathcal{G}(\mathbb{R}^n)$ be the (Dirac) generalized function with the representative $u_\delta(\varphi, x) = \varphi(-x)$, $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and let $F(x) = e^{ix}$, $x \in \mathbb{R}$, so that $F \in \mathcal{O}_M(\mathbb{R}; \mathbb{C})$. If we want to make sense out of the composed generalized function $e^{i\delta}$ having the representative $v(\varphi, x) = e^{i\varphi(-x)}$, then, in the construction of the algebra $\mathcal{G}(\mathbb{R}^n)$ we should restrict ourselves to sets $\mathcal{A}_0(\mathbb{R}^n; \mathbb{R})$ made only of real-valued functions φ. Note also that if $f \in C(\mathbb{R}^n)$, then f is real valued iff the representative $u_f \in \mathcal{E}_M[\mathbb{R}^n]$ of the generalized function $\jmath(f) \in \mathcal{G}(\mathbb{R}^n)$ from (2.12) satisfies the property

$$u_f(\varphi, x) \in \mathbb{R} \quad \forall \varphi \in \mathcal{A}_0(\mathbb{R}^n; \mathbb{R}), \quad \forall x \in \mathbb{R}^n.$$

Thus, it is natural to give the following definition.

Definition 4.4. We say that a generalized function $U \in \mathcal{G}(\mathbb{R}^n)$ is *real valued* if it has a representative $u \in \mathcal{E}_M[\mathbb{R}^n]$ with the property $u(\varphi, x) \in \mathbb{R}$ for all $\varphi \in \mathcal{A}_0(\mathbb{R}^n; \mathbb{R})$ and $x \in \mathbb{R}^n$ (we assume here that the algebra $\mathcal{G}(\mathbb{R}^n)$ is constructed starting from the index set $\mathcal{A}_0(\mathbb{R}^n; \mathbb{R})$). □

On the other hand, we can define a *real* algebra of generalized functions $\mathcal{G}(\mathbb{R}^n; \mathbb{R})$ if we take the algebra $\mathcal{E}[\mathbb{R}^n; \mathbb{R}] = C^\infty(\mathbb{R}^n; \mathbb{R})^{\mathcal{A}_0(\mathbb{R}^n; \mathbb{R})}$ and define, in the usual way, its subalgebra $\mathcal{E}_M[\mathbb{R}^n; \mathbb{R}]$ and an ideal $\mathcal{N}[\mathbb{R}^n; \mathbb{R}]$ in it, and then set

$$\mathcal{G}(\mathbb{R}^n; \mathbb{R}) = \mathcal{E}_M[\mathbb{R}^n; \mathbb{R}]/\mathcal{N}[\mathbb{R}^n; \mathbb{R}].$$

Now the composition $e^{i\delta} \in \mathcal{G}(\mathbb{R}^n)$ can be well defined as follows: since $\delta \in \mathcal{G}(\mathbb{R}^n; \mathbb{R})$, $\delta = u_\delta + \mathcal{N}[\mathbb{R}^n; \mathbb{R}]$, and $F(x) = e^{ix} \in \mathcal{O}_M(\mathbb{R}; \mathbb{C})$, we set $e^{i\delta} = e^{iu_\delta} + \mathcal{N}[\mathbb{R}^n] \in \mathcal{G}(\mathbb{R}^n)$. In this way, one can define many nonlinear operations over generalized functions: if $F \in \mathcal{O}_M(\mathbb{R}^p)$ and $U_1, \ldots, U_p \in \mathcal{G}(\mathbb{R}^n)$ are real-valued generalized functions with corresponding real-valued representatives $u_1, \ldots, u_p \in \mathcal{E}_M[\mathbb{R}^n; \mathbb{R}]$, we set

$$(F(u_1, \ldots, u_p))(\varphi, x) = F(u_1(\varphi, x), \ldots, u_p(\varphi, x)), \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n; \mathbb{R}) \quad x \in \mathbb{R}^n,$$

and define the generalized function $F(U_1, \ldots, U_p) \in \mathcal{G}(\mathbb{R}^n)$ by

$$F(U_1, \ldots, U_p) = F(u_1, \ldots, u_p) + \mathcal{N}[\mathbb{R}^n].$$

Taking into account the fact that all calculations are, in fact, made in $\mathcal{E}[\mathbb{R}^n]$ after an arbitrary choice of representatives of the generalized functions under consideration, it is easy to see that new nonlinear operations over generalized functions satisfy the same rules of calculations as their classical counterparts. For example, if $U \in \mathcal{G}(\mathbb{R}^n)$ is a real-valued generalized function, then one has

$$\partial_{x_j}(\sin U) = (\cos U) \cdot (\partial_{x_j} U) \quad \text{in} \quad \mathcal{G}(\mathbb{R}^n), \quad j = 1, \ldots, n.$$

For real generalized functions and real generalized numbers, one has analogs of Propositions 4.2 and 4.3. Also, it is clear that everything we had said in the case $\Omega = \mathbb{R}^n$ could also be said above in the case of an open set $\Omega \subset \mathbb{R}^n$.

4.2. Composition of generalized functions. Now we define the *composition of generalized functions.* Given a function $\phi \in \mathcal{D}(\mathbb{R})$, we define the n-times tensor product of ϕ with itself (or the n-fold tensor product of ϕ) by

$$\phi^{\otimes n}(x_1, \ldots, x_n) = \prod_{j=1}^{n} \phi(x_j), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{4.6}$$

Consider the set of index functions having the tensor product structure

$$\mathcal{A}_q^{\otimes}(\mathbb{R}^n) = \{\, \varphi = \phi^{\otimes n} \in \mathcal{A}_q(\mathbb{R}^n) \mid \phi \in \mathcal{A}_q(\mathbb{R}) \,\}, \quad q \in \mathbb{N}_0. \tag{4.7}$$

As we have seen above (step 2 in the proof of Lemma 2.1), these sets are nonempty.

Let $\Omega \subset \mathbb{R}^n$ and $\Delta \subset \mathbb{R}^m$ be open sets. Set

$$\mathcal{E}[\Omega;\mathbb{R}^m] = (C^\infty(\Omega;\mathbb{R}^m))^{\mathcal{A}_0^{\otimes}(\mathbb{R}^n)},$$

where $C^\infty(\Omega;\mathbb{R}^m)$ is the space of C^∞ functions from Ω into $\mathbb{R}^m$. Define an algebra of *moderate* elements similarly to (2.6), i.e.,

$$\begin{aligned}\mathcal{E}_M[\Omega;\mathbb{R}^m] = \{u \in \mathcal{E}[\Omega;\mathbb{R}^m] \mid \forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \ \exists N \in \mathbb{N}:\\ \forall \varphi \in \mathcal{A}_N^{\otimes}(\mathbb{R}^n)\ \exists c > 0,\ \eta > 0:\\ \forall \varepsilon \in (0,\eta): \sup_{x\in K} |\partial^\alpha u(\varphi_\varepsilon, x)|_m \le c\varepsilon^{-N}\},\end{aligned} \tag{4.8}$$

where $|\cdot|_m$ is the (Euclidean) norm in $\mathbb{R}^m$. Define an ideal $\mathcal{N}[\Omega;\mathbb{R}^m]$ of *null* elements of $\mathcal{E}_M[\Omega;\mathbb{R}^m]$ similarly to (2.7) with same modifications as in (4.8). Define the algebra of generalized functions $\mathcal{G}(\Omega;\mathbb{R}^m)$ on Ω with values in $\mathbb{R}^m$ in the usual way:

$$\mathcal{G}(\Omega;\mathbb{R}^m) = \mathcal{E}_M[\Omega;\mathbb{R}^m]/\mathcal{N}[\Omega;\mathbb{R}^m].$$

Clearly, an element of $\mathcal{G}(\Omega;\mathbb{R}^{2m})$ can be considered as m (complex valued) elements of $\mathcal{G}(\Omega;\mathbb{C})$ (if we identify $\mathbb{C}$ with $\mathbb{R}^2$).

Finally, denote by $\mathcal{G}_*(\Omega;\Delta)$ the set of elements $U \in \mathcal{G}(\Omega;\mathbb{R}^m)$ such that there is a representative $u \in \mathcal{E}_M[\Omega;\mathbb{R}^m]$ of U with the property

$$\begin{aligned}&\forall K \subset\subset \Omega\ \exists N \in \mathbb{N} \text{ such that } \forall \varphi \in \mathcal{A}_N^{\otimes}(\mathbb{R}^n)\ \exists \eta > 0,\ K_1 \subset\subset \Delta, \text{ such}\\ &\text{that } K_1 \text{ does not depend on } \varphi \text{ and } \{u(\varphi_\varepsilon, x) \mid x \in K,\ \varepsilon \in (0,\eta)\} \subset K_1.\end{aligned} \tag{4.9}$$

Note that if one representative of the generalized function U satisfies property (4.9), then all representatives of this generalized function satisfy this property as well. In general, $\mathcal{G}_*(\Omega;\mathbb{R}^m) \ne \mathcal{G}(\Omega;\mathbb{R}^m)$ since for the element $\delta \in \mathcal{G}(\mathbb{R}^n;\mathbb{R})$, in view of Examples 2.3(1) and 2.8(1), we have $\delta \notin \mathcal{G}_*(\mathbb{R}^n;\mathbb{R})$. Nevertheless, the class $\mathcal{G}_*(\Omega;\Delta)$ of generalized functions is sufficiently large.

Proposition 4.5. *If $f \in C(\Omega;\Delta)$, then the generalized function $\jmath(f)$ is in $\mathcal{G}_*(\Omega;\Delta)$, where the imbedding $\jmath : C(\Omega;\Delta) \longrightarrow \mathcal{G}(\Omega;\mathbb{R}^m)$ is defined similarly to* (2.12) *and* (2.17), *and the index sets $\mathcal{A}_q^{\otimes}(\mathbb{R}^n;\mathbb{R})$ consist of real-valued functions.*

Proof. Let $K \subset\subset \Omega$ and $\varphi \in \mathcal{A}_0^{\otimes}(\mathbb{R}^n;\mathbb{R})$. By (2.18), there is an $\eta_0 = \eta_0(K,\varphi) > 0$ such that, in view of the calculations in the proof of Proposition 1.3(a), we have

$$|u_f(\varphi_\varepsilon, x) - f(x)|_m \le c(\varphi) \sup_{\substack{y\in K\\ \lambda \in B_{\varepsilon\rho(\varphi)}}} |f(y+\lambda) - f(y)|_m, \quad x \in K, \quad \varepsilon \in (0,\eta_0), \tag{4.10}$$

with $c(\varphi) = \displaystyle\int_{B_{\rho(\varphi)}} |\varphi(\mu)|\,d\mu$. Let $0 < r < d$, where $d = \operatorname{dist}(f(K), \partial\Delta)$ (then $d > 0$ since $f(K) \subset\subset \Delta$). Due to (1.3), it follows that $f(K) + B_r^m \subset\subset \Delta$, where B_r^m is the closed ball in $\mathbb{R}^m$ of radius r centered at the origin. Noting that the right-hand side in inequality (4.10) tends to zero as $\varepsilon \longrightarrow +0$, we can find $\eta = \eta(\varphi) > 0$ such that this right-hand side is less than r for all $\varepsilon \in (0,\eta)$. This means that

$$\{u_f(\varphi_\varepsilon, x) \mid x \in K,\ \varepsilon \in (0,\eta)\} \subset f(K) + B_r^m$$

with the compact set on the right hand side of this inclusion independent of φ. □

Theorem 4.6. *Let generalized functions $V \in \mathcal{G}(\Delta)$ and $U \in \mathcal{G}_*(\Omega; \Delta)$ have representatives $v \in \mathcal{E}_M[\Delta]$ and $u \in \mathcal{E}_M[\Omega; \mathbb{R}^m]$, respectively, and let an element $w \in \mathcal{E}[\Omega]$ be defined as follows:*

$$\begin{aligned}
&\forall K \subset\subset \Omega \;\exists N \in \mathbb{N} \text{ such that } \forall \varphi = \phi^{\otimes n} \in \mathcal{A}_N^{\otimes}(\mathbb{R}^n)\\
&\text{with } \phi \in \mathcal{A}_N(\mathbb{R}),\; \exists \eta > 0 \text{ such that}\\
&w(\phi_\varepsilon^{\otimes n}, x) = v(\phi_\varepsilon^{\otimes m}, u(\phi_\varepsilon^{\otimes n}, x)), \quad x \in K, \quad \varepsilon \in (0, \eta).
\end{aligned} \tag{4.11}$$

Then $w \in \mathcal{E}_M[\Omega]$.

The generalized function $V \circ U = w + \mathcal{N}[\Omega] \in \mathcal{G}(\Omega)$ is called the composition of generalized functions V and U; it is well defined in the sense that it is independent of the choice of representatives of the generalized functions V and U.

Proof. 1. First we verify the moderate property of w. Let $K \subset\subset \Omega$, and let N and η be as in (4.9). Then the set $\{u(\phi_\varepsilon^{\otimes n}, x)\}_{x \in K, \, \varepsilon \in (0,\eta)}$ is contained in a compact set $K_1 \subset \Delta$, which is independent of ϕ. Using the moderate property of v, we find an integer $N_1 \in \mathbb{N}$ such that if $\varphi = \phi^{\otimes n} \in \mathcal{A}_{N_1}^{\otimes}(\mathbb{R}^n)$, then there is an $\eta_1 > 0$ such that

$$|w(\phi_\varepsilon^{\otimes n}, x)| \le c\varepsilon^{-N_1}, \quad x \in K, \quad \varepsilon \in (0, \eta_1).$$

Now, the same kind of bounds can be established for partial derivatives $\partial^\alpha w$, $\alpha \in \mathbb{N}_0^n$. For derivatives of the first order, in view of (4.11), we have

$$\partial_{x_k}(w(\phi_\varepsilon^{\otimes n}, x)) = \sum_{j=1}^m (\partial_j v)(\phi_\varepsilon^{\otimes m}, u(\phi_\varepsilon^{\otimes n}, x)) \cdot (\partial_{x_k} u_j)(\phi_\varepsilon^{\otimes n}, x), \quad k = 1, \ldots, n; \tag{4.12}$$

here $\partial_j v$ denotes the partial derivative of v with respect to the jth argument, and u_j is the jth component of the vector $u = (u_1, \ldots, u_m)$. This equality and the moderate property of v and u imply the necessary bound for the derivative $\partial_{x_k}(w(\phi_\varepsilon^{\otimes n}, x))$. The general case of derivatives of arbitrary order follows from Faa di Bruno's formula (see Sec. 4.1).

2. Assume that $\tilde{v}$ and $\tilde{u}$ are other representatives of V and U respectively, so that $v - \tilde{v} \in \mathcal{N}[\Delta]$ and $u - \tilde{u} \in \mathcal{N}[\Omega; \mathbb{R}^m]$. Define an element $\tilde{w}$ similarly to w in (4.11) with v and u replaced by $\tilde{v}$ and $\tilde{u}$, respectively. We have

$$\begin{aligned}
w(\phi_\varepsilon^{\otimes n}, x) - \tilde{w}(\phi_\varepsilon^{\otimes n}, x) = \\
= [v(\phi_\varepsilon^{\otimes m}, u(\phi_\varepsilon^{\otimes n}, x)) - v(\phi_\varepsilon^{\otimes m}, \tilde{u}(\phi_\varepsilon^{\otimes n}, x))] + \\
+ [v(\phi_\varepsilon^{\otimes m}, \tilde{u}(\phi_\varepsilon^{\otimes n}, x)) - \tilde{v}(\phi_\varepsilon^{\otimes m}, \tilde{u}(\phi_\varepsilon^{\otimes n}, x))].
\end{aligned}$$

The first difference in square brackets has the necessary bound for $\mathcal{N}[\Omega]$, which follows from the mean-value theorem (see the proof of Proposition 4.1(b)) and the following conditions: $v \in \mathcal{E}_M[\Delta]$, $u - \tilde{u} \in \mathcal{N}[\Omega; \mathbb{R}^m]$, and (4.9). The second difference also has the necessary bound for $\mathcal{N}[\Omega]$ if we take into account that $v - \tilde{v} \in \mathcal{N}[\Delta]$ and that $\tilde{u} \in \mathcal{E}_M[\Omega; \mathbb{R}^m]$ has the property (4.9).

The same arguments gives the necessary bound for $\mathcal{N}[\Omega]$ for the first-order derivative $\partial_k(w - \tilde{w})$, $\partial_k = \partial_{x_k}$ if we note that (for brevity, in the calculations below, we omit the dependence of functions on $\phi_\varepsilon^{\otimes m}$, $\phi_\varepsilon^{\otimes n}$, and x)

$$\begin{aligned}
\partial_k(w - \tilde{w}) = \partial_k(v(u) - \tilde{v}(\tilde{u})) = \sum_{j=1}^m \Big([(\partial_j v)(u) - (\partial_j v)(\tilde{u})] \cdot \partial_k u_j + \\
+ (\partial_j v)(\tilde{u}) \cdot [\partial_k u_j - \partial_k \tilde{u}_j] + [(\partial_j v)(\tilde{u}) - (\partial_j \tilde{v})(\tilde{u})] \cdot \partial_k \tilde{u}_j \Big).
\end{aligned}$$

In the case of a derivative of arbitrary order $\partial^\alpha(w - \tilde{w})$, we argue using Faa di Bruno's formula. Thus, $w - \tilde{w} \in \mathcal{N}[\Omega]$; this is what we have to prove. □

Remark. Note that if $m = n$, we must define the composition of generalized functions in Theorem 4.6 in the case of the initial index sets $\mathcal{A}_q(\mathbb{R}^n)$ which do not necessarily have the tensor product structure as $\mathcal{A}_q^{\otimes}(\mathbb{R}^n)$. The necessary condition, however, which must hold for a representative of a generalized function $U \in \mathcal{G}_*(\Omega; \Delta)$ is a condition of the form (4.9), where $\varphi \in \mathcal{A}_N^{\otimes}(\mathbb{R}^n)$ is replaced by $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$. □

The concept of the composition of generalized functions generalizes exactly the corresponding concept for C^∞ functions: if $g \in C^\infty(\Delta)$ and $f \in C^\infty(\Omega; \Delta)$, then, in view of Proposition 4.5 (applied to the function f), we have

$$\imath_\Delta(g) \circ \imath_{\Omega,\mathbb{R}^m}(f) = \imath_\Omega(g \circ f) \quad \text{in} \quad \mathcal{G}(\Omega),$$

where we denote by $\imath_\Delta$, $\imath_{\Omega,\mathbb{R}^m}$, and $\imath_\Omega$ the canonical imbeddings $\imath_\Delta : C^\infty(\Delta) \longrightarrow \mathcal{G}(\Delta)$, $\imath_{\Omega,\mathbb{R}^m} : C^\infty(\Omega; \Delta) \longrightarrow \mathcal{G}(\Omega; \mathbb{R}^m)$, and $\imath_\Omega : C^\infty(\Omega) \longrightarrow \mathcal{G}(\Omega)$, which are defined similarly to (2.9). The composition $g \circ f$ of *continuous* functions $g \in C(\Delta)$ and $f \in C(\Omega; \Delta)$ is recovered by means of the concept of the associated distribution (Theorem 8.14).

The formula (4.12) can be rewritten in the form

$$\partial_k(V \circ U) = \sum_{j=1}^{m} (\partial_j V)(U) \cdot \partial_k U_j \quad \text{in} \quad \mathcal{G}(\Omega), \tag{4.13}$$

where $\partial_k = \partial_{x_k}$ ($k = 1, \ldots, n$), $\partial_j V$ is the derivative of V with respect to the jth argument, and U_j is the jth component of the vector $U = (U_1, \ldots, U_m)$. Since formula (4.13) for the differentiation of the composition of generalized functions holds, it follows that Faa di Bruno's formula (see Sec. 4.1) holds in $\mathcal{G}(\Omega)$; this is a natural generalization of the classical Faa di Bruno's formula.

Having at hand the concept of the composition of generalized functions, we are able to establish the formula of *change of variables* in generalized integrals:

Proposition 4.7. *Let Ω, $\Delta \subset \mathbb{R}^n$ be open sets, $U \in \mathcal{G}(\Delta; \mathbb{K})$, and let $h : \Omega \longrightarrow \Delta$ be a C^∞ diffeomorphism. Then for every compact set $K \subset\subset \Omega$ we have*

$$\int_K (U \circ h)(x) \cdot |\det \partial_x h(x)| \, dx = \int_{h(K)} U(y) \, dy \quad \text{in} \quad \overline{\mathbb{K}}, \tag{4.14}$$

where $\partial_x h(x)$ is the Jacobian of the function h at the point $x \in \Omega$.

Proof. It suffices to note that if $u \in \mathcal{E}_M[\Delta; \mathbb{K}]$ is a representative of U, then a representative of the generalized number at the left-hand side in formula (4.14) is of the form

$$\int_K u(\varphi_\varepsilon, h(x)) |\det \partial_x h(x)| \, dx = \int_{h(K)} u(\varphi_\varepsilon, y) \, dy, \quad \varphi \in \mathcal{A}_N^{\otimes}(\mathbb{R}^n), \quad \varepsilon \in (0, \eta);$$

the latter equality is nothing else but the classical formula of change of variables in the Lebesgue integral. □

4.3. Restriction of generalized functions to linear subspaces. The tensor product structure of the sets $\mathcal{A}_q^{\otimes}(\mathbb{R}^n)$ is very convenient in connection with the definition of a *restriction of generalized functions to linear subspaces.* Here we consider the simplest restriction to the subspace $\mathbb{R}^m$, where $m \in \mathbb{N}$ and $m < n$ (in the case $m = n - 1$ of a hyperplane, such a restriction arises naturally in connection with the solution of Cauchy problems for partial differential equations). Let $u \in \mathcal{E}_M[\mathbb{R}^n]$ be a representative of a generalized function $U \in \mathcal{G}(\mathbb{R}^n)$. Given a point $x \in \mathbb{R}^n$, it is convenient to use the following notation: $x = (x', x'')$ with $x' = (x_1, \ldots, x_m)$ and $x'' = (x_{m+1}, \ldots, x_n)$; in particular, $0' = (0, \ldots, 0) \in \mathbb{R}^m$, $0'' = (0, \ldots, 0) \in \mathbb{R}^{n-m}$. The *restriction* $U|\mathbb{R}^m \in \mathcal{G}(\mathbb{R}^m)$ of the *generalized function* U to the *linear subspace* $\mathbb{R}^m = \{ x \in \mathbb{R}^n \mid x = (x', 0'') \}$ is defined by means of a representative $u|\mathbb{R}^m$ of $U|\mathbb{R}^m$ as follows:

$$(u|\mathbb{R}^m)(\phi^{\otimes m}, x_1, \ldots, x_m) = u(\phi^{\otimes n}, x_1, \ldots, x_m, 0, \ldots, 0), \quad \phi \in \mathcal{A}_0(\mathbb{R}),$$

where $(x_1, \ldots, x_m) \in \mathbb{R}^m$, $\phi^{\otimes m} \in \mathcal{A}_0^{\otimes}(\mathbb{R}^m)$, and $\phi^{\otimes n} \in \mathcal{A}_0^{\otimes}(\mathbb{R}^n)$. Clearly, $u|\mathbb{R}^m \in \mathcal{E}[\mathbb{R}^m]$; the fact that the above definition is well defined follows from the fact that if $u \in \mathcal{E}_M[\mathbb{R}^n]$ (resp. $\mathcal{N}[\mathbb{R}^n]$), then $u|\mathbb{R}^m \in \mathcal{E}_M[\mathbb{R}^m]$ (resp. $\mathcal{N}[\mathbb{R}^m]$). For smooth functions, the new restriction to linear subspaces generalizes exactly the classical restriction: if $f \in C^\infty(\mathbb{R}^n)$, then $\imath(f)|\mathbb{R}^m = \imath(f|\mathbb{R}^m)$ in $\mathcal{G}(\mathbb{R}^m)$, where at the left-hand side, we denote by $\imath$ the canonical imbedding of $C^\infty(\mathbb{R}^n)$ into $\mathcal{G}(\mathbb{R}^n)$, and at the right-hand side, we denote by $\imath$ the imbedding of $C^\infty(\mathbb{R}^m)$ into $\mathcal{G}(\mathbb{R}^m)$. For *continuous* functions, the classical restriction to a linear subspace is recovered by means of the association relation (Theorem 8.16).

Note that the restriction map defined above can be considered as the composition as well; in fact, if we denote by $\kappa : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ a C^∞ function defined by $\kappa((x_1, \ldots, x_m)) = (x_1, \ldots, x_m, 0, \ldots, 0) \in \mathbb{R}^n$, then for $U \in \mathcal{G}(\mathbb{R}^n)$, we find that $U|\mathbb{R}^m = U \circ \kappa$ since the composition $U \circ \kappa$ has, in view of (4.11), a representative of the form $u(\phi_\varepsilon^{\otimes n}, x_1, \ldots, x_m, 0, \ldots, 0)$. On the other hand, note that the restriction map from $\mathcal{G}(\mathbb{R}^n)$ into $\mathcal{G}(\mathbb{R}^m)$ is *surjective*: if $V \in \mathcal{G}(\mathbb{R}^m)$ and $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a smooth function defined by $\pi((x_1, \ldots, x_n)) = (x_1, \ldots, x_m)$, then the generalized function $U = V \circ \pi \in \mathcal{G}(\mathbb{R}^n)$ has the property $U|\mathbb{R}^m = V$ in $\mathcal{G}(\mathbb{R}^m)$ since the composition of the above two smooth functions $\pi \circ \kappa = \mathrm{id}_{\mathbb{R}^m}$ is the identity mapping of $\mathbb{R}^m$.

(A more general definition of restrictions of generalized functions to linear subspaces was given by Biagioni [15, § 1.3.5]; see also Aragona and Biagioni [8, § 2.4] and Oberguggenberger [156, III.§ 11].)

5. The Space of Schwartz Distributions

In this section, L. Schwartz's linear distribution theory is studied as a continuation of J.-F. Colombeau's nonlinear theory of generalized functions. A distribution is defined as a generalized function which locally (on every relatively compact open subset) is a partial derivative of a continuous function. It should be noted that this concept of distributions is exactly the one used in the classical distribution theory. This approach to distributions proposed by Colombeau [39] (see also Aragona and Biagioni [8]) seems to be one of the most simple and efficient approaches to the distribution theory: indeed, even for the definition of $\mathcal{G}(\Omega)$ one needs only concepts of open and compact subsets of $\mathbb{R}^n$, C^∞ functions of several real variables, rudiments of integration theory, and quotient structures in commutative algebras of smooth functions, and one does not need the theory of locally convex topological vector spaces, which is usually used in the classical approach in order to present any nontrivial part of the distribution theory (there is, however, another sequential approach to the distribution theory [5]). It is certainly not our purpose to present the distribution theory in its entirety; we develop here a relatively small part of it, which (in our opinion) shows the naturalness and the power of the mentioned approach.

5.1. The definition of Schwartz distributions. In (2.9), (2.12), and Proposition 2.4, we have established the coherence of inclusions

$$C^\infty(\Omega) \subset C(\Omega) \subset \mathcal{G}(\Omega).$$

Since the algebra $\mathcal{G}(\Omega)$ is invariant under partial derivatives and since partial derivatives in $\mathcal{G}(\Omega)$ exactly generalize partial derivatives in $C^k(\Omega)$, it is natural that the following elements of $\mathcal{G}(\Omega)$ attract our attention:

Definition 5.1. An element $T \in \mathcal{G}(\Omega)$ is said to be a *distribution* on an open set $\Omega \subset \mathbb{R}^n$ if for every compact set $K \subset\subset \Omega$, there exist a continuous function $f \in C(\Omega)$ and a multi-index $\alpha \in \mathbb{N}_0^n$ such that we have the representation

$$T|_{K^\circ} = (\partial^\alpha \jmath(f))|_{K^\circ} = \partial^\alpha \jmath(f|_{K^\circ}) \quad \text{in} \quad \mathcal{G}(K^\circ) \tag{5.1}$$

on the interior $K^\circ = \mathrm{int}K$ of K, where we have used the imbedding (2.17) and the restriction property (2.19). □

Remark 5.2. Note that if K, f, and α are as in Definition 5.1, $K_1 \subset\subset \Omega$ is such that $K \subset \mathrm{int}K_1 = K_1^\circ$, and $\zeta \in \mathcal{D}(K_1^\circ)$ is such that $\zeta = 1$ on K, and if we set $f_1 = \zeta f$ (so that $f_1 = 0$ on $\mathbb{R}^n \setminus K_1^\circ$), then we have $f_1 \in C_c(K_1^\circ) \subset C_c(\Omega)$, $f_1|_{K^\circ} = f|_{K^\circ}$ in $C(K^\circ)$, and $T|_{K^\circ} = \partial^\alpha \jmath(f_1|_{K^\circ})$ in $\mathcal{G}(K^\circ)$. Thus, the function f in

Definition 5.1 is not uniquely determined, and (which is very convenient) it can be chosen to be compactly supported on any compact set $K_1 \subset \Omega$ which contains K in its interior. $\square$

The set of all distributions on Ω is denoted by (see Sec. 2.4)

$$\mathcal{D}'(\Omega) = \{ T \in \mathcal{G}(\Omega) \mid \forall K \subset\subset \Omega \;\; \exists f \in C_c(\Omega),\ \alpha \in \mathbb{N}_0^n : \\ T = \partial^\alpha \jmath(f) \;\text{ on }\; K^\circ \}. \tag{5.2}$$

Examples 5.3. (1) For $x \in \mathbb{R}$, set $\operatorname{sign} x = x/|x|$ if $x \neq 0$, $\operatorname{sign} 0 = 0$, $x_+ = \max\{0, x\} = (x + |x|)/2$, and $H(x) \equiv \operatorname{sign}^+ x := (\operatorname{sign} x)_+$. The function H is called the *Heaviside function* on $\mathbb{R}$ (the value $H(0)$ can be considered to be undetermined). In view of imbedding (2.12) and (2.10), for representatives of these functions in the algebra $\mathcal{G}(\mathbb{R})$, we have

$$\begin{aligned} u_H(\varphi, x) &= (H * \check{\varphi})(x) = \int_0^\infty \varphi(\lambda - x)\, d\lambda, \\ u_{x_+}(\varphi, x) &= (x_+ * \check{\varphi})(x) = \int_0^\infty \lambda\varphi(\lambda - x)\, d\lambda, \end{aligned} \qquad \varphi \in \mathcal{A}_0(\mathbb{R}), \quad x \in \mathbb{R}.$$

Integrating by parts, we find that

$$\frac{d}{dx} u_{x_+}(\varphi, x) = -\int_0^\infty \lambda\varphi'(\lambda - x)\, d\lambda = \int_0^\infty \varphi(\lambda - x)\, d\lambda = u_H(\varphi, x),$$

and hence $\jmath(H) = \frac{d}{dx}\jmath(x_+)$ in $\mathcal{G}(\mathbb{R})$, or, in short, $H = \frac{d}{dx}x_+$ in $\mathcal{G}(\mathbb{R})$, so that $H \in \mathcal{D}'(\mathbb{R})$. Differentiating the representative of H, we obtain

$$\frac{d^2}{dx^2} u_{x_+}(\varphi, x) = \frac{d}{dx} u_H(\varphi, x) = -\int_0^\infty \varphi'(\lambda - x)\, d\lambda = \varphi(-x) = u_\delta(\varphi, x).$$

Thus, $\delta = \frac{d}{dx}H = \frac{d^2}{dx^2}x_+ \in \mathcal{D}'(\mathbb{R})$. The generalized function $\delta \in \mathcal{G}(\mathbb{R})$ with representative u_δ is called the *Dirac δ function* (or the *Dirac δ distribution*) on $\mathbb{R}$ (see Proposition 2.3(1) and Example 2.8(1)).

(2) Now we extend example (1) to the case of $\mathbb{R}^n$. For $(x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $x_+ = (x_1)_+ \cdots (x_n)_+$ and $H_n(x) = H^{\otimes n}(x) = H(x_1) \cdots H(x_n)$. If u_{x_+} and u_{H_n} are representatives of x_+ and H_n in the algebra $\mathcal{G}(\mathbb{R}^n)$, then by integration by parts, it follows from (2.10) that if $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, $x = (x_1, \ldots, x_n)$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$, then

$$\partial_{x_1} \cdots \partial_{x_n} u_{x_+}(\varphi, x) = \partial_{x_1} \cdots \partial_{x_n} \int_{\mathbb{R}^n} \lambda_+ \varphi(\lambda - x)\, d\lambda =$$

$$= \partial_{x_1} \cdots \partial_{x_n} \int_0^\infty \ldots \int_0^\infty \lambda_1 \cdots \lambda_n \varphi(\lambda_1 - x_1, \ldots, \lambda_n - x_n)\, d\lambda_1 \ldots d\lambda_n =$$

$$= (-1)^n \int_0^\infty \ldots \int_0^\infty \lambda_1 \cdots \lambda_n (\partial_{\lambda_1} \cdots \partial_{\lambda_n} \varphi)(\lambda_1 - x_1, \ldots, \lambda_n - x_n)\, d\lambda_1 \ldots d\lambda_n =$$

$$\overset{\text{(by parts)}}{=} \int_0^\infty \ldots \int_0^\infty \varphi(\lambda - x)\, d\lambda = \int_{\mathbb{R}^n} H_n(\lambda)\varphi(\lambda - x)\, d\lambda = u_{H_n}(\varphi, x).$$

Thus, $H_n = \partial_{x_1} \cdots \partial_{x_n} x_+$ in $\mathcal{G}(\mathbb{R}^n)$, so that $H_n \in \mathcal{D}'(\mathbb{R}^n)$. Differentiating the representative of H_n, we obtain

$$\partial^2_{x_1} \cdots \partial^2_{x_n} u_{x_+}(\varphi, x) = \partial_{x_1} \cdots \partial_{x_n} u_{H_n}(\varphi, x) = \varphi(-x) = u_\delta(\varphi, x);$$

from this we conclude that the *Dirac δ function* $\delta \in \mathcal{G}(\mathbb{R}^n)$, which has the representative $u_\delta(\varphi, x) = \check{\varphi}(x)$ with $(\varphi, x) \in \mathcal{A}_0(\mathbb{R}^n) \times \mathbb{R}^n$, is a *distribution* on $\mathbb{R}^n$. □

Theorem 5.4. (a) $\mathcal{D}'(\Omega)$ *is a linear space (over the field* $\mathbb{K}$*), but not an algebra.*
(b) $\partial^\beta \mathcal{D}'(\Omega) \subset \mathcal{D}'(\Omega) \quad \forall \beta \in \mathbb{N}_0^n$.

Proof. First we prove (b). If $T \in \mathcal{D}'(\Omega)$, and K, f, and α are such that equality (5.1) holds, then $(\partial^\beta T)|_{K^\circ} = \partial^\beta(T|_{K^\circ}) = \partial^\beta(\partial^\alpha \jmath(f|_{K^\circ})) \dot{=} \partial^{\beta+\alpha}\jmath(f|_{K^\circ})$, and it follows that $\partial^\beta T \in \mathcal{D}'(\Omega)$. Before we prove (a), in the first two steps below we will obtain some auxiliary facts (which will be useful for what follows).

1. If $\Omega \subset \mathbb{R}$ is open and $f \in C_c(\Omega)$, then $\operatorname{supp} f \subset [a, \infty)$ for some $a \in \mathbb{R}$. Set $J^0 = \mathrm{id}$, $(Jf)(x) = \int_a^x f(\xi)\, d\xi$, $x \in \mathbb{R}$, and $J^m = J \circ J^{m-1}$, $m \in \mathbb{N}$. If $\partial^m = d^m/dx^m$, then for $m \in \mathbb{N}$, we have $J^m f \in C^m(\mathbb{R})$, $\partial^m(J^m f) = f$ on $\mathbb{R}$, and, moreover, by induction on m, from the integration by parts formula we find that

$$(J^m f)(x) = \frac{1}{(m-1)!} \int_{-\infty}^{x} (x-\xi)^{m-1} f(\xi)\, d\xi = (E_m * f)(x), \quad x \in \mathbb{R}, \quad m \in \mathbb{N},$$

where

$$E_m(x) = \begin{cases} H(x) = \text{the Heaviside function if } m = 1, \\ (x_+)^{m-1}/(m-1)! \text{ if } 2 \le m \in \mathbb{N}. \end{cases} \tag{5.3}$$

2. In the same way, let us show that if $\Omega \subset \mathbb{R}^n$ is open and $f \in C_c(\Omega)$, then

$$\forall \alpha \in \mathbb{N}_0^n \ \exists f_\alpha \in C^\alpha(\mathbb{R}^n) \text{ such that } \partial^\alpha f_\alpha = f \text{ on } \mathbb{R}^n, \tag{5.4}$$

where $C^\alpha(\mathbb{R}^n) = \{ f \in C(\mathbb{R}^n) \mid \exists \partial^\beta f \in C(\mathbb{R}^n)\ \forall \beta \in \mathbb{N}_0^n, 0 \le \beta \le \alpha \}$, and as usual, we write $\beta \le \alpha$ if $\beta_1 \le \alpha_1, \ldots, \beta_n \le \alpha_n$.

Fix $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Since $f \in C_c(\Omega)$, there is $a \in \mathbb{R}$ such that $\operatorname{supp} f \subset L^n = L \times \cdots \times L$ with $L \doteq [a, \infty) \subset \mathbb{R}$. For $j = 1, \ldots, n$, define an operator J_j by

$$(J_j f)(x) = \int_a^{x_j} f(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots, x_n)\, d\xi, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad J_j^0 = \mathrm{id}.$$

It follows that $\partial_{x_j}(J_j f)(x) = f(x)$ for $x \in \mathbb{R}^n$ and $\operatorname{supp}(J_j f) \subset L^n$, so, in view of the arguments in the first step, we have

$$(J_j^{\alpha_j} f)(x) = \int_{-\infty}^{x_j} \frac{(x_j - \xi)^{\alpha_j - 1}}{(\alpha_j - 1)!} f(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots, x_n)\, d\xi,$$

with $\partial_j^{\alpha_j}(J_j^{\alpha_j} f) = f$ on $\mathbb{R}^n$, and $\operatorname{supp}(J_j^{\alpha_j} f) \subset L^n$. If $J = (J_1, \ldots, J_n)$ we set $J^\alpha = J_1^{\alpha_1} \circ \cdots \circ J_n^{\alpha_n}$ (this is correct since, by Fubini's theorem, the operators J_j commute). It follows that if all $\alpha_j \ne 0$, then

$$(J^\alpha f)(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{(x_1 - \xi_1)^{\alpha_1 - 1}}{(\alpha_1 - 1)!} \cdots \frac{(x_n - \xi_n)^{\alpha_n - 1}}{(\alpha_n - 1)!} f(\xi_1, \ldots, \xi_n)\, d\xi_1 \ldots d\xi_n;$$

otherwise, if (without loss of generality) $\alpha_1 \ne 0, \ldots, \alpha_p \ne 0$, $\alpha_{p+1} = \ldots = \alpha_n = 0$, then

$$(J^\alpha f)(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} \frac{(x_1 - \xi_1)^{\alpha_1 - 1}}{(\alpha_1 - 1)!} \cdots \frac{(x_p - \xi_p)^{\alpha_p - 1}}{(\alpha_p - 1)!} f(\xi_1, \ldots, \xi_p, x_{p+1}, \ldots, x_n)\, d\xi_1 \ldots d\xi_p.$$

In other words, $(J^\alpha f)(x) = (E_\alpha * f)(x)$ for all $x \in \mathbb{R}^n$, where

$$E_\alpha(x) = \begin{cases} \prod_{j=1}^n E_{\alpha_j}(x_j), & \text{if } \alpha_j \geq 1,\ j = 1, \ldots, n, \\ \prod_{j=1}^p E_{\alpha_j}(x_j), & \text{if } \alpha_1 \geq 1, \ldots, \alpha_p \geq 1,\ \alpha_{p+1} = \ldots = \alpha_n = 0, \end{cases} \tag{5.5}$$

with the function E_{α_j} defined in (5.3). Noting that $J^\alpha f \in C^\alpha(\mathbb{R}^n)$ and $\partial^\alpha(J^\alpha f) = f$ on $\mathbb{R}^n$, it will suffice to put $f_\alpha = J^\alpha f$ in (5.4).

3. To prove (a), let $T,\ S \in \mathcal{D}'(\Omega)$, $c \in \mathbb{K}$, and let $K \subset\subset \Omega$. By (5.2), there are $f,\ g \in C_c(\Omega)$ and α, $\beta \in \mathbb{N}_0^n$ such that $T = \partial^\alpha \jmath(f)$ and $S = \partial^\beta \jmath(g)$ on K°. Since $cT = \partial^\alpha \jmath(cf)$ on K° with $cf \in C_c(\Omega)$, it is clear that $cT \in \mathcal{D}'(\Omega)$. Let $f_\beta \in C^\beta(\mathbb{R}^n)$, and let $g_\alpha \in C^\alpha(\mathbb{R}^n)$ be as in step 2, that is, $\partial^\beta f_\beta = f$ and $\partial^\alpha g_\alpha = g$ on $\mathbb{R}^n$. Then, using the linearity of the imbedding $\jmath$ and the commutativity of $\jmath$ with partial derivatives, we have on K°

$$T + S = \partial^\alpha \jmath(f) + \partial^\beta \jmath(g) = \partial^\alpha \jmath(\partial^\beta f_\beta) + \partial^\beta \jmath(\partial^\alpha g_\alpha) = \partial^{\alpha+\beta} \jmath(f_\beta + g_\alpha),$$

with $f_\beta + g_\alpha \in C(\mathbb{R}^n) \subset C(\Omega)$. Thus, $T + S \in \mathcal{D}'(\Omega)$.

4. It will be shown later (Example 8.8(4)) that $\delta^2 \notin \mathcal{D}'(\mathbb{R}^n)$, where, as usual, δ is the Dirac δ function on $\mathbb{R}^n$. □

It is clear that if $f \in C(\Omega)$ or $f \in L^1_{\mathrm{loc}}(\Omega)$, then $\jmath(f) \in \mathcal{D}'(\Omega)$.

Denote by $\mathcal{E}'(\Omega)$ the space of distributions with compact supports,

$$\mathcal{E}'(\Omega) = \{\, T \in \mathcal{D}'(\Omega) \mid \operatorname{supp} T \subset\subset \Omega \,\}.$$

From the proof of Proposition 2.7(a) it is clear that $\mathcal{E}'(\Omega)$ is a linear subspace of $\mathcal{D}'(\Omega)$, and we have the natural inclusion maps:

$$\mathcal{D}(\Omega) \subset C_c(\Omega) \subset \mathcal{E}'(\Omega) \subset \mathcal{G}_c(\Omega).$$

Examples 5.3 and 2.8(1) imply $\delta \in \mathcal{E}'(\mathbb{R}^n)$. Note also that the space $\mathcal{E}'(\Omega)$ is invariant with respect to partial derivatives:

$$\partial^\alpha \mathcal{E}'(\Omega) \subset \mathcal{E}'(\Omega), \quad \alpha \in \mathbb{N}_0^n.$$

The basic result for the sequel is the following theorem, which extends Proposition 3.8(a):

Theorem 5.5. *Let $T \in \mathcal{D}'(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$ or $T \in \mathcal{E}'(\Omega)$ and $\psi \in C^\infty(\Omega)$. Then there is a number in $\mathbb{K}$ denoted by $T(\psi) = \langle T, \psi \rangle$ such that*

$$\int_\Omega (T \cdot \psi)(x)\, dx = \langle T, \psi \rangle \quad \text{in} \quad \overline{\mathbb{K}}, \tag{5.6}$$

where the equality is understood in the sense of the convention in Sec. 3.1, and the dot $\cdot$ denotes the product in $\mathcal{G}(\Omega)$. In particular, in view of Proposition 3.8(a), we have

$$\langle f, \psi \rangle = \int_\Omega f(x)\psi(x)\, dx, \quad f \in C(\Omega) \text{ or } L^1_{\mathrm{loc}}(\Omega), \quad \psi \in \mathcal{D}(\Omega). \tag{5.7}$$

Remark. In this theorem, we distinguish the generalized number at the left-hand side in (5.6) from the classical number $\langle T, \psi \rangle$. Equality (5.6) means that $\langle T, \psi \rangle$ is a representative in $\overline{\mathbb{K}}$ of the generalized number on the left-hand side, so, for brevity, we shall assume that this equality is the definition of the number $\langle T, \psi \rangle$; this convention will not cause any trouble in the sequel. □

Proof. 1. First we prove an auxiliary statement, namely, we show that

$$\begin{gathered} \text{if } U_1,\ U_2 \in \mathcal{G}(\Omega),\ V \in \mathcal{G}_c(\Omega), \text{ and } G \subset \Omega \text{ is an open set such that} \\ \operatorname{supp} V \subset G, \text{ then the equality } U_1|_G = U_2|_G \text{ in } \mathcal{G}(G) \text{ implies} \\ \int_\Omega (U_1 \cdot V)(x)\, dx = \int_\Omega (U_2 \cdot V)(x)\, dx \quad \text{in} \quad \overline{\mathbb{K}}. \end{gathered} \tag{5.8}$$

In fact, let $K \subset\subset G$ be such that $\operatorname{supp} V \subset K^\circ \equiv \operatorname{int} K$. By Proposition 2.7(a), $\operatorname{supp}(U_1 \cdot V) \subset K^\circ$ and $\operatorname{supp}(U_2 \cdot V) \subset K^\circ$, so that, in view of (3.8) and the local property of the integral (see Sec. 3.3), in $\overline{\mathbb{K}}$, we have the following chain of equalities:

$$\int_\Omega U_1 \cdot V = \int_K U_1 \cdot V = \int_K (U_1 \cdot V)|_G = \int_K (U_1|_G) \cdot (V|_G) =$$

$$= \int_K (U_2|_G) \cdot (V|_G) = \int_K (U_2 \cdot V)|_G = \int_K U_2 \cdot V = \int_\Omega U_2 \cdot V.$$

2. Assume that $T \in \mathcal{D}'(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$. By (3.10), the integral in (5.6) is a well-defined generalized number from $\overline{\mathbb{K}}$. Fix a compact set $K \subset \Omega$ such that $\operatorname{supp} \psi \subset K^\circ$. From the definition of the distribution $T \in \mathcal{D}'(\Omega)$, for the compact set K, there are a function $f \in C_c(\Omega)$ and a multi-index $\alpha \in \mathbb{N}_0^n$ such that $T|_{K^\circ} = \partial^\alpha \jmath(f)|_{K^\circ}$ in $\mathcal{G}(K^\circ)$. Using (5.8) and the integration by parts formula (3.9), we have

$$\int_\Omega (T \cdot \psi)(x)\, dx = \int_\Omega (\partial^\alpha \jmath(f) \cdot \psi)(x)\, dx = (-1)^{|\alpha|} \int_\Omega (\jmath(f) \cdot (\partial^\alpha \psi))(x)\, dx =$$

(note that $\partial^\alpha \psi \in \mathcal{D}(\Omega)$ and use Proposition 3.8(a))

$$= (-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \psi)(x)\, dx =: \langle T, \psi \rangle \in \mathbb{K} \quad \text{in} \quad \overline{\mathbb{K}}; \tag{5.9}$$

this completes the proof in our case.

3. Let now $T \in \mathcal{E}'(\Omega)$, and let $\psi \in C^\infty(\Omega)$. If $\zeta \in \mathcal{D}(\Omega)$ is such that $\zeta = 1$ in a neighborhood of $\operatorname{supp} T$, then since $T \in \mathcal{G}_c(\Omega)$, from Proposition 2.7(b), we have $T = \zeta \cdot T = T \cdot \zeta$ in $\mathcal{G}(\Omega)$; hence we have

$$\int_\Omega (T \cdot \psi)(x)\, dx = \int_\Omega ((T \cdot \zeta) \cdot \psi)(x)\, dx = \int_\Omega (T \cdot (\zeta \cdot \psi))(x)\, dx = \int_\Omega (T \cdot (\zeta\psi))(x)\, dx \quad \text{in } \overline{\mathbb{K}}; \tag{5.10}$$

in the last equality we have used (2.9) and the equality $\zeta \cdot \psi = \zeta\psi$ in $\mathcal{G}(\Omega)$, where $\zeta\psi$ is the classical product of C^∞ functions ζ and ψ. Since $\zeta\psi \in \mathcal{D}(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, due to step 2 and (5.10), the integral $\int_\Omega (T \cdot \psi)(x)\, dx$ is an ordinary classical number (in the sense of the convention in Sec. 3.1). The proof is completed. □

Remark. The formulas (5.6) and (5.10) imply that if $T \in \mathcal{E}'(\Omega)$, $\psi \in C^\infty(\Omega)$, and if $\zeta \in \mathcal{D}(\Omega)$ is such that $\zeta = 1$ in a neighborhood of $\operatorname{supp} T$, then the following equality holds:

$$\langle T, \psi \rangle = \langle T, \zeta\psi \rangle \quad \text{in} \quad \mathbb{K}, \tag{5.11}$$

which in the classical distribution theory is taken as the definition of the number $\langle T, \psi \rangle$. □

Let us show by examples the applications of Theorem 5.5 and Proposition 3.8(a).

Examples 5.6. (1) If $\delta \in \mathcal{D}'(\mathbb{R}^n)$ is the Dirac δ function (see Example 5.3(2)), then

$$\langle \partial^\alpha \delta, \psi \rangle = \int_{\mathbb{R}^n} (\partial^\alpha \delta \cdot \psi)(x)\, dx = (-1)^{|\alpha|} (\partial^\alpha \psi)(0) \quad \forall \psi \in \mathcal{D}(\mathbb{R}^n), \tag{5.12}$$

and in particular,

$$\langle \delta, \psi \rangle = \int_{\mathbb{R}^n} (\delta \cdot \psi)(x)\, dx = \psi(0), \quad \psi \in \mathcal{D}(\mathbb{R}^n). \tag{5.13}$$

In fact, from Example 5.3(2) we know that $\delta = \partial_1 \ldots \partial_n H_n$ in $\mathcal{G}(\mathbb{R}^n)$, so in view of the integration by parts formula and Proposition 3.8(a), we find that

$$\int_{\mathbb{R}^n} (\partial^\alpha \delta \cdot \psi)(x)\, dx = \int_{\mathbb{R}^n} (\partial^\alpha \partial_1 \cdots \partial_n \jmath(H_n) \cdot \psi)(x)\, dx =$$

$$= (-1)^{|\alpha|+n} \int_{\mathbb{R}^n} (\jmath(H_n) \cdot (\partial_1 \cdots \partial_n \partial^\alpha \psi))(x)\, dx = (-1)^{|\alpha|+n} \int_{\mathbb{R}^n} H_n(x) \partial_1 \cdots \partial_n (\partial^\alpha \psi)(x)\, dx =$$

$$= (-1)^{|\alpha|+n} \int_0^\infty \!\!\dots\! \int_0^\infty \frac{\partial^n (\partial^\alpha \psi)(x_1, \dots, x_n)}{\partial x_1 \cdots \partial x_n}\, dx_1 \dots dx_n = (-1)^{|\alpha|} (\partial^\alpha \psi)(0).$$

(2) From (5.13) we have $\delta \notin \jmath(L^1_{\text{loc}}(\mathbb{R}^n))$; that is, the Dirac δ function is not "generated" by a locally integrable function. In fact, if $\delta = \jmath(f)$ for some function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x)\psi(x)\, dx = \int_{\mathbb{R}^n} (\jmath(f) \cdot \psi)(x)\, dx = \int_{\mathbb{R}^n} (\delta \cdot \psi)(x)\, dx = \psi(0) \quad \forall \psi \in \mathcal{D}(\mathbb{R}^n); \tag{5.14}$$

in particular, this implies

$$\int_{\mathbb{R}^n} (|x|^2 f(x))\psi(x)\, dx = \int_{\mathbb{R}^n} f(x)(|x|^2 \psi(x))\, dx = |x|^2 \psi(x)|_{x=0} = 0.$$

Since ψ is arbitrary, the latter equality yields $|x|^2 f(x) = 0$ for almost all $x \in \mathbb{R}^n$, so that $f = 0$ almost everywhere on $\mathbb{R}^n$. However, the latter property is noncompatible with (5.14). □

5.2. The classical presentation of distributions. Now we can establish the equivalence of Definition 5.1 and the classical definition from the distribution theory. To this end, let us recall the concept of *bounded sets* in $\mathcal{D}(\Omega)$:

Definition 5.7. A set of functions $B \subset \mathcal{D}(\Omega)$ is said to be *bounded* if there are a compact set $K \subset \Omega$ and a sequence $\{M_n\}_{n=1}^\infty \subset (0, \infty)$ such that

(a) $\forall \psi \in B$, $\operatorname{supp} \psi \subset K$;

(b) $\forall \psi \in B\ \forall \alpha \in \mathbb{N}_0^n$, $\sup_{x \in \mathbb{R}^n} |(\partial^\alpha \psi)(x)| \le M_{|\alpha|}$. □

Theorem 5.8. *Let $T \in \mathcal{D}'(\Omega)$ (be fixed). Then the linear mapping $L_T : \mathcal{D}(\Omega) \longrightarrow \mathbb{K}$ defined by $L_T(\psi) = \langle T, \psi \rangle$ for all $\psi \in \mathcal{D}(\Omega)$ is* bounded *in the sense that if B is a bounded set in $\mathcal{D}(\Omega)$, then its image $L_T(B)$ is a bounded set in $\mathbb{K}$.*

Proof. The linearity of L_T follows from (5.6) and the linearity of the integral. Let B be a bounded set in $\mathcal{D}(\Omega)$, let $K \subset\subset \Omega$, and let $\{M_n\}_{n=1}^\infty$ be as in (a) and (b) of Definition 5.7. If $K_1 \subset\subset \Omega$ and $K \subset K_1^\circ$, there are $f \in C_c(\Omega)$ and $\alpha \in \mathbb{N}_0^n$ such that $T = \partial^\alpha \jmath(f)$ on K_1°. In view of the equality (5.9), for all $\psi \in B$, we have

$$|\langle T, \psi \rangle| = |(-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \psi)(x)\, dx| \le \int_K |f(x)||(\partial^\alpha \psi)(x)|\, dx \le M_{|\alpha|} \int_K |f(x)|\, dx,$$

and the theorem follows. □

The next result shows that every distribution T is completely characterized by the bounded linear mapping L_T:

Theorem 5.9. *If $T \in \mathcal{D}'(\Omega)$, then the mapping $u_T : \mathcal{A}_0(\mathbb{R}^n) \times \Omega \longrightarrow \mathbb{K}$ defined by*

$$u_T(\varphi, x) = \int_\Omega (T \cdot \ell(\varphi) \cdot \tau_x \varphi)(\lambda)\, d\lambda = \langle T, \ell(\varphi)\tau_x \varphi \rangle, \quad (\varphi, x) \in \mathcal{A}_0(\mathbb{R}^n) \times \Omega, \tag{5.15}$$

is in $\mathcal{E}_M[\Omega]$ and is a representative of the distribution T in $\mathcal{G}(\Omega)$; here the dot $\cdot$ denotes the product in $\mathcal{G}(\Omega)$, and the function $\ell(\varphi) \in \mathcal{D}(\Omega)$ is defined in (2.16).

In particular, if $\Omega = \mathbb{R}^n$, *then* $T \in \mathcal{D}'(\mathbb{R}^n)$ *also has the mapping*

$$u_T(\varphi, x) = \int_{\mathbb{R}^n} (T \cdot \tau_x\varphi)(\lambda)\, d\lambda = \langle T, \tau_x\varphi\rangle, \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \tag{5.16}$$

as its representative.

Proof. By Theorem 5.5, the mapping u_T is well defined since, given $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ and $x \in \Omega$, we have

$$\int_{\Omega} (T \cdot \ell(\varphi) \cdot \tau_x\varphi)(\lambda)\, d\lambda = \int_{\Omega} (T \cdot (\ell(\varphi)\tau_x\varphi))(\lambda)\, d\lambda = \langle T, \ell(\varphi)\tau_x\varphi\rangle \in \mathbb{K}.$$

Let $K_1 \subset\subset \Omega$ be such that $\operatorname{supp} \ell(\varphi) \subset K_1^\circ$. Since $T \in \mathcal{D}'(\Omega)$, there are $f \in C_c(\Omega)$ and $\alpha \in \mathbb{N}_0^n$ such that $T = \partial^\alpha \jmath(f)$ on K_1°, and in view of (5.9), it follows that

$$u_T(\varphi, x) = \langle T, \ell(\varphi)\tau_x\varphi\rangle = (-1)^{|\alpha|} \int_{\Omega} f(\lambda)\partial_\lambda^\alpha(\ell(\varphi)\tau_x\varphi)(\lambda)\, d\lambda, \quad x \in \Omega. \tag{5.17}$$

From here and from Proposition 1.2 we readily have that $u_T(\varphi, \cdot) \in C^\infty(\Omega)$.

Let us show now that $u_T \in \mathcal{E}_M[\Omega]$. Fix a compact set $K \subset\subset \Omega$ and choose $K_1 \subset\subset \Omega$ such that $K \subset K_1^\circ$ and $T = \partial^\alpha \jmath(f)$ on K_1° for some $f \in C_c(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. Given $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, by properties of $\ell(\varphi)$ (cf. Sec. 2.3), there is a number $\eta = \eta(K, K_1, \varphi) > 0$ such that $K_{\varepsilon\rho(\varphi)} := K + B_{\varepsilon\rho(\varphi)} \subset K_1^\circ \subset \Omega(\check{\varphi}_\varepsilon)$ and $\ell(\varphi_\varepsilon) = 1$ on $K_{\varepsilon\rho(\varphi)}$ for all $\varepsilon \in (0, \eta)$. From (5.17), for all $x \in K$ and $\varepsilon \in (0, \eta)$, we have

$$\begin{aligned} u_T(\varphi_\varepsilon, x) &= (-1)^{|\alpha|} \int_{K_{\varepsilon\rho(\varphi)}} f(\lambda)\partial_\lambda^\alpha(\tau_x\varphi_\varepsilon)(\lambda)\, d\lambda = \int_{K_{\varepsilon\rho(\varphi)}} f(\lambda)\partial_x^\alpha(\tau_x\varphi_\varepsilon)(\lambda)\, d\lambda = \\ &= \partial_x^\alpha \int f(\lambda)(\tau_x\varphi_\varepsilon)(\lambda)\, d\lambda = \partial_x^\alpha(f * \check{\varphi}_\varepsilon)(x) = \partial_x^\alpha u_f(\varphi_\varepsilon, x). \end{aligned} \tag{5.18}$$

Since $u_f \in \mathcal{E}_M[\Omega]$, in view of (2.18) and (2.11), we obtain $u_T \in \mathcal{E}_M[\Omega]$.

On the other hand, if we fix $K_1 \subset\subset \Omega$, then the above arguments also show that if $U_T \in \mathcal{G}(\Omega)$ is a generalized function with the representative u_T, then

$$U_T|_{K_1^\circ} = T|_{K_1^\circ} \quad \text{in} \quad \mathcal{G}(K_1^\circ) \quad \text{for all} \quad K_1 \subset\subset \Omega, \tag{5.19}$$

since the equality $u_T(\varphi_\varepsilon, x) = \partial^\alpha u_f(\varphi_\varepsilon, x)$, $x \in K$, $\varepsilon \in (0, \eta)$, which was proved above for all $K \subset\subset K_1^\circ$, means that $u_T - \partial^\alpha u_f \in \mathcal{N}[K_1^\circ]$, where $\partial^\alpha u_f$ is a representative of T on K_1°. From Theorem 2.6(c) and (5.19) it follows that $U_T = T$ in $\mathcal{G}(\Omega)$. Q.E.D. □

Note that we have simultaneously proved the following property of the representative u_T of a distribution $T \in \mathcal{D}'(\Omega)$ which is analogous to (2.18):

$$\begin{aligned} &\forall K \subset\subset \Omega,\ \forall \varphi \in \mathcal{A}_0(\mathbb{R}^n)\ \exists f \in C_c(\Omega),\ \alpha \in \mathbb{N}_0^n,\ \eta > 0, \text{ such that} \\ &K \subset \Omega(\check{\varphi}_\varepsilon) \text{ and } u_T(\varphi_\varepsilon, x) = \partial_x^\alpha u_f(\varphi_\varepsilon, x) = \partial_x^\alpha(f * \check{\varphi}_\varepsilon),\ x \in K,\ \varepsilon \in (0, \eta). \end{aligned} \tag{5.20}$$

Corollary 5.10. *Let* $T, T_1, T_2 \in \mathcal{D}'(\Omega)$. *Then*

(a) $T = 0$ *in* $\mathcal{G}(\Omega) \iff u_T \in \mathcal{N}[\Omega]$;

(b) $T_1 = T_2$ *in* $\mathcal{G}(\Omega) \iff \langle T_1, \psi\rangle = \langle T_2, \psi\rangle$ *in* $\mathbb{K}$ $\forall \psi \in \mathcal{D}(\Omega)$.

Proof. Both statements follow immediately from Theorem 5.9. □

We note that if a distribution T is a continuous function, then its representative from (5.15) coincides with the representative of a continuous function from (2.17); in fact, if $f \in C(\Omega)$ (or $f \in L^1_{loc}(\Omega)$) and $T = \jmath(f) \in \mathcal{D}'(\Omega)$, then by Proposition 3.8(a), we have

$$\int_\Omega (\jmath(f) \cdot \ell(\varphi) \cdot \tau_x\varphi)(\lambda)\, d\lambda = \int_\Omega \jmath(f) \cdot (\ell(\varphi)\tau_x\varphi)(\lambda)\, d\lambda = \int_\Omega f(\lambda)(\ell(\varphi)\tau_x\varphi)(\lambda)\, d\lambda =$$

$$= \int_\Omega (f\ell(\varphi))(\lambda)(\tau_x\varphi)(\lambda)\, d\lambda = ((\ell(\varphi)f) * \check{\varphi})(x) = u_f(\varphi, x).$$

Hence, the imbedding $\jmath$ from (2.17) can be extended from the space $C(\Omega)$ of continuous functions to the space of distributions $\mathcal{D}'(\Omega)$. Thus, for functions $f \in L^1_{loc}(\Omega)$, we can and will write $f \in \mathcal{G}(\Omega)$ instead of $\jmath(f) \in \mathcal{G}(\Omega)$. Moreover, by the integration by parts formula, we have the equality

$$\partial_x^\alpha u_T(\varphi, x) = u_{\partial^\alpha T}(\varphi, x), \quad T \in \mathcal{D}'(\Omega), \quad \alpha \in \mathbb{N}_0^n, \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n), \quad x \in \Omega.$$

We say that two distributions $T_1, T_2 \in \mathcal{D}'(\Omega)$ are *equal in* $\mathcal{D}'(\Omega)$, and write $T_1 = T_2$ in $\mathcal{D}'(\Omega)$, if $\langle T_1, \psi\rangle = \langle T_2, \psi\rangle$ in $\mathbb{K}$ for all $\psi \in \mathcal{D}(\Omega)$. In view of this fact, Corollary 5.10(b) claims that ($T_1 = T_2$ in $\mathcal{G}(\Omega)$) $\iff$ ($T_1 = T_2$ in $\mathcal{D}'(\Omega)$).

Denote by $L(\mathcal{D}(\Omega)) = L(\mathcal{D}(\Omega); \mathbb{K})$ the *linear* space (with natural operations) of *bounded* (in the sense of Theorem 5.8) *linear* mappings from $\mathcal{D}(\Omega)$ into $\mathbb{K}$. Then in view of Theorem 5.8, we have the following inclusion mapping:

$$J : \mathcal{D}'(\Omega) \longrightarrow L(\mathcal{D}(\Omega)), \quad J(T) = L_T = \langle T, \cdot\rangle \quad \text{for} \quad T \in \mathcal{D}'(\Omega).$$

The mapping J is linear (due to the linearity of the integral, see (5.6)) and injective (by Corollary 5.10(b)). In the classical presentation, a distribution is defined as an element of the space $L(\mathcal{D}(\Omega))$, and then one proves that any distribution locally is a partial derivative of a continuous function in the sense (5.2) [179, § III.6; 176, Thm. 6.28]. This result shows that a mapping J is also *surjective*. In the proof of this result, one uses the Hahn–Banach theorem in the context of infinite-dimensional topological vector spaces (TVS). We note that in the presentation of distributions starting from the Colombeau's algebra of generalized functions $\mathcal{G}(\Omega)$, no TVS are used at all.

So, we have shown that distributions defined within the framework of $\mathcal{G}(\Omega)$ exactly coincide with classical distributions. In the next section, we will consider some classical properties of distributions which supplement the material of the present section.

6. Classical Properties of Distributions

In this section, we present the classical properties of the Schwartz distributions, many of which are definitions in the distribution theory. Here we touch upon a rather small part of the distribution theory as a continuation of the study of generalized functions from $\mathcal{G}(\Omega)$. The properties we consider allow us to exhibit some features in the construction of the Colombeau algebra.

6.1. Distributions as continuous linear functionals on $\mathcal{D}(\Omega)$. If $T \in \mathcal{D}'(\Omega)$, then

$$\langle T, c_1\psi_1 + c_2\psi_2\rangle = c_1\langle T, \psi_1\rangle + c_2\langle T, \psi_2\rangle, \quad c_1, c_2 \in \mathbb{K}, \quad \psi_1, \psi_2 \in \mathcal{D}(\Omega);$$

$$\forall K \subset\subset \Omega \ \exists C > 0, \ k \in \mathbb{N}_0 : |\langle T, \psi\rangle| \le C \sum_{|\alpha| \le k} \sup_K |\partial^\alpha \psi|, \quad \psi \in \mathcal{D}(K). \tag{6.1}$$

The first property (the linearity of T) follows from the linearity of integral (5.6); the second property (the continuity of T) is an immediate consequence of representation (5.9). The continuity property guarantees that a distribution behaves well when applied to functions smoothly depending on parameters, that is, the following analog of Proposition 1.2 holds: under the hypotheses of this proposition, if $T \in \mathcal{D}'(Y)$ and $F(x) = \langle T, \Phi(x, \cdot)\rangle = \int\limits_Y (T(y) \cdot \Phi(x,y))\, dy$, $x \in \Omega$, then $F : X \longrightarrow \mathbb{K}$, $F \in C^\infty(X)$, and $\partial_x^\alpha F(x) = \langle T, \partial_x^\alpha \Phi(x, \cdot)\rangle$ for all $x \in \Omega$ and $\alpha \in \mathbb{N}_0^n$.

The *convolution of a distribution* $T \in \mathcal{D}'(\mathbb{R}^n)$ *and a function* $\varphi \in \mathcal{D}(\mathbb{R}^n)$ (or $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in C^\infty(\mathbb{R}^n)$) is defined by

$$(T * \varphi)(x) := \langle T, \tau_x \check{\varphi}\rangle = \int\limits_{\mathbb{R}^n} (T \cdot \tau_x \check{\varphi})(\lambda)\, d\lambda, \quad x \in \mathbb{R}^n, \tag{6.2}$$

so that (see (1.7) and (5.16))

$$T * \varphi \in C^\infty(\mathbb{R}^n) \quad \text{and} \quad \partial^\alpha (T * \varphi) = T * (\partial^\alpha \varphi) = (\partial^\alpha T) * \varphi \text{ for all } \alpha \in \mathbb{N}_0^n. \tag{6.3}$$

Analogously to (1.6) and (1.7), one can define the convolution of a distribution $T \in \mathcal{D}'(\Omega)$ and a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ on the open set $\Omega(\check{\varphi}) \subset \Omega$.

The continuity property of a distribution can be reformulated as a sequential continuity. We say that a sequence of functions $\{\varphi_\nu\} = \{\varphi_\nu\}_{\nu=1}^\infty \subset \mathcal{D}(\Omega)$ *converges in* $\mathcal{D}(\Omega)$ *to a function* $\varphi \in \mathcal{D}(\Omega)$ if there is a compact set $K \subset\subset \Omega$ such that $\operatorname{supp} \varphi_\nu \subset K$ for all $\nu \in \mathbb{N}$ and $\sup_\Omega |\partial^\alpha(\varphi_\nu - \varphi)| \longrightarrow 0$ as $\nu \longrightarrow \infty$ for all $\alpha \in \mathbb{N}_0^n$ (this is written as $\varphi_\nu \longrightarrow \varphi$ in $\mathcal{D}(\Omega)$). Taking into account the continuity property (6.1) and the definition of the convergence in $\mathcal{D}(\Omega)$, we obtain at once that if $\varphi_\nu \longrightarrow \varphi$ in $\mathcal{D}(\Omega)$, then $\langle T, \varphi_\nu\rangle \longrightarrow \langle T, \varphi\rangle$ as $\nu \longrightarrow \infty$. Let us give another example of a convergence in $\mathcal{D}(\mathbb{R}^n)$. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$, where the unit is on the jth place, then

$$\frac{1}{h}(\tau_{x+he_j}\varphi - \tau_x \varphi) \longrightarrow -\tau_x(\partial_j \varphi) \quad \text{in} \quad \mathcal{D}(\mathbb{R}^n) \quad \text{as} \quad \mathbb{R} \ni h \longrightarrow 0. \tag{6.4}$$

6.2. Differentiation of distributions. If $T \in \mathcal{D}'(\Omega)$, $\psi \in \mathcal{D}(\Omega)$, and $\alpha \in \mathbb{N}_0^n$, then we have the formula

$$\langle \partial^\alpha T, \psi\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha \psi\rangle, \tag{6.5}$$

which follows from the integration by parts (3.9) for generalized functions

$$\langle \partial^\alpha T, \psi\rangle = \int\limits_\Omega (\partial^\alpha T) \cdot \psi = (-1)^{|\alpha|} \int\limits_\Omega T \cdot (\partial^\alpha \psi) = (-1)^{|\alpha|}\langle T, \partial^\alpha \psi\rangle.$$

Note that formula (5.12) is a consequence of (5.13) and the above rule for the differentiation of distributions:

$$\langle \partial^\alpha \delta, \psi\rangle = (-1)^{|\alpha|}\langle \delta, \partial^\alpha \psi\rangle = (-1)^{|\alpha|}(\partial^\alpha \psi)(0).$$

It is clear that partial derivatives of distributions generalize the corresponding derivatives in the space $C^k(\Omega)$ for $k \in \mathbb{N}$.

6.3. Restriction of distributions to open subsets. Let $T \in \mathcal{D}'(\Omega)$, and let $G \subset \Omega$ be an open subset. By definition (5.2), it is clear that $T|_G \in \mathcal{D}'(G)$, and, moreover, we have

$$\langle T|_G, \psi\rangle = \langle T, \psi\rangle \quad \forall \psi \in \mathcal{D}(G),$$

since, using the local property of the integral (see Sec. 3.3), which is marked in the equalities below by "loc," for $\psi \in \mathcal{D}(G)$, we have

$$\langle T|_G, \psi\rangle = \int\limits_G (T|_G) \cdot \psi \overset{(3.10)}{=} \int\limits_{\operatorname{supp}\psi} (T|_G) \cdot \psi \overset{\text{loc}}{=} \int\limits_{\operatorname{supp}\psi} T \cdot \psi \overset{(3.10)}{=} \int\limits_\Omega T \cdot \psi = \langle T, \psi\rangle.$$

The above formula for the restriction of distributions is taken as the definition in the distribution theory. We say that two distributions T_1, $T_2 \in \mathcal{D}'(\Omega)$ are *equal on* G (and write $T_1 = T_2$ on G) if $\langle T_1|_G, \psi\rangle = \langle T_2|_G, \psi\rangle$

$\forall \psi \in \mathcal{D}(G)$. We have shown just now that $T_1 = T_2$ on G iff $\langle T_1, \psi\rangle = \langle T_2, \psi\rangle$ for all $\psi \in \mathcal{D}(G)$. From Corollary 5.10(b) we conclude that $T_1 = T_2$ on G iff $T_1|_G = T_2|_G$ in $\mathcal{G}(G)$. In the classical distribution theory, the support $\operatorname{supp} T$ of a distribution T is defined by

$$\operatorname{supp} T = \Omega \setminus \Omega_0'(T), \quad \text{where} \quad \Omega_0'(T) = \cup\{\, G \subset \Omega \mid G \text{ is open, and } T = 0 \text{ on } G\,\}.$$

From the above and (2.20), it follows that the definition of the support of a distribution in the sense of $\mathcal{G}(\Omega)$ coincides with the classical definition from the distribution theory (see also Sec. 6.5 below). For instance, if δ is the Dirac δ function on $\mathbb{R}^n$, then from the formula (5.13) and the definition of the support of a distribution it follows immediately that $\operatorname{supp}\delta = \{0\}$.

6.4. Multiplication of a distribution by a smooth function. Let $a \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. In the distribution theory [179, § V.1], Schwartz proposed the following definition of the product $aT = Ta \in \mathcal{D}'(\Omega)$ (here there is no dot between a and T!):

$$\langle aT, \psi\rangle = \langle T, a\psi\rangle, \quad \psi \in \mathcal{D}(\Omega). \tag{6.6}$$

In other words, if a representative of the generalized function $aT \in \mathcal{G}(\Omega)$ is defined according to (5.15) by the formula

$$u_{aT}(\varphi, x) = \langle T, a\,\ell(\varphi)\tau_x\varphi\rangle, \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n), \quad x \in \Omega,$$

then actually $aT \in \mathcal{D}'(\Omega)$, and the equality (6.6) holds. In view of the analog of Proposition 1.2 (see Sec.6.1), it is clear that $u_{aT} \in \mathcal{E}[\Omega]$. To prove that u_{aT} is a moderate element, let $K, K_1 \subset\subset \Omega$ be such that $K \subset K_1^\circ$, and let $f \in C_c(\Omega)$ and $\alpha \in \mathbb{N}_0^n$ be such that $T = \partial^\alpha \jmath(f)$ on K_1°. It follows that if $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, then there exists $\eta = \eta(K, K_1, \varphi) > 0$ such that, analogous to (5.17) and (5.18) with regard to Leibnitz's rule, we obtain

$$u_{aT}(\varphi_\varepsilon, x) = \sum_{0\le\beta\le\alpha} \partial_x^\beta (f_\beta * \check{\varphi}_\varepsilon)(x) = \sum_{0\le\beta\le\alpha} \partial_x^\beta u_{f_\beta}(\varphi_\varepsilon, x), \quad x \in K, \quad \varepsilon \in (0,\eta), \tag{6.7}$$

where $f_\beta = (-1)^{|\alpha|-|\beta|}\binom{\alpha}{\beta} f\,\partial^{\alpha-\beta}a \in C_c(\Omega)$. Hence $u_{aT} \in \mathcal{E}_M[\Omega]$, and simultaneously (see step 3 in the proof of Theorem 5.4(a)), $aT \in \mathcal{D}'(\Omega)$. Now let $\psi \in \mathcal{D}(\Omega)$ be such that $\operatorname{supp}\psi \subset K^\circ$; then using the fact that $T = \partial^\alpha \jmath(f)$ on K_1°, and also (6.7) and (5.9), we have

$$\begin{aligned} \langle aT, \psi\rangle &= \int_\Omega (aT)\cdot\psi = \sum_{0\le\beta\le\alpha} (-1)^{|\beta|} \int_\Omega f_\beta(\partial^\alpha\psi) = \\ &= (-1)^{|\alpha|} \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} \int_\Omega f\,(\partial^{\alpha-\beta}a)(\partial^\beta\psi) = (-1)^{|\alpha|} \int_\Omega f\,(\partial^\alpha(a\psi)) = \langle T, a\psi\rangle. \end{aligned} \tag{6.8}$$

As an example, consider the product of the Dirac δ function $\delta \in \mathcal{D}'(\mathbb{R})$ and a function $a \in C^\infty(\mathbb{R})$. Due to (5.16) and (5.13), a representative of $a\delta \in \mathcal{D}'(\mathbb{R}^n)$ is of the form

$$u_{a\delta}(\varphi, x) = \langle \delta, a\,\tau_x\varphi\rangle = (a\,\tau_x\varphi)(0) = a(0)\varphi(-x) = a(0)u_\delta(\varphi, x)$$

for all $\varphi \in \mathcal{A}_0(\mathbb{R})$ and $x \in \mathbb{R}$, so that we have $a\,\delta = a(0)\delta$ in $\mathcal{D}'(\mathbb{R})$. In particular, if $a(x) = x^m$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, then

$$x^m\delta = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}), \quad \text{whereas} \quad x^m\cdot\delta \neq 0 \quad \text{in} \quad \mathcal{G}(\mathbb{R}) \tag{6.9}$$

(see Example 2.8(2)). Nevertheless, the products $x^m\delta$ and $x^m\cdot\delta$ are related by means of a weaker equality (in the sense of generalized distributions, see Theorem 8.3, and (8.1)).

It is interesting to compare Proposition 3.12 with the following result:

Proposition 6.1. *Let $T \in \mathcal{D}'(\mathbb{R})$, then the equality $xT = 0$ in $\mathcal{D}'(\mathbb{R})$ is equivalent to $T = c\,\delta$ with some constant $c \in \mathbb{K}$.*

Proof. Since $xT = 0$, we have $\langle T, x\varphi\rangle = \langle xT, \varphi\rangle = 0$ $\forall\varphi \in \mathcal{D}(\mathbb{R})$. Fix $\zeta \in \mathcal{D}(\mathbb{R})$ such that $\zeta(0) = 1$. Any function $\psi \in \mathcal{D}(\mathbb{R})$ is represented in the form $\psi = \psi(0)\zeta + \chi$, where $\chi = \psi - \psi(0)\zeta$, so that $\chi(0) = 0$. It follows that

$$\langle T, \psi\rangle = \psi(0)\,\langle T, \zeta\rangle + \langle T, x\,\frac{\chi(x)}{x}\rangle = \langle T, \zeta\rangle\,\psi(0) = \langle c\delta, \psi\rangle,$$

with $c = \langle T, \zeta\rangle$. (This proposition can be generalized, cf. [179, § V.4; 89, Thm. 3.1.16].) □

Consider a continuous function $f(x) = x(\log|x| - 1)$, $x \in \mathbb{R}$, where $\log x = \int_1^x \frac{dt}{t}$ for $x > 0$, and at $x = 0$ the function f has, by definition, the value 0 (l'Hospital's rule). In the classical sense, $f''(x) = (\log|x|)' = 1/x$ if $x \neq 0$. On the other hand, one readily verifies that if $\varphi \in \mathcal{A}_0(\mathbb{R})$ and $x \in \mathbb{R}$, then

$$\partial_x u_f(\varphi, x) = \partial_x(f * \check{\varphi})(x) = (\log|x| * \check{\varphi})(x) = u_{\log|x|}(\varphi, x).$$

The second derivative $\partial^2 \jmath(f)$ of the generalized function $\jmath(f) \in \mathcal{G}(\mathbb{R})$ is a distribution on $\mathbb{R}$ which is denoted by $\mathrm{vp}\,\frac{1}{x}$ and which is such that

$$\langle \mathrm{vp}\,\frac{1}{x}, \psi\rangle = \mathrm{vp}\int_{\mathbb{R}} \frac{\psi(x)}{x}\,dx = \lim_{\varepsilon\to+0}\Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\Big)\frac{\psi(x)}{x}\,dx, \quad \psi \in \mathcal{D}(\mathbb{R}).$$

It is seen that $x\,\mathrm{vp}\,\frac{1}{x} = 1$ in $\mathcal{D}'(\mathbb{R})$. In the theory of distributions, one concludes that a "good" multiplication of distributions which is associative and commutative is impossible since one has the following contradictory chain of equalities (here for the sake of clarity the Schwartz product in $\mathcal{D}'(\mathbb{R})$ is denoted by the point):

$$\delta = \delta\,.\,1 = \delta\,.\,(x\,.\,\mathrm{vp}\,\frac{1}{x}) = (\delta\,.\,x)\,.\,\mathrm{vp}\,\frac{1}{x} = (x\,.\,\delta)\,.\,\mathrm{vp}\,\frac{1}{x} = 0\,.\,\mathrm{vp}\,\frac{1}{x} = 0. \tag{6.10}$$

In the algebra of generalized functions, this result is interpreted differently, namely, the above chain of equalities imposes a restriction to an imbedding of the space $\mathcal{D}'(\mathbb{R})$ into an associative algebra, more precisely, all three formulas $x\,.\,\delta = 0$, $x\,.\,\mathrm{vp}\,\frac{1}{x} = 1$, and $\delta\,.\,1 = \delta$ cannot hold simultaneously in such an algebra, and the product $C^\infty.\,\mathcal{D}'$ is necessarily changed (cf. (6.9) and Example 9.4).

Remark 6.2. Using product (6.6) and definition (6.2), the formula (5.15) for the representative of a distribution $T \in \mathcal{D}'(\Omega)$ in the algebra $\mathcal{G}(\Omega)$ can be written as follows:

$$u_T(\varphi, x) = \langle T, \ell(\varphi)\tau_x\varphi\rangle = \langle \ell(\varphi)T, \tau_x\varphi\rangle = ((\ell(\varphi)T) * \check{\varphi})(x), \tag{6.11}$$

where $\ell(\varphi)T \in \mathcal{E}'(\Omega)$ is the product of $\ell(\varphi) \in \mathcal{D}(\Omega)$ and T in the sense of (6.6). Taking into consideration the properties of $\ell(\varphi)$, the definition of the convolution of a distribution T and a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ on the open set $\Omega(\check{\varphi})$ (which was mentioned in Sec. 6.1), and the property (5.20), for the representative u_T, we obtain a property analogous to (2.18):

$$\begin{aligned}&\forall K \subset\subset \Omega\ \forall\varphi \in \mathcal{A}_0(\mathbb{R}^n)\ \exists\eta = \eta(K, \varphi) > 0 \ \text{ such that}\\ &K \subset \Omega(\check{\varphi}_\varepsilon) \text{ and } u_T(\varphi_\varepsilon, x) = (T * \check{\varphi}_\varepsilon)(x),\ x \in K,\ \varepsilon \in (0, \eta).\quad\Box\end{aligned} \tag{6.12}$$

Remark 6.3. In view of Theorem 5.5 and property (6.6), for $T \in \mathcal{D}'(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$, or $T \in \mathcal{E}'(\Omega)$ and $\psi \in C^\infty(\Omega)$, we have an equality refining Proposition 3.8(a):

$$\int_\Omega (T \cdot \psi)(x)\,dx = \int_\Omega (\psi T)(x)\,dx \quad \text{in} \quad \overline{\mathbb{K}},$$

where $T \cdot \psi$ is the product in $\mathcal{G}(\Omega)$ and ψT is the product in $\mathcal{D}'(\Omega)$ in the sense of Schwartz. In fact,

$$\int_\Omega (T \cdot \psi)(x)\,dx = \langle T, \psi \rangle = \langle \psi T, 1 \rangle = \int_\Omega ((\psi T) \cdot 1)(x)\,dx = \int_\Omega (\psi T)(x)\,dx.$$

In particular, if $T \in \mathcal{E}'(\Omega)$, then

$$\int_\Omega T(x)\,dx = \langle T, 1 \rangle \quad \text{in} \quad \overline{\mathbb{K}}.$$

Since the product $(T \cdot \psi)(x)$ can be naturally written as $T(x) \cdot \psi(x)$, the integral $\int (T \cdot \psi)(x)\,dx$ can be written as $\int T(x) \cdot \psi(x)\,dx$. The latter notation, used, as a rule, heuristically by physicists, was replaced by the dual notation $\langle T, \psi \rangle$ in the Schwartz distribution theory. In Colombeau's theory of generalized functions, the role of duality between $\mathcal{D}'$ and $\mathcal{D}$ is not so noticeable as in the distribution theory, so again this leads to notation of the kind $\int T(x) \cdot \psi(x)\,dx$, which has a natural meaning.

6.5. Localization principle for distributions. Just as for Colombeau's generalized functions, for Schwartz's distributions Theorem 2.6 holds in which the symbol $\mathcal{G}$ is replaced everywhere by $\mathcal{D}'$. This means that $\mathcal{D}'$ is a *sheaf*. The proof in this case is similar to the above, cf. [179, § I.3; 70, 5.6.2]. A recent study of the space $\mathcal{D}'(\Omega)$ from the point of view of the sheaf theory is due to Damyanov [66], where he has shown that $\mathcal{D}'$ is a sheaf of Hausdorff topological $\mathbb{C}$-vector spaces.

6.6. Distributions with compact supports. In the following proposition, we describe the structure of distributions from $\mathcal{E}'(\Omega)$ having compact supports in Ω:

Proposition 6.4. *Let $T \in \mathcal{E}'(\Omega)$. Then for every $K \subset\subset \Omega$ such that* $\operatorname{supp} T \subset K^\circ$, *there are $k \in \mathbb{N}$ and $f_\alpha \in C_c(K^\circ)$ with $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le k$, such that*

$$T = \sum_{|\alpha| \le k} \partial^\alpha \jmath(f_\alpha) \quad \text{in} \quad \mathcal{G}(\Omega).$$

(This theorem is analogous to Rudin [176, Thm. 6.27], and Colombeau [39, 2.4.9].)

Proof. If K is as in the proposition, then $T = \partial^\alpha \jmath(f)$ on K° for some $f \in C_c(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. On the other hand, if $a \in \mathcal{D}(K^\circ)$ and $a = 1$ in a neighborhood of $\operatorname{supp} T$, then, in view of Proposition 2.7(b), $T = a \cdot T$ in $\mathcal{G}(\Omega)$. Taking into account Proposition 3.8(a) for an arbitrary $\psi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} \langle T, \psi \rangle &\overset{(5.11)}{=} \langle T, a\psi \rangle \overset{(6.8)}{=} \sum_{0 \le \beta \le \alpha} (-1)^{|\beta|} \int_\Omega f_\beta (\partial^\beta \psi) = \\ &= \sum_{0 \le \beta \le \alpha} (-1)^{|\beta|} \int_\Omega \jmath(f_\beta) \cdot \partial^\beta \psi \overset{(3.9)}{=} \int_\Omega \Big(\sum_{0 \le \beta \le \alpha} \partial^\beta \jmath(f_\beta) \Big) \cdot \psi = \\ &= \Big\langle \sum_{0 \le \beta \le \alpha} \partial^\beta \jmath(f_\beta), \psi \Big\rangle, \end{aligned} \tag{6.13}$$

with the same functions $f_\beta \in C_c(K^\circ)$ as those following (6.7). Now it suffices to take into consideration Corollary 5.10(b). □

6.7. Distributions supported at a point.

Theorem 6.5. *Let $T \in \mathcal{E}'(\Omega)$ and* $\operatorname{supp} T = \{0\}$ *with $0 \in \Omega$. Then there are $k \in \mathbb{N}$ and a unique collection of numbers $\{c_\alpha\}_{|\alpha| \le k} \subset \mathbb{K}$ such that*

$$T = \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n). \tag{6.14}$$

Conversely, any distribution T of the form (6.14) *has the point* 0 *as its support* (*except for the case where all* c_α *are equal to zero.*)

(See, for instance, Rudin [176, Thm. 6.25] or Schwartz [179, § III.10]; the idea of the proof was taken from Folland [79, p. 265].)

Proof. The converse is clear; so we prove the main part of the theorem. First, assume that $\Omega = \mathbb{R}^n$.

1. $\exists\, k \in \mathbb{N}, C > 0 : \forall \psi \in C^\infty(\Omega) : |\langle T, \psi\rangle| \le C \sum\limits_{|\alpha|\le k} \sup_{B_1} |\partial^\alpha \psi|$. This bound follows from Proposition 6.4 applied to the unit ball $K = B_1$ and from calculations (6.13), so that the constant $C = \max_{|\beta|\le k} \| f_\beta \|_{L^1(B_1)} > 0$ is independent of ψ.

2. Let $\psi \in C^\infty(\mathbb{R}^n)$ be such that $(\partial^\alpha\psi)(0) = 0$ for $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le k$. If we set $\psi_\nu(x) = \psi(x)(1 - \zeta(\nu x))$, $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}$, where $\zeta \in \mathcal{D}(\mathbb{R}^n)$ is such that $\zeta(x) = 1$ if $|x| \le 1$, and $\zeta(x) = 0$ if $|x| \ge 2$, then

$$\partial^\alpha\psi_\nu \longrightarrow \partial^\alpha\psi \text{ as } \nu \longrightarrow \infty \text{ uniformly on any } K \subset\subset \mathbb{R}^n \text{ for all } |\alpha| \le k. \tag{6.15}$$

(If, in addition, $\psi \in \mathcal{D}(\mathbb{R}^n)$, then the above convergence is uniform on $\mathbb{R}^n$.) First, we note that if $|\beta| \le k$ and $K \subset\subset \mathbb{R}^n$, then there is $c(\psi, \beta, K) > 0$ such that

$$|(\partial^\beta\psi)(x)| \le c(\psi,\beta,K)|x|^{k+1-|\beta|}, \quad x \in K, \tag{6.16}$$

since, from Taylor's formula applied to the function $\partial^\beta\psi$ at the point $x = 0$ and conditions on ψ, it follows that

$$(\partial^\beta\psi)(x) = \sum_{|\gamma|=k+1-|\beta|} \frac{k+1-|\beta|}{\gamma!} \int_0^1 (1-t)^{k-|\beta|}(\partial^{\gamma+\beta}\psi)(tx)\, dt \cdot x^\gamma, \quad x \in K.$$

Using Leibnitz's rule, in view of (6.16) and the properties of ζ, we have

$$|\partial^\alpha(\psi_\nu - \psi)(x)| = |-\partial_x^\alpha(\psi(x)\zeta(\nu x))| \le \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} \nu^{|\alpha|-|\beta|} |(\partial^\beta\psi)(x)| \cdot |(\partial^{\alpha-\beta}\zeta)(\nu x)| \le$$

$$\le \nu^{|\alpha|-k-1} \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} c(\psi,\beta,K)|\nu x|^{k+1-|\beta|} |(\partial^{\alpha-\beta}\zeta)(\nu x)| \le$$

$$\le \frac{1}{\nu} \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} c(\psi,\beta,K) 2^{k+1-|\beta|} \sup_{|x|\le 2} |(\partial^{\alpha-\beta}\zeta)(x)|;$$

this implies (6.15).

3. If ψ is such as in step 2, then $\langle T, \psi\rangle = 0$. In fact, since $\operatorname{supp}(T\cdot\psi) \subset \{0\}$ and $\psi_\nu = 0$ for $|x| \le 1/\nu$, due to (3.8), we have

$$\langle T, \psi_\nu\rangle = \int_{\mathbb{R}^n} (T\cdot\psi_\nu)(x)\, dx = \int_{|x|\le 1/\nu} (T\cdot\psi_\nu)(x)\, dx = 0 \quad \forall\, \nu \in \mathbb{N},$$

so that the estimate of step 1 and (6.15) imply

$$|\langle T, \psi\rangle| = |\langle T, \psi_\nu - \psi\rangle| \le C \sum_{|\alpha|\le k} \sup_{B_1} |\partial^\alpha\psi_\nu - \partial^\alpha\psi| \longrightarrow 0, \quad \nu \longrightarrow \infty.$$

4. Assume now that $\psi \in C^\infty(\mathbb{R}^n)$ is arbitrary. From Taylor's formula it follows that

$$\psi(x) = \sum_{|\alpha|\le k} \frac{1}{\alpha!}(\partial^\alpha\psi)(0)x^\alpha + R_k(x), \quad x \in \mathbb{R}^n,$$

where $R_k \in C^\infty(\mathbb{R}^n)$ is Taylor's remainder, so that $(\partial^\alpha R_k)(0) = 0$ for all $|\alpha| \le k$. Then, by step 3, $\langle T, R_k \rangle = 0$, so that using (5.12), we find that

$$\langle T, \psi \rangle = \sum_{|\alpha| \le k} \frac{1}{\alpha!} (\partial^\alpha \psi)(0) \langle T, x^\alpha \rangle = \langle \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta, \psi \rangle,$$

where $c_\alpha = (-1)^{|\alpha|} \langle T, x^\alpha \rangle / \alpha!$, $|\alpha| \le k$. In view of Corollary 5.10(b), we come to (6.14).

5. In the general case of $\Omega \subset \mathbb{R}^n$, we note that $C^\infty(\mathbb{R}^n) \subset C^\infty(\Omega)$, so that we can consider $T \in \mathcal{E}'(\Omega)$ as an element of $\mathcal{E}'(\mathbb{R}^n)$. Let $\psi \in C^\infty(\Omega)$, and let $\zeta \in \mathcal{D}(\Omega)$ be such that $\zeta = 1$ in a neighborhood of $\{0\} = \operatorname{supp} T$. Since $\zeta\psi \in \mathcal{D}(\Omega)$, by virtue of (5.11), we have

$$\langle T, \psi \rangle = \langle T, \zeta\psi \rangle = \langle \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta, \zeta\psi \rangle = \langle \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta, \psi \rangle.$$

6. To prove the uniqueness, assume that $T = \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta = 0$ in $\mathcal{D}'(\Omega)$. Then for multi-indices of length $|\beta| \le k$, we find that

$$0 = \langle \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta, x^\beta \rangle = \sum_{|\alpha| \le k} c_\alpha \langle \partial^\alpha \delta, x^\beta \rangle = \sum_{|\alpha| \le k} c_\alpha (-1)^{|\alpha|} (\partial^\alpha x^\beta)(0) = c_\beta (-1)^{|\beta|} \beta!,$$

whence $c_\beta = 0$. (Simultaneously, it was proved that derivatives of the Dirac δ function are linearly independent.) □

6.8. Convolution of distributions. In Sec. 6.1, the convolution (6.2) of a distribution and a smooth function was defined which generalizes the convolution of functions (1.6). At the end of Sec. 2, the translation operator for generalized functions from $\mathcal{G}(\mathbb{R}^n)$ was defined. Let $T \in \mathcal{D}'(\mathbb{R}^n)$, and let $y \in \mathbb{R}^n$. It is clear that the translation $\tau_y T$ (in the sense of generalized functions) is in $\mathcal{D}'(\mathbb{R}^n)$, and we have the equality

$$\langle \tau_y T, \psi \rangle = \langle T, \tau_{-y} \psi \rangle, \quad \psi \in \mathcal{D}(\mathbb{R}^n), \tag{6.17}$$

since, in view of the formula of change of variables in integral (4.14), we have

$$\langle \tau_y T, \psi \rangle = \int_{\mathbb{R}^n} ((\tau_y T) \cdot \psi)(\lambda)\, d\lambda = \int_{\mathbb{R}^n} T(\lambda - y) \cdot \psi(\lambda)\, d\lambda =$$

$$= \int_{\mathbb{R}^n} T(\mu) \cdot \psi(\mu + y)\, d\mu = \int_{\mathbb{R}^n} (T \cdot (\tau_{-y}\psi))(\mu)\, d\mu = \langle T, \tau_{-y}\psi \rangle.$$

In the distribution theory, the formula (6.17) is taken as the definition of the translation of a distribution T. For $\psi = \tau_x \varphi$, this implies that the representatives u_T and $u_{\tau_y T}$ of distributions T and $\tau_y T$ are related as follows:

$$u_{\tau_y T}(\varphi, x) = \langle \tau_y T, \tau_x \varphi \rangle = \langle T, \tau_{x-y} \varphi \rangle = u_T(\varphi, x - y) = (\tau_y u_T)(\varphi, x),$$

where, as usual, $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. We have the following simple correlation between the convolution and the translation:

$$\langle \tau_x T, \varphi \rangle = \langle T, \tau_{-x} \varphi \rangle = \langle T, \tau_{-x} (\check{\varphi})^\vee \rangle = (T * \check{\varphi})(-x), \tag{6.18}$$

$$(T * \varphi)(x) = \langle T, \tau_x \check{\varphi} \rangle = \langle \tau_{-x} T, \check{\varphi} \rangle, \quad x \in \mathbb{R}^n, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{6.19}$$

The distribution $\delta_x := \tau_x \delta$ is called the *Dirac δ function concentrated at the point* $x \in \mathbb{R}^n$ (note that $\operatorname{supp} \delta_x = \{x\}$); it applies to test functions according to the rule

$$\langle \delta_x, \psi \rangle = \psi(x), \quad \psi \in \mathcal{D}(\mathbb{R}^n),$$

so that

$$\delta_x * \varphi = \tau_x\varphi, \quad \varphi \in \mathcal{D}(\mathbb{R}^n), \quad x \in \mathbb{R}^n.$$

In general, besides (6.3), the convolution of $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ has the properties

$$\tau_x(T * \varphi) = (\tau_x T) * \varphi = T * (\tau_x\varphi), \quad x \in \mathbb{R}^n,$$

$$T * (\varphi * \psi) = (T * \varphi) * \psi, \quad \psi \in \mathcal{D}(\mathbb{R}^n).$$

The following (well known in distribution theory [176, 6.29–6.37]) theorem asserts that the convolution $*$ is the only continuous bilinear operation which commutes with translations and differentiations:

Theorem 6.6. *Let $L : \mathcal{D}(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ be a continuous linear mapping. If at least one of the two conditions below holds*

(a) $\tau_x L = L\tau_x \ \forall\, x \in \mathbb{R}^n$, *or*

(b) $\partial^\alpha L = L\partial^\alpha \ \forall\, \alpha \in \mathbb{N}_0^n$,

then there is a unique distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$L(\varphi) = T * \varphi \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{6.20}$$

In the theory of distributions, it is shown that Theorem 6.6 holds for more general operators L satisfying (a) or (b) (see for instance [179, § VI.3 and VI.7]). In this connection, it is interesting to note that any mapping of $\mathcal{D}(\mathbb{R})$ into itself commuting with translations is automatically continuous (Meisters [138]), and hence, according to Theorem 6.6, it is represented by means of the convolution.

Proof. (a) Let L commute with translations. If a distribution T as in (6.20) exists, then, due to (6.18), we must have

$$\langle T, \varphi\rangle = (T * \check{\varphi})(0) = (L\,\check{\varphi})(0) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{6.21}$$

So, for any test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we set $\langle T, \varphi\rangle := (L\,\check{\varphi})(0)$. Since L is linear and continuous, $\varphi \longmapsto \check{\varphi}$ is a linear continuous mapping from $\mathcal{D}(\mathbb{R}^n)$ into itself, and since "the value at the point 0" is a linear continuous functional on $C^\infty(\mathbb{R}^n)$, we see that T is the composition of linear continuous mappings

$$\begin{array}{ccccccc} \mathcal{D}(\mathbb{R}^n) & \longrightarrow & \mathcal{D}(\mathbb{R}^n) & \longrightarrow & C^\infty(\mathbb{R}^n) & \longrightarrow & \mathbb{K} \\ \varphi & \longmapsto & \check{\varphi} & \longmapsto & L\,\check{\varphi} & \longmapsto & (L\,\check{\varphi})(0) \end{array}$$

so that $T \in \mathcal{D}'(\mathbb{R}^n)$. Furthermore, for all $x \in \mathbb{R}^n$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$(L\varphi)(x) = \tau_{-x}(L\varphi)(0) = (\tau_{-x}L\varphi)(0) = (L\tau_{-x}\varphi)(0) =$$

$$= L(\tau_{-x}\varphi)(0) = \langle T, (\tau_{-x}\varphi)^\vee\rangle = \langle T, \tau_x\check{\varphi}\rangle = (T * \varphi)(x).$$

The uniqueness of the distribution T follows if we note that if $T * \varphi = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $\langle T, \psi\rangle = (T * \check{\psi})(0) = 0 \ \forall \psi \in \mathcal{D}(\mathbb{R}^n)$, so that $T = 0$ in $\mathcal{D}'(\mathbb{R}^n)$.

(b) The case where L commutes with differentiations reduces to the above case as follows. Let ∂_j be the partial differential operator with respect to x_j, $j = 1, \ldots, n$. Consider an auxiliary function $b(x) = (L\tau_x\varphi)(x)$, $x \in \mathbb{R}^n$, and note that due to (6.4) one has

$$(\partial_j b)(x) = (\partial_j L\tau_x\varphi)(x) - (L\tau_x\partial_j\varphi)(x) = 0, \quad j = 1, \ldots, n,$$

where the latter equality is fulfilled due to conditions $\partial_j L = L\partial_j$. Hence $b(x) = b(0)$ for all $x \in \mathbb{R}^n$. But since $b(x) = (\tau_{-x}L\tau_x\varphi)(0)$, we obtain $\tau_{-x}L\tau_x = L$ or $L\tau_x = \tau_x L$. □

Now let us define the *convolution of two distributions T and S* such that

either (1) $T \in \mathcal{D}'(\mathbb{R}^n)$ and $S \in \mathcal{E}'(\mathbb{R}^n)$, or (2) $T \in \mathcal{E}'(\mathbb{R}^n)$ and $S \in \mathcal{D}'(\mathbb{R}^n)$.

To this end, we define a mapping $L : \mathcal{D}(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ by

$$L(\varphi) = T * (S * \varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

This is correct since $S * \varphi \in \mathcal{D}(\mathbb{R}^n)$ in case (1) and $S * \varphi \in C^\infty(\mathbb{R}^n)$ in case (2). Moreover, $\tau_x L = L\tau_x$ for all $x \in \mathbb{R}^n$, so, by Theorem 6.6, there is a unique distribution $R \in \mathcal{D}'(\mathbb{R}^n)$ such that $L(\varphi) = R * \varphi$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

The distribution R is called the *convolution of distributions T and S* and is denoted by $T * S$. Thus, the convolution $T * S$ is completely characterized by the formula

$$(T * S) * \varphi = T * (S * \varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{6.22}$$

Now, using (6.18), (6.22), and (6.19), let us calculate the value of the distribution $T * S$ on a test function $\psi \in \mathcal{D}(\mathbb{R}^n)$:

$$\begin{aligned}\langle T * S, \psi \rangle &= ((T * S) * \check{\psi})(0) = (T * (S * \check{\psi}))(0) = \\ &= \langle T, \tau_0 (S * \check{\psi})^\vee \rangle = \langle T_x, (S * \check{\psi})^\vee(x) \rangle;\end{aligned}$$

here the subscript x in T_x shows the variable in the test function to which the distribution T applies. Noting that

$$\begin{aligned}(S * \check{\psi})^\vee(x) &= (S * \check{\psi})(-x) = \langle S, \tau_{-x}(\check{\psi})^\vee \rangle = \langle S, \tau_{-x}\psi \rangle = \\ &= \langle S_y, (\tau_{-x}\psi)(y) \rangle = \langle S_y, \psi(y + x) \rangle,\end{aligned}$$

we obtain

$$\langle T * S, \psi \rangle = \langle T_x, \langle S_y, \psi(x + y) \rangle \rangle, \quad \psi \in \mathcal{D}(\mathbb{R}^n). \tag{6.23}$$

(The right-hand side in the latter equality is equal to $\langle T_x \otimes S_y, \psi(x + y) \rangle$, where the distribution $T_x \otimes S_y \in \mathcal{D}'(\mathbb{R}^{2n})$ is the *tensor product of T and S*, so that the convolution of two distributions is *commutative* when it exists.)

From (6.23) we easily have

$$\left.\begin{aligned}T * \delta_x &= \delta_x * T = \tau_x T, & x &\in \mathbb{R}^n, \\ T * (\partial^\alpha \delta) &= (\partial^\alpha \delta) * T = \partial^\alpha T, & \alpha &\in \mathbb{N}_0^n,\end{aligned}\right. \quad T \in \mathcal{D}'(\mathbb{R}^n).$$

In particular, $T * \delta = \delta * T = T$, that is, δ is the *unit* element in $\mathcal{D}'(\mathbb{R}^n)$ with respect to the convolution. Thus, convolutions are characterized by the fact that they commute with translations and differentiations, and that differentiations can be considered as convolutions with derivatives of the Dirac δ function.

6.9. Approximation of distributions. The space $\mathcal{D}'(\Omega)$ is endowed with the *weak topology* (also called weak-$*$ topology), which is defined by a family of seminorms $\{p_\varphi\}_{\varphi \in \mathcal{D}(\Omega)}$ such that $p_\varphi(T) = |\langle T, \varphi \rangle|$, $T \in \mathcal{D}'(\Omega)$. For example, the *convergence of a sequence $T_\nu \longrightarrow T$ in $\mathcal{D}'(\Omega)$ as $\nu \longrightarrow \infty$* means that $\langle T_\nu, \psi \rangle \longrightarrow \langle T, \psi \rangle$ as $\nu \longrightarrow \infty$ for all $\psi \in \mathcal{D}(\Omega)$.

Theorem 6.7. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$, and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then $T * \varphi_\varepsilon$, $T * \check{\varphi}_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\varepsilon > 0$, and*

$$T * \varphi_\varepsilon \ \text{ and } \ T * \check{\varphi}_\varepsilon \longrightarrow \Big(\int \varphi\Big) T \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \longrightarrow +0. \tag{6.24}$$

Proof. 1. First, let us show that

$$\psi * \check{\varphi}_\varepsilon \longrightarrow \Big(\int \varphi\Big) \psi \quad \text{in} \quad \mathcal{D}(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \longrightarrow +0. \tag{6.25}$$

To this end, one has to verify the following two conditions:

$$\exists K \subset\subset \mathbb{R}^n,\ \eta > 0,\ \forall \varepsilon \in (0,\eta) : \operatorname{supp}(\psi * \check{\varphi}_\varepsilon) \subset K; \text{ and}$$
$$\forall \alpha \in \mathbb{N}_0^n : \lim_{\varepsilon \to +0} \sup_K |\partial^\alpha(\psi * \check{\varphi}_\varepsilon) - \Big(\int \varphi\Big)\partial^\alpha \psi| = 0.$$

The first of the above conditions follows from the inclusions

$$\operatorname{supp}(\psi * \check{\varphi}_\varepsilon) \subset \operatorname{supp}\psi + \operatorname{supp}\check{\varphi}_\varepsilon \subset \operatorname{supp}\psi + B_1 =: K$$

if $\varepsilon\rho(\varphi) = \rho(\varphi_\varepsilon) < 1$. Since we have the equality $\partial^\alpha(\psi * \check{\varphi}_\varepsilon) = (\partial^\alpha \psi) * \check{\varphi}_\varepsilon$, it suffices to verify the second condition only in the case $\alpha = 0$, but this follows at once from Propositions 1.3(a) and 1.1(e).

2. Applying (6.21) and using the associativity of the convolution, we have

$$\langle T * \varphi_\varepsilon, \psi \rangle = ((T * \varphi_\varepsilon) * \check{\psi})(0) = (T * (\varphi_\varepsilon * \check{\psi}))(0) = \langle T, (\varphi_\varepsilon * \check{\psi})^\vee \rangle.$$

Since for $x \in \mathbb{R}^n$, we have

$$(\varphi_\varepsilon * \check{\psi})^\vee(x) = (\varphi_\varepsilon * \check{\psi})(-x) = \tau_x(\varphi_\varepsilon * \check{\psi})(0) = ((\tau_x\varphi_\varepsilon) * \check{\psi})(0) =$$
$$= \langle \tau_x\varphi_\varepsilon, \psi \rangle = \langle \psi, \tau_x\varphi_\varepsilon \rangle = (\psi * \check{\varphi}_\varepsilon)(x),$$

where in the next to last equality we have taken into account (5.7), using the continuity of T and (6.25), we find that

$$\langle T * \varphi_\varepsilon, \psi \rangle = \langle T, \psi * \check{\varphi}_\varepsilon \rangle \longrightarrow \langle T, \Big(\int \varphi\Big)\psi \rangle = \langle \Big(\int \varphi\Big)T, \psi \rangle, \quad \varepsilon \longrightarrow +0.$$

Finally, in view of Proposition 1.1(e), we have

$$T * \check{\varphi}_\varepsilon \longrightarrow \Big(\int \check{\varphi}\Big)T = \Big(\int \varphi\Big)T \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \longrightarrow +0. \quad \square$$

Corollary 6.8. *If $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, then*

(a) $\displaystyle\lim_{\varepsilon \to +0} \int_{\mathbb{R}^n} (T * \varphi_\varepsilon)(x)\psi(x)\,dx = \lim_{\varepsilon \to +0} \int_{\mathbb{R}^n} \langle T, \tau_x\varphi_\varepsilon\rangle\psi(x)\,dx = \langle T, \psi \rangle \ \forall \psi \in \mathcal{D}(\mathbb{R}^n).$

(b) *$\tau_x\varphi_\varepsilon$ and $\tau_x\check{\varphi}_\varepsilon \longrightarrow \delta_x$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \longrightarrow +0$ for all $x \in \mathbb{R}^n$ (since $\tau_x\varphi_\varepsilon = \delta_x * \varphi_\varepsilon$, $\tau_x\check{\varphi}_\varepsilon = \delta_x * \check{\varphi}_\varepsilon$).*

(c) *φ_ε and $\check{\varphi}_\varepsilon \longrightarrow \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \longrightarrow +0$.* $\quad \square$

Note that items (b) and (c) of this corollary, which follow from item (a), are already contained in Proposition 1.3(a)! From (6.24) it follows that the space $\mathcal{D}'(\mathbb{R}^n)$ could be defined as the completion of $C^\infty(\mathbb{R}^n)$ in the weak topology of the space $\mathcal{D}'(\mathbb{R}^n)$ (the sequential approach, cf. [5]).

The result of Theorem 6.7 can be sharpened, which we are going to do by proving an asymptotic formula for the convolution of a distribution and a smooth function (cf. also Todorov [192], Christov and Damyanov [32]).

Let $\Omega \subset \mathbb{R}^n$ be an open set, $T \in \mathcal{D}'(\Omega)$, $\psi \in \mathcal{D}(\Omega)$, and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Let K, $K_1 \subset\subset \Omega$ be such that $\operatorname{supp}\psi \subset K^\circ$ and $K \subset K_1^\circ$. Then there is $\eta = \eta(K, K_1, \varphi) > 0$ such that

$$\operatorname{supp}\tau_x\varphi_\varepsilon = x + \varepsilon \operatorname{supp}\varphi \subset K + B_{\varepsilon\rho(\varphi)} \subset\subset K_1^\circ, \quad \varepsilon \in (0,\eta), \quad x \in K.$$

Hence, for all $x \in K$ and $\varepsilon \in (0,\eta)$, the following number is well defined:

$$\langle T, \tau_x\varphi_\varepsilon \rangle = (T * \check{\varphi}_\varepsilon)(x).$$

Bearing in mind that the convolution of a distribution and a smooth function is a smooth function and using equality (6.21) as well as the associativity and commutativity of the convolution, for $\varepsilon \in (0,\eta)$, we have

$$\int_\Omega \langle T, \tau_x\varphi_\varepsilon\rangle\psi(x)\,dx = \int_\Omega (T * \check{\varphi}_\varepsilon)(x)\psi(x)\,dx = \langle T * \check{\varphi}_\varepsilon, \psi\rangle =$$

$$= ((T * \check{\varphi}_\varepsilon) * \check{\psi})(0) = (T * (\check{\varphi}_\varepsilon * \check{\psi}))(0) = (T * (\check{\psi} * \check{\varphi}_\varepsilon))(0) =$$

$$= ((T * \check{\psi}) * \check{\varphi}_\varepsilon)(0) = \langle T * \check{\psi}, \varphi_\varepsilon\rangle = \int (T * \check{\psi})(\lambda)\varphi_\varepsilon(\lambda)\,d\lambda =$$

$$= \int (T * \check{\psi})(\varepsilon\mu)\varphi(\mu)\,d\mu,$$

where in the last equality we have used the formula of change of variables in the integral. In view of (6.19), we find that

$$(T * \check{\psi})(\varepsilon\mu) = \langle T, \tau_{\varepsilon\mu}(\check{\psi})^\vee\rangle = \langle T, \tau_{\varepsilon\mu}\psi\rangle = \langle T_x, (\tau_{\varepsilon\mu}\psi)(x)\rangle = \langle T_x, \psi(x-\varepsilon\mu)\rangle;$$

therefore,

$$\int_\Omega \langle T, \tau_x\varphi_\varepsilon\rangle\psi(x)\,dx = \int \langle T_x, \psi(x-\varepsilon\mu)\rangle\varphi(\mu)\,d\mu.$$

Expanding the function ψ at the point x according to Taylor's formula up to order $q \in \mathbb{N}$ and applying the distribution T to this expansion in the variable x, we obtain

$$\langle T_x, \psi(x-\varepsilon\mu)\rangle = \langle T_x, \psi(x)\rangle + \sum_{|\alpha|=1}^{q} \frac{\varepsilon^{|\alpha|}}{\alpha!}(-1)^{|\alpha|}\langle T_x, (\partial^\alpha\psi)(x)\rangle\mu^\alpha +$$

$$+(-\varepsilon)^{q+1}\sum_{|\alpha|=q+1}\frac{q+1}{\alpha!}\int_0^1 (1-t)^q\langle T_x, (\partial^\alpha\psi)(x-t\varepsilon\mu)\rangle\,dt\cdot\mu^\alpha =$$

$$= \langle T, \psi\rangle + \sum_{|\alpha|=1}^{q}\frac{\varepsilon^{|\alpha|}}{\alpha!}\langle\partial^\alpha T, \psi\rangle\mu^\alpha +$$

$$+\varepsilon^{q+1}\sum_{|\alpha|=q+1}\frac{q+1}{\alpha!}\int_0^1 (1-t)^q\langle(\partial^\alpha T)_x, \psi(x-t\varepsilon\mu)\rangle\,dt\cdot\mu^\alpha.$$

Consequently,

$$\int_\Omega \langle T, \tau_x\varphi_\varepsilon\rangle\psi(x)\,dx = \int_\Omega (T * \check{\varphi}_\varepsilon)(x)\psi(x)\,dx =$$

$$= \Big(\int \varphi(\mu)\,d\mu\Big)\langle T, \psi\rangle + \sum_{|\alpha|=1}^{q}\frac{\varepsilon^{|\alpha|}}{\alpha!}\Big(\int \mu^\alpha\varphi(\mu)\,d\mu\Big)\langle\partial^\alpha T, \psi\rangle +$$

$$+\varepsilon^{q+1}\sum_{|\alpha|=q+1}\frac{q+1}{\alpha!}\int_{B_{\rho(\varphi)}}\Big(\int_0^1 (1-t)^q\langle(\partial^\alpha T)_x, \psi(x-t\varepsilon\mu)\rangle\,dt\Big)\mu^\alpha\varphi(\mu)\,d\mu.$$

This implies the following asymptotic formula, in which we use the notation (2.1):

$$\begin{gathered}\int_\Omega \langle T, \tau_x\varphi_\varepsilon\rangle\psi(x)\,dx = \int_\Omega (T * \check{\varphi}_\varepsilon)(x)\psi(x)\,dx = \\ = M^0(\varphi)\langle T, \psi\rangle + \sum_{|\alpha|=1}^{\infty}\frac{M^\alpha(\varphi)}{\alpha!}\langle\partial^\alpha T, \psi\rangle\,\varepsilon^{|\alpha|}, \quad \varepsilon \to +0;\end{gathered} \tag{6.26}$$

this is understood in the sense that for any $q \in \mathbb{N}$, we have

$$\int_{\Omega} \langle T, \tau_x \varphi_\varepsilon \rangle \psi(x)\, dx - M^0(\varphi)\langle T, \psi \rangle - \sum_{|\alpha|=1}^{q} \frac{M^\alpha(\varphi)}{\alpha!} \langle \partial^\alpha T, \psi \rangle\, \varepsilon^{|\alpha|} = o(\varepsilon^q), \quad \varepsilon \to +0. \tag{6.27}$$

Note that if $T = f \in C^\infty(\Omega)$, then it follows from the proof of Proposition 2.4 that

$$(f * \check{\varphi}_\varepsilon)(x) = M^0(\varphi)f(x) + \sum_{|\alpha|=1}^{\infty} \frac{M^\alpha(\varphi)}{\alpha!} (\partial^\alpha f)(x)\, \varepsilon^{|\alpha|}, \quad x \in \Omega, \quad \varepsilon \to +0.$$

In particular, if $q \in \mathbb{N}_0$ and $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$, then (6.27) implies

$$\int_{\Omega} \langle T, \tau_x \varphi_\varepsilon \rangle \psi(x)\, dx = \langle T, \psi \rangle + o(\varepsilon^q), \quad \varepsilon \to +0; \tag{6.28}$$

this again gives the results of Theorem 6.7 and Corollary 6.8. □

Remark 6.9. As was shown above, a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ has in $\mathcal{G}(\mathbb{R}^n)$ a representative of the form $u_T(\varphi_\varepsilon) = T * \check{\varphi}_\varepsilon$, which converges in $\mathcal{D}'(\mathbb{R}^n)$ to the distribution T itself. In particular, the Dirac δ function has a representative $u_\delta(\varphi_\varepsilon) = \check{\varphi}_\varepsilon$; which according to Corollary 6.8(c) is a delta net of smooth functions. All the information about the distribution T is already contained in its representative, including its nonlinear properties. The deep distinction between the theories of $\mathcal{G}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ is that in the distribution theory one *passes* to a limit of the kind $T * \check{\varphi}_\varepsilon \longrightarrow T$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \longrightarrow +0$ for $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, and as a result, which is quite natural, the important information about the "object" T contained in its representative is lost. In Colombeau's theory, no limit procedure of the above kind is used, so that the representative $T * \check{\varphi}$ contains much more information about the "limit" object T than this object itself considered in $\mathcal{D}'(\mathbb{R}^n)$. Moreover, there are elements of $\mathcal{G}(\mathbb{R}^n)$ which *leave no* information in $\mathcal{D}'(\mathbb{R}^n)$ at all: for example, a representative $u(\varphi) = (\check{\varphi})^2$ of δ^2; later it will be shown that δ^2 is not a distribution and it has no associated distribution (Example 8.8(4)). □

The above formulas for the asymptotic expansions in ε of expressions $\langle T * \check{\varphi}_\varepsilon, \psi \rangle$ with $T \in \mathcal{D}'(\Omega)$, and $(f * \check{\varphi}_\varepsilon)(x)$ with $f \in C^\infty(\Omega)$, show the natural character of Colombeau's construction of $\mathcal{G}(\Omega)$. The convolution with smooth functions naturally enters the construction: on the one hand, it allows us to form nets of regularizations for distributions, while on the other hand, it commutes with differentiations (cf. Theorem 6.6). The latter property means that the distributional derivatives (formula (6.5)) can be extended to the set of nets of regularizations. At the same time, bounds defining the ideal $\mathcal{N}[\Omega]$ emerge at once from the requirement that $C^\infty(\Omega)$ be a subalgebra in $\mathcal{G}(\Omega)$, for which one makes use of Taylor's formula in order to identify $f * \check{\varphi}_\varepsilon$ and f in the case where $f \in C^\infty(\Omega)$.

We note that originally Colombeau had found the ideal $\mathcal{N}[\Omega]$ from somewhat different considerations, namely, starting from the algebra $\mathcal{E}(\mathcal{D}(\Omega)) = C^\infty(\mathcal{D}(\Omega))$ of all infinitely Silva-differentiable mappings on $\mathcal{D}(\Omega)$ with values in $\mathbb{C}$ (a detailed description of this construction is contained in the original papers of Colombeau [33–37]). Later [39] the original definition was replaced by an elementary one; this new elementary definition was taken as the basis in the present work.

We also note that the comparison of nets of smooth functions modulo nets which decrease faster than any power of ε as $\varepsilon \longrightarrow +0$ (as in the ideal $\mathcal{N}[\Omega]$) naturally arises in the theory of asymptotic expansions. A factorization of the above kind has been used by Maslov and Tsupin [135, 136]; in the framework of the theory of asymptotic expansions, Maslov and Omel'yanov [132–134], as well as Todorov [190, 191], have considered asymptotic nets of smooth functions, identified as above, as generalized functions of a new type.

6.10. The Malgrange–Ehrenpreis theorem. We will complete this section with a remark on fundamental solutions of a linear partial differential operator $P(\partial) = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$ with constant complex coefficients a_α

(Ehrenpreis [73], Malgrange [129]): every nonzero operator $P(\partial)$ has a fundamental solution on $\mathbb{R}^n$, that is, there exists a distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$P(\partial)E = \delta \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n).$$

A very elementary proof of this result in which neither the Fourier transform nor even distribution theory are used was given by Rosay [166].

Note that the function E_m in (5.3) and the function E_α from (5.5) are fundamental solutions of the differential operators d^m/dx^m on $\mathbb{R}$ and ∂_x^α on $\mathbb{R}^n$, respectively. In fact, if $\psi \in \mathcal{D}(\mathbb{R}^n)$, then

$$\langle \partial^\alpha E_\alpha, \psi \rangle = ((\partial^\alpha E_\alpha) * \check{\psi})(0) = (\partial^\alpha (E_\alpha * \check{\psi}))(0) = \check{\psi}\,(0) = \psi(0) = \langle \delta, \psi \rangle. \quad \square$$

7. Difficulties in Multiplication of Distributions

7.1. Examples of difficulties and ambiguities. Simple arguments below show the difficulties that arise when one tries to define a multiplication in the space of distributions $\mathcal{D}'$ as well as when one tries to embed the space $\mathcal{D}'$ into an algebra $\mathbb{A}$ (cf. also Colombeau [37, Chap. 1, 2], Oberguggenberger [156, Chap. 1]), Rosinger [169, Part 1, Chap. 2].

If an *associative multiplication* with the unit $1 \in C^\infty(\mathbb{R})$ is defined in $\mathcal{D}'(\mathbb{R})$, then, as we have seen earlier, one has a contradictory chain of equalities (6.10). Furthermore, if H is the Heaviside function on $\mathbb{R}$, and $\operatorname{sign}(x) = 2H(x) - 1$, $x \in \mathbb{R}$, then we have $\operatorname{sign}^2(x) = 1$ for $x \in \mathbb{R} \setminus \{0\}$. If we assume that the derivative D in $\mathcal{D}'(\mathbb{R})$ satisfies Leibnitz's rule (see Sec. 2.1), then we obtain $0 = D(\operatorname{sign}^2(x)) = 2\delta \cdot \operatorname{sign}(x) + 2\operatorname{sign}(x) \cdot \delta$, that is,

$$\operatorname{sign}(x) \cdot \delta = -\delta \cdot \operatorname{sign}(x).$$

This means that an associative product in $\mathcal{D}'(\mathbb{R})$ must be *noncommutative*, since otherwise we have

$$\operatorname{sign}(x) \cdot \delta = \delta \cdot \operatorname{sign}(x) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}). \tag{7.1}$$

Let us consider one more example (Colombeau [45]). For the Heaviside function H, we have $H^2 = H$ and $H^3 = H$ in $L^1_{\text{loc}}(\mathbb{R})$; if a product in $\mathcal{D}'(\mathbb{R})$ is *commutative*, then differentiating these equalities in $\mathcal{D}'(\mathbb{R})$ with regard to Leibnitz's rule, we find that (by setting $H' = D(H)$):

$$D(H^2) = H' \cdot H + H \cdot H' = 2H \cdot H',$$

$$D(H^3) = H' \cdot H^2 + H \cdot (H^2)' = 3H^2 \cdot H',$$

whence $2H \cdot H' = H'$ and $3H^2 \cdot H' = H'$. Multiplying the first equality by $2H$ and noting that $4H^2 \cdot H' = 2H \cdot H' = H'$, we get

$$\frac{1}{3}H' = \frac{1}{4}H' \quad \text{in} \quad \mathcal{D}'(\mathbb{R}), \tag{7.2}$$

which is an absurdity, since $H' = \delta \neq 0$ in $\mathcal{D}'(\mathbb{R})$.

From these examples it can seem that the *imbedding*

$$\mathcal{D}'(\mathbb{R}) \subset \mathbb{A} \tag{7.3}$$

is *impossible*, where $\mathbb{A}$ is a "good enough algebra." In fact, if $\mathbb{A}$ is associative, then $\delta = 0$ in $\mathbb{A}$ (by (6.10)), while if $\mathbb{A}$ is commutative then one has (7.1) and (7.2) with $H' = \delta$. But since $\delta \neq 0$ in $\mathcal{D}'(\mathbb{R})$ and (7.3) is an *imbedding*, we have $\delta \neq 0$ in $\mathbb{A}$.

However, the above difficulties disappear and the imbedding (7.3) becomes "possible," if the multiplication in $\mathbb{A}$ is such that

$$x \cdot \operatorname{vp}\frac{1}{x} \neq 1 \in \mathbb{A} \quad \text{or} \quad x \cdot \delta \neq 0 \quad \text{in} \quad \mathbb{A}, \tag{7.4}$$

and it even does not matter if the algebra $\mathbb{A}$ is associative or not. Analogously if

$$\operatorname{sign}^2(x) \neq 1 \quad \text{in} \quad \mathbb{A}, \tag{7.5}$$

then the imbedding (7.3) into a commutative algebra $\mathbb{A}$ becomes possible, where (7.1) is replaced by

$$\operatorname{sign}(x) \cdot \delta = \delta \cdot \operatorname{sign}(x) \neq 0 \quad \text{in} \quad \mathbb{A}.$$

We have precisely this situation concerning multiplication in the Colombeau algebra $\mathbb{A} = \mathcal{G}(\mathbb{R})$, which is an associative and commutative differential algebra. The most important point in the above difficulties is that for a given imbedding (7.3), the product in the algebra $\mathbb{A}$ *cannot coincide* with the usual product of functions (or distributions, as in (6.6)) unless both multipliers are smooth enough.

One might suspect that algebras $\mathbb{A}$ containing distributions (7.3), in which (7.4) and (7.5) take place, can have such *strange* multiplications that the corresponding imbeddings (7.3) turn out to be useless. Fortunately, this is not the case, as can be seen from Colombeau's theory. The multiplication in the algebra $\mathbb{A} = \mathcal{G}(\mathbb{R})$ can again be *successfully* linked up with the usual products of functions and distributions because of the specific concept of an *associated distribution* (Sec. 8.2 below). In particular, in Colombeau's theory, relations of the kind $x \cdot \operatorname{vp}\frac{1}{x} = 1$, $x \cdot \delta = 0$, and $\operatorname{sign}^2(x) = 1$ can be successfully interpreted (see Theorem 8.3 and the remark following Theorem 8.12).

7.2. Schwartz's impossibility result. The result of L. Schwartz [178] which we consider below indicates further difficulties that come into view when one tries to find nonlinear extensions of the space of distributions.

Theorem 7.1. (Schwartz's impossibility result). *Let $\mathbb{A}$ be an associative algebra over the field $\mathbb{R}$ with a product denoted by $\odot$, and let $D : \mathbb{A} \longrightarrow \mathbb{A}$ be a derivation in $\mathbb{A}$ (that is, D is a linear mapping satisfying Leibnitz's rule, Sec.* 2.1*). Assume that the following conditions hold:*

(a) *continuous functions* 1, x, $x(\log|x|-1)$, *and* $x^2(\log|x|-1)$ *belong to* $\mathbb{A}$ *(the last two functions are defined to be equal to* 0 *at the point* $x = 0$*);*

(b) *the constant function* 1 *is the unit element of the algebra* $\mathbb{A}$*;*

(c) $(x(\log|x|-1)) \odot x = x^2(\log|x|-1)$ *in* $\mathbb{A}$*;*

(d) *the derivative* D *of continuously differentiable functions* 1, x, *and* $x^2(\log|x|-1)$ *coincides with the usual derivative, that is,* $D(1) = 0$, $D(x) = 1$, $D(x^2(\log|x|-1)) = 2x(\log|x|-1) + x$.

Then the linear mapping $L : \mathbb{A} \longrightarrow \mathbb{A}$, defined by $L(a) = x \odot a$ for all $a \in \mathbb{A}$, is injective. (In other words, there is no element $0 \neq \delta \in \mathbb{A}$ such that $x \odot \delta = 0$ in $\mathbb{A}$, where δ corresponds to the Dirac δ function.)

If, in addition to the above conditions, we have

(e) *continuous functions* $|x|$, $x|x|$ *belong to* $\mathbb{A}$*;*

(f) $x \odot |x| = x|x|$ *in* $\mathbb{A}$*;*

(g) *the derivative* D *of* $x|x|$ *coincides with the usual one:* $D(x|x|) = 2|x|$,

then $x \odot (D^2|x|) = 0$ in $\mathbb{A}$ (so that $D^2|x| = 0$ in $\mathbb{A}$, and there is an element in $\mathbb{A}$ whose derivative is zero and which is not a constant function).

Proof. 1. The idea of the proof is simple: due to (c) we find an element $x^{-1} \in \mathbb{A}$ such that $x^{-1} \odot x = 1$ in $\mathbb{A}$ (x^{-1} is a "left inverse" to $x \in \mathbb{A}$), and then, by the associativity of the product $\odot$ and (b), we obtain the injectivity of L: if $\delta \in \mathbb{A}$ and $x \odot \delta = 0$ in $\mathbb{A}$, then (cf. (6.10)!)

$$\delta = 1 \odot \delta = (x^{-1} \odot x) \odot \delta = x^{-1} \odot (x \odot \delta) = x^{-1} \odot 0 = 0 \quad \text{in} \quad \mathbb{A}. \tag{7.6}$$

As a left inverse for the element x one could take $x^{-1} = \dfrac{1}{x}$, but in this case one should also assume that the singular function $\dfrac{1}{x}$ is in $\mathbb{A}$, which would be a weakening of the theorem. However, the presence in $\mathbb{A}$ of such singular functions is possible due to the differentiation $D : \mathbb{A} \longrightarrow \mathbb{A}$ and conditions (a)–(d); note that if $f = x(\log|x|-1)$ the second classical derivative $\dfrac{d^2}{dx^2}f = \dfrac{1}{x}$, $x \in \mathbb{R}\setminus\{0\}$, and for the distributional derivative D in $\mathcal{D}'(\mathbb{R})$ one has

$$Df = \log|x|, \qquad D^2 f = \mathrm{vp}\,\frac{1}{x}.$$

2. Let us prove the first part of the theorem. First, let us show that the element $D^2 f \in \mathbb{A}$, where $f := x(\log|x| - 1) \in \mathbb{A}$, is a left inverse in $\mathbb{A}$ to the function $x \in \mathbb{A}$, that is,

$$(D^2 f)\odot x = 1 \quad \text{in} \quad \mathbb{A}. \tag{7.7}$$

Using Leibnitz's rule, the linearity of D, and conditions (d) and (b), we have

$$D(f\odot x) = (Df)\odot x + f\odot(Dx),$$

$$\begin{aligned} D^2(f\odot x) &= (D^2 f)\odot x + 2(Df)\odot(Dx) + f\odot(D^2 x) = \\ &= (D^2 f)\odot x + 2(Df). \end{aligned}$$

On the other hand, in view of (c) and (d) and the linearity of D, we find that

$$D^2(f\odot x) = D^2(x^2(\log|x| - 1)) = D(2f + x) = 2(Df) + 1.$$

Comparing the last two expressions, we come to (7.7). Now setting $x^{-1} := D^2 f \in \mathbb{A}$ and using the associativity of $\odot$, we prove the injectivity of L as in (7.6).

3. Let us prove the second part of the theorem. In view of Leibnitz's rule, the linearity of D, and (b), (d), (e), we have in $\mathbb{A}$

$$D^2(x\odot|x|) = 2(D|x|) + x\odot(D^2|x|). \tag{7.8}$$

On the other hand, by virtue of (f) and (g), we have

$$D^2(x\odot|x|) = D^2(x|x|) = D(2|x|) = 2(D|x|).$$

Consequently, $x\odot(D^2|x|) = 0$ in $\mathbb{A}$, so that by the first part of the theorem, $D^2|x| = 0$, where $|x| = (1/2)D(x|x|) \in \mathbb{A}$. (Note that $|x|$ is not a polynomial of degree ≤ 1. The equality $D^2|x| = 0$ is surely in contrast to $D^2|x| = 2\delta \neq 0$ in $\mathcal{D}'(\mathbb{R})$.) □

Remark 7.2. In $\mathcal{D}'(\mathbb{R})$, we have the equality $x\delta = 0$ (cf. (6.9)), which gives an upper bound of the singularity of δ at $x = 0$. However, Theorem 7.1 by no means states that differential algebras $\mathbb{A}$ from (7.3) cannot contain the Dirac distribution δ. This theorem asserts only that in $\mathbb{A}$ an element δ (representing the Dirac δ function) no longer satisfies the condition $x\odot\delta = 0$, that is, δ will have at the point $x = 0$ a singularity of order not less than the function $1/x$. The Dirac function δ also has many other important properties, for example, connected with its definition — $\langle\delta,\psi\rangle = \psi(0)$, $\psi \in \mathcal{D}(\mathbb{R})$, and $D^2(x_+) = D(H) = \delta$ in $\mathcal{D}'(\mathbb{R})$ — and these very properties are preserved in differential algebras $\mathbb{A}$ containing the space of distributions $\mathcal{D}'(\mathbb{R})$, and in particular, this is true for the algebra $\mathcal{G}(\mathbb{R})$. □

Under stronger additional hypotheses, we have the following important corollary of Theorem 7.1:

Corollary 7.3. *Let $\mathbb{A}$ be an associative differential algebra over $\mathbb{R}$ with a product $\odot$ and a differentiation $D : \mathbb{A} \longrightarrow \mathbb{A}$. If*

(a) *$C(\mathbb{R})$ is a subalgebra in $\mathbb{A}$;*

(b) *the constant function 1 is the unit element in $\mathbb{A}$;*

(c) *$D|_{C^1(\mathbb{R})}$ coincides with the usual differentiation in $C^1(\mathbb{R})$,*

then there is no element $\delta \in \mathbb{A}$, $\delta \neq 0$, such that $x \odot \delta = 0$ in $\mathbb{A}$, and the equality $x \odot (D^2|x|) = 0$ implies $D^2|x| = 0$ in $\mathbb{A}$. □

This corollary shows that if we want an associative differential algebra containing the space $C(\mathbb{R})$ to have an element $0 \neq \delta \in \mathbb{A}$ corresponding to the Dirac δ function, then at least one of conditions (a) or (c) will be violated. Therefore, in the algebra $\mathbb{A}$ the following three conditions are incompatible:

(1) the product in $\mathbb{A}$ exactly generalizes the product in $C(\mathbb{R})$,

(2) the differentiation in $\mathbb{A}$ exactly generalizes the differentiation in $C^1(\mathbb{R})$,

(3) there is an element $0 \neq \delta \in \mathbb{A}$ with the property $x \odot \delta = 0$ in $\mathbb{A}$.

Thus, the difficulties have not so much to do with the multiplication of distributions as with the multiplication of continuous functions and the usual differentiation. However, Schwartz's result (1954) has given rise to the saying that "multiplication of distributions is impossible." A closer look at this result makes it clear that the multiplication and the differentiation cannot simultaneously be extended onto associative algebras of generalized functions with the preservation of all their classical properties.

The following theorem concerns imbeddings of $\mathcal{D}'(\mathbb{R})$ into differential algebras $\mathbb{A}$ (Colombeau [37, I.2.4]):

Theorem 7.4. *Let $\mathbb{A}$ be an associative differential algebra over $\mathbb{R}$ with a product $\odot$ and a differentiation $D : \mathbb{A} \longrightarrow \mathbb{A}$. If*

(a) *$\mathcal{D}'(\mathbb{R})$ is a linear subspace in $\mathbb{A}$;*

(b) *the constant function 1 is the unit element in $\mathbb{A}$;*

(c) *$D|_{\mathcal{D}'(\mathbb{R})}$ coincides with the differentiation in $\mathcal{D}'(\mathbb{R})$,*

then the product $\odot$ in $\mathbb{A}$ restricted to $C(\mathbb{R})$ does not coincide with the usual product of continuous functions from $C(\mathbb{R})$.

Proof. Note that in $\mathcal{D}'(\mathbb{R})$ we have the equalities $D(x) = 1$, $D(1) = 0$, $D^2(|x|) = 2\delta$, and $D(x^2(\log|x| - 1)) = 2x(\log|x| - 1) + x$; hence, in view of (c), these equalities hold in $\mathbb{A}$ as well. From (a) it follows that x, $|x| \in \mathbb{A}$; therefore, as in (7.8), we get $D^2(x \odot |x|) = 2D(|x|) + 2x \odot \delta$. On the other hand, $x|x| \in \mathbb{A}$, so that $D^2(x|x|) = 2D(|x|)$. Now if we assume that the product in $\mathbb{A}$ of continuous functions coincides with their usual product in $C(\mathbb{R})$ then, in particular,

$$x \odot |x| = x|x| \quad \text{and} \quad (x(\log|x| - 1)) \odot x = x^2(\log|x| - 1) \quad \text{in} \quad \mathbb{A}.$$

Consequently,

$$2D(|x|) + 2x \odot \delta = D^2(x \odot |x|) = D^2(x|x|) = 2D(|x|),$$

whence $x \odot \delta = 0$ in $\mathbb{A}$, and $\delta = 0$ in $\mathbb{A}$ by Theorem 7.1. However, $\delta \neq 0$ in $\mathcal{D}'(\mathbb{R})$; hence $\delta \neq 0$ in $\mathbb{A}$, a contradiction. □

Now it is seen that Colombeau's achievement is in that he has explicitly constructed an associative and commutative differential algebra of generalized functions $\mathbb{A} = \mathcal{G}(\mathbb{R})$ which possesses the following *optimal* properties: properties (a), (b), (c) of Theorem 7.4 and

$$C^\infty(\mathbb{R}) \text{ is a subalgebra in } \mathbb{A}.$$

Recall that for any $k \in \mathbb{N}_0$, the algebra $C^k(\mathbb{R})$ is not a subalgebra in $\mathbb{A} = \mathcal{G}(\mathbb{R})$ (Example 2.5(1)). As for nonassociative algebras, in 1953–1955 König [107, 108] produced an algebra $\mathbb{A}$ satisfying simultaneously conditions (a), (b), (c) of Theorem 7.4 and condition (a) of Corollary 7.3 such that the product in $\mathbb{A}$ generalizes Schwartz's product (6.6) as well.

Let us summarize the above. In the linear space $C(\mathbb{R})$, the multiplication is always possible, but differentiation is not always possible and there is no element $\delta \neq 0$ with the property $x \cdot \delta = 0$. In the linear space $\mathcal{D}'(\mathbb{R})$ differentiation is always possible and there is an element $\delta \neq 0$ with the property $x \cdot \delta = 0$, but multiplication is not always possible. In the linear space $\mathcal{G}(\mathbb{R})$, the multiplication and differentiation are always possible, but there is no element $\delta \neq 0$ with the property $x \cdot \delta = 0$ (Proposition 3.12). Finally, one cannot simultaneously have the multiplication as in $C(\mathbb{R})$, the differentiation as in $C^1(\mathbb{R})$, and an element $\delta \neq 0$ (corresponding to the Dirac δ function) with the property $x \cdot \delta = 0$.

7.3. Degeneracy results when imbedding $\mathcal{D}'$ into an algebra. When one embeds $\mathcal{D}'(\mathbb{R})$, or some subspace therein, into a differential algebra $\mathbb{A}$, certain products containing the Dirac δ distribution or its derivatives may vanish. We are going to give examples of such products (Rosinger [167, I. § 11; 168, I. § 11; 169, 1.2.4]), although products of this kind are not always desirable in applications.

Theorem 7.5. *Let $\mathbb{A}$ be an associative and commutative differential algebra with a product $\odot$ and a differentiation $D : \mathbb{A} \longrightarrow \mathbb{A}$ such that*

(a) *real-valued polynomials on $\mathbb{R}$ belong to $\mathbb{A}$;*

(b) *distributions from $\mathcal{E}'(\mathbb{R})$ supported at a finite number of points belong to $\mathbb{A}$;*

(c) *multiplication $\odot$ in $\mathbb{A}$ induces on polynomials from* (a) *the usual multiplication of polynomials;*

(d) *the polynomial* 1 *is the unit element in $\mathbb{A}$;*

(e) *the derivative D of polynomials from* (a) *and of distributions from* (b) *coincides with the derivative in $\mathcal{D}'(\mathbb{R})$;*

(f) *$x \odot \delta = 0$ in $\mathbb{A}$.*

Then the following formulas hold in the algebra $\mathbb{A}$:

$$p,\ q \in \mathbb{N}_0,\ p > q \quad \Longrightarrow \quad x^p \odot D^q\delta = 0, \tag{7.9}$$

$$p \in \mathbb{N}_0 \quad \Longrightarrow \quad (p+1)D^p\delta + x \odot D^{p+1}\delta = 0, \tag{7.10}$$

$$p,\ q \in \mathbb{N}_0,\ q \geq 2 \quad \Longrightarrow \quad x^p \odot (D^p\delta)^q = 0, \tag{7.11}$$

$$\delta^2 = \delta \odot D\delta = 0. \tag{7.12}$$

Proof. 1. Using (f), (e), and (d), by Leibnitz's rule, we get

$$0 = D(x \odot \delta) = (Dx) \odot \delta + x \odot (D\delta) = \delta + x \odot D\delta. \tag{7.13}$$

Multiplying (7.13) by x with regard for (f) and (c), we find that

$$0 = x \odot (\delta + x \odot D\delta) = x \odot \delta + x \odot (x \odot D\delta) = x^2 \odot D\delta. \tag{7.14}$$

Apply the operator D to the last equality and multiply the result by x:

$$0 = D(x^2 \odot D\delta) = D(x^2) \odot D\delta + x^2 \odot D^2\delta \overset{(e)}{=} 2x \odot D\delta + x^2 \odot D^2\delta \quad \Longrightarrow$$

$$0 = x \odot (2x \odot D\delta + x^2 \odot D^2\delta) \overset{(c)}{=} 2x^2 \odot D\delta + x^3 \odot D^2\delta \overset{(7.14)}{=} x^3 \odot D^2\delta. \tag{7.15}$$

Continuing this process of application of D and multiplication of the result by x and taking into account (f), (7.14), (7.15), we get (7.9).

2. Since (7.13) is (7.10) with $p = 0$, we have, applying D to (7.13) and using (e) and (d),

$$0 = D(\delta + x \odot D\delta) = 2D\delta + x \odot D^2\delta.$$

Applying successively the operator D starting with the last equality, we come to (7.10).

3. Multiplying (7.10) by x^p, in view of (c), we have

$$(p+1)x^p \odot D^p\delta + x^{p+1} \odot D^{p+1}\delta = 0 \quad \text{in} \quad \mathbb{A}.$$

Multiplying the last equality by $(D^p\delta)^{q-1} = (D^p\delta) \odot (D^p\delta)^{q-2}$, $q \geq 2$, and taking into consideration (7.9), we find that

$$0 = (p+1)x^p \odot (D^p\delta)^q + (x^{p+1} \odot D^p\delta) \odot (D^p\delta)^{q-2} \odot D^{p+1}\delta = (p+1)x^p \odot (D^p\delta)^q.$$

4. Finally, from (7.11) with $p = 0$ and $q = 2$ we have the equality

$$\delta^2 = 0 \quad \text{in} \quad \mathbb{A},$$

to which we apply the derivative:

$$0 = D(\delta \odot \delta) = D\delta \odot \delta + \delta \odot D\delta = 2\delta \odot D\delta. \qquad \square$$

The degeneracy result (7.12) is not consistent with other results encountered in the literature, especially in those theories connected with multiplication of distributions in which differential algebras contain distributions: — in such theories one has $\delta^2 \notin \mathcal{D}'(\mathbb{R})$, so that $\delta^2 \neq 0$ (see, for example, Antosik, Mikusiński, and Sikorski [5], Colombeau [37], Rosinger [167, 168, 171], Mikusiński [141], Egorov [72], Ivanov [94]).

8. The Association Relation in $\mathcal{G}(\Omega)$

In order to recover classical values (of products, nonlinear operations, compositions, restrictions to linear subspaces) two special equivalence relations $\simeq$ and $\approx$ are defined in the algebra $\mathcal{G}(\Omega)$, which are called respectively the ***equality in the sense of generalized distributions*** (or the semiweak equality) and the ***association*** (or the weak equality). These new equivalence relations and the usual equality (between equivalence classes) in $\mathcal{G}(\Omega)$, together with ***unrestricted multiplication*** and ***unrestricted differentiation*** in $\mathcal{G}(\Omega)$, are the essence of ***Colombeau's associated analysis.***

To motivate somehow the definitions below, we mention one of the most important features of the linear Schwartz distribution theory, namely, that any distribution $T \in \mathcal{D}'(\Omega)$ is completely characterized by its "mean" values $\langle T, \psi \rangle = \int T \cdot \psi$ on test functions $\psi \in \mathcal{D}(\Omega)$. In the distribution theory, to an arbitrary distribution T no value $T(x)$ at a point $x \in \Omega$ is assigned in general, so the above characteristics play a special role, meaning, in particular, that $T = 0$ in $\mathcal{D}'(\Omega) \iff \int T \cdot \psi = 0$ in $\overline{\mathbb{K}}$ $\forall \psi \in \mathcal{D}(\Omega)$. It is convenient to extend this property from $\mathcal{D}'(\Omega)$ onto $\mathcal{G}(\Omega)$ if one takes into account that $T = 0$ in $\mathcal{G}(\Omega) \iff T = 0$ in $\mathcal{D}'(\Omega)$. This is the idea for introducing the equality in the sense of generalized distributions, with which we start this section. Note that at present Colombeau's associated calculus is one of the simplest and most efficient ways of overcoming those restrictions and difficulties that any nonlinear theory of generalized functions is likely to encounter.

8.1. The equality in the sense of generalized distributions.

Definition 8.1. We say that a generalized function $U \in \mathcal{G}(\Omega)$ is *equal to zero in the sense of generalized distributions* and write $U \simeq 0$ in $\mathcal{G}(\Omega)$ if (cf. (3.10))

$$\forall \psi \in \mathcal{D}(\Omega), \quad \int_\Omega U \cdot \psi = 0 \quad \text{in} \quad \overline{\mathbb{K}}.$$

Two generalized functions U_1, $U_2 \in \mathcal{G}(\Omega)$ are said to be *equal in the sense of generalized distributions*, which is written as $U_1 \simeq U_2$ in $\mathcal{G}(\Omega)$, if $U_1 - U_2 \simeq 0$ in $\mathcal{G}(\Omega)$.

In terms of representatives, the equality $U_1 \simeq U_2$ means that for some (and then for any) respective representatives u_1 and u_2 of these generalized functions, we have

$$\Big[\mathcal{A}_0(\mathbb{R}^n) \ni \varphi \longmapsto \int (u_1(\varphi, x) - u_2(\varphi, x))\psi(x)dx \in \mathbb{K}\Big] \in \mathcal{N}_0 \quad \forall \psi \in \mathcal{D}(\Omega). \quad \square$$

In view of the properties of the integral it is clear that $\simeq$ is an equivalence relation on $\mathcal{G}(\Omega)$. Obviously, if $U_1 = U_2$ in $\mathcal{G}(\Omega)$, then $U_1 \simeq U_2$ in $\mathcal{G}(\Omega)$. But on $\mathcal{D}'(\Omega)$ both equalities $=$ and $\simeq$ coincide:

Proposition 8.2. *If $T_1, T_2 \in \mathcal{D}'(\Omega)$, then*

$$(\,T_1 \simeq T_2 \text{ in } \mathcal{G}(\Omega)\,) \iff (\,T_1 = T_2 \text{ in } \mathcal{D}'(\Omega)\,) \iff (\,T_1 = T_2 \text{ in } \mathcal{G}(\Omega)\,).$$

Proof. This follows at once from Definition 8.11 and Corollary 5.10b). $\square$

The main motivation for introducing the equality $\simeq$ is that by means of it one can *recover* the classical Schwartz product (6.6):

Theorem 8.3. *If $a \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, then $a \cdot T \simeq aT$ in $\mathcal{G}(\Omega)$.*

Proof. Let $\psi \in \mathcal{D}(\Omega)$. Since $C^\infty(\Omega)$ is a subalgebra in $\mathcal{G}(\Omega)$, we have $(a\cdot T)\cdot\psi = (T\cdot a)\cdot\psi = T\cdot(a\cdot\psi) = T\cdot(a\,\psi)$ in $\mathcal{G}(\Omega)$, so that, in view of (6.6), we obtain

$$\int (a \cdot T)\cdot\psi = \int T\cdot(a\,\psi) = \langle T, a\,\psi\rangle = \langle a\,T, \psi\rangle = \int (a\,T)\cdot\psi \quad \text{in } \overline{\mathbb{K}}. \quad \square$$

Taking into account (6.9), we have

$$x^m \cdot \delta \neq 0 = x^m\delta, \qquad x^m \cdot \delta \simeq 0 = x^m\delta \quad \text{in} \quad \mathcal{G}(\mathbb{R}). \tag{8.1}$$

This shows that the relation $\simeq$ is weaker than the equality $=$ in $\mathcal{G}(\Omega)$. On the other hand, (8.1) and Proposition 8.2 mean that $x^m \cdot \delta \notin \mathcal{D}'(\mathbb{R})$ (recall that $x^m\delta \in \mathcal{D}'(\mathbb{R})$ due to Sec. 6.4).

A generalization of the Schwartz product is the product of distributions with disjoint singular supports. If $T \in \mathcal{D}'(\Omega)$ and $G \subset \Omega$ is open, we say that $T|_G \in C^\infty(G)$ if $T = \jmath(f)$ on G for some $f \in C^\infty(\Omega)$. Denote by $\Omega^\infty(T)$ the largest open subset of Ω such that $T|_{\Omega^\infty(T)} \in C^\infty(\Omega^\infty(T))$. The set $\operatorname{sing\,supp} T := \Omega \setminus \Omega^\infty(T)$ is called the *singular support* of T. For example, $\operatorname{sing\,supp} \delta_x = \{x\}$.

Let T_1, $T_2 \in \mathcal{D}'(\Omega)$ and $\operatorname{sing\,supp} T_1 \cap \operatorname{sing\,supp} T_2 = \varnothing$. Then $\Omega = \Omega^\infty(T_1) \cup \Omega^\infty(T_2)$, so that by the theorem on C^∞ partitions of unity, there are a_1, $a_2 \in C^\infty(\Omega)$ such that $\operatorname{supp} a_1 \subset \Omega^\infty(T_1)$, $\operatorname{supp} a_2 \subset \Omega^\infty(T_2)$, and $a_1 + a_2 = 1$ on Ω. The product $T_1T_2 = T_2T_1 \in \mathcal{D}'(\Omega)$ is defined by

$$T_1T_2 = (a_1T_1)T_2 + T_1(a_2T_2) \quad \text{in} \quad \mathcal{D}'(\Omega),$$

where a_1T_1, $a_2T_2 \in C^\infty(\Omega)$. As a corollary of Theorem 8.3, we have $T_1 \cdot T_2 \simeq T_1T_2$ in $\mathcal{G}(\Omega)$.

The equality in the sense of generalized distributions is preserved by linear operations, multiplication by a smooth function, and differentiation:

Proposition 8.4. *Let U, V, U_1, $V_1 \in \mathcal{G}(\Omega)$, $U \simeq V$, $U_1 \simeq V_1$ in $\mathcal{G}(\Omega)$, and let $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. Then*

$$U + U_1 \simeq V + V_1, \quad f\cdot U \simeq f\cdot V, \quad \partial^\alpha U \simeq \partial^\alpha V \quad \text{in} \quad \mathcal{G}(\Omega).$$

Proof. These properties follow from the linearity of the integral, the fact that $C^\infty(\Omega)$ is a subalgebra in $\mathcal{G}(\Omega)$, and the formula of integration by parts. $\square$

Example 8.5. Generally speaking, the equality $\simeq$ is not compatible with multiplication. Set $U = x \cdot \delta$ in $\mathcal{G}(\mathbb{R})$. Then $U \simeq 0$. A representative of $U^2 = x^2 \cdot \delta^2$ is the mapping $u(\varphi_\varepsilon, x) = x^2(1/\varepsilon^2)\varphi^2(-x/\varepsilon)$ with $\varphi \in \mathcal{A}_0(\mathbb{R})$, $x \in \mathbb{R}$, and $\varepsilon > 0$; hence if $\psi \in \mathcal{D}(\mathbb{R})$, then a representative of the generalized number

$$\int_{\mathbb{R}} ((x^2 \cdot \delta^2) \cdot \psi)(x)\, dx \in \overline{\mathbb{K}}$$

is the mapping

$$I(\varphi_\varepsilon) = \int_{\mathbb{R}} x^2 \frac{1}{\varepsilon^2} \varphi^2\left(-\frac{x}{\varepsilon}\right)\psi(x)\, dx = \varepsilon \int_{\mathbb{R}} y^2 \varphi^2(y)\psi(-\varepsilon y)\, dy. \tag{8.2}$$

Since

$$\lim_{\varepsilon \to +0} \frac{I(\varphi_\varepsilon)}{\varepsilon} = \psi(0) \int_{\mathbb{R}} y^2 \varphi^2(y)\, dy,$$

from (3.3) it follows at once that $I \notin \mathcal{N}_0$, that is, $U^2 = x^2 \cdot \delta^2 \not\simeq 0$ in $\mathcal{G}(\mathbb{R})$. □

So, with the help of the equality $\simeq$ in the sense of generalized distributions, we have recovered the classical Schwartz product; however, it is *not always* possible to use this equality to recover the pointwise product of continuous functions. Hence we are going to introduce a more general (and weaker than $=$ and $\simeq$) equality *in the sense of the association*: as a hint, let us recall that in the algebra of generalized numbers there is a weaker kind of equivalence relation, which is called the association (Definition 3.1).

8.2. The equality in the sense of the association.

Definition 8.6. We say that a generalized function $U \in \mathcal{G}(\Omega)$ is *equal to zero in the sense of the association*, and we write $U \approx 0$ in $\mathcal{G}(\Omega)$ if (Definition 3.1)

$$\forall \psi \in \mathcal{D}(\Omega) \quad : \quad \int_{\Omega} U \cdot \psi \approx 0 \quad \text{in} \quad \overline{\mathbb{K}}.$$

Two generalized functions $U_1, U_2 \in \mathcal{G}(\Omega)$ are said to be *equal in the sense of the association* (or, in short, are *associated* to each other), which is written as $U_1 \approx U_2$ in $\mathcal{G}(\Omega)$, if $U_1 - U_2 \approx 0$ in $\mathcal{G}(\Omega)$. The relation $\approx$ on $\mathcal{G}(\Omega)$ will be called the *association*. We say that a generalized function $U \in \mathcal{G}(\Omega)$ *has an associated distribution* $T \in \mathcal{D}'(\Omega)$ if $U \approx T$ in $\mathcal{G}(\Omega)$ (in other words, if for every $\psi \in \mathcal{D}(\Omega)$ the generalized number $\int_{\Omega} U \cdot \psi$ has $\langle T, \psi \rangle = \int_{\Omega} T \cdot \psi$ as an associated ordinary number in the sense of Definition 3.1).

In terms of representatives, the last definition appears as follows: $U \approx T$ in $\mathcal{G}(\Omega)$ if for some (and then for any) representative $u \in \mathcal{E}_M[\Omega]$ of the generalized function U, one has

$$\begin{gathered} \forall \psi \in \mathcal{D}(\Omega)\ \exists N \in \mathbb{N} \text{ such that } \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) : \\ \lim_{\varepsilon \to +0} \int_{\Omega} u(\varphi_\varepsilon, x)\psi(x)\, dx = \langle T, \psi \rangle. \end{gathered} \tag{8.3}$$

An associated distribution T does not depend on a representative of U and is unique (if it exists). □

In view of the properties of the association $\approx$ on $\overline{\mathbb{K}}$, it is clear that $\approx$ is an equivalence relation on $\mathcal{G}(\Omega)$. Obviously, in $\mathcal{G}(\Omega)$ we have implications:

$$U_1 = U_2 \implies U_1 \simeq U_2 \implies U_1 \approx U_2.$$

But on $\mathcal{D}'(\Omega)$ all the three equalities in $\mathcal{G}(\Omega)$ coincide (see Proposition 8.2):

Proposition 8.7. *If $T_1, T_2 \in \mathcal{D}'(\Omega)$, then*

$$(T_1 \approx T_2 \text{ in } \mathcal{G}(\Omega)) \iff (T_1 = T_2 \text{ in } \mathcal{D}'(\Omega)) \iff (T_1 = T_2 \text{ in } \mathcal{G}(\Omega)).$$

In particular, every distribution has itself as an associated distribution. □

Examples 8.8. (1) If $U \in \mathcal{G}(\Omega)$ and $U \approx T \in \mathcal{D}'(\Omega)$, then, in general, $U \neq T$ in $\mathcal{G}(\Omega)$. This is the case, for example, for the function $x \cdot \delta \in \mathcal{G}(\mathbb{R})$: $x \cdot \delta \approx 0 \in \mathcal{D}'(\mathbb{R})$, but $x \cdot \delta \neq 0$ in $\mathcal{G}(\mathbb{R})$. This shows that the association $\approx$ is weaker than the equality $=$ in $\mathcal{G}(\Omega)$.

(2) The association is also weaker than the equality in the sense of generalized distributions: if $\mathrm{P} \in \mathcal{G}(\mathbb{R}^n)$ is a constant generalized function from Example 3.2(3), then in $\mathcal{G}(\mathbb{R}^n)$, we have $\mathrm{P} \neq 0$, $\mathrm{P} \not\simeq 0$, and $\mathrm{P} \approx 0$. The function $x^2 \cdot \delta^2 \in \mathcal{G}(\mathbb{R})$ from Example 8.5 has the same properties: $x^2 \cdot \delta^2 \not\simeq 0$, $x^2 \cdot \delta^2 \approx 0$ in $\mathcal{G}(\mathbb{R})$, where the last property follows from (8.2). Calculations as in Example 8.5, however, show that $x \cdot \delta^2 \not\approx 0$ in $\mathcal{G}(\mathbb{R})$.

(3) Let $F \in \mathcal{O}_M(\mathbb{R})$ be a bounded function and let $\delta \in \mathcal{G}(\mathbb{R}^n; \mathbb{R})$, where the algebra $\mathcal{G}(\mathbb{R}^n; \mathbb{R})$ is constructed starting from the sets $\mathcal{E}[\mathbb{R}^n; \mathbb{R}] = C^\infty(\mathbb{R}^n; \mathbb{R})^{\mathcal{A}_0(\mathbb{R}^n; \mathbb{R})}$. Then

$$F(\delta) \approx F(0) \quad \text{in} \quad \mathcal{G}(\mathbb{R}^n),$$

since by Lebesgue's dominated convergence theorem, one has

$$\lim_{\varepsilon \to +0} \int_{\mathbb{R}^n} F(\check{\varphi}_\varepsilon(x))\psi(x)\,dx = \int_{\mathbb{R}^n} F(0)\psi(x)\,dx, \quad \psi \in \mathcal{D}(\mathbb{R}^n).$$

In particular, $\cos\delta \approx 1$, $\sin\delta \approx 0$, and $e^{i\delta} \approx 1$.

(4) Not all generalized functions have associated distributions: consider $\delta^2 \in \mathcal{G}(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\psi = 1$ in a neighborhood of $0 \in \mathbb{R}^n$; then for all sufficiently small $\varepsilon > 0$, we have

$$\int \frac{1}{\varepsilon^{2n}} \varphi^2\Big(-\frac{x}{\varepsilon}\Big)\psi(x)\,dx = \frac{1}{\varepsilon^{2n}} \int \varphi^2\Big(-\frac{x}{\varepsilon}\Big)\,dx = \frac{1}{\varepsilon^n} \int \varphi^2(\lambda)\,d\lambda;$$

the last integral does not converge for $\varphi \in \mathcal{A}_0(\mathbb{R}^n; \mathbb{R})$. According to Proposition 8.7, this also means that $\delta^2 \notin \mathcal{D}'(\mathbb{R}^n)$. □

The equality in the sense of the association is compatible with linear operations, multiplication by a smooth function, and differentiation:

Proposition 8.9. *If $U, V, U_1, V_1 \in \mathcal{G}(\Omega)$, $U \approx V$, $U_1 \approx V_1$ in $\mathcal{G}(\Omega)$, and $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, then*

$$U + U_1 \approx V + V_1, \quad f \cdot U \approx f \cdot V, \quad \partial^\alpha U \approx \partial^\alpha V \quad \text{in} \quad \mathcal{G}(\Omega). \quad □$$

In general, the equality $\approx$ is not preserved by multiplication since, for example, in $\mathcal{G}(\mathbb{R})$ we have $x \cdot \delta \approx 0$, but $x \cdot \delta^2 \not\approx 0$ (Example 8.8(2)).

The set of all generalized functions from $\mathcal{G}(\Omega)$ having associated distributions will be denoted by

$$\mathcal{G}_{\mathrm{A}}(\Omega) = \{\, U \in \mathcal{G}(\Omega) \mid \exists T \in \mathcal{D}'(\Omega) \;:\; U \approx T \quad \text{in} \quad \mathcal{G}(\Omega) \,\}.$$

Clearly, $\mathcal{G}_{\mathrm{A}}(\Omega)$ is a linear subspace in $\mathcal{G}(\Omega)$, and one has the following natural imbeddings of linear spaces:

$$\mathcal{D}'(\Omega) \subsetneq \mathcal{G}_{\mathrm{A}}(\Omega) \subsetneq \mathcal{G}(\Omega).$$

On the other hand, a mapping from $\mathcal{G}_{\mathrm{A}}(\Omega)$ into $\mathcal{D}'(\Omega)$ which to a generalized function relates its associated distribution is linear and surjective, but not injective. The square of this mapping coincides with itself, thus can be considered as a kind of a *projection* of the set $\mathcal{G}_{\mathrm{A}}(\Omega)$ onto $\mathcal{D}'(\Omega)$.

Now we are able to recover the pointwise product of continuous functions:

Theorem 8.10. *If $f,\, g \in C(\Omega)$, then $\jmath(f)\cdot\jmath(g) \approx \jmath(fg)$ in $\mathcal{G}(\Omega)$.*

Proof. Let $\psi \in \mathcal{D}(\Omega)$, $K \subset\subset \Omega$, and let $\operatorname{supp}\psi \subset K^\circ$. First, note that in view of Proposition 3.8(a), one has

$$\int \jmath(fg)\cdot\psi = \int f(x)g(x)\psi(x)\,dx.$$

Given $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, by the property (2.18), for all $\varepsilon > 0$ small enough, one finds that

$$I(\varphi_\varepsilon) := \int [\,u_f(\varphi_\varepsilon, x)u_g(\varphi_\varepsilon, x) - f(x)g(x)\,]\psi(x)\,dx =$$

$$= \iiint [\,f(x+\varepsilon\lambda)g(x+\varepsilon\mu) - f(x)g(x)\,]\varphi(\lambda)\varphi(\mu)\psi(x)\,d\lambda\,d\mu\,dx. \tag{8.4}$$

Since $I(\varphi_\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow +0$, our proposition follows.

Remark. This theorem is also valid in the case where $f \in L^\infty_{\text{loc}}(\Omega)$ and $g \in C(\Omega)$: in fact, changing variables in the integral (8.4), one obtains

$$I(\varphi_\varepsilon) = \int_K \int_{B_{\rho(\varphi)}} \int_{B_{\rho(\varphi)}} f(x)[\,g(x+\varepsilon\mu-\varepsilon\lambda)\psi(x-\varepsilon\lambda) - g(x)\psi(x)\,]\varphi(\lambda)\varphi(\mu)\,d\lambda\,d\mu\,dx\,;$$

now it suffices to note that $I(\varphi_\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow +0$ by Lebesgue's dominated convergence theorem. □

Thus, although the new product in $\mathcal{G}(\Omega)$ does not coincide algebraically with the classical product of continuous functions, it can be "projected" onto $\mathcal{D}'(\Omega)$ to obtain the classical product. Therefore, the use of the new product and the classical product lead to the same results as far as "natural" calculations are concerned. Hence the new product can be considered as a generalization of classical products in a somewhat weaker, but quite acceptable, sense than the "strong algebraic equality," which leads to the Schwartz impossibility result.

Example 8.11. In Theorem 8.10, it is essential even for functions $f,\, g \in C^k(\Omega)$ that we have used the association $\approx$ instead of $\simeq$. Consider an example (see also Rosinger [169, 2.1.6]). Let $f,\, g \in C^k(\mathbb{R})$ be the functions from Example 2.5(1). As we have seen, $\jmath(f)\cdot\jmath(g) \neq 0 = \jmath(fg)$ in $\mathcal{G}(\mathbb{R})$, but at the same time $\jmath(f)\cdot\jmath(g) \approx 0$ in view of Theorem 8.10. So let us show that $\jmath(f)\cdot\jmath(g) \not\simeq 0$. Let $\varphi \in \mathcal{A}_q(\mathbb{R})$, where $q \geq 2k+3$, and let $\operatorname{supp}\varphi \subset [a,b]$, where $b < 0$ (the role of these conditions will be clear in what follows). For representatives of our functions and $x \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$u_f(\varphi_\varepsilon, x) = \int_{-x/\varepsilon}^{\infty} (x+\varepsilon\mu)^{k+1}\varphi(\mu)\,d\mu, \quad u_g(\varphi_\varepsilon, x) = \int_{-\infty}^{-x/\varepsilon} (x+\varepsilon\mu)^{k+1}\varphi(\mu)\,d\mu.$$

Given $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi(0) = 1$, we prove that the generalized number with the representative

$$I(\varphi_\varepsilon) = \int_{\mathbb{R}} u_f(\varphi_\varepsilon, x)u_g(\varphi_\varepsilon, x)\psi(x)\,dx$$

is not zero. Noting that $u_f(\varphi_\varepsilon, x) = 0$ if $-x/\varepsilon \geq b$, and $u_g(\varphi_\varepsilon, x) = 0$ if $-x/\varepsilon \leq a$, we find that

$$I(\varphi_\varepsilon) = \int_{-\varepsilon b}^{-\varepsilon a}\Big(\int_{-x/\varepsilon}^{b} (x+\varepsilon\mu)^{k+1}\varphi(\mu)\,d\mu\Big)\Big(\int_a^{-x/\varepsilon} (x+\varepsilon\mu)^{k+1}\varphi(\mu)\,d\mu\Big)\psi(x)\,dx =$$

$$= \varepsilon^{2k+3}\int_a^b\Big(\int_y^b (y-\mu)^{k+1}\varphi(\mu)\,d\mu\Big)\Big(\int_a^y (y-\mu)^{k+1}\varphi(\mu)\,d\mu\Big)\psi(-\varepsilon y)\,dy,$$

whence

$$\lim_{\varepsilon\to+0}\frac{I(\varphi_\varepsilon)}{\varepsilon^{2k+3}} = I_0\cdot\psi(0) = I_0,$$

where

$$I_0 = \int_a^b \Big(\int_y^b (y-\mu)^{k+1}\varphi(\mu)\,d\mu\Big)\Big(\int_a^y (y-\mu)^{k+1}\varphi(\mu)\,d\mu\Big)\,dy.$$

If we show that $I_0 \neq 0$, then $I \notin \mathcal{N}_0$ by virtue of (3.3), and we are through. Setting $\theta(y) = \int_a^y (y-\mu)^{k+1}\varphi(\mu)\,d\mu$, $y \in \mathbb{R}$, and noting that $\varphi \in \mathcal{A}_{k+1}(\mathbb{R})$, we obtain that $\int_y^b (y-\mu)^{k+1}\varphi(\mu)\,d\mu = y^{k+1} - \theta(y)$, so that

$$I_0 = \int_a^b y^{k+1}\theta(y)\,dy - \int_a^b \theta^2(y)\,dy.$$

Computing the first integral (changing the order of integration in the double integral, using the binomial theorem and the condition $\varphi \in \mathcal{A}_{2k+3}(\mathbb{R})$), we see that it is equal to $b^{2k+3}/(2k+3)$. Since $b < 0$, $I_0 < 0$. □

Theorem 8.10 is generalized to the case of arbitrary nonlinear operations:

Theorem 8.12. *Let $F \in \mathcal{O}_M(\mathbb{K}^p)$, and let $f_1, \ldots, f_p \in C(\Omega)$. Then*

$$F(\jmath(f_1), \ldots, \jmath(f_p)) \approx \jmath(F(f_1, \ldots, f_p)) \qquad in \qquad \mathcal{G}(\Omega).$$

Proof. Let $\psi \in \mathcal{D}(\Omega)$, and let $K \subset\subset \Omega$ be such that $\operatorname{supp}\psi \subset K^\circ$. By Proposition 3.8(a), we have

$$\int \jmath(F(f_1, \ldots, f_p))\cdot\psi = \int F(f_1(x), \ldots, f_p(x))\psi(x)\,dx.$$

Given $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, by the property (2.18), for representatives of generalized functions $\jmath(f_j)$ and small enough $\varepsilon > 0$, we find that

$$I(\varphi_\varepsilon) := \int [\,F(\{(f_j * \check{\varphi}_\varepsilon)(x)\}_{j=1}^p) - F(\{f_j(x)\}_{j=1}^p)\,]\,\psi(x)\,dx. \tag{8.5}$$

Since ψ has a compact support, by Lebesgue's dominated convergence theorem, $I(\varphi_\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow +0$, which is what required. □

Remark 8.13. A function $g : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be *piecewise continuous* if, on every finite segment $[a,b] \subset \mathbb{R}$, it has only a finite number of points of discontinuities such that at each such point the function has one-sided finite limits from the left and from the right. The set of all such functions is denoted by $C_f(\mathbb{R})$. Theorem 8.12 is valid for functions $f_1, \ldots, f_p \in C_f(\mathbb{R})$ as well [38, 30]. In fact, let $x_1, \ldots, x_k$ be points of discontinuities of f_j on the segment $K = [-\rho(\psi), \rho(\psi)]$. In view of Example 3.4(4), for points x in K we have as $\varepsilon \longrightarrow +0$

$$(f_j * \check{\varphi}_\varepsilon)(x) \longrightarrow f_j(x), \quad x \in K \setminus \{x_1, \ldots, x_k\},$$

$$(f_j * \check{\varphi}_\varepsilon)(x_i) \longrightarrow f_j(x_i - 0)\int_{-\infty}^0 \varphi(\mu)\,d\mu + f_j(x_i+0)\int_0^\infty \varphi(\mu)\,d\mu, \quad i = 1, \ldots, k,$$

and from Proposition 1.3(e), for $0 < \varepsilon < 1$, we have the estimate

$$\sup_{x\in K} |(f_j * \check{\varphi}_\varepsilon)(x)| \le \| f_j \|_{L^\infty(K_{\rho(\varphi)})} \| \varphi \|_{L^1(\mathbb{R})} \equiv C(f_j, \varphi, \psi) < \infty,$$

where $K_{\rho(\varphi)} = [-\rho(\varphi) - \rho(\psi), \rho(\varphi) + \rho(\psi)]$. Hence, the net of functions $F(\{(f_j * \check{\varphi}_\varepsilon)(x)\})$ is uniformly bounded on K for small $\varepsilon > 0$ and converges pointwise to $F(\{f_j(x)\})$ almost everywhere on K. It follows that the integral (8.5) tends to zero as $\varepsilon \longrightarrow +0$ by Lebesgue's bounded convergence theorem. □

The generalized composition (Theorem 4.6) of continuous functions coincides with the usual composition of continuous functions in the sense of the association:

Theorem 8.14. *Let $\Omega \subset \mathbb{R}^n$, and let $\Delta \subset \mathbb{R}^m$ be open sets. If $g \in C(\Delta)$ and $f \in C(\Omega; \Delta)$, then*

$$\jmath_\Delta(g) \circ \jmath_{\Omega,\mathbb{R}^m}(f) \approx \jmath_\Omega(g \circ f) \quad \text{in} \quad \mathcal{G}(\Omega),$$

where we denote by $\jmath_\Delta$, $\jmath_\Omega$, $\jmath_{\Omega,\mathbb{R}^m}$ the imbeddings $\jmath_\Delta : C(\Delta) \longrightarrow \mathcal{G}(\Delta)$, $\jmath_\Omega : C(\Omega) \longrightarrow \mathcal{G}(\Omega)$, and $\jmath_{\Omega,\mathbb{R}^m} : C(\Omega; \Delta) \longrightarrow \mathcal{G}(\Omega; \mathbb{R}^m)$ defined in (2.17).

Proof. Since $V = \jmath_\Delta(g) \in \mathcal{G}(\Delta)$, and $U = \jmath_{\Omega,\mathbb{R}^m}(f) \in \mathcal{G}_*(\Omega; \Delta)$ by Proposition 4.5, the composition $W = V \circ U \in \mathcal{G}(\Omega)$ is well defined by Theorem 4.6. Let $\psi \in \mathcal{D}(\Omega)$, $K \subset\subset \Omega$, and $\operatorname{supp} \psi \subset K^\circ$. First, note that

$$\int_\Omega \jmath_\Omega(g \circ f) \cdot \psi = \int_\Omega g(f(x))\psi(x)\, dx$$

(Proposition 3.8(a)). Given $\varphi = \phi^{\otimes n} \in \mathcal{A}_0^\otimes(\mathbb{R}^n; \mathbb{R})$, in view of the property (2.18), a representative u of U is of the form

$$u(\phi_\varepsilon^{\otimes n}, x) = f * (\phi^{\otimes n})_\varepsilon^\vee(x) = \int f(x + \varepsilon\mu)\phi^{\otimes n}(\mu)\, d\mu, \quad x \in K, \quad \varepsilon \in (0, \eta),$$

for some $\eta > 0$. From Proposition 4.5 and property (4.9) it follows that there are $K_1 \subset\subset \Delta$ and $\eta_1 \in (0, \eta)$ such that $u(\phi_\varepsilon^{\otimes n}, x) \in K_1$ for all $x \in K$ and $\varepsilon \in (0, \eta_1)$ (and the compact set K_1 is independent of ϕ). Again, by property (2.18), for a representative v of V, we have

$$v(\phi_\varepsilon^{\otimes m}, y) = g * (\phi^{\otimes m})_\varepsilon^\vee(y) = \int g(y + \varepsilon\lambda)\phi^{\otimes m}(\lambda)\, d\lambda, \quad y \in K_1, \quad \varepsilon \in (0, \eta_1).$$

Hence a representative w of W is, by definition, of the form

$$w(\phi_\varepsilon^{\otimes n}, x) = \int g\Big(\int f(x + \varepsilon\mu)\phi^{\otimes n}(\mu)\, d\mu + \varepsilon\lambda\Big)\phi^{\otimes m}(\lambda)\, d\lambda, \quad x \in K, \quad \varepsilon \in (0, \eta_1).$$

It follows that the expression

$$I(\phi_\varepsilon^{\otimes n}) = \iint \Big[g\Big(\int f(x + \varepsilon\mu)\phi^{\otimes n}(\mu)\, d\mu + \varepsilon\lambda\Big) - g(f(x))\Big]\phi^{\otimes m}(\lambda)\, \psi(x)\, d\lambda\, dx$$

tends to zero as $\varepsilon \longrightarrow +0$ by Lebesgue's dominated convergence theorem (note that $\operatorname{supp} \psi \subset\subset \Omega$). The proof is complete. □

Example 8.15. For $T \in \mathcal{D}'(\Delta)$ and $f \in C(\Omega; \Delta)$, the composition $T \circ f \in \mathcal{G}(\Omega)$ is well defined (for brevity, we set $T \circ f = T \circ \jmath_{\Omega,\mathbb{R}^m}(f)$), but in general it is not a distribution on Ω and even can have no associated distribution. For example, let $T = \delta \in \mathcal{D}'(\mathbb{R}^n)$, and let $f \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $f \equiv 0$. Then a representative w of the composition $\delta \circ f$ is of the form

$$w(\varphi_\varepsilon, x) = \frac{1}{\varepsilon^n}\varphi\Big(-\frac{1}{\varepsilon}\int f(x + \varepsilon\mu)\varphi(\mu)\, d\mu\Big) = \frac{1}{\varepsilon^n}\varphi(0).$$

It follows that the composition $\delta \circ f = \delta \circ 0$ has no associated distribution (hence it is not a distribution itself), and $\delta \circ 0$ is a constant generalized function equal to the generalized number $\delta(0)$. □

The next result shows that restrictions of a continuous function to linear subspaces in the generalized sense and in the classical sense coincide in the sense of the association (cf. also [156, Chap. III, § 11; 15, 1.6.11]):

Theorem 8.16. *Let $f \in C(\mathbb{R}^n)$ and let $m < n$. Then $\jmath(f)|\mathbb{R}^m \approx \jmath(f|\mathbb{R}^m)$ in $\mathcal{G}(\mathbb{R}^m)$, where at the left and at the right we have used the same symbol $\jmath$ to denote, respectively, the imbeddings $C(\mathbb{R}^n) \longrightarrow \mathcal{G}(\mathbb{R}^n)$ and $C(\mathbb{R}^m) \longrightarrow \mathcal{G}(\mathbb{R}^m)$, defined in* (2.12) *and* (2.10).

Proof. We use the notation from the end of Sec. 4. Let $\psi \in \mathcal{D}(\mathbb{R}^m)$. Then by Proposition 3.8(a), we get

$$\int_{\mathbb{R}^m} \jmath(f|\mathbb{R}^m) \cdot \psi = \int_{\mathbb{R}^m} f(x', 0'')\psi(x')\, dx'.$$

Let $\phi \in \mathcal{A}_0(\mathbb{R})$. By definition, a representative $u_f|\mathbb{R}^m$ of the generalized function $\jmath(f)|\mathbb{R}^m$ is of the form

$$(u_f|\mathbb{R}^m)(\phi_\varepsilon^{\otimes m}, x') = \int_{\mathbb{R}^n} f((x', 0'') + \varepsilon\mu)\phi^{\otimes n}(\mu)\, d\mu, \quad x' \in \mathbb{R}^m, \quad \varepsilon > 0,$$

and it suffices to note that the expression

$$I(\phi_\varepsilon^{\otimes m}) := \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} [\, f((x', 0'') + \varepsilon\mu) - f(x', 0'')\,]\phi^{\otimes n}(\mu)\psi(x')\, d\mu\, dx'$$

tends to zero as $\varepsilon \longrightarrow +0$. □

Example 8.17. If $T \in \mathcal{D}'(\mathbb{R}^n)$, then $T|\mathbb{R}^m \in \mathcal{G}(\mathbb{R}^m)$, but in general this restriction cannot be a distribution and even cannot have an associated distribution. Consider the Dirac δ function $\delta \in \mathcal{D}'(\mathbb{R}^2)$. A representative $u_\delta|\mathbb{R}$ of its restriction $\delta|\mathbb{R}$ is of the form

$$(u_\delta|\mathbb{R})(\phi_\varepsilon, x) = \phi_\varepsilon^{\otimes 2}\Big(-\frac{x}{\varepsilon}, 0\Big) = \frac{1}{\varepsilon^2}\phi\Big(-\frac{x}{\varepsilon}\Big)\phi(0), \quad \phi \in \mathcal{A}_0(\mathbb{R}), \quad x \in \mathbb{R}, \quad \varepsilon > 0.$$

For a representative I of the generalized number $\int_{\mathbb{R}} (\delta|\mathbb{R}) \cdot \psi$, we then have

$$I(\phi_\varepsilon) := \int_{\mathbb{R}} \frac{1}{\varepsilon^2}\phi\Big(-\frac{x}{\varepsilon}\Big)\phi(0)\psi(x)\, dx = \frac{1}{\varepsilon}\phi(0)\int_{\mathbb{R}} \phi(y)\psi(-\varepsilon y)\, dy.$$

The last expression tends to $+\infty$ as $\varepsilon \longrightarrow +0$, if, for example, $\phi(0) = 1$ and $\psi(0) = 1$. Hence, $\delta|\mathbb{R}$ has no associated distribution, and $\delta|\mathbb{R} \notin \mathcal{D}'(\mathbb{R})$. □

In order to better understand how the information contained in elements of $\mathcal{G}(\Omega)$ is transmitted by means of the association into the space of distributions $\mathcal{D}'(\Omega)$, consider the following representation of $\mathcal{D}'(\Omega)$ as a quotient linear space (Rosinger [169], Oberguggenberger [156]). Denote by $S(\Omega)$ the linear subspace in $\mathcal{E}_M[\Omega]$ defined as follows:

$$\begin{aligned} S(\Omega) = \{\, u \in \mathcal{E}_M[\Omega] \mid \exists T \in \mathcal{D}'(\Omega) :\ & \\ & \forall \psi \in \mathcal{D}(\Omega)\ \exists N \in \mathbb{N} : \\ & \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) : \lim_{\varepsilon \to +0} \int_\Omega u(\varphi_\varepsilon, x)\psi(x)\, dx = \langle T, \psi \rangle \,\}. \end{aligned}$$

In $S(\Omega)$, consider the following linear subspace:

$$\begin{aligned} V(\Omega) = \{\, u \in \mathcal{E}_M[\Omega] \mid \forall \psi \in \mathcal{D}(\Omega)\ \exists N \in \mathbb{N} :\ & \\ & \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) : \lim_{\varepsilon \to +0} \int_\Omega u(\varphi_\varepsilon, x)\psi(x)\, dx = 0 \,\}. \end{aligned}$$

The spaces $\mathcal{D}'(\Omega)$ and $S(\Omega)/V(\Omega)$ are linearly isomorphic; the isomorphism is defined by

$$J : \mathcal{D}'(\Omega) \longrightarrow S(\Omega)/V(\Omega), \quad J(T) = u_T + V(\Omega) \quad \text{for} \quad T \in \mathcal{D}'(\Omega),$$

where the representative $u_T \in \mathcal{E}_M[\Omega]$ is defined in (6.11) as

$$u_T(\varphi) = (\ell(\varphi)T) * \check{\varphi}, \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n).$$

(The linearity and the surjectivity of J are obvious, the injectivity of J follows from (6.12) and (6.28).) Thus, the objects of $\mathcal{D}'(\Omega)$ are determined up to elements of $V(\Omega)$, whereas the objects of $\mathcal{G}(\Omega)$ are determined up to elements of a much smaller set $\mathcal{N}[\Omega]$. Two elements of $\mathcal{G}(\Omega)$ are associated to each other iff the difference of their representatives belongs to $V(\Omega)$, that is, the information contained in the nets weakly (in $\mathcal{D}'(\Omega)$) convergent to zero is neglected. One may say that the equality in $\mathcal{G}(\Omega)$ expresses the "microscopic" behavior of elements of $\mathcal{G}(\Omega)$, whereas the association expresses the "macroscopic" behavior of these elements.

The following proposition is interesting as compared to Proposition 3.5 (see also Biagioni [15, 1.10.2] and Aragona and Biagioni [8, 6.3]):

Proposition 8.18. *Let $U \in \mathcal{G}(\mathbb{R}^n)$, and let $\alpha \in \mathbb{N}_0^n$. The following two conditions are equivalent:*

(a) $\partial^\alpha U \approx 0$ *in* $\mathcal{G}(\mathbb{R}^n)$;

(b) *there is a* $V \in \mathcal{G}(\mathbb{R}^n)$ *such that* $\partial^\alpha V = 0$ *and* $U \approx V$ *in* $\mathcal{G}(\mathbb{R}^n)$.

In particular, if $n = 1$ and $\alpha = 1$, then the equality $U' \approx 0$ in $\mathcal{G}(\mathbb{R})$ is equivalent to the existence of a generalized number $Z \in \overline{\mathbb{K}}$ such that $U \approx Z$ in $\mathcal{G}(\mathbb{R})$.

Proof. (b)$\Longrightarrow$(a). This follows from Proposition 8.9.

(a)$\Longrightarrow$(b). 1. First, let $n = 1$, and let $\alpha = 1$, that is, $U' \approx 0$ in $\mathcal{G}(\mathbb{R})$. Let $\psi \in \mathcal{D}(\mathbb{R})$. Fix $\psi_0 \in \mathcal{A}_0(\mathbb{R})$ and set $\varphi_1 = \psi - \Big(\int_{\mathbb{R}} \psi\Big)\psi_0 \in \mathcal{D}(\mathbb{R})$. Since $\int_{\mathbb{R}} \varphi_1 = 0$, there is a function $\psi_1 \in \mathcal{D}(\mathbb{R})$ such that $\psi_1' = \varphi_1$ on $\mathbb{R}$. For a representative u of U, $\varphi \in \mathcal{A}_0(\mathbb{R})$, and $\varepsilon > 0$, we have

$$\int u(\varphi_\varepsilon, x)\psi(x)\,dx = \Big(\int_{\mathbb{R}} \psi\Big) \int u(\varphi_\varepsilon, x)\psi_0(x)\,dx + \int u(\varphi_\varepsilon, x)\varphi_1(x)\,dx =$$

$$= \int \Big(\int u(\varphi_\varepsilon, y)\psi_0(y)\,dy\Big)\psi(x)\,dx - \int u'(\varphi_\varepsilon, x)\psi_1(x)\,dx.$$

By the assumption, there is $N = N(\psi_1) \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_N(\mathbb{R})$, the last integral tends to zero as $\varepsilon \longrightarrow +0$, so that from the above equality we conclude that

$$\int U \cdot \psi \approx \int \Big(\int U \cdot \psi_0\Big) \cdot \psi \quad \text{in} \quad \overline{\mathbb{K}},$$

whence $U \approx \int U \cdot \psi_0 =: Z$ in $\mathcal{G}(\mathbb{R})$.

If $U'' \approx 0$, then $U' \approx Z$ in view of the above, where $Z \in \overline{\mathbb{K}}$. But as $(U - Zx)' \approx 0$ we have $U - Zx \approx Z_1 \in \overline{\mathbb{K}}$. Hence, (b) is valid with $V := Zx + Z_1$. The case of an arbitrary $\alpha \in \mathbb{N}$ follows by induction.

2. Let $n \in \mathbb{N}$, $\alpha = (1, 0, \dots, 0) \in \mathbb{N}_0^n$, and let $\partial^\alpha U = \partial_1 U \approx 0$ in $\mathcal{G}(\mathbb{R}^n)$. The proof in this case is as above; so we will indicate only the main points. Fix a $\psi_0 \in \mathcal{A}_0(\mathbb{R})$ and set

$$v(\varphi, x) = v(\varphi, \widehat{x_1}) = \int_{\mathbb{R}} u(\varphi, t, \widehat{x_1})\psi_0(t)\,dt, \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n),$$

where $x = (x_1, \widehat{x_1}) \in \mathbb{R}^n$, $\widehat{x_1} = (x_2, \dots, x_n)$, and u is a representative of U. It is clear that $v \in \mathcal{E}_M[\mathbb{R}^n]$, and if $V = v + \mathcal{N}[\mathbb{R}^n]$, then $\partial_1 V = 0$ in $\mathcal{G}(\mathbb{R}^n)$. Let us show that $U \approx V$. Let $\psi \in \mathcal{D}(\mathbb{R}^n)$. Set

$$\varphi_1(x) = \psi(x) - \Big(\int_{\mathbb{R}} \psi(t, \widehat{x_1})\,dt\Big)\psi_0(x_1), \quad x = (x_1, \widehat{x_1}) \in \mathbb{R}^n.$$

Since $\varphi_1 \in \mathcal{D}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi_1(x)\,dx = 0$, there is a function $\psi_1 \in \mathcal{D}(\mathbb{R}^n)$ such that $\partial_1\psi_1 = \varphi_1$ on $\mathbb{R}^n$. Consequently,

$$\int u(\varphi_\varepsilon, x)\psi(x)\,dx = \iint u(\varphi_\varepsilon, x)\Big(\int_{\mathbb{R}} \psi(t, \widehat{x_1})\,dt\Big)\psi_0(x_1)\,dx_1 d\widehat{x_1} + \int u(\varphi_\varepsilon, x)\varphi_1(x)\,dx =$$

$$= \int v(\varphi_\varepsilon, x)\psi(x)\,dx - \int \frac{\partial u(\varphi_\varepsilon, x)}{\partial x_1}\psi_1(x)\,dx.$$

The general case follows by induction. □

9. Further Properties of the Association

9.1. Multiplication by the Dirac δ function. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a piecewise continuously differentiable function, i.e., outside a finite set of points $A = \{x_1 < \ldots < x_k\}$, the function f is continuous, has continuous derivative f', and at the points of A, functions f and f' have finite one-sided limits from the left and from the right. By the integration by parts formula, for a representative of f, one has

$$\partial_x u_f(\varphi, x) = -\Big(\int_{-\infty}^{x_1} + \sum_{i=2}^{k} \int_{x_{i-1}}^{x_i} + \int_{x_k}^{\infty}\Big) f(\lambda)\varphi'(\lambda - x)\,d\lambda =$$

$$= \sum_{i=1}^{k} [f]_{x_i}\varphi(x_i - x) + \Big(\int_{-\infty}^{x_1} + \sum_{i=2}^{k} \int_{x_{i-1}}^{x_i} + \int_{x_k}^{\infty}\Big) f'(\lambda)\varphi(\lambda - x)\,dx, \tag{9.1}$$

where $[f]_{x_i} = f(x_i + 0) - f(x_i - 0)$ is the jump of the function f at the point x_i. From here we obtain the well-known formula from the distribution theory (for example, Vladimirov [201, 1.2.3])

$$\jmath(f)' = \{f'\} + \sum_{i=1}^{k} [f]_{x_i}\delta_{x_i} \quad \text{in} \quad \mathcal{G}(\mathbb{R}),$$

where a representative of the generalized function $\{f'\}$ is defined by the second summand in (9.1). Differentiating both sides of the formula $\jmath(f)^m \approx \jmath(f^m)$, $m \in \mathbb{N}$, which is valid due to Remark 8.13, we find that

$$m\jmath(f)^{m-1} \cdot \Big(\{f'\} + \sum_{i=1}^{k} [f]_{x_i}\delta_{x_i}\Big) \approx \{(f^m)'\} + \sum_{i=1}^{k} [f^m]_{x_i}\delta_{x_i}.$$

In particular, if $H := \jmath(H) \in \mathcal{G}(\mathbb{R})$ is the Heaviside function (Example 5.3), then for any $m, k \in \mathbb{N}$, $m \neq k$, we have the following relations in $\mathcal{G}(\mathbb{R})$:

$$H^m \approx H \approx H^k, \qquad H^m \neq H^k, \qquad H^m \not\approx H^k, \tag{9.2}$$

$$H^{m-1} \cdot H' = H^{m-1} \cdot \delta \approx \frac{1}{m}\,\delta. \tag{9.3}$$

Note also that

$$H^m|_{(-\infty,0)} = 0 \quad \text{in} \quad \mathcal{G}(-\infty, 0), \qquad H^m|_{(0,\infty)} = 1 \quad \text{in} \quad \mathcal{G}(0, \infty),$$

$$H^m|_{\mathbb{R}\setminus\{0\}} = H^k|_{\mathbb{R}\setminus\{0\}} \quad \text{in} \quad \mathcal{G}(\mathbb{R} \setminus \{0\}).$$

We are going to verify only the last two relations in (9.2) since all the others have already been ascertained. Assume that $H^m = H^k$ for some $m \neq k$; then $H^{m+1} = H^{k+1}$, and by Leibnitz's rule in $\mathcal{G}(\mathbb{R})$, we have

$$H^{m+k-1} \cdot H' = H^k \cdot (H^{m-1} \cdot H') = \frac{k}{m}\, H^{2k-1} \cdot H' \approx \frac{k}{m}\frac{1}{2k}\,\delta,$$

$$H^{m+k-1} \cdot H' = H^{k-1} \cdot (H^m \cdot H') = \frac{k+1}{m+1}\, H^{2k-1} \cdot H' \approx \frac{k+1}{m+1}\frac{1}{2k}\,\delta,$$

which imply $\dfrac{k}{m} = \dfrac{k+1}{m+1}$, or $m = k$.

Now let us show that $H^m \not\approx H^k$ if $m \neq k$. To this end, it suffices to prove that $mH^{m-1} \cdot \delta \not\approx kH^{k-1} \cdot \delta$. Let $\psi \in \mathcal{D}(\mathbb{R})$. A representative of the generalized number

$$\int\limits_{\mathbb{R}} ((mH^{m-1} \cdot \delta - kH^{k-1} \cdot \delta) \cdot \psi)(x)\, dx$$

is the mapping

$$I(\varphi_\varepsilon) = \int\limits_{\mathbb{R}} \Big[m\Big(\int\limits_{-x/\varepsilon}^{\infty} \varphi(\mu)\, d\mu \Big)^{m-1} - k\Big(\int\limits_{-x/\varepsilon}^{\infty} \varphi(\mu)\, d\mu \Big)^{k-1} \Big] \frac{1}{\varepsilon} \varphi\Big(-\frac{x}{\varepsilon}\Big) \psi(x)\, dx =$$

$$= \int\limits_a^b [m\theta(y)^{m-1} - k\theta(y)^{k-1}] \varphi(y) \psi(-\varepsilon y)\, dy,$$

where

$$\varphi \in \mathcal{A}_0(\mathbb{R}), \quad \operatorname{supp} \varphi \subset [a, b], \quad \text{and} \quad \theta(y) = \int\limits_y^b \varphi(\mu)\, d\mu, \quad y \in \mathbb{R}. \tag{9.4}$$

Since $\theta(a) = 1$ and $\theta(b) = 0$,

$$\int\limits_a^b [m\theta(y)^{m-1} - k\theta(y)^{k-1}] \varphi(y)\, dy = 0,$$

and therefore,

$$\lim_{\varepsilon \to +0} \frac{I(\varphi_\varepsilon)}{\varepsilon} = -\psi'(0) \int\limits_a^b [m\theta(y)^{m-1} - k\theta(y)^{k-1}] \varphi(y) y\, dy =$$

$$= \psi'(0) \int\limits_a^b y \frac{d}{dy} [\theta(y)^m - \theta(y)^k]\, dy = -\psi'(0) \int\limits_a^b [\theta(y)^m - \theta(y)^k]\, dy.$$

Taking into account (3.3), we conclude at once that $I \notin \mathcal{N}_0$. □

The formula (9.3) is generalized as follows [30]:

Proposition 9.1. *Let $f : \mathbb{R} \setminus \{x_0\} \longrightarrow \mathbb{K}$ be a continuous function, and at the point x_0, there exist finite one-sided limits from the left $f(x_0-)$ and from the right $f(x_0+)$, and let $F \in \mathcal{O}_M(\mathbb{K})$. Then*

(a) *if $f(x_0-) = f(x_0+) =: f(x_0)$, and in particular, if f is continuous at x_0, then*

$$F(\jmath(f)) \cdot \delta_{x_0} \approx F(f(x_0))\, \delta_{x_0} \quad \text{in} \quad \mathcal{G}(\mathbb{R}); \tag{9.5}$$

(b) *if $f(x_0-) \neq f(x_0+)$, then*

$$F(\jmath(f)) \cdot \delta_{x_0} \approx \Big(\frac{1}{f(x_0+) - f(x_0-)} \int\limits_{f(x_0-)}^{f(x_0+)} F(z)\, dz \Big) \delta_{x_0} \quad \text{in} \quad \mathcal{G}(\mathbb{R}). \tag{9.6}$$

(Clearly, this proposition is of local character.)

Proof. Given $\psi \in \mathcal{D}(\mathbb{R})$, a representative of the corresponding generalized number is the mapping (we assume that (9.4) holds)

$$I(\varphi_\varepsilon) = \int F\Big(\int\limits_a^b f(x + \varepsilon\mu) \varphi(\mu)\, d\mu \Big) \frac{1}{\varepsilon} \varphi\Big(\frac{x_0 - x}{\varepsilon} \Big) \psi(x)\, dx =$$

$$= \int\limits_a^b F\Big(\Big[\int\limits_a^y + \int\limits_y^b \Big] f(x_0 + \varepsilon\mu - \varepsilon y) \varphi(\mu)\, d\mu \Big) \varphi(y) \psi(x_0 - \varepsilon y)\, dy.$$

In view of Lebesgue's bounded convergence theorem and conditions $\theta(a) = 1$ and $\theta(b) = 0$ (note that $\varphi \in \mathcal{A}_0(\mathbb{R})$), we have

$$\lim_{\varepsilon\to+0} I(\varphi_\varepsilon) = -\psi(x_0)\int_a^b F\Big(f(x_0-) + \theta(y)(f(x_0+) - f(x_0-))\Big)\frac{d}{dy}\theta(y)\,dy =$$

$$= \psi(x_0)\int_0^1 F\Big(f(x_0-) + t(f(x_0+) - f(x_0-))\Big)\,dt.$$

The assertion (a) follows at once from the last formula. If $f(x_0-) \neq f(x_0+)$, then using Hadamard's formula, we come to (b). □

In particular, if f is as in Proposition 9.1 and $F(x) = x$, then

$$\jmath(f)\cdot\delta_{x_0} \approx \frac{f(x_0+) + f(x_0-)}{2}\delta_{x_0} \quad \text{in} \quad \mathcal{G}(\mathbb{R}). \tag{9.7}$$

Formulas of this kind are encountered in Raju [164], where they are calculated from the point of view of nonstandard analysis (Stroyan and Luxemburg [187]), and the symbol $\approx$ in (9.7) is understood in the sense that the difference of the left- and right-hand sides of this equality is an infinitely small distribution; also they were established by Fisher [78], where they were found by the method of regularization and passage to the limit, which was defined in his earlier paper [75]. In connection with (9.6), note that in view of (9.7), we have

$$\jmath(F\circ f)\cdot\delta_{x_0} \approx \frac{F(f(x_0+)) + F(f(x_0-))}{2}\delta_{x_0}.$$

Formulas (9.2) and (9.3) are also generalized to the case of $\mathbb{R}^n$.

Example 9.2. (1) Let $H_n \in \mathcal{G}(\mathbb{R}^n)$ be the n-dimensional Heaviside function (Example 5.3). Then

$$(H_n)^m \approx H_n \quad \text{in} \quad \mathcal{G}(\mathbb{R}^n) \quad \forall m \in \mathbb{N}. \tag{9.8}$$

In fact, a representative of H_n in $\mathcal{G}(\mathbb{R}^n)$ is the mapping $u = u_{H_n}$ given by

$$u(\varphi_\varepsilon, x) = \int_{-x_1/\varepsilon}^{\infty}\dots\int_{-x_n/\varepsilon}^{\infty}\varphi(\mu_1,\dots,\mu_n)\,d\mu_1\dots d\mu_n, \quad \varphi\in\mathcal{A}_0(\mathbb{R}^n), \quad x = (x_1,\dots,x_n)\in\mathbb{R}^n,$$

from which we get the uniform boundedness of $|u(\varphi_\varepsilon, x)|$ and $|(u^m - u)(\varphi_\varepsilon, x)|$:

$$|u(\varphi_\varepsilon, x)| \le M, \quad |(u^m - u)(\varphi_\varepsilon, x)| \le M^m + M, \quad x\in\mathbb{R}^n, \quad \varepsilon > 0, \quad M := \int|\varphi|.$$

In order to apply Lebesgue's bounded convergence theorem, let us find the pointwise limit $\lim_{\varepsilon\to+0} u(\varphi_\varepsilon, x)$. If $x_1 > 0, \dots, x_n > 0$, then this limit is equal to 1, and if at least one $x_i < 0$, then it equals 0. In the other cases (i.e., if some of the x_i are greater than zero and the others are zero, so that the set of such points is of Lebesgue's n-dimensional measure zero) this limit exists, is finite, and depends on φ. Hence, $u(\varphi_\varepsilon, x)$ converges almost everywhere on $\mathbb{R}^n$ to the classical n-dimensional Heaviside function $H_n(x)$, and the same is valid for the function $u^m(\varphi_\varepsilon, x)$, which is a representative of $(H_n)^m$. Thus, $u^m(\varphi_\varepsilon, x) - u(\varphi_\varepsilon, x) \longrightarrow 0$ as $\varepsilon \longrightarrow +0$ for almost all $x \in \mathbb{R}^n$; this implies (9.8).

(2) If the algebra $\mathcal{G}(\mathbb{R}^n)$ is constructed starting from the index set $\mathcal{A}_0^{\otimes}(\mathbb{R}^n)$, then

$$(H_n)^{m-1}\cdot\delta_n \approx \frac{1}{m^n}\delta_n \quad \text{in} \quad \mathcal{G}(\mathbb{R}^n), \quad m\in\mathbb{N}, \tag{9.9}$$

where δ_n denotes the n-dimensional Dirac δ function from $\mathcal{G}(\mathbb{R}^n)$. Representatives of the Heaviside and the Dirac functions are related as follows:

$$u_{H_n}(\phi^{\otimes n}, x) = \prod_{i=1}^{n} u_H(\phi, x_i), \quad u_{\delta_n}(\phi^{\otimes n}, x) = \prod_{i=1}^{n} u_\delta(\phi, x_i), \quad \phi \in \mathcal{A}_0(\mathbb{R}),$$

where u_H and u_δ are respective representatives of the Heaviside and the Dirac functions from $\mathcal{G}(\mathbb{R})$. These relations can be rewritten in the form $u_{H_n} = (u_H)^{\otimes n}$ and $u_{\delta_n} = (u_\delta)^{\otimes n}$, and in this sense, in the algebra $\mathcal{G}(\mathbb{R}^n)$, we have

$$H_n = H^{\otimes n} \quad \text{and} \quad \delta_n = \delta^{\otimes n}.$$

Clearly, the equality (9.8) holds in the algebra $\mathcal{G}(\mathbb{R}^n)$ as well, so that differentiating, we find that

$$\partial_1 \ldots \partial_n (H_n)^m \approx \partial_1 \ldots \partial_n H_n = \delta_n.$$

Writing the operator $\partial_1 \ldots \partial_n$ in the form $\partial^{\otimes n}$, where ∂ is the derivative in $\mathcal{G}(\mathbb{R})$, it remains to note that

$$\partial^{\otimes n}(H^{\otimes n})^m = \partial^{\otimes n}(H^m)^{\otimes n} = (\partial(H^m))^{\otimes n} = (mH^{m-1} \cdot \delta)^{\otimes n} =$$

$$= m^n (H^{m-1})^{\otimes n} \cdot \delta^{\otimes n} = m^n (H^{\otimes n})^{m-1} \cdot \delta^{\otimes n} = m^n (H_n)^{m-1} \cdot \delta_n. \qquad \square$$

9.2. Multiplication of distributions. The following theorem (Jelínek [95]) characterizes the case where the product of two distributions in the algebra $\mathcal{G}(\mathbb{R}^n)$ has an associated distribution:

Theorem 9.3. *Let T, S, $R \in \mathcal{D}'(\mathbb{R}^n)$. Then the association relation*

$$T \cdot S \approx R \quad \text{in} \quad \mathcal{G}(\mathbb{R}^n) \tag{9.10}$$

is equivalent to the condition

$$\forall \psi \in \mathcal{D}(\mathbb{R}^n) \ \exists N \in \mathbb{N} \ \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n):$$

$$\langle R, \psi \rangle = \frac{1}{2^{n+1}} \lim_{\varepsilon \to +0} \langle T_x \otimes S_y, \psi\Big(\frac{x+y}{2}\Big)\Big[\varphi_\varepsilon\Big(\frac{x-y}{2}\Big) + \varphi_\varepsilon\Big(\frac{y-x}{2}\Big)\Big]\rangle; \tag{9.11}$$

here the tensor product $T_x \otimes S_y \in \mathcal{D}'(\mathbb{R}^{2n})$ of distributions T and S acts on a test function $\chi = \chi(x, y) \in \mathcal{D}(\mathbb{R}^{2n})$ according to the rule

$$\langle T_x \otimes S_y, \chi(x,y) \rangle = \langle T_x, \langle S_y, \chi(x,y) \rangle \rangle = \langle S_y, \langle T_x, \chi(x,y) \rangle \rangle.$$

Proof. In fact, we do not prove this theorem here; we only reformulate the main result of [95] in the convenient form (9.11), recalling some relevant definitions along the way. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is said to *have value $c \in \mathbb{C}$ in the sense of Lojasiewicz* [126] *at a point $x \in \mathbb{R}^n$* if

$$\lim_{\varepsilon \to +0} \langle T, \tau_x \varphi_\varepsilon \rangle = \lim_{\varepsilon \to +0} (T * \check{\varphi}_\varepsilon)(x) = c \quad \forall \varphi \in \mathcal{A}_0(\mathbb{R}^n);$$

this is written as $T(x) = c$. For example, the Heaviside function H and the Dirac δ function do not have values at $x = 0$, and any continuous function on $\mathbb{R}^n$ has a value in the sense of Lojasiewicz at every point, which coincides with the classical value. Not only continuous functions have values at points; for example, the distribution on $\mathbb{R}$ of the form $T_r = \sum_{m=1}^{\infty} \frac{1}{m^r}(\tau_{\frac{1}{m}}\delta)$, $r > 0$, which acts on a test function as

$$\langle T_r, \psi \rangle = \sum_{m=1}^{\infty} \frac{1}{m^r} \psi\Big(-\frac{1}{m}\Big), \quad \psi \in \mathcal{D}(\mathbb{R}),$$

has the following property: $T_r(0) = 0 \iff r > 2$ (for details, see the book by Oberguggenberger [156, 7.10–7.11]).

A distribution $T = T_{x,y} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$ is said to *have section* $T|\{y = 0\} \in \mathcal{D}'(\mathbb{R}^n)$ *in the sense of Lojasiewicz* [127] *at* $y = 0 \in \mathbb{R}^m$ if for every function $\psi \in \mathcal{D}(\mathbb{R}^n)$, the distribution $T_\psi(y) := \langle T_{x,y}, \psi(x)\rangle \in \mathcal{D}'(\mathbb{R}^m)$ has a value in the sense of Lojasiewicz at the point $y = 0$; in other words, the section $T|\{y = 0\}$ is defined by its action on a test function $\psi \in \mathcal{D}(\mathbb{R}^n)$ as follows:

$$\langle T|\{y=0\}, \psi\rangle = T_\psi(0) = \lim_{\varepsilon\to+0} \langle T_{x,y}, \psi(x)\varphi_\varepsilon(y)\rangle \quad \forall \varphi \in \mathcal{A}_0(\mathbb{R}^m). \tag{9.12}$$

The main result of [95, Thm. 1] asserts that the association relation (9.10) is equivalent to the equality

$$R_x = \frac{1}{2}[T_{x-y} \otimes S_{x+y} + T_{x+y} \otimes S_{x-y}]|\{y = 0\} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n). \tag{9.13}$$

The first tensor product $T_{x-y} \otimes S_{x+y} \in \mathcal{D}'(\mathbb{R}^{2n})$ from (9.13) acts on a test function $\chi = \chi(x,y) \in \mathcal{D}(\mathbb{R}^{2n})$ according to the rule (in the second equality below we change variables $(x-y, x+y) \longmapsto (x,y)$ with the Jacobian equal to $(1/2)^n$):

$$\begin{aligned}\langle T_{x-y} \otimes S_{x+y}, \chi(x,y)\rangle &= \iint T(x-y)\cdot S(x+y)\cdot\chi(x,y)\,dx\,dy = \\ = \frac{1}{2^n}\iint T(x)\cdot S(y)\cdot \chi\Big(\frac{x+y}{2}, \frac{y-x}{2}\Big)\,dx\,dy &= \frac{1}{2^n}\langle T_x \otimes S_y, \chi\Big(\frac{x+y}{2}, \frac{y-x}{2}\Big)\rangle.\end{aligned} \tag{9.14}$$

Computing analogously the second tensor product from (9.13) but changing variables $(x+y, x-y) \longmapsto (x,y)$, we find that

$$\langle T_{x+y} \otimes S_{x-y}, \chi(x,y)\rangle = \frac{1}{2^n}\langle T_x \otimes S_y, \chi\Big(\frac{x+y}{2}, \frac{x-y}{2}\Big)\rangle. \tag{9.15}$$

Setting $\chi(x,y) = \psi(x)\varphi_\varepsilon(y)$ in (9.14) and (9.15), where $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, and using (9.13) and (9.12), we come to (9.11). □

Examples 9.4. Using Theorem 9.3, let us show that

$$\mathrm{vp}\frac{1}{x}\cdot\delta \approx -\frac{1}{2}\delta' \quad \text{in} \quad \mathcal{G}(\mathbb{R}). \tag{9.16}$$

(In a different way this equality was obtained by Kamiński [100], Colombeau [37, 2.1.4], and Oberguggenberger [156, 7.7 and 10.5(b)]. The historically first equality of this kind is due to Gonzalez Domingues and Scarfiello [81].) Given $\psi \in \mathcal{D}(\mathbb{R})$ and $\varphi \in \mathcal{A}_0(\mathbb{R})$, we set

$$\chi_\varepsilon(x,y) = \psi\Big(\frac{x+y}{2}\Big)\Big[\varphi_\varepsilon\Big(\frac{x-y}{2}\Big) + \varphi_\varepsilon\Big(\frac{y-x}{2}\Big)\Big];$$

then

$$\begin{aligned}\frac{1}{4}\langle \mathrm{vp}\frac{1}{x}, \langle\delta(y), \chi_\varepsilon(x,y)\rangle\rangle &= \frac{1}{4}\langle \mathrm{vp}\frac{1}{x}, \chi_\varepsilon(x,0)\rangle = \\ &= \frac{1}{4}\lim_{a\to+0}\int_{|y|\ge a}\frac{\psi(y)-\psi(0)}{y}[\varphi_\varepsilon(y)+\varphi_\varepsilon(-y)]\,dy = \\ = \frac{1}{4}\int_{\mathbb{R}}\Big(\int_0^1 \psi'(ty)\,dt\Big)[\varphi_\varepsilon(y) + \check{\varphi}_\varepsilon(y)]\,dy &\underset{\varepsilon\to+0}{\longrightarrow} \frac{1}{2}\psi'(0) = \langle -\frac{1}{2}\delta', \psi\rangle;\end{aligned}$$

this yields (9.16). Since $x\cdot\delta \approx 0 \Longrightarrow x\cdot\delta' + \delta \approx 0$, from (9.16), in view of Proposition 8.9, we have $x\cdot\mathrm{vp}\frac{1}{x}\cdot\delta \approx \frac{1}{2}\delta$; this implies $x\cdot\mathrm{vp}\frac{1}{x} \not\approx 1$ and $x\cdot\delta \not\approx 0$ in $\mathcal{G}(\mathbb{R})$.

One easily verifies that Theorem 9.3 implies (9.3), (9.7), and Theorem 8.10. Consider a more interesting example of relations for the *Heisenberg distributions*

$$\delta_+ = \frac{1}{2}(\delta + \frac{1}{\pi i}\operatorname{vp}\frac{1}{x}), \quad \delta_- = \frac{1}{2}(\delta - \frac{1}{\pi i}\operatorname{vp}\frac{1}{x}),$$

which are often used in quantum mechanics. The relations are of the form (in a different way they were obtained by Mikusiński [141])

$$(\delta_\pm)^2 \approx \mp\frac{1}{4\pi i}\delta' - \frac{1}{4\pi^2}\operatorname{Pf}\frac{1}{x^2},$$

where the distribution $\operatorname{Pf}\frac{1}{x^2} \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle \operatorname{Pf}\frac{1}{x^2}, \psi\rangle = \operatorname{vp}\int_{\mathbb{R}} \frac{\psi(x) - \psi(0)}{x^2}\,dx, \quad \psi \in \mathcal{D}(\mathbb{R}). \qquad \square$$

In general, most of the particular products of distributions that were defined earlier in the framework of the distribution theory are recovered by means of the association relation from their product in the Colombeau algebra. Here is a list of papers (which is far from complete) in this direction (see also references in the cited papers): Ambrose [3], Berg [12, 13], Cerutti [26], Cheng and Fisher [27, 76, 77], Itano [90, 91, 92], Kamiński [98–103], Li Bang-He [116], Lodder [124, 125], Mikusiński [139, 140], Oberguggenberger [145, 150, 156], Panzone [159], Shiraishi [181, 182], Tysk [197], Wagner [202], Wawak [203–206].

In the definition of an associated distribution (8.3), it is required that the limit equality hold for all $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$. But as we have already mentioned (Remark 2.2), if $N \geq 2$, then elements of the set $\mathcal{A}_N(\mathbb{R}^n)$ cannot be nonnegative only or nonpositive only. The following lemma, which is a very particular case of the results in Oberguggenberger [144], shows that if the limit equality in definition (8.3) in the case of the product of distributions holds for all $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ with $\varphi \geq 0$, then it holds also for all $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$:

Lemma 9.5. *Let T_1, T_2, $S \in \mathcal{D}'(\Omega)$. Assume that for all $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, $\varphi \geq 0$, we have*

$$\lim_{\varepsilon \to +0}\int_\Omega (T_1 * \check{\varphi}_\varepsilon)(x)(T_2 * \check{\varphi}_\varepsilon)(x)\psi(x)\,dx = \langle S, \psi\rangle \quad \forall \psi \in \mathcal{D}(\Omega). \tag{9.17}$$

Then equality (9.17) *holds for all $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$, so that*

$$T_1 \cdot T_2 \approx S \quad \text{in} \quad \mathcal{G}(\Omega).$$

Proof. First of all, note that for small $\varepsilon > 0$, the convolution $T_i * \check{\varphi}_\varepsilon$, $i = 1, 2$, is a well-defined C^∞ function on a neighborhood of the compact set $\operatorname{supp}\psi$. Let $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. Choose $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi \geq 0$ and $\varphi + \chi \geq 0$, and let $c = \int \chi(x)\,dx$. Then $\varphi + 2\chi \geq 0$. In the equality

$$a_1a_2 = 2b_1b_2 + 2(a_1 + b_1)(a_2 + b_2) - (a_1 + 2b_1)(a_2 + 2b_2),$$

set $a_i = (T_i * \check{\varphi}_\varepsilon)(x)$ and $b_i = (T_i * \check{\chi}_\varepsilon)(x)$, $i = 1, 2$. Then by assumption (9.17), the integral

$$\int (T_1 * \check{\varphi}_\varepsilon)(T_2 * \check{\varphi}_\varepsilon)\psi = 2\int (T_1 * \check{\chi}_\varepsilon)(T_2 * \check{\chi}_\varepsilon)\psi +$$

$$+2\int [T_1 * (\varphi + \chi)_\varepsilon^\vee][T_2 * (\varphi + \chi)_\varepsilon^\vee]\psi - \int [T_1 * (\varphi + 2\chi)_\varepsilon^\vee][T_2 * (\varphi + 2\chi)_\varepsilon^\vee]\psi$$

tends to

$$[2c^2 + 2(1 + c)^2 - (1 + 2c)^2]\langle S, \psi\rangle = \langle S, \psi\rangle,$$

as $\varepsilon \longrightarrow +0$; this is what is required. $\square$

Let us consider an example of the product of functions with infinite discontinuities.

Example 9.6. Let x_+ be the function defined in Example 5.3, and let $x_- = (-x)_+$. If $0 < \alpha < 1$, the functions $x_+^{-\alpha}$ and $x_-^{\alpha-1}$ are locally integrable on $\mathbb{R}$ and their classical product is 0 (outside the point $x = 0$). Let us compute the product of these functions in $\mathcal{G}(\mathbb{R})$ up to association, namely, let us show that

$$x_+^{-\alpha} \cdot x_-^{\alpha-1} \approx \frac{\pi}{2} \frac{1}{\sin \pi\alpha} \delta \quad \text{in} \quad \mathcal{G}(\mathbb{R}).$$

Given $\varphi \in \mathcal{A}_0(\mathbb{R})$, $\varepsilon > 0$, and $\psi \in \mathcal{D}(\mathbb{R})$, we have

$$I(\varphi_\varepsilon) = \langle (x_+^{-\alpha} * \varphi_\varepsilon)(x_-^{\alpha-1} * \varphi_\varepsilon), \psi \rangle =$$

$$= \iiint (x-\lambda)_+^{-\alpha} \frac{1}{\varepsilon} \varphi\left(\frac{\lambda}{\varepsilon}\right)(x-\mu)_-^{\alpha-1} \frac{1}{\varepsilon} \varphi\left(\frac{\mu}{\varepsilon}\right) \psi(x)\, d\lambda\, d\mu\, dx =$$

$$= \int \psi(\varepsilon x)\, dx \int_{-\infty}^{x} d\lambda \int_{x}^{\infty} \varphi(\lambda)\varphi(\mu)(x-\lambda)^{-\alpha}(\mu - x)^{\alpha-1} d\mu \underset{\varepsilon \to +0}{\longrightarrow}$$

$$\underset{\varepsilon \to +0}{\longrightarrow} \psi(0) \int dx \int_{x}^{\infty} d\mu \int_{-\infty}^{x} \varphi(\mu)\varphi(\lambda)(x-\lambda)^{-\alpha}(\mu - x)^{\alpha-1} d\lambda \equiv \psi(0) \cdot I_0.$$

Changing the order of integrations in I_0 and then changing variables, we obtain

$$I_0 = \int \varphi(\mu)\, d\mu \int_{-\infty}^{\mu} \varphi(\lambda)\, d\lambda \int_{\lambda}^{\mu} (x-\lambda)^{-\alpha}(\mu - x)^{\alpha-1} dx =$$

$$= \int \varphi(\mu)\, d\mu \int_{-\infty}^{\mu} \varphi(\lambda)\, d\lambda \cdot \int_{0}^{1} t^{-\alpha}(1-t)^{\alpha-1} dt = \frac{1}{2}\, \mathrm{B}(1-\alpha, \alpha),$$

where B is Euler's beta function. It remains to note that

$$\mathrm{B}(1-\alpha, \alpha) = \Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin \pi\alpha},$$

and replace φ by $\check{\varphi}$ in the above. Note that from different considerations the obtained formula was also proved by Fisher [78]. □

9.3. Heaviside and Dirac generalized functions.

Definition 9.7. A generalized function $Y \in \mathcal{G}(\mathbb{R})$ is said to be a *Heaviside generalized function* (in short H.g.f.) if it has a representative $u_Y \in \mathcal{E}_M[\mathbb{R}]$ with the property

$\forall \varphi \in \mathcal{A}_0(\mathbb{R})\ \exists a : (0, \infty) \longrightarrow (0, \infty),\ \lim_{\varepsilon \to +0} a(\varepsilon) = 0$, and $\exists \eta > 0$ such that

(a) $u_Y(\varphi_\varepsilon, x) = 0$ if $\varepsilon > 0$ and $x < -a(\varepsilon)$;

(b) $u_Y(\varphi_\varepsilon, x) = 1$ if $\varepsilon > 0$ and $x > a(\varepsilon)$;

(c) $\sup_{\varepsilon \in (0,\eta)} \sup_{x \in [-a(\varepsilon), a(\varepsilon)]} |u_Y(\varphi_\varepsilon, x)| < \infty$.

(Note that the definition of a H.g.f. which imitates the behavior of the classical Heaviside function is not completely fixed, and it depends on the context. For other definitions and examples, see Aragona and Biagioni [8], Biagioni [15, 1.9], Colombeau [43, II § 5], Colombeau and Le Roux [60], Egorov [72, § 6.4], and [31].) □

Clearly, the Heaviside function $H \in \mathcal{G}(\mathbb{R})$ is a H.g.f. Another example is the function $Y \in \mathcal{G}(\mathbb{R})$ with representative $u_Y(\varphi) = H * \Phi_{\rho(\varphi)}$, $\varphi \in \mathcal{A}_0(\mathbb{R})$, where $\Phi \in \mathcal{A}_0(\mathbb{R})$ is a fixed function and the mapping ρ is defined in Sec. 1.

In order to give one more example of a H.g.f., note that

$$\text{if } \varphi,\, \psi \in \mathcal{D}(\mathbb{R}^n), \text{ then } \varphi * \psi \in \mathcal{D}(\mathbb{R}^n) \text{ and}$$

$$M^\alpha(\varphi * \psi) = \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} M^{\alpha-\beta}(\varphi) M^\beta(\psi), \quad \alpha \in \mathbb{N}_0^n,$$

$$(\varphi * \psi)_\varepsilon = \varphi_\varepsilon * \psi_\varepsilon, \quad \varepsilon > 0;$$

this, in particular, implies

$$\varphi,\, \psi \in \mathcal{A}_q(\mathbb{R}^n) \implies \varphi * \psi \in \mathcal{A}_q(\mathbb{R}^n), \; q \in \mathbb{N}_0.$$

An element of $\mathcal{G}(\mathbb{R})$ with the representative $u(\varphi) = H * (\varphi^{*m})$, $\varphi \in \mathcal{A}_0(\mathbb{R})$, is a H.g.f., where we denote by $\varphi^{*m} := \varphi * \cdots * \varphi$ the m-fold convolution of φ with $m \in \mathbb{N}$ fixed.

If $Y \in \mathcal{G}(\mathbb{R})$ is a H.g.f. with a representative u_Y, then for any $\varphi \in \mathcal{A}_0(\mathbb{R})$, we have

$$\lim_{\varepsilon \to +0} \int u_Y(\varphi_\varepsilon, x)\psi(x)\, dx = \int_0^\infty \psi(x)\, dx, \quad \psi \in \mathcal{D}(\mathbb{R}).$$

Hence, $Y \approx H$ in $\mathcal{G}(\mathbb{R})$, so that all H.g.f.s are associated and the derivative Y' of Y is associated to the Dirac δ function:

$$\int \Big(\frac{d}{dx} u_Y(\varphi_\varepsilon, x)\Big)\psi(x)\, dx = -\int u_Y(\varphi_\varepsilon, x)\psi'(x)\, dx \underset{\varepsilon \to +0}{\longrightarrow} -\int_0^\infty \psi'(x)\, dx = \psi(0) = \langle \delta, \psi \rangle.$$

Any natural power Y^m of a H.g.f. $Y \in \mathcal{G}(\mathbb{R})$ is also a H.g.f., and so differentiating the equalities $Y^m \approx H \approx Y$, we obtain

$$mY^{m-1} \cdot Y' \approx \delta \approx Y'. \tag{9.18}$$

As in (9.2) one can verify that $Y^m \neq Y^k$ and $Y^m \not\approx Y^k$ if $m \neq k$.

Analogously we can define Dirac generalized functions:

Definition 9.8. A generalized function $D \in \mathcal{G}(\mathbb{R}^n)$ is said to be a *Dirac generalized function* (in short D.g.f.) if it has a representative $d \in \mathcal{E}_M[\mathbb{R}^n]$ with the property

$$\forall \varphi \in \mathcal{A}_0(\mathbb{R}) \; \exists a : (0,\infty) \longrightarrow (0,\infty), \; \lim_{\varepsilon \to +0} a(\varepsilon) = 0, \text{ and } \exists \eta > 0 \text{ such that}$$

(a) $d(\varphi_\varepsilon, x) = 0$ if $\varepsilon > 0$ and $|x| > a(\varepsilon)$;

(b) $\int_{\mathbb{R}^n} d(\varphi_\varepsilon, x) = 1$ if $\varepsilon > 0$;

(c) $\sup_{\varepsilon \in (0,\eta)} \int_{\mathbb{R}^n} |d(\varphi_\varepsilon, x)|\, dx < \infty$.

(The same remark as in Definition 9.7 holds.) □

As examples of D.g.f.s, we have the following elements of $\mathcal{G}(\mathbb{R}^n)$ with representatives: $u(\varphi) = \check{\varphi}$ (the usual Dirac δ function), $u(\varphi) = \Psi_{\rho(\varphi)}$ ($\Psi \in \mathcal{A}_0(\mathbb{R}^n)$ is fixed), $u(\varphi) = \varphi^{*m}$ or $u(\varphi) = (\check{\varphi})^{*m}$ ($m \in \mathbb{N}$) with $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$.

Any two D.g.f.s are associated since for a representative $d(\varphi_\varepsilon, x)$ of such a generalized function we have

$$\int d(\varphi_\varepsilon, x)\psi(x)\, dx \longrightarrow \psi(0), \quad \varepsilon \longrightarrow +0, \quad \psi \in \mathcal{D}(\mathbb{R}^n), \quad \varphi \in \mathcal{A}_0(\mathbb{R}^n).$$

Thus, in Colombeau's theory, unlike the distribution theory, there are many generalized functions which are similar to the Heaviside function and there are many generalized functions which are similar to the Dirac δ function.

Example 9.9. Let $n = 1$ and $D \in \mathcal{G}(\mathbb{R})$ be a D.g.f. with a representative $d(\varphi_\varepsilon, x)$. The mapping

$$u(\varphi_\varepsilon, x) = \int_{a_0}^{x} d(\varphi_\varepsilon, t)\, dt \quad \text{(for some } \ a_0 < 0)$$

defines a H.g.f. $Y \in \mathcal{G}(\mathbb{R})$ such that $Y' = D$ in $\mathcal{G}(\mathbb{R})$. For any $a > 0$ we show that (Biagioni [15, 1.9.6])

$$D(x^2 - a^2) \approx \frac{1}{2a}(\tau_a\delta + \tau_{-a}\delta) \quad \text{in} \quad \mathcal{G}(\mathbb{R}).$$

In particular, this equality holds for the distribution $D = \delta \in \mathcal{D}'(\mathbb{R})$.

First, note that the composition $D(x^2 - a^2)$ is well defined according to Theorem 4.6. Let $\varphi \in \mathcal{A}_0(\mathbb{R})$ and let $\eta > 0$ be such that $a(\varepsilon) < a^2$ for $\varepsilon \in (0, \eta)$. Setting $s_y = \sqrt{y + a^2}$, for $\psi \in \mathcal{D}(\mathbb{R})$ and $\varepsilon \in (0, \eta)$, we have

$$\int_{\mathbb{R}} d(\varphi_\varepsilon, x^2 - a^2)\psi(x)\, dx = \int_{\mathbb{R}} u'(\varphi_\varepsilon, x^2 - a^2)\psi(x)\, dx =$$

$$= \frac{1}{2}\int_{-a^2}^{\infty} u'(\varphi_\varepsilon, y)\frac{\psi(s_y) + \psi(-s_y)}{s_y}\, dy \quad \text{(integrate by parts)} =$$

$$= -\frac{1}{4}\int_{-a(\varepsilon)}^{\infty} u(\varphi_\varepsilon, y)\Big[\frac{\psi'(s_y) - \psi'(-s_y)}{(s_y)^2} - \frac{\psi(s_y) + \psi(-s_y)}{(s_y)^3}\Big]\, dy =$$

$$= -\frac{1}{4}\Big(\int_{-a(\varepsilon)}^{a(\varepsilon)} + \int_{a(\varepsilon)}^{\infty}\Big)u(\varphi_\varepsilon, y)\Big[\frac{\psi'(s_y)}{(s_y)^2} - \frac{\psi(s_y)}{(s_y)^3} - \frac{\psi'(-s_y)}{(s_y)^2} - \frac{\psi(-s_y)}{(s_y)^3}\Big]\, dy.$$

The first of the above integrals tends to zero as $\varepsilon \longrightarrow +0$ since ψ has compact support and $\int_{-a(\varepsilon)}^{a(\varepsilon)} |u(\varphi_\varepsilon, x)|\, dx \longrightarrow 0$ as $\varepsilon \longrightarrow +0$. In view of condition (b) of Definition 9.7, it now suffices to consider the integral

$$-\frac{1}{4}\int_{a(\varepsilon)}^{\infty}\Big(\frac{\psi'(s_y)}{(s_y)^2} - \frac{\psi(s_y)}{(s_y)^3} - \frac{\psi'(-s_y)}{(s_y)^2} - \frac{\psi(-s_y)}{(s_y)^3}\Big)\, dy =$$

$$= -\frac{1}{2}\int_{s_{a(\varepsilon)}}^{\infty}\Big(\frac{\psi'(z)}{z} - \frac{\psi(z)}{z^2} - \frac{\psi'(-z)}{z} - \frac{\psi(-z)}{z^2}\Big)\, dz =$$

$$= \frac{\psi(s_{a(\varepsilon)}) + \psi(-s_{a(\varepsilon)})}{2s_{a(\varepsilon)}} \quad \longrightarrow \quad \frac{\psi(a) + \psi(-a)}{2a} \quad \text{as} \ \ \varepsilon \longrightarrow +0. \quad \square$$

From (9.3), we have $H \cdot \delta \approx (1/2)\delta$, where H and δ are the classical Heaviside and Dirac functions. In view of (9.18), this is also true for generalized functions. Sometimes in real physical problems, the coefficient 1/2 of the δ function in the formulas above does not reflect correctly the physical phenomenon. Thus, we note that if we take different H.g.f.s and D.g.f.s, then this coefficient can assume any value (in physical considerations this is usually a number between -1 and 1). For example, let a H.g.f. $Y \in \mathcal{G}(\mathbb{R})$, and let a D.g.f. $D \in \mathcal{G}(\mathbb{R})$ have respective representatives (Oberguggenberger [156, 10.5])

$$u_Y(\varphi) = H * \Phi_{\rho(\varphi)}, \quad d(\varphi) = \Psi_{\rho(\varphi)}, \quad \varphi \in \mathcal{A}_0(\mathbb{R}),$$

where Φ, $\Psi \in \mathcal{A}_0(\mathbb{R})$ are fixed. Then it is easily seen that

$$Y \cdot D \approx \Big(\int\limits_{\mathbb{R}} \int\limits_{-\infty}^{y} \Phi(x)\,dx \Psi(y)\,dy\Big)\delta \quad \text{in} \quad \mathcal{G}(\mathbb{R}),$$

and with the appropriate choice of Φ and Ψ one can have the coefficient of δ as an arbitrary number. □

9.4. Discontinuous solutions of the equation $u_t + f(u)_x \approx 0$. Consider the following first-order quasilinear equation ([165, 183, 31]):

$$u_t(x,t) + f(u(x,t))_x = 0, \quad (x,t) \in S := \mathbb{R} \times (0,\infty), \tag{9.19}$$

where $f \in \mathcal{O}_M(\mathbb{R})$, $u_t(x,t) = \partial_t u(x,t)$, $f(u(x,t))_x = \partial_x f(u(x,t))$. We are interested in discontinuous solutions of this equation corresponding to the step-like initial data:

$$u(x,0) = \begin{cases} u_L & \text{if} \quad x < 0, \\ u_R & \text{if} \quad x > 0, \end{cases} \tag{9.20}$$

where u_L, $u_R \in \mathbb{R}$, $u_L \neq u_R$ (this corresponds to the Riemann problem). Let $H \in \mathcal{G}(\mathbb{R})$ be a H.g.f. (not necessarily a distribution). Equation (9.19) is naturally interpreted in the form

$$u_t + f(u)_x \approx 0 \quad \text{in} \quad \mathcal{G}(S), \tag{9.21}$$

and we search for solutions to this equation of the form

$$u(x,t) = \Delta u H(x - vt) + u_L, \quad \Delta u := u_R - u_L, \tag{9.22}$$

where $v \in \mathbb{R}$ is an unknown number (the velocity of the shock wave or the speed of the discontinuity). Solutions of the form (9.22) will be called *traveling waves*. Substituting the function $u(x,t)$ into Eq. (9.21) and noting that $u_t(x,t) = -v\Delta u H'(x - vt)$, we find that

$$-v\Delta u H'(x - vt) + f(\Delta u H(x - vt) + u_L)_x \approx 0 \quad \text{in} \quad \mathcal{G}(S).$$

We will show below that in $\mathcal{G}(\mathbb{R})$, the following equality holds:

$$f(\Delta u H(x - vt) + u_L)_x \approx (f(u_R) - f(u_L))H'(x - vt); \tag{9.23}$$

assume now that it is fulfilled. Then

$$-v\Delta u H'(x - vt) + (f(u_R) - f(u_L))H'(x - vt) \approx 0.$$

Since $H'(x - vt) \neq 0$ in $\mathcal{G}(\mathbb{R})$ and $\Delta u \neq 0$, we have

$$v = \frac{f(u_R) - f(u_L)}{u_R - u_L}. \tag{9.24}$$

Relation (9.24) is called the *jump condition*; in gas dynamics it is known as the *Rankine–Hugoniot condition* [183, Part III]. Thus, we have shown that

> Equation (9.21) has generalized solutions from $\mathcal{G}(\mathbb{R})$ in the form of traveling waves (9.22) provided the values v, u_L and u_R are tied together by the Rankine–Hugoniot condition (9.24).

Now we prove (9.23). Let $\psi = \psi(x,t) \in \mathcal{D}(S)$, and let $h \in \mathcal{E}_M[\mathbb{R}]$ be a representative of the H.g.f. H. Integrating by parts, for $\varphi \in \mathcal{A}_0(\mathbb{R})$ and $\varepsilon > 0$, we have

$$I(\varphi_\varepsilon) := \lim_{\varepsilon \to +0} \int\limits_0^\infty \int\limits_{\mathbb{R}} f(\Delta u h(\varphi_\varepsilon, x - vt) + u_L)_x \cdot \psi(x,t)\,dx\,dt =$$

$$= -\lim_{\varepsilon\to+0}\Big(\iint\limits_{S\cap\{x\ge vt\}} + \iint\limits_{S\cap\{x\le vt\}}\Big) f(\Delta u h(\varphi_\varepsilon, x-vt)+u_L)\psi_x(x,t)\,dx\,dt =$$

$$= -f(u_R)\int_0^\infty dt \int_{vt}^\infty \psi_x(x,t)\,dx - f(u_L)\int_0^\infty dt \int_{-\infty}^{vt} \psi_x(x,t)\,dx,$$

so that in view of the relation between the integral and the derivative, we have

$$I(\varphi_\varepsilon) = (f(u_R)-f(u_L))\int_0^\infty \psi(vt,t)\,dt. \tag{9.25}$$

On the other hand,

$$\lim_{\varepsilon\to+0}\int_0^\infty\int_{\mathbb{R}} h'(\varphi_\varepsilon, x-vt)\psi(x,t)\,dx\,dt =$$

$$= -\lim_{\varepsilon\to+0}\Big(\iint\limits_{S\cap\{x\ge vt\}} + \iint\limits_{S\cap\{x\le vt\}}\Big) h(\varphi_\varepsilon, x-vt)\psi_x(x,t)\,dx\,dt =$$

$$= -\int_0^\infty dt\int_{vt}^\infty \psi_x(x,t)\,dx = \int_0^\infty \psi(vt,t)\,dt,$$

hence, taking into account (9.25), we obtain (9.23).

Note that (Colombeau [45]) Eq. (9.19) with the strong equality = in $\mathcal{G}(S)$ has no discontinuous solutions in the form of traveling waves (9.22). For simplicity, assume that $f(u) = u^2/2$. Then $v = (u_R+u_L)/2$ by virtue of (9.24). Multiplying the equation $u_t + uu_x = 0$ by u, we find that

$$\partial_t\Big(\frac{u^2}{2}\Big) + \partial_x\Big(\frac{u^3}{3}\Big) = 0 \quad \text{in} \quad \mathcal{G}(S).$$

Substituting function (9.22) into this equation and arguing as above, we obtain

$$v = \frac{2}{3}\,\frac{u_R^2 + u_R u_L + u_L^2}{u_R+u_L},$$

but this value is inconsistent with the above value v. The effect here is that multiplication by a singular function u does not preserve the association:

$$u_t + uu_x \approx 0 \quad \not\Longleftrightarrow \quad uu_t + u^2u_x \approx 0.$$

In different (sub)spaces of the space of distributions, Burgers' equation (see (9.19), (9.21)) with the flux function $f(u) = u^2/2$ with or without viscosity was studied by many authors; let us mention here the classical work of Hopf [87] and recent papers of Dix [69] and the author [28, 29], for the viscous Burgers' equation. In algebras of Colombeau's generalized functions, Burgers' equation and its regularizations were studied by Biagioni and Oberguggenberger [20, 21]; see also the book by Oberguggenberger [156, Chap. V].

9.5. Traveling waves and delta waves. Consider the following quasilinear first-order system of conservation laws [48, 183, 31]:

$$\begin{aligned} u_t + f(u)_x &\approx 0 \quad \text{in} \quad \mathcal{G}(S),\\ \rho_t + (u\rho)_x &\approx 0 \quad \text{in} \quad \mathcal{G}(S). \end{aligned} \tag{9.26}$$

Solutions of this system are sought in the form of a traveling wave (9.22) and in the form of a combination of a traveling wave and a delta wave:

$$\rho(x,t) = \Delta\rho K(x-vt) + \rho_L + \alpha\, t\,\delta(x-vt), \tag{9.27}$$

where $\Delta\rho = \rho_R - \rho_L \neq 0$, $\alpha \in \mathbb{R}$, v is the velocity of the shock (9.24), H and $K \in \mathcal{G}(\mathbb{R})$ are (possibly different) H.g.f.s and $\delta \in \mathcal{G}(\mathbb{R})$ is a D.g.f. Note that solutions (9.22) and (9.27) correspond to the step-like initial data (9.20) and

$$\rho(x,0) = \begin{cases} \rho_L & \text{if } x < 0, \\ \rho_R & \text{if } x > 0, \end{cases}$$

in accordance with the Riemann problem. Noting that

$$\rho_t(x,t) = -v\Delta\rho K'(x-vt) + \alpha\,\delta(x-vt) - v\alpha\,t\,\delta'(x-vt),$$

$$\rho_x(x,t) = \Delta\rho K'(x-vt) + \alpha\,t\,\delta'(x-vt),$$

and substituting (9.27) and (9.22) into the second equation of system (9.26), we find that the relation

$$\rho_t + u_x\rho + u\rho_x \approx 0$$

is of the form (for brevity we omit the argument $x - vt$)

$$(-v\Delta\rho K' + \alpha\,\delta - v\alpha\,t\,\delta') + \Delta u H' \cdot (\Delta\rho K + \rho_L + \alpha\,t\,\delta) +$$

$$+ (\Delta u H + u_L) \cdot (\Delta\rho K' + \alpha\,t\,\delta') \approx 0.$$

Regrouping the terms in the last equation, we have

$$\begin{aligned} &(u_L - v)\Delta\rho K' + \rho_L\Delta u H' + \alpha\,\delta + \Delta u\Delta\rho(KH)' + \\ &+\alpha\,t\,(\Delta u(H\delta)' + (u_L - v)\delta') \approx 0. \end{aligned} \tag{9.28}$$

Consider the last term in parentheses. Set $\delta = M'$, where $M \in \mathcal{G}(\mathbb{R})$ is a H.g.f. having the property

$$H \cdot \delta \approx \beta M' \approx \beta\delta, \quad \text{with} \quad \beta := \frac{v - u_L}{\Delta u}. \tag{9.29}$$

Since $(H\delta)' \approx \beta\delta'$, the term in the parentheses vanishes if

$$\beta\Delta u + u_L - v = 0. \tag{9.30}$$

Noting that $KH \approx H \approx K$ and $(KH)' \approx H' \approx K' \approx \delta$, from the remaining terms in (9.28) we obtain the condition on α

$$(u_L - v)\Delta\rho + \rho_L\Delta u + \alpha + \Delta u\Delta\rho = 0.$$

Evaluating α with regard to (9.30) and (9.29), we have

$$\alpha = v\Delta\rho - (u_R\rho_R - u_L\rho_L). \tag{9.31}$$

Thus,

> system (9.26) has generalized solutions from $\mathcal{G}(S)$ in the form of the traveling wave (9.22) and in the form of a combination of the traveling wave and the delta wave (9.27) provided a D.g.f. $\delta = M'$ is chosen so that (9.29) holds and α is of the form (9.31).

For example, if $f(u) = u^2/2$ (Burgers' equation (9.21) without viscosity), we have $v = (u_R + u_L)/2$, and the generalized function δ can be chosen to be equal to H' (i.e., $M = H$), where H is the H.g.f. from (9.22), so that $\beta = 1/2$, $HH' \approx (1/2)H'$, and $\alpha = -\Delta u(\rho_R + \rho_L)/2$.

Note that calculations above with function (9.27) have no analog in the distribution theory. Note also that if $\alpha = 0$, i.e., the *Rankine–Hugoniot jump condition* for the second equation of the system (9.26) holds

$$v = \frac{u_R\rho_R - u_L\rho_L}{\rho_R - \rho_L},$$

then the system (9.26) has generalized solutions from $\mathcal{G}(S)$ in the form of traveling waves (9.22) and $\rho(x,t) = \Delta\rho K(x - vt) + \rho_L$. □

We have considered rather a simple example of a system in conservative form and shown that such systems have discontinuous solutions and solutions in the form of delta waves. It is interesting to note that Colombeau's theory of generalized functions has its important applications in the theory of systems in nonconservative form: from the point of view of distribution theory such systems were studied by Le Floch and his collaborators [65, 112–114], and in algebras of generalized functions nonconservative systems were studied by Colombeau and his collaborators [24, 25, 48, 59, 60–62] and Oberguggenberger [155].

Bibliographical notes.

Here we list some recent papers (known to the author) not mentioned in the body of this paper, which contribute to Colombeau's theory and related theories of generalized functions and their applications.

Colombeau's theory of generalized functions: Aragona [6, 7], Aragona and Colombeau [9, 10], Biagioni and Colombeau [17–19], Colombeau [46], Colombeau and Galé [49, 50], Colombeau and Heibig [51], Kiselman [106], Pilipović [160].

Generalized functions and nonstandard analysis: Li Bang-He [116, 117], Li Bang-He and Li Yaqing [118–120], Li Yaqing [123], Oberguggenberger [150, 156], Todorov [193–196].

Partial differential equations of evolution type: Biagioni [16], Colombeau [40], Colombeau and Heibig [52], Colombeau, Heibig, and Oberguggenberger [53–55], Heibig and Moussaoui [86], Colombeau and Langlais [56, 57], Langlais [110], Rosinger [170, 172], Zhao Baoheng [207].

Hyperbolic systems: Biagioni [14], Colombeau and Oberguggenberger [63], Colombeau [44], Gramchev [82, 83], Lafon and Oberguggenberger [109], Oberguggenberger [146–149, 151–154], Shelkovich [180].

Conservation laws: Colombeau and Oberguggenberger [64], Gramchev [85], Oberguggenberger and Wang [158].

Ordinary differential equations: Colombeau [37, 38] Egorov [71], Kim [105], Ligęza [121, 122], Radyno and Ngo Fu Tkhan' [161], Radyno and Nguen Hoĭ Ngia [163], Rubel [173–175].

Colombeau type algebras of generalized functions: Antonevich and Radyno [4], Gramchev [83], Oberguggenberger [157], Takači [188], Takači and Takači [189].

Products of distributions: Lysik [128], Kadłubowska and Wawak [97], Keller [104], Embacher, Grübl, and Oberguggenberger [74], Jelínek [96], Ivanov [93], Christov and Damyanov [32], Vinokurov [200], Wawak [203].

Fourier transform and convolution in algebras of generalized functions: Colombeau [37, 39], Nedeljkov and Pilipovič [143], Nedeljkov [142], Radyno, Ngo Fu Tkhan', and Sabra Ramadan [162].

Periodic generalized functions: Valmorin [198, 199].

Numerical methods: Adamczewski, Colombeau, and Le Roux [1], Barka, Colombeau, and Perrot [11], Colombeau, Laurens, and Perrot [58], Laurens [111].

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